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Article
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Cogent Economics & Finance

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Simulating the market coefficient of relative risk aversion

Samih Antoine Azar1* and Vera Karaguezian-Haddad1

Abstract: In this paper, expected utility, defined by a Taylor series expansion around expected wealth, is maximized. The coefficient of relative risk aversion (CRRA) that is commensurate with a 100% investment in the risky asset is simulated. The following parameters are varied: the riskless return, the market standard deviation, the market stock premium, and the skewness and the kurtosis of the risky return. Both the high extremes and the low extremes are considered. With these figures, the upper bound of the market CRRA is 3.021 and the lower bound is 0.466. Log utility, which corresponds to a CRRA of 1, is not excluded.

Subjects: Finance; Insurance; Investment & Securities

Keywords: relative risk aversion; expected utility maximization; Taylor series expansion; 100% investment in the risky asset; normal distribution; skewness; kurtosis; simulation

JEL classifications: D81; G11; C15

1. Introduction

The purpose of this paper is to estimate the market coefficient of relative risk aversion (CRRA). The market CRRA is defined as the CRRA that corresponds to a 100% investment in the risky asset, which is chosen to be the stock market portfolio. In many capital asset pricing theories, like the CAPM, this portfolio is nothing else but the tangency portfolio, at which the average risk is borne, and from which the average return is earned. In the CAPM, this average risk is the market systematic risk, and the average return is the return on an average-risk capital asset. The market CRRA measures the
amount of risk aversion needed to invest in the whole market, and not only in the financial market, notwithstanding Roll’s critique (Roll, 1977). It can be said that it is the average risk aversion of the whole economy in addition to that of the typical or representative investor. Because of that it is essential to identify a plausible range for its value. Another important issue, debated in the literature, is whether the market CRRA is the same across countries. Economic theory does not tell much about this. Since the statistical parameters of the financial markets of the United States are not materially different from the parameters of other developed countries, it is expected that the bounds, identified herein, apply also to these countries.

One concern is whether the CRRA is indeed constant. Friend and Blume (1975) are among the first to test for the constancy of this parameter. They find support for such constancy. Chiappori and Paiella (2011) confirm this constancy when studying the effect of wealth on the CRRA. Nonetheless, Das and Sarkar (2010) find strong evidence for time variability of the CRRA in their GARCH-M models.

Friend and Blume (1975) conclude that the estimate of the CRRA is likely to exceed 2, while the 95% confidence interval obtained from Pindyck’s study on risk aversion, Pindyck (1988), puts the CRRA in a range between 1.57 and 5.32. French, Schwert, and Stambaugh (1987) obtain an estimate of 2.41 for the CRRA, based on a relation due to Merton (1980), and which is:

$$\frac{E_t - R_f}{\sigma^2} = C$$

where $R_m$ is the stock market return, $R_f$ is the risk-free return, $\sigma^2$ is the stock market variance, $C$ is the CRRA, and $E_{t-1}$ is the expectation operator from the previous period. Azar (2006) finds an average market CRRA of 4.5. Finally, Tödter (2008), by bootstrapping the actual data for the stock market returns, finds a mean CRRA of 3.51 with a standard deviation of 1.41. It is unclear whether this standard deviation should be adjusted to obtain a standard error or not. Nonetheless, Tödter (2008) reports that the range of the CRRA that is between the 2.5% percentile and the 97.5% percentile is from 1.4 to 7.1. Again, it is unclear whether this range is for the individual CRRA or for the mean CRRA.

However, there is an ambiguity about the CRRA of the average consumer, and that of the market portfolio. For example, Chetty (2006) and Leonard (2012) calculate the average CRRA of consumers by using labor supply elasticities. The average consumer CRRA is not necessarily equal to the market CRRA since many consumers do not hold investments in the risky asset or in the market portfolio. The estimates and the ranges of the CRRA in Chetty (2006) and Leonard (2012) are surprisingly lower than those herein although they contain individuals who do not invest in the stock market and are expected to have high risk aversion.

In a related field, the elasticity of marginal utility of consumption is a crucial input in the determination of the social discount rate. This elasticity is nothing else but the CRRA. Evans (2004a) estimates a CRRA of 1.6 for the United Kingdom and an estimate of 1.3–1.4 for France (Evans, 2004b). In Evans and Sezer (2004), the estimates are between 1.25 and 1.6 for six major countries. Azar (2007) finds evidence for log utility in a dividend-CCAPM model, at least for the period posterior to the year 1938. Log utility has a CRRA of 1.

The procedure followed in this paper assumes a riskless asset and one risky asset, the latter being the market portfolio. The utility has a power functional form. The analysis starts by approximating expected utility by a Taylor series expansion evaluated around expected wealth. Then it is shown that maximizing expected utility does not depend on initial wealth. This allows the analysis to take into consideration the first four statistical moments of the distribution of the risky return: the average, the standard deviation, the skewness, and the kurtosis. In general, the actual skewness and kurtosis tend to reduce the demand for the risky asset. Therefore, in order to hold the market portfolio, risk aversion must decrease relative to the case when a normal distribution is the reference. Besides taking stock market data, and accommodating higher statistical moments for the risky return, the paper
contributes to the literature by simulating values for three fundamental parameters, the riskless return, the market risk premium, and the volatility of the risky asset, and assessing the impact of these changes on the market CRRA. Taking into account lower and upper extremes, the range of the market CRRA is estimated to lie between 0.466 and 3.021. Log utility, with a CRRA of 1, is not excluded.

The paper concludes that these results show that holding the risky market portfolio requires a significant amount of risk, even in high risk circumstances. This finding contributes to the literature by simulating values for three fundamental parameters, the riskless return, the market risk premium, and the volatility of the risky asset, and assessing the impact of these changes on the market CRRA. Taking into account lower and upper extremes, the range of the market CRRA is estimated to lie between 0.466 and 3.021. Log utility, with a CRRA of 1, is not excluded.

2. The theory

The purpose is to maximize expected utility $E(U)$, where $E(.)$ is the expectation operator, $U(.)$ is the utility function, $W_0$ is the initial wealth, $\alpha$ is the share of wealth invested in the risky asset, $\bar{r}$ is the risky return, $r_f$ is the risk-free return, $\gamma$ is the constant CRRA, and $\alpha^*$ is the optimal share of wealth:

$$\alpha^* \in \arg \max \quad E(U) \quad \text{where} \quad U = \frac{(W_0(1 + rf + \alpha (\bar{r} - r_f)))^{1-\gamma}}{1-\gamma}$$

Equation 2 assumes that there exist only two financial assets: a riskless asset with a fixed return $r_f$ and a risky asset, with a stochastic return $\bar{r}$. A Taylor series expansion of Equation 2 around expected wealth, $E(\bar{W})$, with higher order elements omitted, produces the following (Fabozzi, Focardi, & Kolm, 2006, pp. 134–136; Jondeau & Rockinger, 2006, pp. 33–34):

$$E(U) = E \left( \frac{U(E(\bar{W})) + (\bar{W} - E(\bar{W})) U'(E(\bar{W})) + \frac{(\bar{W} - E(\bar{W}))^2 U''(E(\bar{W}))}{2} + \frac{(\bar{W} - E(\bar{W}))^3 U'''(E(\bar{W}))}{2 \times 3} + \frac{(\bar{W} - E(\bar{W}))^4 U''''(E(\bar{W}))}{2 \times 3 \times 4} }{2} \right)$$

where $U'$, $U''$, $U'''$, and $U''''$ are, respectively, the first, the second, the third, and the fourth derivatives of $U(.)$ with respect to $\alpha$, and where:

$$\bar{W} = W_0(1 + rf + \alpha (\bar{r} - r_f)), E(\bar{W}) = W_0(1 + rf + \alpha (\bar{r} - r_f))$$

In turn the following relations hold:

$$E((\bar{W} - E(\bar{W}))^2) = W_0^2 \sigma^2 E(\bar{r} - E(\bar{r}))^2 = W_0^2 \sigma^2$$

$$E((\bar{W} - E(\bar{W}))^3) = W_0^2 \sigma^2 E(\bar{r} - E(\bar{r}))^3 = W_0^2 \sigma^2 \sigma_1$$

$$E((\bar{W} - E(\bar{W}))^4) = W_0^2 \sigma^4 E(\bar{r} - E(\bar{r}))^4 = W_0^2 \sigma^4 \sigma_2$$

where $\sigma_1$ and $\sigma_2$ are, respectively, the skewness and the kurtosis, and the following are true (Hill, Griffiths, & Lim, 2012):

$$\sigma^2 = E(\bar{r} - E(\bar{r}))^2$$

$$\theta_1 = \frac{E(\bar{r} - E(\bar{r}))^3}{\sigma^3}$$

$$\theta_2 = \frac{E(\bar{r} - E(\bar{r}))^4}{\sigma^4}$$
Replacing Equations 4–7 in Equation 3, noting that the expectation of the second term in Equation 3 is zero, using the utility functional in Equation 2 one obtains:

\[
E(U) = \left( 1 + \alpha \left( E(\tilde{r}) - rf \right) \right)^{1-\gamma} - \frac{\gamma \alpha^2 \sigma^2 W_0^3 \gamma^{-1} \left( 1 + \alpha \left( E(\tilde{r}) - rf \right) \right)^{1-\gamma}}{2} \]

Factoring out \( W_0^{1-\gamma} \) one gets:

\[
E(U) = W_0^{1-\gamma} \left( 1 + \alpha \left( E(\tilde{r}) - rf \right) \right)^{1-\gamma} - \frac{\gamma \alpha^2 \sigma^2 \left( 1 + \alpha \left( E(\tilde{r}) - rf \right) \right)^{1-\gamma}}{2} \frac{\gamma}{2 - \gamma} \theta_1 + \frac{2}{\gamma} \frac{3}{2 - \gamma} \theta_2
\]

Equation 10 is to be maximized relative to \( \alpha \). In this regard, the factor \( W_0^{1-\gamma} \) can be disregarded from the maximization. Therefore, the result to be maximized is independent of initial wealth, and is scale-preserving. The procedure is to fix \( rf \) (the risk-free rate), \( \sigma \) (the volatility), \( E(\tilde{r}) - rf \) (the equity premium), \( \theta_1 \) (the skewness), and \( \theta_2 \) (the kurtosis), and find, by trial and error, the CRRA (\( \theta \)) that corresponds to an \( \alpha = 1 \) by maximizing Equation 10 relative to \( \alpha \) using the solver command in Excel. The procedure is repeated for other values of \( rf \), \( \sigma \), \( E(\tilde{r}) - rf \), \( \theta_1 \), and \( \theta_2 \). An \( \alpha = 1 \) corresponds to a 100% share of wealth in the risky asset, i.e., an investment totally in the stock market portfolio, and the respective CRRA is the market CRRA.

3. The empirical results
According to Brealey, Myers, and Allen (2014) \( rf \), the riskless return, amounts to 3.9%, the standard deviation of a portfolio of common stocks \( \sigma \) is 20%, and the market premium, \( E(\tilde{r}) - rf \), is 7.3%. Other characteristics of the market portfolio are needed. For this, the stock market variable “Total Share Prices for All Shares for the United States” is selected from the website of the Federal Reserve Bank of Saint Louis. The data span the monthly period between January 1957 and December 2013. The series are logged and first differences are calculated to compute the continuously compounded growth rate. The sample size is 683 observations. The sample skewness is -1.123120 and the sample kurtosis is 7.891023. The Jarque–Bera normality test, Jarque and Bera (1980, 1981, 1987), rejects the null of normality at very low marginal significance levels, less than 0.000001. The Quandt–Andrews unknown breakpoint test, Andrews (1993) and Andrews and Ploberger (1994), fails to find any break point in the above series, although 546 breaks are compared. The minimum p-value for this test under the null of no breaks is 0.5395. Moreover, the Bai–Perron test, Bai (1997), and Bai and Perron (1998), fails to select any calendar breakpoint. This is evidence that the series is surprisingly homogeneous, despite the long time span of the data.

Table 1 presents the results of the simulations. The demand for the risky asset \( \alpha \) increases ceteris paribus with the risk-free return \( rf \), with the market risk premium \( E(\tilde{r}) - rf \), and decreases with the market volatility \( \sigma \). In addition, assuming a normal distribution increases also this demand for the risky asset. The actual negative skewness and the actual leptokurtic probability distribution jointly decrease this demand.

An interval range for \( rf \), the riskless return, is assumed to be between 0% and 8%, and an interval range for \( E(\tilde{r}) - rf \), the equity premium, is assumed to be between 3.3 and 11.3%. The latter range is approximately a 95% confidence interval with a critical t-statistic of 2.0 and a standard error of...
\[ \frac{0.2}{\sqrt{100}}, \text{ where 20\% is the standard deviation and 100 is the number of years of data points in the estimation of the equity premium. The interval range of the standard deviation is assumed between 17 and 23\%, based upon the lower and upper 2.5\% marginal significance levels of the } \chi^2 \text{ distribution with 100 years of degrees of freedom (see Keller, 2012).} \]

A market CRRA of 4.5 is obtained using the upper values of \( r_f \) and \( E(\bar{r}) - r_f \), and \( \sigma \) (see Table 1). This shows how overstated the results in Azar (2006) are. Moreover, with the average values of \( r_f, E(\bar{r}) - r_f \), and \( \sigma \), the market CRRA is much lower at 1.892. This means that the CRRA that matches the characteristics of the market portfolio is only 1.892, way below the required CRRA of 50 found in studies of the equity premium puzzle (Mehra & Prescott, 1985), where consumption characteristics are used.

When the actual skewness and kurtosis are assumed, the demand for the risky asset falls and hence the market CRRA at the average values of the riskless return \( r_f \), the equity premium \( E(\bar{r}) - r_f \), and the volatility \( \sigma \) become 1.385, very close to log utility which has a CRRA of 1. Risk aversion must decrease to account for a lower demand for the risky asset. Taking the upper range value for the riskless return \( r_f \) increases ceteris paribus the market CRRA to 1.447. Taking approximately two standard deviations above the mean for the equity premium, \( E(\bar{r}) - r_f \), increases ceteris paribus the market CRRA to 2.098. Taking the lower 2.5\% limit for the standard deviation decreases ceteris paribus the market CRRA to 1.925. Making the above three changes jointly increases the market CRRA to 3.021. This can be considered to be the upper bound of the market CRRA.

With a zero riskless return \( r_f \), the market CRRA changes to 2.099 ceteris paribus. With two standard deviations below the mean for the equity premium, \( E(\bar{r}) - r_f \), the market CRRA changes to a figure of 1.037. Taking the upper 2.5\% limit for the standard deviation decreases ceteris paribus the market CRRA to 0.660. Making the above three changes jointly decreases the market CRRA to 0.466. This can be considered to be the lower bound of the market CRRA.

Hence, the market CRRA has an upper bound of 3.021 and a lower bound of 0.466. These bounds assume extreme situations which are very unlikely to happen. Azar (2010) finds an upper bound of 3.74 and a lower bound of 3.01. It seems that these figures are overstated. This overstatement is due

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<th>( r_f )</th>
<th>( \sigma )</th>
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<th>( E(\bar{r}) )</th>
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perhaps because of assuming normal probability distributions in the Monte Carlo simulations in Azar (2010). When bootstrapping is undertaken in Azar (2010), the 95% confidence interval for the market CRRA is bounded between 1.61 and 4.90. Therefore, this letter finds much lower upper bounds and much inferior lower bounds.

In real life, there is a relation between the required return and the minimum risk acceptable for that return. The market volatility is 20%. A volatility of 17% would imply an 85% investment in the market risky asset. This would decrease the actual premium to 6.205% assuming the same riskless return of 3.9%. The required market CRRA for such a premium falls from 1.925 to 1.659 relative to the case where the premium remains the same.

Similarly, a volatility of 23% would imply a 115% investment in the market risky asset. This would increase the actual premium to 8.395% assuming the same return of 3.9% for the riskless asset. The required market CRRA for such a premium increases from 1.037 to 1.185 relative to the case where the premium does not change. Hence, in real life, where the magnitudes of the parameters co-vary together, the range between the upper bound and the lower bound is smaller.

4. Conclusion
This paper starts by maximizing expected utility, approximated by a Taylor series expansion around expected wealth, and calculates the CRRA that is commensurate with a 100% investment in the market portfolio. A constant CRRA is more in line with theoretical expectations than assuming a constant coefficient of absolute risk aversion, as in (Jondeau & Rockinger, 2006). The following parameters are changed: rf (the riskless return), σ (the market standard deviation), E(r) − rf (the market stock premium), θ (the skewness), and η (the kurtosis). The high extremes are rf = 8%, σ = 17%, and θ = 11.3%. The low extremes are rf = 0%, σ = 23%, and θ = 3.3%. With these figures, the upper bound of the market CRRA is 3.021 and the lower bound is 0.466. Log utility, which corresponds to a CRRA of 1, is not excluded. In addition, most estimates of the CRRA used in the public finance literature, especially in the determination of the social discount rate, lie within the above range. An avenue for future research is to undertake sensitivity analysis by varying the assumed skewness and the kurtosis of the risky return, despite the fact that the data series on the risky return is found to lack any calendar break.

Funding
The authors received no direct funding for this research.

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Citation information
Cite this article as: Simulating the market coefficient of relative risk aversion, S.A. Azar & V. Karaguezian-Haddad, Cogent Economics & Finance (2014), 2: 990742.

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