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Combinatorial Properties of Strength Groups in Round Robin Tournaments

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No. 611

Combinatorial Properties of Strength Groups in Round Robin Tournaments

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Abstract

A single round robin tournament (RRT) consists of a set T of n teams (n even) and a set P of $n - 1$ periods. The teams have to be scheduled such that each team plays exactly once against each other team and such that each team plays exactly once per period. In order to establish fairness among teams we consider a partition of teams into strength groups. Then, the goal is to avoid a team playing against extremely weak or extremely strong teams in consecutive periods. We propose two concepts ensuring different degrees of fairness. One question arising here is whether a single RRT exists for a given number of teams n and a given partition of the set of teams into strength groups or not. In this paper we examine this question. Furthermore, we analyse the computational complexity of cost minimization problems in the presence of strength group requirements.

Keywords: Round robin tournaments, fairness, partition of teams, strength groups

1 Introduction

A single round robin tournament (RRT) based on a set T ($|T| = n$, n even) of teams is a schedule of matches where a match is a competition between two teams such that

- each team $i \in T$ plays against each other team $j \in T$, $j \neq i$, exactly once and such that
- each team plays exactly once per period.

This structure results in a set P ($|P| = n - 1$) of periods and can be arranged for each even number of teams. In addition to other issues fairness according to the strength of opponents played by a specific team in consecutive periods is one of the major requirements in real world sports leagues as outlined in [2]. As proposed in [1] we consider the set S of strength groups as a partition $S = \{S_0, \dots, S_{|S|-1}\}$ of T . We restrict ourselves to the case that all strength groups having identical size, hence $|S_s| = \frac{n}{|S|}$, $s \in \{0, \dots, |S| - 1\}$. Without loss of generality we let $S_s = \{s \frac{n}{|S|} + k \mid k \in \{0, \dots, \frac{n}{|S|} - 1\}\}$ for each $s \in \{0, \dots, |S| - 1\}$.

In the following we introduce two concepts establishing different degrees of fairness among all teams. Obviously, a team playing twice against teams of the same strength group in two consecutive periods is unfair, especially when both are extremely strong or extremely weak, respectively. Therefore, we propose a class of single RRTs being fair according to the strength of opponents in consecutive periods.

Definition 1. *A single RRT where no team plays against teams of the same strength group in two consecutive periods is called group-changing.*

Although group-changing single RRTs guarantee a certain degree of fairness we can reasonably strengthen it. Note that in group-changing single RRTs with $|S| = 4$ it is possible that a specific team exclusively plays against teams of two strength groups (in alternating order) in the first half of the tournament and against teams of the remaining two strength groups (in alternating order) in the second half. This can be prevented by considering the class of single RRTs defined in the following.

Definition 2. *A single RRT where no team plays more than once against teams of the same strength group within $|S|$ consecutive periods is called group-balanced.*

A natural question arising here is whether a group-changing single RRT and a group-balanced single RRT, respectively, can be arranged for each n and $|S|$ where $\frac{n}{|S|}$ is integer.

Establishing fairness in RRTs has been the topic of several papers. Strength groups are introduced in [1] and [2]. Carry-over effects among teams are considered in [12] and [13]. Another major concern when considering fairness relates to breaks – a topic which has been analysed in [7], [9], [10], and [11], for example.

Before we address in sections 3 and 4 the question given above we introduce several aspects concerning 1-factorization of complete graphs in section 2. In section 5 we introduce two optimization problems and proof their complexity, respectively. Finally, some conclusive remarks and an outlook to further research are given in section 6.

2 Factorizations

First, for the sake of convenience we introduce some short notations. We denote the strength group of team i by $S(i)$. The opponent of team i in period p is denoted by $o_{i,p}$.

Next, we focus on graph theoretical aspects. A 1-factor of a graph $G := (V, E)$ is a set of edges $E' \subseteq E$ such that each node $i \in V$ is incident to exactly one $e \in E'$. A 1-factorization of G is a partition of its edges into 1-factors. For details we refer the reader to [14], for example.

A near-1-factor of G is a set of edges $E' \subseteq E$ such that each node but one is incident to exactly one $e \in E'$. This node is incident to no $e \in E'$. A near-1-factorization of G is a partition of its edges into near-1-factors.

An ordered 1-factorization is a 1-factorization where the 1-factors are ordered. An ordered near-1-factorization is defined analogously.

2.1 Ordered 1-Factorization of $K_{k,k}$

The complete balanced bipartite graph $K_{k,k}$, $k \in \mathbb{N}$, is well known to have an ordered 1-factorization F^{bip} , as proposed for example in [5]. Let $V_0 := \{i \mid i \in \{0, \dots, k-1\}\}$ and $V_1 := \{i \mid i \in \{k, \dots, 2k-1\}\}$ be the partition of V under consideration. Then

$$F^{bip} = \{F_0^{bip}, \dots, F_{k-1}^{bip}\}, \text{ where}$$

$$F_l^{bip} = \{[m, k + (m + l) \bmod k] \mid m \in \{0, \dots, k-1\}\} \forall l \in \{0, \dots, k-1\}.$$

Here, $[i, j]$, $i, j \in V$, denotes the edge incident to i and j . Note that differences $i - j$ and $j - i$ are not equal to 1 in F_0^{bip} unless $k = 1$. An example with $k = 4$ is given in figure 1. We emphasize that in $F_{\frac{k}{2}}^{bip}$ no edge $[m, k + n]$, $m, n < \frac{k}{2}$, and no edge $[m, k + n]$, $m, n \geq \frac{k}{2}$, is contained if k is even.

2.2 Ordered 1-Factorizations of K_k

It is well known that there is an ordered 1-factorization consisting of $k - 1$ 1-factors of each K_k , k even. The most popular ordered 1-factorization of K_k , k even, might be the canonical one as defined in the following (all indices being taken modulo $k - 1$):

$$F^c = \{F_0^c, \dots, F_{k-2}^c\}, \text{ where}$$

$$F_l^c = [l, k-1] \cup \left\{ [l-m, l+m] \mid m \in \left\{ 1, \dots, \frac{k}{2} - 1 \right\} \right\} \forall l \in \{0, \dots, k-2\}.$$

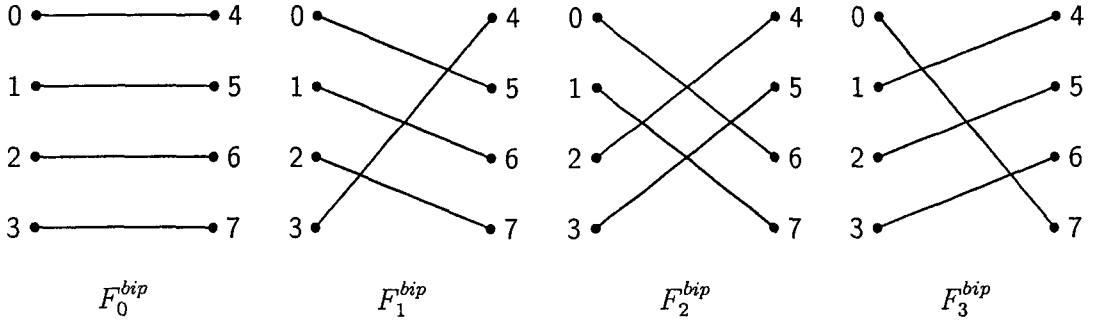


Figure 1: 1-Factorization of $K_{4,4}$

If not stated otherwise we refer to F^c as 1-factorization of K_k in the remainder. Note that we can force $F_{\frac{k}{2}-1}^c$ to contain each edge of form $[i, i+1]$, i even, by a simple mapping $\sigma : V \rightarrow V$. An illustration of the canonical 1-factorization of K_6 is given in figure 2.

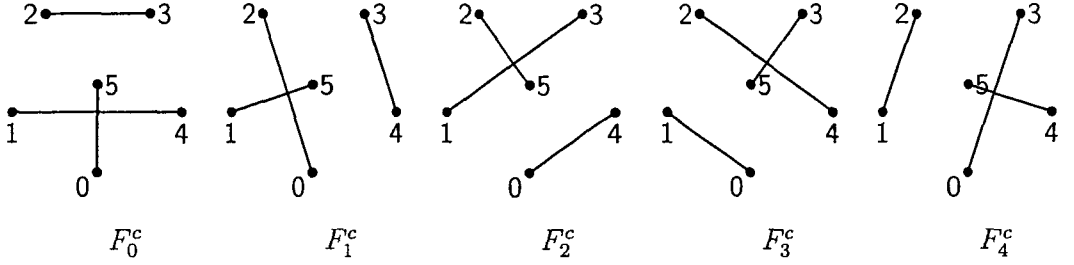


Figure 2: 1-Factorization of K_6

If k is odd we can construct a near-1-factorization consisting of k near-1-factors by simply letting the node matched with k in F^c of K_{k+1} be unmatched in each F_l^c , $l \in \{0, \dots, k-1\}$.

$$nF^c = \{aF_0^c, \dots, aF_{k-1}^c\}, \text{ where}$$

$$nF_l^c = \left\{ [l-m, l+m] \mid m \in \left\{ 1, \dots, \frac{k}{2} - 1 \right\} \right\} \forall l \in \{0, \dots, k-1\}.$$

Since F^c and nF^c are so called started induced 1-factorization and near-1-factorization, respectively, each number in $\{1, \dots, 2k-2\}$ can be found as a difference $i-j$ or $j-i$ of an edge $[i, j]$ in each 1-factor in F^c . Next, we introduce 1-factorizations and near-1-factorizations for K_k , $k > 3$, where not each of those numbers is contained in each 1-factor.

The binary 1-factorization as proposed in [5] can be constructed for K_{2k} if k even. Let $V_0 := \{i \mid i \in \{0, \dots, k-1\}\}$ and $V_1 := \{i \mid i \in \{k, \dots, 2k-1\}\}$ be a partition of V . Then, 1-factor $F_l^{b,e}$ is set to F_l^{bip} as introduced in section 2.1 for each $l \in \{0, \dots, k-1\}$. Hence, each edge between V_0 and V_1 is contained in 1-factors $F_0^{b,e}$ to $F_{k-1}^{b,e}$. Additionally, 1-factors $F_k^{b,e}$ to $F_{2k-2}^{b,e}$ are constructed as 1-factorization according to V_0 and V_1 , respectively. Then,

1-factorization $F^{b,e}$ is defined as follows:

$$F^{b,e} = \{F_0^{b,e}, \dots, F_{2k-2}^{b,e}\}, \text{ where}$$

$$F_l^{b,e} = \{[m, k + (m + l) \bmod k] \mid m \in \{0, \dots, k-1\}\} \forall l \in \{0, \dots, k-1\},$$

$$F_l^{b,e} = [l, k-1] \cup [l+k, 2k-1] \cup \\ \{[(l-k-m) \bmod k, (l-k+m) \bmod k] \mid m \in \{0, \dots, k-1\}\} \cup \\ \{[k+(l-k-m) \bmod k, k+(l-k+m) \bmod k] \mid m \in \{0, \dots, k-1\}\} \\ \forall l \in \{k, \dots, 2k-2\}.$$

Since $F_0^{b,e}$ is based on F_0^{bip} none of the differences $i-j$ or $j-i$ is equal to 1 if $k > 1$. Figure 3 represents the binary 1-factorization of K_8 .

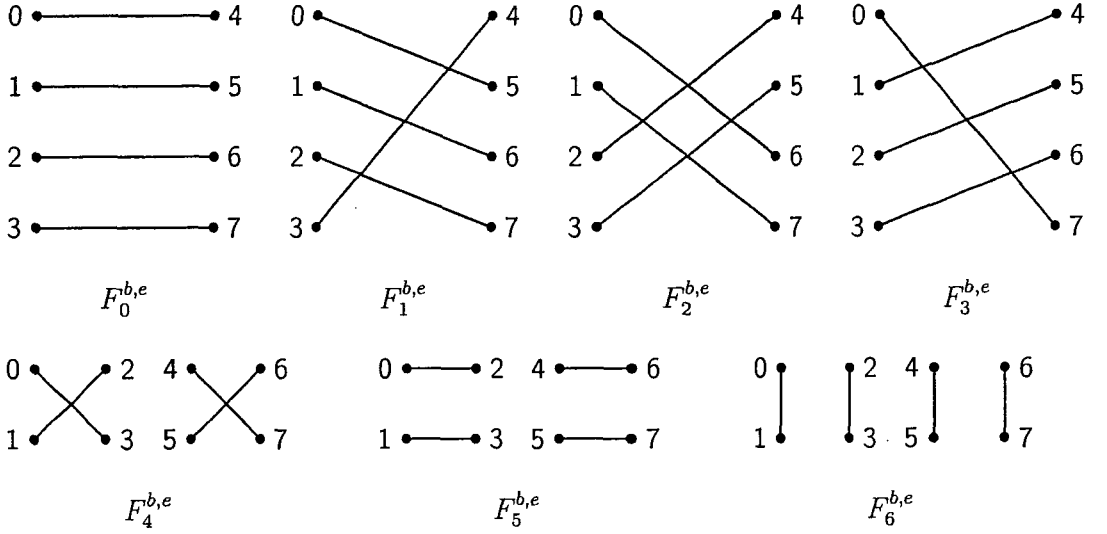


Figure 3: Binary 1-Factorization of K_8

We extend the binary 1-factorization of K_{2k} to the case where k is odd. 1-factors $F_0^{b,o}$ to $F_{k-2}^{b,o}$ are defined as 1-factors F_0^{bip} to F_{k-2}^{bip} according to V_0 and V_1 . 1-factors $F_{k-1}^{b,o}$ to $F_{2k-2}^{b,o}$ are composed of near-1-factors according to V_0 and V_1 . Note that one node of both, V_0 and V_1 , is unmatched. Those nodes are matched and form the edges between V_0 and V_1 missing from $F_0^{b,o}$ to $F_{k-2}^{b,o}$.

$$F^{b,o} = \{F_0^{b,o}, \dots, F_{2k-2}^{b,o}\}, \text{ where}$$

$$F_l^{b,o} = \{[m, k + (m + l) \bmod k] \mid m \in \{0, \dots, k-1\}\} \forall l \in \{0, \dots, k-2\}$$

$$F_l^{b,o} = [l-k+1, k+(l-k) \bmod k] \cup \\ \{[(l-k+1-m) \bmod k, (l-k+1+m) \bmod k] \mid m \in \{0, \dots, k-1\}\} \cup \\ \{[k+(l-k-m) \bmod k, k+(l-k+m) \bmod k] \mid m \in \{0, \dots, k-1\}\} \\ \forall l \in \{k-1, \dots, 2k-2\}$$

Theorem 1. $F^{b,o}$ is a 1-factorization of K_{2k} , k odd.

Proof. We show that $F_l^{b,o}$ is a 1-factor for each $l \in \{0, \dots, 2k-2\}$ and that $F^{b,o}$ is a partition of edges of K_{2k} .

Obviously, $F_l^{b,o}$ is a 1-factor for $l \in \{0, \dots, k-2\}$ since it is defined by 1-factors of a partition of the set of edges. $F_l^{b,o}$, $l \in \{0, \dots, k-2\}$, is defined by near-1-factors of both, V_0 and V_1 . Therefore all but nodes $l-k+1$ and $k+(l-k) \bmod (k-1)$ are matched implicitly. These two nodes are matched explicitly.

Each edge between V_0 and V_1 but $[m, (m-1) \bmod (k-1)]$, $m \in \{0, \dots, k-1\}$ is contained exactly once in 1-factors $F_l^{b,o}$, $l \in \{0, \dots, k-2\}$. Edges $[m, (m-1) \bmod (k-1)]$, $m \in \{0, \dots, k-1\}$ are added to both near-1-factors of V_0 and V_1 in $F_l^{b,o}$, $l \in \{k-1, \dots, 2k-2\}$. Edges within V_0 and V_1 , respectively, are contained exactly once in $F_l^{b,o}$, $l \in \{k-1, \dots, 2k-2\}$ by definition of near-1-factors. □

Again, $F_0^{b,o}$ is based on F_0^{bip} and, therefore, none of the differences $i-j$ or $j-i$ is equal to 1 if $k > 1$. Figure 4 illustrates the binary 1-factorization with $k = 5$.

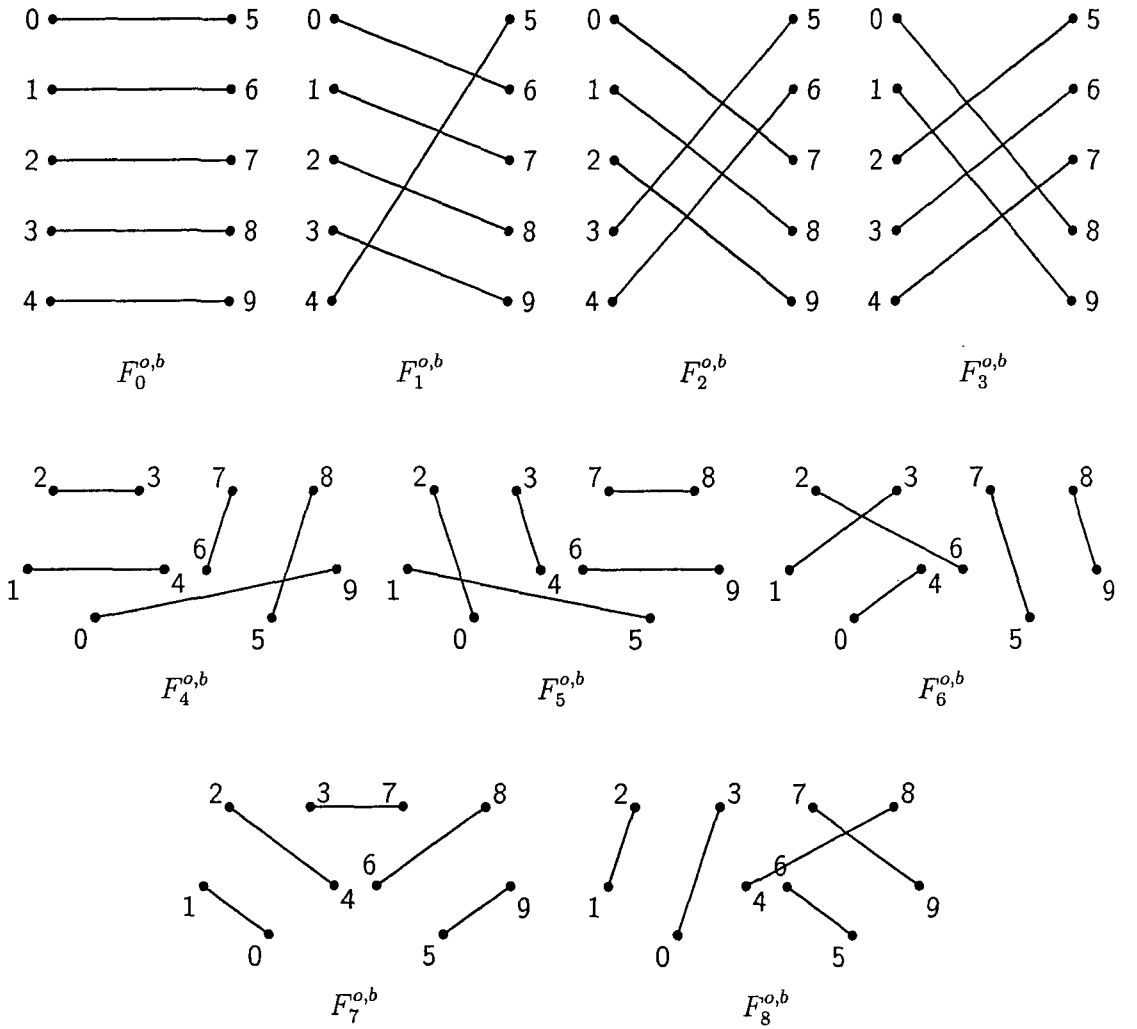


Figure 4: Binary 1-Factorization of K_{10}

We can construct near-1-factorizations according to $F^{b,e}$ and $F^{b,o}$ of K_{4k-1} and K_{4k+1} by simply adding a dummy node, constructing the corresponding 1-factorization of K_{4k} and K_{4k+2} , and considering each node matched with the dummy node as unmatched. Again, no difference in the first near-1-factor is equal to 1 if and only if $k > 3$.

2.3 Ordered symmetric 2-Factorization of $2K_{2k+1}$

A 2-factor of a graph $G := (V, E)$ is a set of edges $E' \subseteq E$ such that each node $i \in V$ is incident to exactly two $e, e' \in E'$, $e \neq e'$. A 2-factorization of G is a partition of its edges into 2-factors (see [8] for details). An ordered 2-factorization is a 2-factorization having its 2-factors ordered.

The complete multi-graph $2K_n$ is a graph on $|V| = n$ nodes having exactly two edges incident with each pair of edges. K_n , n odd, is known to have a 2-factorization as outlined in [4]. Hence, $2K_n$, n odd, has one, as well. An oriented 2-factorization is a 2-factorization where each edge $e \in E$ is given an orientation.

Definition 3. A symmetric 2-factorization of $2K_k$, k odd, is an oriented 2-factorization where edges corresponding to the same pair of nodes are given opposite orientations.

We can construct a symmetric 2-factorization of $2K_k$, k odd, as follows:

$$\begin{aligned} 2F &= \{2F_0, \dots, 2F_{k-2}\}, \text{ where} \\ 2F_l &= \{[m, (m+1+l) \bmod k]_0 \mid m \in \{0, \dots, \frac{k-1}{2} - 1\}\}, \\ 2F_{l+\frac{k-1}{2}} &= \{[(m+1+l) \bmod k, m]_1 \mid m \in \{0, \dots, \frac{k-1}{2} - 1\}\}. \end{aligned}$$

Here, $[i, j]_k$, $i, j \in V$, $k \in \{0, 1\}$, identifies the edge between nodes i and j and having index k being oriented $i \rightarrow j$.

Theorem 2. $2F$ is a symmetric 2-factorization of $2K_k$, k odd.

Proof. We show that $\{2F_0, \dots, 2F_{\frac{k-1}{2}-1}\}$ forms a 2-factorization of $G' := (V, \{[i, j]_0 \mid i, j \in V, i < j\})$.

Obviously, each node has degree equal to two in 2-factor $2F_l$ unless

$$(m+1+l) \bmod k = (m-l-1) \bmod k \Leftrightarrow l = \frac{k}{2} - 1$$

holds which is impossible since k is odd and l is integer.

For each pair $i, j \in V$, $i < j$, either $[i, j]_0$ is contained in $2F_{j-i-1}$ if $j-i \leq \frac{k-1}{2}$ or $[j, i]_0$ is contained in $2F_{k-1-(j-i)}$ if $j-i > \frac{k-1}{2}$.

Consequently, $\{2F_{\frac{k-1}{2}}, \dots, 2F_{k-2}\}$ forms a 2-factorization of $G' := (V, \{[i, j]_1 \mid i, j \in V, i < j\})$. Obviously, both edges incident to a pair $i, j \in V$ have opposite orientations by definition. □

Note that each pair $i, j \in V$ being matched in $2F_0$ has difference $|i-j| = 1$ as can be observed in figure 5. Furthermore, note that each node i is incident to exactly one ingoing edge $[j, i]_k$, $j \in V, k \in \{0, 1\}$, and to exactly one outgoing edge $[i, j]_k$, $j \in V, k \in \{0, 1\}$, in each 2-factor $2F_l$, $l \in \{0, \dots, k-2\}$. Hence, each 2-factor consists of oriented circles.

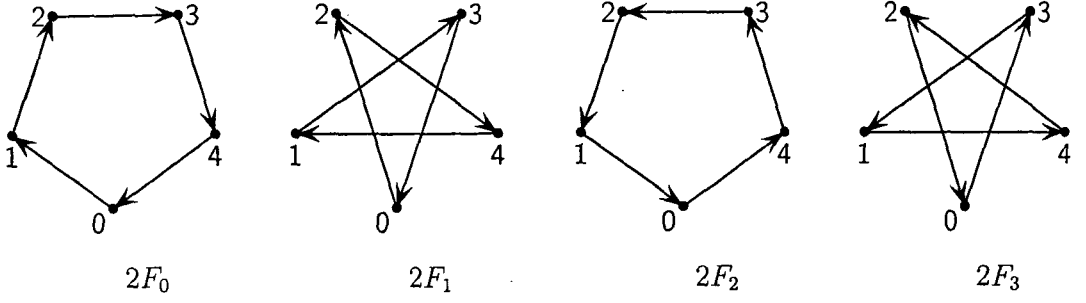


Figure 5: Symmetric 2-factorization of $2K_5$

3 Group-Balanced Single Round Robin Tournaments

In the following we provide several characteristics of group-balanced single RRTs. Additionally, we give a necessary and sufficient condition for n and $|S|$ such that a corresponding group-balanced single RRT exists.

Observation 1. *The set of group-balanced single RRTs having n teams and $|S|$ strength groups is a subset of the set of group-changing single RRTs having n teams and $|S|$ strength groups. Hence, given n and $|S|$ such that there is no corresponding group-changing single RRT then there neither is a corresponding group-balanced single RRT.*

Theorem 3. *In a group-balanced single RRT the difference of two periods $p_{\bar{j}}$ and $p_{\underline{j}}$ where team i plays against teams \bar{j} and \underline{j} with $S(\bar{j}) = S(\underline{j})$, respectively, is $|p_{\bar{j}} - p_{\underline{j}}| = k|S|$ with $0 \leq k < \frac{n}{|S|}$.*

Proof. Suppose team i plays against teams \underline{j} and \bar{j} with $S(\underline{j}) = S(\bar{j})$ in periods $p_{\underline{j}}$ and $p_{\bar{j}}$ with $p_{\underline{j}} < p_{\bar{j}}$ and $p_{\bar{j}} - p_{\underline{j}} = k|S| + l$ with $0 \leq k < \frac{n}{|S|}$ and $0 < l < |S|$. Then one of the following two cases holds.

- I There are less than $\frac{p_{\bar{j}} - p_{\underline{j}}}{|S|} - 1$ matches of team i against teams of $S_{S(\bar{j})}$ in periods p with $p_{\underline{j}} < p < p_{\bar{j}}$. Then, there is at least one pair (\underline{p}, \bar{p}) with $p_{\underline{j}} \leq \underline{p} < \bar{p} \leq p_{\bar{j}}$ such that team i plays against teams in $S(\bar{j})$ in \underline{p} and \bar{p} , team i does not play against any team in $S(\bar{j})$ in any period p' with $\underline{p} < p' < \bar{p}$, and $\bar{p} - \underline{p} > |S|$. Hence, team i plays more than once against teams of at least one strength group S_k , $k \neq S(\bar{j})$, in periods p'' with $\underline{p} + 1 \leq p'' \leq \underline{p} + |S| < \bar{p}$.
- II There are more than $\frac{p_{\bar{j}} - p_{\underline{j}}}{|S|} - 1$ matches of team i against teams of $S_{S(\bar{j})}$ in periods p with $p_{\underline{j}} < p < p_{\bar{j}}$. Then, there is at least one pair (\underline{p}, \bar{p}) with $p_{\underline{j}} \leq \underline{p} < \bar{p} \leq p_{\bar{j}}$ such that team i plays against teams in $S(\bar{j})$ in \underline{p} and \bar{p} and $\bar{p} - \underline{p} < |S|$.

□

Theorem 4. *In a group-balanced single RRT each match of team i against team j with $S(i) = S(j)$ is carried out in period $p = k|S| - 1$, $0 < k < \frac{n}{|S|}$.*

Proof. According to theorem 3 the first period p containing a match between team i and an other team of strength group $S_{S(i)}$ determines the set of periods containing all matches between team i and teams of strength group $S_{S(i)}$. If $p \neq |S| - 1$ then one of the following two cases holds.

- I If $p > |S| - 1$ then team i plays twice against a team of at least one strength group S_k , $k \neq S(i)$, in periods p' with $0 \leq p' \leq |S| - 1$.
- II If $p < |S| - 1$ then team i plays twice against a team of at least one strength group S_k , $k \neq S(i)$, in periods p' with $n - |S| - 1 \leq p' \leq n - 2$.

□

Theorem 5. *There is no group-balanced single RRT where $\frac{n}{|S|}$ is odd.*

Proof. According to theorem 4 in periods p with $p = k|S| - 1$, $0 < k < \frac{n}{|S|}$, only matches between teams i and j with $S(i) \neq S(j)$ are carried out. If the number of teams $\frac{n}{|S|}$ in a strength group S_k is odd then no more than $\frac{n}{|S|} - 1$ teams can play in those periods.

□

Theorem 6. *In a group-balanced single RRT $S(o_{i,p}) = S(o_{j,p})$ holds for each period p and for each pair of teams (i, j) with $S(i) = S(j)$.*

Proof. Assume there are teams i and j , $S(i) = S(j)$, and a period p such that $S(o_{i,p}) \neq S(o_{j,p})$. Then, exactly one team of $S_{S(o_{i,p})}$ plays against team j in period p' with $\max(0, p - |S| + 1) \leq p' \leq p + \max(0, |S| - p - 1)$, $p' \neq p$, according to theorem 3. Obviously, $|p' - p| < |S|$ and, hence, $|p'' - p| \neq k|S|$, $k \in \mathbb{N}$, holds for each period p'' where j plays against a team of $S_{S(o_{i,p})}$ according to theorem 3. Then, $o_{i,p}$ plays against i and j in two periods having distance not equal to $k|S|$ for any $k \in \mathbb{N}$ which is infeasible since $S(i) = S(j)$.

□

Definition 4. *A pairing of strength groups is a mapping $\sigma : S \rightarrow S$ such that $\sigma(\sigma(S_k)) = S_k$ for each $k \in \{0, \dots, |S| - 1\}$.*

Theorem 7. *There is no group-balanced single RRT where $|S|$ is odd.*

Proof. According to theorem 6 there is a pairing σ_p in each period p such that for two strength groups S_k, S_l , $\sigma_p(S_k) = S_l$, each team in S_k plays against a team in S_l in p . According to theorem 4 $\sigma_p(S_k) \neq S_k$ in period $p \neq k|S| - 1$, $k \in \{1, \dots, \frac{n}{|S|} - 1\}$. Then, no σ_p exists if $|S|$ is odd.

□

In order to construct a single RRT we have to arrange matches between each pair of teams and, therefore, pairings of strength groups such that each strength group is paired with each other strength group. This can be represented as 1-factorization of the complete graph $K_{|S|}$ where nodes correspond to strength groups and a 1-factor corresponds to a pairing.

Two strength groups S_k, S_l , $k \neq l$, have to be paired exactly the amount of times needed to let each team of S_k play against each team of S_l . This number is known to be $\frac{n}{|S|}$ from the cardinality of a 1-factorization of the complete bipartite graph $K_{\frac{n}{|S|}, \frac{n}{|S|}}$ introduced in section 2.1. Furthermore, $\sigma_p(S_l) = S_l$ for each $p = k|S| - 1$, $k \in \{1, \dots, \frac{n}{|S|} - 1\}$.

$l \in \{0, \dots, |S| - 1\}$ (since only matches between teams of identical strength groups can be carried out in these periods).

Accordingly, the construction scheme proposed in the following has two stages. In the first stage a schedule is constructed which prescribes teams of a specific strength S_k group to play against teams of an other strength group S_l , $l \neq k$, or to play against teams of the same strength group S_k , respectively. The result is exemplarily represented in table 1.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	3	2	1	0	3	2	1	0	3	2	1	0	3	2	1
1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0
2	1	0	3	2	1	0	3	2	1	0	3	2	1	0	3
3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2

Table 1: Strength group schedule for $n = 16$, $|S| = 4$

Line 2 to 5 correspond to strength groups S_0 to S_3 . For each strength group S_k , $k \in \{0, \dots, 3\}$, the strength group S_l , $l \in \{0, \dots, 3\}$, being paired with S_k in period p is given in the line corresponding to S_k and in the column corresponding to p . In the second stage we arrange matches between teams according to 1-factorizations introduced in section 2.

Theorem 8. *A group-balanced single RRT can be arranged if and only if $|S|$ is even and $\frac{n}{|S|}$ is even.*

Proof. We show that a group-balanced single RRT can be arranged if $|S|$ is even and if $\frac{n}{|S|}$ is even. Then, using theorems 5 and 7 theorem 8 follows.

We first construct the pairing σ_p of strength groups for each period p . According to theorem 4 $\sigma_p(S_l) = S_l$ for each $p = k|S| - 1$, $k \in \{1, \dots, \frac{n}{|S|} - 1\}$, $l \in \{0, \dots, |S| - 1\}$.

Additionally, we arrange a 1-factorization of $K_{|S|}$ in periods 0 to $|S| - 2$. This is possible if and only if $|S|$ is even. Naturally, the 1-factorization structure means each strength group being paired with each other strength group exactly once and each strength group being contained in each pairing σ_p , $p \in \{0, \dots, |S| - 2\}$ exactly once. Next, we set $\sigma_{p+k|S|} := \sigma_p$, for each $p \in \{0, \dots, |S| - 2\}$ and $k \in \{1, \dots, \frac{n}{|S|} - 1\}$. Hence, we obtain exactly $\frac{n}{|S|}$ pairings containing a specific pair S_k, S_l , $k \neq l$, of strength groups.

For each pair of strength groups S_k, S_l , $k \neq l$, we arrange all matches between teams of S_k and S_l in a way representable as a 1-factorization of $K_{\frac{n}{|S|}, \frac{n}{|S|}}$ as shown in section 2.1.

Next, we arrange all group inherent matches in periods p with $p = k|S| - 1$, $k \in \{1, \dots, \frac{n}{|S|} - 1\}$ by arranging 1-factorization for the complete graph $K_{\frac{n}{|S|}}$ corresponding to each strength group S_k where nodes represent teams of S_k . This is possible if and only if $\frac{n}{|S|}$ is even (see section 2.2 for details).

The result is a group-balanced single RRT since:

- Each strength group is contained in each σ_p , $p \in \{0, \dots, n - 2\}$. Due to the 1-factorization structure according to teams of one strength group and teams of two paired strength groups, respectively, each team plays exactly once per period.
- Each pair of teams meets exactly once due to the 1-factorization structure according to teams of one strength groups and teams of two paired strength groups, respectively.

- No team plays more than once against teams of the same strength group within $|S|$ consecutive periods since identical pairings have distance of $k|S|$, $k \in \{0, \dots, \frac{n}{|S|} - 1\}$ periods by construction.

□

4 Group–Changing Single Round Robin Tournaments

In this section we discuss cases of n and $|S|$ where no group-balanced single RRTs exists and give construction schemes for other cases. We adopt the basic idea of pairings of strength groups from section 3. However, we have to extend the concept in order to allow fairness according to strength groups for more than those cases given in section 3.

Observation 2. *If $|S|$ is even and $\frac{n}{|S|}$ is even a group-changing single RRT can be arranged by construction proposed for group-balanced single RRTs.*

Observation 3. *If $|S| = 2$ a group-changing single RRT is group-balanced as well and, therefore, no group-changing single RRT with $|S| = 2$ exists if $\frac{n}{2}$ is odd.*

Theorem 9. *If $|S| = 4k + 3$, $k \in \mathbb{N}^+$, a group-changing single RRT can be arranged.*

Proof. We construct a group-changing single RRT given n and $|S| = 4k + 3$, $k \in \mathbb{N}^+$. First, we construct a binary near-1-factorization according to $F^{b,e}$ as introduced in section 2.2 on the complete graph $K_{|S|}$.

We interpret each near-1-factor as a pairing σ of strength groups (see definition 4). Additionally, we define $\sigma(S_k) = S_k$ if the node not matched in the near-1-factor corresponds to S_k .

The pairing resulting from $F_l^{b,e}$ is assigned to period $p = k|S| - 1 - l$, $k \in \{1, \dots, \frac{n}{|S|} - 1\}$. Thus, periods 0 to $n - |S| - 1$ are assigned to pairings containing each pair (S_i, S_j) exactly $\frac{n}{|S|} - 1$ times.

We arrange matches in periods 0 to $n - |S| - 1$ according to the pairing σ_p assigned to p , i.e. a match of i against j can not be carried out in period p if $S(i)$ is not paired with $S(j)$ in p .

We arrange matches according to the 1-factorization of $K_{\frac{n}{|S|}, \frac{n}{|S|}}$ as given in section 2.1. Since we can arrange only $\frac{n}{|S|} - 1$ 1-factors we leave $F_{\frac{n}{2|S|}}^{bip}$ out for each pair of strength groups.

Furthermore, each strength group is paired with itself exactly $\frac{n}{|S|} - 1$ times. Hence, we can arrange all matches of teams of the same strength group according to a 1-factorization of $K_{\frac{n}{|S|}}$ (see section 2.2).

Finally, all matches contained in 1-factors $F_{\frac{n}{2|S|}}^{bip}$ of $K_{\frac{n}{|S|}, \frac{n}{|S|}}$ corresponding to each pair of strength groups have to be arranged in periods $n - |S|$ to $n - 2$. We construct a symmetric 2-factorization of $2K_{|S|}$ according to section 2.3 and assign factor $2F_p$ to period $n - |S| + p$, $p \in \{0, \dots, |S| - 2\}$. If $[i, j]_k$ is contained in 2-factor $2F_p$, $p \in \{0, \dots, |S| - 2\}$, we arrange matches between the first $\frac{n}{2|S|}$ teams of S_i and the last $\frac{n}{2|S|}$ teams of S_j in period $n - |S| + p$. Note that $\frac{n}{|S|}$ is even since $|S|$ is odd while n is even. Now, we have a single RRT with changing opponent strength:

- Each team plays exactly once per period. For period p , $p \in \{0, \dots, n - |S| - 1\}$, this is obvious due to the 1-factor structure within pairs of strength groups. In periods p , $p \in \{n - |S|, \dots, n - 2\}$, each team plays exactly once since each 2-factor is composed of oriented circles (see section 2.3). Hence, for each strength group the outgoing arc covers the first half of teams while the ingoing arc covers the second half.

- Each pair of teams meets exactly once. Obviously, no pair of teams plays twice in periods p , $p \leq n - |S| - 1$, due to the 1-factorization structure between each pair of strength groups and within single strength groups, respectively. The 1-factors $F_{\frac{n}{2|S|}}^{bip}$ between each pair of strength groups missing from periods 0 to $n - |S| - 1$ are exactly covered by both arcs between a pair of strength groups in $2K_{|S|}$.
- No team plays against teams of the same strength group in two consecutive periods. In each time window $[p, p + |S| - 1]$, $0 \leq p \leq n - 2|S|$, each team plays exactly once against each strength group due to repeating sequence of pairings. In periods p , $p \in \{n - |S|, \dots, n - 2\}$, each strength group is paired twice with each other strength group. However, corresponding arcs have opposite orientation and, hence, the set of teams involved in both pairs are disjoint.

Therefore, in periods p , $p \in \{n - |S|, \dots, n - 2\}$ each team plays exactly once against each other strength group and, hence, there is no violation of changing opponent strength in periods p and $p + 1$, $p \in \{0, \dots, n - |S| - 2\}$ and $p \in \{n - |S|, \dots, n - 2\}$, respectively.

Note that 1-factor $F_0^{b,e}$ chosen for $p = n - |S| - 1$ does not contain any pair $(S_i, S_{(i+1) \bmod |S|})$ (see section 2.2) if $|S| > 3$. $2F_0$ chosen for period $p = n - |S|$ exclusively contains pairs of this form. Therefore, no team can play against the same strength group in periods $n - |S| - 1$ and $n - |S|$.

□

Theorem 10. *If $|S| = 4k + 1$, $k \in \mathbb{N}^+$ a group-changing single RRT can be arranged.*

Proof. The proof is analogous to the proof of theorem 9. The only difference is employing the binary 1-factorization of K_{2k} , k odd, $F^{b,o}$ given in section 2.2 instead of the binary 1-factorization in order to establish pairings for periods 0 to $n - |S| - 1$. Each conclusion follows as above.

□

Observation 4. *According to theorems 9 and 10 a group-changing single RRT can be arranged if $|S| > 3$ and odd.*

Theorem 11. *If $\frac{n}{|S|}$ is odd and $|S| > 2$ a group-changing single RRT can be arranged.*

Proof. Given n and $|S| > 2$ with $\frac{n}{|S|}$ odd we construct a group-changing single RRT. First, we construct an ordered 1-factorization of $K_{|S|}$ and associate 1-factors with pairings such that we obtain pairing $\sigma_{\frac{|S|}{2}-1}$ having S_k paired with S_{k+1} for each $k \in \{2l \mid l \in \{0, \dots, \frac{|S|}{2}\}\}$. Then we assign σ_p to periods $p + k|S| + 1$ for $k \in \{0, \dots, \frac{n}{|S|} - 2\}$, $p \in \{0, \dots, |S| - 2\}$. Additionally, we assign σ_p to period $p + \left(\frac{n}{|S|} - 1\right)|S| + 1$ for each $p \in \{0, \dots, \frac{|S|}{2} - 2\}$ and we assign $\sigma_{p'}$ to period $p' + \left(\frac{n}{|S|} - 1\right)|S|$ for each $p' \in \{\frac{|S|}{2}, \dots, |S| - 2\}$. Hence, each pair of strength groups except those contained in $\sigma_{\frac{|S|}{2}-1}$ is arranged exactly $\frac{n}{|S|}$ times. We can construct 1-factorization F^{bip} according to section 2.1 for the teams contained in each of those pairs of strength groups. Pairing $\sigma_{\frac{|S|}{2}-1}$ is contained exactly $\frac{n}{|S|} - 1$ times. Therefore, we can arrange all 1-factors of F^{bip} but $F_{\frac{|S|}{2}-1}^{bip}$. Consequently, all matches between pairs of

strength groups are arranged except between teams i ($i = S(i)\frac{n}{|S|} + k$, $k \in \{0, \dots, \frac{n}{|S|}\}$) and j ($j = \frac{n}{|S|} + (j-1)\text{mod } \frac{n}{|S|}$) with $S(i)$ even.

To the remaining periods p , $p = k|S|$, $k \in \{0, \dots, \frac{n}{|S|} - 1\}$, we assign the pairing having each strength group paired with itself. Since $\frac{n}{|S|}$ is odd we can construct a near-1-factorization with $\frac{n}{|S|}$ 1-factors for the set of teams in each strength group according to section 2.2. Naturally, in each strength group each team is not contained in a near-1-factor exactly once. Team $i \in S_k$, k even, not contained in the near-1-factor assigned to period $p = l|S|$, $l \in \{0, \dots, \frac{n}{|S|} - 1\}$ is arranged to play against team $j \in S_{k+1}$ not contained in the near-1-factor assigned to period p . Formally, we arrange 1-factors $F_l^{b,o}$, $l \in \{k-1, \dots, 2k-2\}$, according to section 2.2 for each pair (S_m, S_{m+1}) , m even.

This results in a group-changing single RRT:

- Each team plays exactly once per period. For period p , $p \neq k|S|$, $k \in \{0, \dots, \frac{n}{|S|} - 1\}$, this is obvious due to the 1-factor structure within pairs of strength groups. In periods p , $p = k|S|$, $k \in \{0, \dots, \frac{n}{|S|} - 1\}$, each team but one per strength group plays exactly once due to the near-1-factor structure within each strength group. The team not playing against teams of the same strength group in periods p , $p = k|S|$, $k \in \{0, \dots, \frac{n}{|S|} - 1\}$, plays against each other. Since $|S|$ is even (n even, $\frac{n}{|S|}$ odd) each of them plays exactly once.
- Each pair of teams meets exactly once. This is obvious for matches between teams i and j , $S(i) < S(j)$, $(S(i) \text{ odd} \vee S(i) + 1 \neq S(j))$, of different strength groups due to the 1-factorization structure between each pair of strength groups. Furthermore, matches between teams i and j , $S(i) = S(j)$, are carried out exactly once due to the 1-factorization structure within strength groups. Matches of teams i and j ($S(i)$ even $\wedge S(i) + 1 = S(j)$) are composed of F_k^{bip} , $k \neq \frac{|S|}{2} - 1$, in periods p , $p \in \left\{ \frac{|S|}{2} + l|S| \mid l \in \{0, \dots, \frac{n}{|S|} - 2\} \right\}$, and $F_{\frac{|S|}{2}-1}^{bip}$ arranged between pairs of unmatched nodes in near-1-factors in periods p , $p \in \{l|S| \mid l \in \{0, \dots, \frac{n}{|S|} - 1\}\}$.
- No team plays against teams of the same strength group in two consecutive periods. This is obvious for pairs S_k, S_l , $k < l$, $(k \text{ odd} \vee k+1 \neq l)$ of strength groups since those are arranged in periods having distance no less than $|S| - 1$ by construction. $|S| - 1 > 1$ for $|S| > 2$. Matches within pairs S_k, S_l , $k < l$ ($k \text{ even} \wedge k+1 = l$), of strength groups are arranged in periods $p \in \left\{ \frac{|S|}{2} + l|S| \mid l \in \{0, \dots, \frac{n}{|S|} - 2\} \right\} \cup \left\{ l|S| \mid l \in \{0, \dots, \frac{n}{|S|} - 1\} \right\} = \left\{ l\frac{|S|}{2} \mid l \in \{0, \dots, \frac{2n}{|S|} - 2\} \right\}$. Obviously, pairwise distance is no less than 2 if $|S| > 2$.

□

Observation 5. *By enumeration: If $n = 6$ and $|S| = 3$ no group-changing single RRT exists but if $n \in \{12, 18\}$ and $|S| = 3$ a group-changing single RRT can be arranged.*

We conjecture that a group-changing single RRT exists for $|S| = 3$ and each $n = 6(k+1)$, $k \in \mathbb{N}^+$.

5 Complexity

As outlined in [3] there are several applications for associating cost $c_{i,j,p}$ with each match of team i at home against team j in period p . The minimum cost single RRT problem is defined as follows.

Definition 5. Given a set T , $|T|$ even, of teams, a set of periods P , $|P| = |T| - 1$, and cost $c_{i,j,p}$ associated with each match of team $i \in T$ at home against team $j \in T$, $j \neq i$, in period $p \in P$, the minimum cost single RRT problem is to find the single RRT having the minimum sum of arranged matches cost.

The minimum cost single RRT problem has been proven to be NP-hard independently in [6] and [3]. In the following we introduce two minimum cost problems corresponding to strength group requirements as introduced in section 1.

Definition 6. Given a set T , $|T|$ even, of teams, a set of periods P , $|P| = |T| - 1$, a number $|S|$ of strength groups and cost $c_{i,j,p}$ associated with each match of team $i \in T$ at home against team $j \in T$, $j \neq i$, in period $p \in P$ the group-balanced single RRT problem is to find the group-balanced single RRT having the minimum sum of arranged matches cost.

Definition 7. Given a set T , $|T|$ even, of teams, a set of periods P , $|P| = |T| - 1$, a number $|S|$ of strength groups and cost $c_{i,j,p}$ associated with each match of team $i \in T$ at home against team $j \in T$, $j \neq i$ in period $p \in P$ the group-changing single RRT problem is to find the group-changing single RRT having the minimum sum of arranged matches cost.

Theorem 12. The group-balanced single RRT problem is NP-hard even if $|S|$ is fixed.

We proof theorem 12 by reduction from minimum cost single RRT problem.

Proof. Given a minimum cost single RRT problem by a set of teams T' , $|T'| = n'$, a set of periods P' , and cost $c'_{i',j',p'}$, $i', j' \in T'$, $i' \neq j'$, $p' \in P'$, we construct a group-balanced single RRT problem with n teams and $|S|$ strength groups as follows. Let $n = n'|S|$. We follow the idea of pairings of strength groups in each period given in section 3. We set cost

$$c_{i,j,p} = \begin{cases} M & \text{for } S(i) = S(j) = 0, p \neq k|S|, k \in \{1, \dots, \frac{n}{|S|} - 1\}, \\ c'_{i',j',p'} & \text{for } S(i) = S(j) = 0, p \neq p'|S|, p' \in P' \\ 0 & \text{otherwise} \end{cases}$$

with $M = \sum_{i' \in T'} \sum_{j' \in T', j' \neq i'} \sum_{p' \in P'} c'_{i',j',p'}$.

Obviously, a group-changing single RRT having cost less than M is provided by the construction scheme given in the proof of theorem 8. Each solution having cost less than M provides a single RRT s of teams of S_0 in periods p with $p = k|S|$, $k \in \{1, \dots, \frac{n}{|S|} - 1\}$.

Next, it can be easily seen that s is optimal for the original minimum cost single RRT problem by contradiction. If there is a single RRT s' having less cost than s according to the minimum cost single RRT problem we can exchange s by s' in the group-balanced single RRT. Trivially, this leads to a group-balanced single RRT having less cost. \square

Theorem 13. The group-changing single RRT problem is NP-hard even if $|S| > 3$ is odd and fixed.

Again, we give a reduction from minimum cost single RRT problem. The idea is quite the same as for theorem 12. Hence, a sketch of the proof suffices.

Proof. Given a minimum cost single RRT problem we construct a group-changing single RRT problem with n teams and $|S| > 3$, $|S|$ odd, strength groups as follows. Let $n = n'|S|$. We follow the idea of pairings of strength groups by near-1-factorizations according to $F^{e,b}$ and $F^{o,b}$ (depending on the number of strength groups $|S|$).

We set cost

$$c_{i,j,p} = \begin{cases} M & \text{for } S(i) = S(j) = 0, p \neq (p' + 1)|S| - 1, p' \in \left\{0, \dots, \frac{n}{|S|} - 2\right\}, \\ c'_{i,j,p'} & \text{for } S(i) = S(j) = 0, p = (p' + 1)|S| - 1, p' \in \left\{0, \dots, \frac{n}{|S|} - 2\right\}, \\ 0 & \text{otherwise} \end{cases}$$

with $M = \sum_{i' \in T'} \sum_{j' \in T', j' \neq i'} \sum_{p' \in P'} c_{i',j',p'}$.

Obviously, a group-balanced single RRT having cost less than M is provided by the construction scheme given in the proof of theorems 9 and 10, respectively. Each solution having cost less than M provides a single RRT s of teams of S_0 in periods $p = (p' + 1)|S| - 1$ with $p' \in \left\{0, \dots, \frac{n}{|S|} - 2\right\}$.

Obviously, s is optimal for the original minimum cost single RRT problem. \square

Theorem 14. *The group-changing single RRT problem is NP-hard even if $|S|$ is even and fixed.*

Again, we give a reduction from minimum cost single RRT problem. The idea is quite the same as for theorem 12.

Proof. Given a minimum cost single RRT problem we construct a group-changing single RRT problem with n teams and $|S|$, $|S|$ odd, strength groups as follows. Let $n = n'|S|$. We follow the idea of pairings of strength groups by 1-factorizations according to F^c .

We set cost

$$c_{i,j,p} = \begin{cases} M & \text{for } \sigma_p(S(i)) \neq S(j), \\ 0 & \text{for } \sigma_p(S(i)) = S(j), p \neq k|S|, k \in \left\{1, \dots, \frac{n}{|S|} - 1\right\} \\ 0 & \text{for } S(i) = S(j), S(i) \neq 1, p = k|S|, k \in \left\{1, \dots, \frac{n}{|S|} - 1\right\} \\ c'_{i',j',p'} & \text{for } S(i) = S(j) = 1, p = p'|S| \end{cases}$$

with $M = \sum_{i' \in T'} \sum_{j' \in T', j' \neq i'} \sum_{p' \in P'} c_{i',j',p'}$.

Obviously, a group-balanced single RRT having cost less than M is provided by the construction scheme given in the proof of theorems 9 and 10, respectively. Note that each group-balanced single RRT is group-changing. Each single RRT which is group-changing but not group-balanced has cost no less than M . Therefore, the optimal solution to the group-changing single RRT problem provides a single RRT s of teams of S_0 in periods p with $p = k|S|$, $k \in \left\{1, \dots, \frac{n}{|S|} - 1\right\}$.

Obviously, s is optimal for the original minimum cost single RRT problem. \square

Observation 6. *According to theorems 13 and 14 the group-changing single RRT problem is NP-hard even if $|S| \neq 3$ is fixed.*

6 Conclusions and Outlook

In this paper we pick up a common idea to achieve fairness among teams competing in a single RRT. Although strength groups have already been proposed in several works there is no answer to the question for which values of n and $|S|$ fair schedules can be constructed. We investigate two degrees of fairness: group-changing single RRTs and group-balanced single RRTs.

We proof a necessary and sufficient condition for n and $|S|$ to allow a group-balanced single RRT. Furthermore, we show how to decide for almost all cases whether a group-changing single RRT is possible or not. The remaining cases are $n = 6k$, $k \in \mathbb{N}$, and $|S| = 3$ and we strongly conjecture a group-changing RRT to be possible if and only if $k > 1$.

In this paper we have analysed almost all cases covered by the considered structure. Nevertheless, several generalizations and variations of the strength group concept might be of interest:

- We assume all strength groups to have identical sizes. This assumption can be relaxed and arbitrary sizes can be considered. In fact, this make sense in terms of real world tournaments if there are both only a few excellent teams and few weak teams while the remaining teams can be considered middle level.
- We suppose the number of teams to be even. Clearly, if n is odd not each team can play in each period. Moreover, if all strength groups have identical sizes $|S|$ as well as $\frac{n}{|S|}$ have to odd. This case is not considered in the paper at hand.
- If we consider RRTs having more than one round, double RRTs for example, the assumption that each team plays against a team of the same strength group in periods $k|S| - 1$, $k \in \{1, \dots, \frac{n}{|S|} - 1\}$ is no longer valid.

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