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Article

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How does the no-Ponzi game condition work in an optimal consumption problem?

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In an optimal consumption choice problem, in which households have assets yielding interest rates, it is difficult to guarantee the existence of a solution without some restrictions for the consumption space, if the assumed utility function is unbounded. In this article, we formally state how the no-Ponzi game condition is used to guarantee an existence of optimal solutions. Furthermore, we provide the condition in which a solution attains the finite intertemporal utility.

I. Introduction

Economists often assume that infinitely lived households behave such that they maximize their lifetime utility subject to an intertemporal budget constraint. However, in solving this maximization problem, there are cases when one needs to use certain techniques to guarantee the existence of a solution path if the assumed utility function is unbounded. In particular, this is the case when households have financial assets yielding an interest rate. This is because the domain of an objective function is not compact in the usual norm.

Stokey and Lucas (1989) describe how we confirm the existence of an optimal path in a capital accumulation problem in which the utility function is unbounded. In this case, the domain of an objective function can be bounded. However, in asset choice problems, this approach is difficult to apply. Boyd (1990) copes with the unboundedness using the specific norm for a domain. Later, this approach is made applicable to a dynamic programming technique by Duran (2000). For other approaches of bounded returns, see Alvarez and Stokey (1998), Streufert (1990), Le Van and Morhaim (2002), Rincon-Zapatero and Rodriguez-Palmero (2003) and Le Van and Vailakis (2005).

These works tend towards generality, but we limit our attention to financial asset choice problems. By doing this, we can show that the no-Ponzi game condition is sufficient for showing the existence of an optimal path under the usual assumptions imposed on a temporal utility function. In this article, we formally state how the no-Ponzi game condition is used to guarantee an existence of optimal solutions.

The rest of this article is organized as follows. In Section 2, we describe the model setting and state the main theorem. In Section 3, we apply our theorem to a problem with the Constant relative risk aversion (hereafter, CRRA) utility function. Finally, in Section 4, we conclude the article.

II. The Existence Theorem

Consider a sequence of consumption, ${}_0c = (c_0, c_1, \dots) \in \mathbb{R}_+^\infty$, where \mathbb{R} denotes the set of real numbers plus infinity and \mathbb{R}_+ represents non-negative part of \mathbb{R} . We define a real-valued intertemporal utility function $U : \mathbb{R}_+^\infty \rightarrow \mathbb{R}$ as follows:

$$U({}_0c) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where u represents a real-valued temporal utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ and β is a discount factor. The maximization problem we are interested in is as follows:

$$\sup U({}_0c), \quad (P)$$

$$\text{s.t. } A_{t+1} \leq R_t A_t + w_t - c_t,$$

$$\lim_{t \rightarrow \infty} \frac{A_t}{\prod_{i=1}^t R_i} = 0,$$

$$\text{given } \{R_t\}, \{w_t\}, \text{ and } A_0$$

where A_t represents a financial asset yielding interest rate denoted by R_t and w_t denotes income. The third of the constraints is the no-Ponzi game condition. We assume that the sequences $\{R_t\}$ and $\{w_t\}$ are bounded.

In a maximization problem like P given above, it is often difficult to guarantee the existence of a solution. However, with the no-Ponzi game condition and a suitable norm, this becomes an easy task.

Our approach is to first construct infeasible consumption sequences that include the feasible sequences. To do so, we define the following notation: $\underline{R} = \inf \{R_t\}$; $\bar{R} = \sup \{R_t\}$; $\bar{w} = \sup \{w_t\}$. The no-Ponzi game condition, $\lim_{t \rightarrow \infty} A_t / (\prod_{i=1}^t R_i) = 0$, and the

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budget constraint yield¹

$$\begin{aligned} c_0 + \frac{c_1}{R_1} + \dots &\leq w_0 + \frac{w_1}{R_1} + \frac{w_2}{R_1 R_2} + \dots + R_0 A_0, \\ &\leq \bar{w} + \frac{\bar{w}}{\underline{R}} + \frac{\bar{w}}{(\underline{R})^2} + \dots + R_0 A_0 \end{aligned}$$

This inequality implies that the maximum consumption at the period 0 is constant and finite: $c_0 \leq M$, where $M = \bar{w} + \bar{w}/\underline{R} + \dots + R_0 A_0$. Next, we consider the maximum consumption level at period 1. It can be obtained when the consumption at period 0 is 0. Therefore, we have $c_1 \leq \bar{R}M$. The maximum consumption level at period 2 can be constructed in a similar manner: $c_2 \leq (\bar{R})^2 M$. Continuing this procedure, we can confirm that for any t , $c_t \leq (\bar{R})^t M$.² Thus, the no-Ponzi game condition yields the maximum consumption plan denoted by ${}_0\bar{c}$, which is infeasible: ${}_0\bar{c} = (M, \bar{R}M, \dots, (\bar{R})^t M, \dots)$.

To facilitate our analysis, we employ the following sup-norm for a consumption sequence ${}_0^t c = (c_0, c_1, \dots, c_t)$, as in Boyd (1990):

$${}_0^t c = \sup_{s \in \{0, 1, \dots, t\}} \left| \frac{c_s}{(\bar{R})^s} \right|$$

Note that ${}_0\bar{c}$ and therefore, ${}_0 c$ is finite for any plan of feasible consumption plans. Consider the set of the feasible consumption plans denoted by C . We denote by l_∞^d the consumption space that is finite for any ${}_0 c$ in the norm of $\lim_{t \rightarrow \infty} {}_0^t c$. Since C is not compact in the sup-norm,³ we consider the weak topology that is generated by the set of all bounded linear functionals on l_∞^d .

Since $(\bar{R})^t M \rightarrow \infty$, we have the possibility of $U({}_0 c) = \infty$. To cope with this in the following main theorem, we define by $U({}_0^t c)$ the partial sums of the discounted temporal utility function denoted as follows:

$$U({}_0^t c) = \sum_{s=0}^t \beta^s u(c_s)$$

In the following, a consumption plan ${}_0 c$ is optimal if $U({}_0 c) \geq U({}_0 c')$ for all ${}_0 c' \in C$. Here, we state the main theorem.

THEOREM *Suppose that the no-Ponzi game condition holds. If $u(c_t)$ is a non-decreasing weak upper-semi continuous function, then there exists an optimal consumption plan for the problem P .*

Proof If $U({}_0 c) = -\infty$ for every ${}_0 c \in C$, then those plans are trivially optimal. Therefore, consider that we have elements such that $U({}_0 c^+) > -\infty$ for some ${}_0 c^+ \in C$. Denote by C^* the set of feasible consumption plans that exclude a plan attaining $U({}_0 c) = -\infty$. Furthermore, to facilitate the proof, we define asymptotic cones as follows: for any ${}_0 c \in C^*$

$$U_\infty({}_0 c) = \left\{ d \in \mathbb{R}_+ \mid \lim_{t \rightarrow \infty} \left| \frac{U({}_0^t c)}{a_t} \right| = d \text{ for some } |a_t| \rightarrow +\infty \right\}$$

If one considers the case of a'' with $\gamma > 1$, then $\lim_{t \rightarrow \infty} |U({}_0^t c)/a''|$ for any ${}_0 c \in C^*$. Therefore, $U_\infty({}_0 c)$ is non-empty for

¹ Note that $\lim_{t \rightarrow \infty} A_t / (\prod_{i=1}^t R_i) \neq 0$ is allowed if it is finite.

² We assume $\underline{R} > 1$. If $\underline{R} \leq 1$, then we can say $c_t \leq M$ for any t .

³ See Majumdar (1975).

any ${}_0 c \in C^*$. Note that we have two cases:

- (i) $U_\infty({}_0 c) = \{0\}$ for any ${}_0 c \in C^*$;
- (ii) $U_\infty({}_0 c') \neq \{0\}$ for some ${}_0 c' \in C^*$, and $U_\infty({}_0 c) = \{0\}$ for any ${}_0 c \in C^*/\{0 c'\}$.⁴

In case (i), since $U({}_0 c)$ is bounded, $\lim_{t \rightarrow \infty} \beta^t |u(c_t)| = 0$ for any $c_t \in C^*$. Since $|U({}_0 c) - U({}_0^t c)| = \sum_{s=t+1}^{\infty} \beta^s |u(c_s)| \rightarrow 0$ for any $c_t \in C^*$, $U({}_0^t c)$ uniformly converges to $U({}_0 c)$. Therefore, $U({}_0 c)$ is weak upper-semi continuous. With this upper-semi continuity, there exists a neighbourhood $\{0 c^\varepsilon\}$ of ${}_0 c^+$ such that

$$\{0 c^\varepsilon\} = \{0 c \in C \mid U({}_0 c^+) + \varepsilon \geq U({}_0 c)\} \text{ for any } \varepsilon > 0.$$

Here, define the following set

$$C^\varepsilon = \{0 c \in \mathbb{R}_+^\infty \mid U({}_0 c^\varepsilon) \leq U({}_0 c) \text{ and } 0 c \in C\}$$

The compactness of C and the upper-semi continuity of U imply that C^ε is compact. The extreme-value theorem guarantees the existence of an optimal consumption plan.

In case (ii), there exists consumption plans $\{0 c'\}$ such that $\lim_{t \rightarrow \infty} U({}_0^t c')/|a_t|$ becomes a convergent sequence since u is non-decreasing; for example, $a'_t = \beta^t u(c'_t)$. Since $U({}_0 c')$ is infinite, $0 c'$ is trivially the optimal plan in the problem P .

If a solution attains infinite utility, it would be meaningless in terms of a policy analysis. Hence, our interest is in case (i) in the above proof. Whether the optimization problem is case (i) or case (ii) can be checked by confirming $\beta^t u((\bar{R})^t M) \rightarrow 0$. If it is satisfied, then we are in case (i). Now we provide the following useful proposition. ■

PROPOSITION *Suppose that the no-Ponzi game condition holds. If $u(c_t)$ is a non-decreasing weak upper-semi continuous function and $\beta^t u((\bar{R})^t M) \rightarrow 0$, then there exists an optimal consumption plan for the problem P and the solution attains finite utility.*

III. Examples

In this section, we introduce an example in which the assumption of the above proposition is satisfied. Suppose that a temporal utility function is the CRRA type, $u(c_t) = (c_t)^{1-\delta}/(1-\delta)$. In this case, we can easily check when the assumption of the above proposition is satisfied:

Case (I): $\delta = 1$ (that is, $u(c_t) = \log c_t$)

In this case, since $\beta^t \log(\bar{R})^t M \rightarrow 0$, we have

$$U({}_0\bar{c}) = \frac{\log M}{1-\beta} + \left(\frac{\beta}{1-\beta} \right)^2 \log \bar{R} < \infty$$

Case (II): $\delta \neq 1$

If $\beta(\bar{R})^{1-\delta} < 1$, then $\beta^t [(\bar{R})^t M]^{1-\delta}/(1-\delta) \rightarrow 0$.

In this case,

$$U({}_0\bar{c}) = \left(\frac{M^{1-\delta}}{1-\delta} \right) \left(\frac{1}{1-\beta(\bar{R})^{1-\delta}} \right) < \infty$$

Note that if $\beta < 1$ and $\bar{R} > 1$, then $\beta(\bar{R})^{1-\delta} < 1$ for any $\delta > 0$.

⁴ If $U_\infty({}_0 c') \neq \{0\}$, then $U({}_0 c') = \infty$.

IV. Conclusion

In this article, we show the sufficiency of the no-Ponzi game condition for guaranteeing the existence of optimal solutions for asset choice problems, using the specific norm. Further, we provide a sufficient condition in which a solution attains the finite intertemporal utility.

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