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Robustness in combinatorial optimization and scheduling theory: An extended annotated bibliography

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Manuskripte
aus den
Instituten für Betriebswirtschaftslehre
der Universität Kiel

No. 606

**Robustness in Combinatorial Optimization and Scheduling Theory:
An Extended Annotated Bibliography ¹**

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Working Paper, August 2006

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Abstract

This work is an up-to-date-extension of a previous annotated bibliography (2004) which covered 40 references only. It focuses on what has been published during the last ten years in the area of combinatorial optimization and scheduling theory concerning robustness and other similar techniques dealing with worst case optimization under uncertainty and non-accuracy of problem data.

Keywords: robustness – tolerance – worst-case scenario – input data uncertainty – non-accuracy – flexibility of solution – maximal regret

1 Introduction

One of the most interesting branches of combinatorial optimization that has emerged over the past 20 - 30 years is robust optimization. Since the early 1970s there has been an increasing interest in the use of worst-case optimization models. The theory of robustness is a relatively new and quickly developing area of combinatorial optimization. It deals with uncertainty of problem parameters. The presence of such parameters in optimization models is caused by inaccuracy of initial data, non-adequacy of models to real processes, errors of numerical methods, errors of rounding off and other factors. So it appears to be important to identify classes of models in which small changes of input data lead to small changes of the result under worst possible scenario of distribution of problem parameters. The models with such properties are called robust counterpart. It is obvious that any optimization problem arising in practice can hardly be correctly formulated and solved without use of results of the theory of robustness and post-optimal analysis. During the last ten years many authors concentrate on robust optimization and related approaches in which one optimizes against the worst instances that might arise by using min-max (or some other) objective.

This survey gives an idea of the variety of modern techniques and may serve as a short introduction into the theory of robustness. The main goal of this annotated bibliography is to collect all existing papers together in order to present a complete description of new avenues that have not been explored earlier.

In the article annotations the original phrases and abstract fragments are used in order to be as close as possible to authors descriptions. Of course, the author remains responsible for reformulation errors or omissions that might exist. Unfortunately, we were not able to take into account papers which have been written in any language different from English. If there are some authors whose articles were not included in the bibliography but their papers are closely related to the topic, please let know.

2 Terms and concepts

We consider a general optimization problem in the following standard form:

$$\text{minimize } f(c, x)$$

subject to

$$\begin{aligned} g_i(a, x) &\leq 0, \\ x &\in X, \quad i = 1, \dots, n, \end{aligned}$$

where

- x is the vector of variables
- X is the set of feasible solutions
- f and g_i are convex function
- c and a are problem parameters (uncertain).

Uncertainty may be related to coefficients of objective function, coefficients of restrictions or both of them. It seems very naturally to give a definition of a robust solution as follows: an optimal solution is robust if it remains optimal under any realization (scenario) of the data (problem parameters). But this definition can hardly be regarded as desirable, because it is too restrictive. Most unlikely such a solution exists. Another definition may be considered more appropriate: our subject is to find a robust solution which minimizes maximum regret (minimizes worst case scenario). The major part of the papers presented in this bibliography deals with problems like that. Authors of most papers attempt to answer to the following closely related questions: How can one represent uncertainty? What is a robust solution? How to calculate a robust solution? How to interpret worst case realization under uncertainty? and others. Different answers to these questions lead to different approaches and directions. Bibliographical analysis provides us with a list of contributors who proposed several main avenues in the theory of robustness:

- Averbakh [11] – [12] (minmax regret optimization)
- Ben-Tal and Nemirovski [1] – [6] (ellipsoidal uncertainty)
- Bertsimas and Sim [14] – [19] (robust optimization with control of conservatism of a solution)
- Kouvelis and Yu [27] (minmax regret)
- Mulvey, Vanderbei and Zenios [9] – [10] (worst possible scenario and penalty functions)
- Yaman, Karasan and Pinar [40] – [42] as well as Montemanni and Gambardella [32] – [36] (absolute and relative robustness for minimum spanning tree and shortest path problems).

References

Part I. Robustness in Convex Optimization and Linear Programming

In recent years a large number of papers is appearing under the name of robust optimization in which playing against the worst instances that might arise by using a min-max objective or some others is considered. We start our survey with some papers dealing with continuous optimization problems since they are historically the first for which robustness has been analyzed. Papers [1] – [10] present different approaches to deal with uncertainty for some convex optimization and linear programming problems. However, as the resulting robust formulations involve conic quadratic problems and other difficulties, such methods cannot be directly applied to discrete optimization. To the best of our knowledge, the only attempt to develop the concept of robustness which is suitable for both convex and combinatorial optimization has been done by Bertsimas et al. in [17] – [19].

- [1] **Ben-Tal A. and Nemirovski A.** (1998). Robust convex optimization. *Mathematics of Operations Research* 23, 769 – 805.

Ben-Tal and Nemirovski address the over conservatism of robust solutions by allowing the uncertainty sets for the data to be ellipsoids, and propose efficient algorithms to solve convex optimization problems under data uncertainty.

- [2] **Ben-Tal A. and Nemirovski A.** (1999). Robust solutions to uncertain programs. *Operations Research Letters* 25, 1 – 13.
- [3] **Ben-Tal A. and Nemirovski A.** (2000). Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming* 88, 411 – 424.
- [4] **Ben-Tal A., El Ghaoui L. and Nemirovski A.** (2000). Robust semidefinite programming. In Saigal R., Vandenberghe L. and Wolkowicz H. editors, *Handbook of Semidefinite programming and applications*, Kluwer Academic Publishers, Waterloo.
- [5] **Ben-Tal A., Nemirovski A. and Roos C.** (2002). Robust solutions to uncertain quadratic and conic-quadratic problems. *SIAM Journal on Optimization* 13, 535 – 560.
- [6] **Ben-Tal A., Goryashko A., Guslitzer E. and Nemirovski A.** (2004). Adjustable robust solutions of uncertain linear programs. *Mathematical Programming* 99, 351 – 376.

A number of important formulations as well as applications are introduced in [1] – [6] and some other papers of these authors. A detailed analysis of the robust optimization framework in linear programming is provided.

[7] **El Ghaoui L. and Lebret H.** (1997). Robust solution to least- squares problems to uncertain data. *SIAM Journal Matrix Analysis Appl.* 18, 1035 – 1064.

[8] **El Ghaoui L., Oustry F. and Lebret H.** (1998). Robust solutions to uncertain semidefinite programs. *SIAM Journal Optimization* 9, 33 – 52.

El Ghaoui et al. derived in [7] – [8] results similar to [1] – [4]. In particular, they deal with robust reformulation of optimization model by adapting robust control techniques under the assumption that the coefficient matrix data may vary inside ellipsoidal uncertainty set. The robust counterpart of some important problems are either exactly or approximately tractable problems that are efficiently solvable with interior point methods. However, the difficulty of the robust problems increases.

[9] **Mulvey J., Vanderbei R. and Zenios S.** (1995). Robust optimization of large-scale systems. *Operations Research* 43, 264 – 281.

Mulvey et al. present an approach that integrates goal programming formulations with scenario-based description of the problem data. They use penalty functions to develop robust models to hedge against the worst possible scenario.

[10] **Mulvey J. and Vanderbei R.** (1995). Robust optimization of large-scale systems: an emerging new technology. *Scientific Report A204992, Princeton University.*

A generalized interior point algorithm for convex objective functions has been developed and applied to various large scale problems.

Part II. Robustness in Combinatorial Optimization

A general combinatorial optimization problem can be stated as follows. Let $E = \{e_1, e_2, \dots, e_n\}$ be a finite ground set and let $X \subseteq 2^E \setminus \{\emptyset\}$ be a non-empty family of subsets of E known also as the power set of E . Note that any feasible solution $x \in X$ is a subset of E , $x \subset E$. Let $c : E \rightarrow \mathbf{Z}$ be a cost function on the elements of E , which w.l.o.g. assumes integer values. The problem consists in minimizing integer-valued objective function $f(c, x) : X \rightarrow \mathbf{Z}$ over the set of feasible solutions X .

Combinatorial problems are integer (in particular binary) programming problems. Defining $x \in \{0, 1\}^n$ by $x_i = 1$ if $e_i \in x$ and 0 otherwise for any $i = 1, \dots, n$, we can identify subsets $x \subset E$ and binary vectors. On the other hand, binary programs are combinatorial ones. General integer programs, where variables are supposed to have non-negative integer values, are also combinatorial while we assume that the set of feasible solutions is bounded and therefore finite.

If input problem parameters, in particular, cost function values, are crisp and fixed, then we say that our problem is deterministically stated, otherwise we assume some sort of uncertainty. There are several general alternative approaches in combinatorial optimization how to model uncertainty: the probabilistic and stochastic approaches, the use of fuzzy numbers, the scenario-based realization of parameters, the interval representation of input data. In scheduling theory, one can often apply critical chain scheduling and buffer management as well. Below one can find a list of references which are related to these and some other approaches which are so popular nowadays. Modelling with fuzzy numbers is a big and relatively independent theory which is not covered in the bibliography. The very general idea which the approaches have in common is the following; in robustness theory, instead of solving the original problem itself, one should construct a robust counterpart problem with one or several objectives which somehow guarantee that an obtained solution is robust or stable with respect to uncertainty of input data.

- [11] **Averbakh I.** (2001). On the complexity of a class of combinatorial optimization problems with uncertainty. *Mathematical Programming* 90, 263 – 272.

In this paper the minmax regret version of the problem of selecting p objects of minimum total weights out of set of m objects is considered. There are two types of uncertainty of problem parameters considered in the paper. In the case of scenario-represented uncertainty (even if there are only two possible scenarios) it is shown that the robust version of the problem of finding the minimum weight base of a uniform matroid of rank p on a ground set of cardinality m is NP-hard. The second case of uncertainty, so-called interval uncertainty, is traditionally more complicated to analyze. All weights can take on any values from some interval. It means that the set of possible realizations has a form of rectangle box in the space of problem parameters. However, in that case a new polynomial algorithm with complexity $O((\min(p, m - p))2m)$ is proposed. The author claims that it is the first known example of a minmax regret combinatorial optimization problem that is polynomially solvable in the case of interval representation of uncertainty while being NP-hard in the case of scenario represented uncertainty. This is a very interesting result and it may give an additional stimulus to researchers for finding polynomial algorithms

for the large variety of robust combinatorial optimization problems where complexity in the practically important case of interval representation of uncertainty is still open.

- [12] **Averbakh I. and Lebedev V.** (2004). Interval data minmax regret network optimization problems. *Discrete Applied Mathematics* 138, 289 – 301.

The authors consider the minimum spanning tree and the shortest path problems on a network with uncertain lengths of edges. In particular, for any edge of the network, only an interval estimate of the length of the edge is known, and it is assumed that the length of each edge can take on any value from the corresponding interval of uncertainty, regardless of the values taken by the lengths of other edges. It is required to find a minmax regret solution. It is proven that both problems are NP-hard even if the bounds of all intervals of uncertainty belong to $(0, 1)$. The interval data minmax regret shortest path problem is NP-hard even if the network is directed, acyclic, and has a layered structure. Nevertheless it was shown that the problems are polynomially solvable in the practically important case where the number of edges with uncertain lengths is fixed or is bounded by the logarithm of a polynomial function of the total number of edges.

The following papers are closely related to Averbakh's approach:

- **Averbakh I.** (2000). Minmax regret solutions for minimax optimization problems with uncertainty. *Operations Research Letters* 27, 57 – 65.
- **Averbakh I. and Berman O.** (2000). Minmax regret median location on a network under uncertainty. *INFORMS Journal on Computing* 12, 104 – 110.
- **Averbakh I. and Berman O.** (2000). Algorithms for the robust 1-center problem on a tree. *European Journal of Operational Research* 123, 292 – 302.
- **Averbakh I.** (2003). Complexity of robust single-facility location problems on networks with uncertain lengths of edges. *Discrete Applied Mathematics* 127, 505 – 522.
- **Averbakh I. and Berman O.** (2003). An improved algorithm for the minmax regret median problem on a tree. *Networks* 41, 97 – 103.
- **Averbakh I.** (2004). Minmax regret linear resource allocation problems. *Operations Research Letters* 32, 174 – 180.
- **Averbakh I. and Lebedev V.** (2005). On the complexity of minmax regret linear programming. *European Journal of Operational Research* 160, 227 – 231.

- **Lebedev V. and Averbakh I.** Complexity of minimizing the total flow time with interval data and minmax regret criterion. (to appear in *Discrete Applied Mathematics*).
- **Averbakh I.** The minmax relative regret median problem on networks. (to appear in *INFORMS Journal on Computing*).
- **Averbakh I.** The minmax regret permutation flow shop problem with two jobs. (to appear in *European Journal of Operational Research*).

- [13] **Aron I. and P. van Hentenryck.** (2004). On the complexity of the robust spanning tree problem with interval data. *Operations Research Letters* 32, 36 – 40.

This paper is focused on the robust spanning tree problem where the edge costs are given by intervals under the robust deviation framework (minimization of the maximum deviation of total cost from the costs of the minimum spanning tree for all possible realizations of the edge cost within the given intervals). This paper proves a conjecture, which originally stated in [27]. It claims that there is no polynomial algorithm for such type of robustness. The authors prove that the robust spanning tree problem is at least as hard as the central tree problem (it consists in finding a tree in graph G such that the rank of its cospanning tree is minimal over all the cospanning trees of G), the NP-completeness of which is well-known. Furthermore, it is shown that the robust spanning tree problem remains hard on complete graphs, even though the central tree can be found in polynomial time on such graphs.

- [14] **Bertsimas D. and Sim M.** (2002). Robust discrete optimization. *Working Paper. Operations Research Center, MIT.*

A robust version (counterpart) of integer programming problems is proposed for the case when both the cost coefficients and the data constraints are subject to uncertainty. When only the cost coefficients are subject to uncertainty and the problem is 0 – 1 discrete optimization problem on n variables the procedure of solving the robust counterpart by solving $n + 1$ instances of the original problem is described. As the consequence a very interesting fact is stated: if the original problem is polynomially solvable, then the robust counterpart problem also remains polynomially solvable. It means that robust versions of such well-known problems as matching, spanning tree, shortest path, matroid intersection etc. are polynomially solvable. Some results concerning the α -approximation of 0 – 1 discrete optimization problems are given.

- [15] **Bertsimas D. and Sim M.** (2003). Robust discrete optimization and network flows. *Mathematical Programming* 98, 49 – 71.

Additionally to the results of [14], the authors propose an algorithm for robust network flows that solves the robust counterpart by solving a polynomial number of nominal minimum cost flow problems in a modified network.

- [16] **Bertsimas D. and Sim M.** (2004). The price of robustness. *Operations Research* 52, 35 – 53.

A robust approach to solving linear optimization problems with uncertain data was proposed in the early 1970s and has recently been extensively studied and extended. The authors propose another approach which is different from previous ones in order to control the level of conservatism in the solution. This approach has the advantage that it leads to a linear optimization model and it can be directly applied to discrete optimization problems (it was done later in [14]).

- [17] **Bertsimas D. and Sim M.** (2004). Robust conic optimization. *Working Paper. Operations Research Center, MIT.*

In earlier proposals the robust counterpart of a conic optimization problem exhibits an increase in complexity, i.e. robust linear programming problems [3] become second order cone problems, robust second order cone problems [1] become semidefinite programming problems and robust semidefinite programming problems become [4] NP-hard. In this paper a relaxed robust counterpart for general cone optimization problems that preserves the computational tractability of the nominal problem is proposed. Namely, under this concept the robust cone optimization retains the original structure, i.e. robust linear programming problems remain LPs, robust second order cone programming problems remain SCCPs and robust semidefinite programming problems remain SDP. Moreover, when that data entries are independently distributed, the size of the proposed robust problem especially under l_2 norm is practically the same as the original problem.

- [18] **Bertsimas D. and Sim M.** (2004). Robust discrete optimization under ellipsoidal uncertainty sets. *Working Paper. Operations Research Center, MIT.*

It is probably the first attempt to investigate robust discrete optimization under ellipsoidal uncertainty sets. It is shown that the robust counterpart of a discrete optimization problem with correlated objective function data is NP-hard even though the original problem is polynomially solvable. For uncorrelated and identically distributed data, it is proved that the robust counterpart retains the complexity of the original problem. A generalization of the robust discrete optimization approach proposed earlier is given which presents the tradeoff between robustness and optimality.

- [19] **Bertsimas D., Pachamanova D. and Sim M.** (2004). Robust linear optimization under general norms. *Operations Research Letters* 32, 510 – 516.

The explicit characterization of the robust counterpart of a linear programming problem with uncertainty set is described by an arbitrary norm. This approach encompasses several approaches from the literature and provides guarantees for constraint violation under probabilistic models that allow arbitrary dependencies in the distribution of the uncertain coefficients.

- [20] **Burkard R. and Dollani H.** (2001). Robust location problem with positive/negative weights on a tree. *Networks* 38, 102 – 113.
- [21] **Burkard R. and Dollani H.** (2002). A note on the robust 1-center problem on trees. *Annals of Operations Research* 110, 69 – 82.

In this paper the authors consider different aspects of robust 1-median problems on a tree network with uncertain or dynamically changing edge lengths and vertex weights which can also take negative values. The dynamic nature of a parameter is modeled by a linear function of time. A linear algorithm is designed for the absolute dynamic robust 1-median problem on a tree.

- [22] **Carrizosa E. and Nickel S.** (2003). Robust facility location. *Mathematical Methods in Operations Research* 58, 331 – 349.
- [23] **Chen B. and Lin C.** (1998). Minmax-regret robust 1-median location on a tree. *Networks* 31, 93 – 103.
- [24] **Deineko V. and Woeginger G.** (2006). On the robust assignment problem under a fixed number of cost scenarios. *Operations Research Letters* 34, 175 – 179.

The authors investigate the complexity of the min/max assignment problem under a fixed number of scenarios. It is proven that this problem is polynomially-time equivalent to the exact perfect matching problem in bipartite graph, an unstudied combinatorial optimization problem of unknown computational complexity.

- [25] **Hites R., De Smet Y., Risse N., Salazar-Neumann M. and Vincke P.** (2003). A comparison between multicriteria and robustness frameworks, IS-MG 2003/16, Universite Libre de Bruxelles.

A parallelism between multicriteria optimization and robustness concepts is established. New problems like multicriteria multiscenarios problem and multicriteria evaluation of robustness are discussed.

- [26] **Kasperski A. and Zielinski P.** (2006). The robust shortest path problem in series-parallel multidigraphs with interval data. *Operations Research Letters* 34, 69 – 76.

The robust shortest path problem in edge series-parallel multidigraph with interval costs is examined. A pseudopolynomial algorithm for the problem with maximal regret criterion is applied to calculate the optimal solution.

- [27] **Kouvelis P. and Yu G.** (1997). *Robust discrete optimization and its application*. Kluwer Academic Publishers, Norwell, M.A.

A comprehensive treatment of the state of the art (up to 1997) in robust discrete optimization and extensive references are presented in this work. However, there still are more open problems than solved ones. Most of the known results correspond to scenario-represented models of uncertainty, i.e. where there exists a finite number of possible scenarios each of which is given explicitly by listing the corresponding values of parameters. It is shown that most classical polynomially solvable combinatorial optimization problems lose this nice property and become NP-hard in a robust version with scenario-represented uncertainty (to appear).

- [28] **Kouvelis P. and Sayin S.** (2005). *Algorithm robust for the bicriteria discrete optimization problem*. *Annals of Operations Research*.

The authors study various definitions of robustness in a discrete scenario discrete optimization setting. It was shown that a generalized definition of robustness into which scenario weights are introduced can be used to identify the efficient solutions of multiple objective discrete optimization problems. It is proven that the solution of a pair of optimization problems, with the first of them being a robust optimization one, is always an efficient solution. Moreover, any efficient solution can be obtained as an optimal solution to a pair of such problems.

- [29] **Kozina G. and Perepelitsa V.** (1993). *Interval discrete models and multiobjectivity complexity estimates*. *Interval Computations 1*, 51 – 59.

- [30] **Kozina G. and Perepelitsa V.** (1994). *Interval spanning tree problem: solvability and computational complexity*. *Interval Computations 1*, 42 – 50.

The author study a special case of minimum spanning tree problem where the edge costs (weights) are not fixed but take their values from some intervals. No stochastic distribution is given inside intervals. The interval function is defined as the sum of interval weights over all edges of feasible spanning tree. Contrary to the classical minimum spanning tree problem which can be easily solved by the algorithms of Kruskal (1956) or Prim (1957), minimum spanning trees of the interval problem depend on weights realization and optimal objective value generally is not unique. Therefore, the authors of [30] propose to introduce the relation on the set of intervals, which gives the possibility to transform the problem into a special bicriteria counterpart. The Pareto set of the counterpart, which can be generated by standard multiobjective methods, is taken to be the solution of the interval problem. It is shown that the counterpart

problem is intractable, and it follows that the interval problem is also very hard to solve.

- [31] **Montemanni R. and Gambardella L.** (2005). A branch and bound algorithm for the robust spanning tree problem with interval data. *European Journal of Operational Research* 161, 771 – 779.
- [32] **Montemanni R. and Gambardella L.** (2004). An exact algorithm for the robust shortest path problem with interval data. *Computers and Operations Research* 31, 1667 – 1680.
- [33] **Montemanni R., Gambardella L. and Donati A.V.** (2004). A branch and bound algorithm for the robust shortest path problem with interval data. *OR Letters* 32, 225 – 232.

Montemanni et al. proposed a branch and bound algorithm for the robust spanning tree and the robust shortest path problem in [31] and [32], [33] respectively. The method embeds the extension of some result previously known in literature and some new original elements. It is claimed that the technique proposed is up to 210 faster than methods recently appeared in literature.

- [34] **Montemanni R.** (2005). A Benders decomposition approach for the robust spanning tree problem with interval data. *European Journal of Operational Research* (to appear).
- [35] **Montemanni R. and Gambardella L.** (2005). The robust shortest path problem with interval data via Benders decomposition. *4OR* 3, 315 – 328 .

Montemanni and Gambardella propose a new exact algorithm, based on Benders decomposition, for the robust spanning tree and the robust shortest path problem in [34] and [35] respectively. Computational results highlight the efficiency of the new method. It was shown that the technique is very fast on all the benchmarks considered, especially on those that were harder to solve for the methods previously known.

- [36] **Montemanni R., Barta j. and Gambardella L.** (2005). The robust traveling salesman problem with interval data. Technical Report IDSIA-20-05.

The authors present a new extension to the basic problem, where travel times are specified as a range of possible values. The robust deviation criterion is applied to drive optimization over the interval data problem so obtained. Some interesting theoretical properties of the new optimization problems are identified and presented, together with a new mathematical formulation and some exact algorithms. The exact methods are compared from an experimental point of view. The methodology proposed can be used to attack the robust counterpart of other NP-hard combinatorial optimization problems too.

- [37] **Y. Nikulin** (2005). Simulated annealing algorithm for the robust spanning tree problem, Manuskripte aus den Instituten für Betriebswirtschaftslehre No. 591, Christian-Albrechts-Universität zu Kiel.

This paper addresses the robust spanning tree problem with interval data, i.e. the case of classical minimum spanning tree problem when edge weights are not fixed but take their values from some intervals associated with edges. The problem consists in finding a spanning tree that minimizes so-called robust deviation, i.e. deviation from an optimal solution under the worst case realization of interval weights. As it was proven in [27], [12] the problem is NP-hard, therefore it is of great interest to tackle it with some metaheuristic approach, namely simulated annealing, in order to calculate an approximate solution for large scale instances efficiently. The author describes theoretical aspects and presents the results of computational experiments. This is the first attempt to develop a metaheuristic approach for solving the robust spanning tree problem.

- [38] **Y. Nikulin**. (2005). The robust shortest path problem with interval data: a probabilistic metaheuristic, Manuskripte aus den Instituten für Betriebswirtschaftslehre No. 597, Christian-Albrechts-Universität zu Kiel.

The metaheuristic approach of [37] is applied to the robust shortest path problem with interval data.

- [39] **Pinar M.** (2004). A note on robust 0 – 1 optimization with uncertain cost coefficients. *4OR* 2, 309 – 316.

Based on the approach of Bertsimas and Sim [15], [18] to robust optimization in the presence of data uncertainty, an easily computable bound on the probability that robust solution gives and objective function value worse than the robust objective function value, under the assumption that only cost coefficients are subject to uncertainty.

- [40] **Yaman H., Karasan O. and Pinar M.** (2001). The robust spanning tree problem with interval data. *Operations Research Letters* 29, 31 – 40.

A robust version of the minimum spanning tree problem under interval type of uncertainty (each edge cost can take any value in its interval, independent of the other edge) is considered. The authors propose to compute a robust spanning tree. Two types of robustness are studied. A spanning tree whose absolute worst case scenario (i.e. scenario in which the cost of this spanning tree is maximum) is minimum is called an absolute robust spanning tree. Under the second concept, a robust spanning tree whose total cost minimizes the maximum deviation from the optimal spanning tree over all realizations of the edge costs is called a relative robust spanning tree. In this paper it is proven that the absolute robust spanning tree problem can be solved in polynomial time. Additionally, a mixed-integer programming reformulation of the relative robust spanning tree problem is presented.

- [41] **Yaman H., Karasan O. and Pinar M.** (2001). The robust shortest path problem with interval data. *Comput.&Oper. Res.* (revised, 2004, available online: www.optimization-online.org/DB_HTML/2001/08/361.html)

Motivated by telecommunication applications, the author investigate the shortest path problem on directed acyclic graphs under arc length uncertainties represented as interval numbers. Using a minimax-regret criterion the authors define and identify robust paths via mixed-integer programming and exploiting interesting structural properties of the problem.

- [42] **Yaman H., Karasan O. and Pinar M.** (2004). Restricted robust optimization for maximization over uniform matroid with interval data uncertainty. *Bilkent University*.

For the problem of selecting p items with uncertain (interval) objective function coefficients so as to maximize total profit (maximization over uniform matroid) the authors introduce the r -restricted robust deviation criterion and seek solutions that minimize it. This new criterion increases the modeling power of the robust deviation (minmax regret) criterion by reducing the level of conservativeness of the robust solution. It is shown that r -restricted robust deviation solutions can be computed efficiently.

- [43] **Yu G.** (1996). On the max-min knapsack problem with robust optimization applications. *Operations Research* 44, 407 – 415.

- [44] **Yu G. and Yang J.** (1997). On the robust shortest path problem. *Comput& Oper. Res.* 25, 457 – 468.

In this paper a scenario approach is adopted to characterize uncertainties. Two robustness criteria are specified: the absolute robust criterion and the robust deviation criterion. It is shown that under both criteria the robust shortest path problem is NP-complete even for much more restricted layered networks of width 2, and with only 2 scenarios. A pseudo-polynomial algorithm is devised to solve the robust shortest path problem in general networks under bounded number of scenarios.

- [45] **Zielinski P.** (2004). The computational complexity of the relative robust shortest path problem with interval data. *European Journal of Operational Research* 158, 570 – 576.

The author shows that the robust deviation path problem is NP-hard even for planar graphs of degree 3.

Part III. Robustness in Scheduling Theory

Uncertainty in scheduling may have many reasons:

- equipment (machine) breakdowns
- activity (job) disruption
- earliness or tardiness
- changes of processing or duration time
- changes of release data, due dates or deadlines
- nature and human factors and others.

No one can guarantee the reliability of input data in the contemporary non-static permanently changing world. That is one reason why almost all modern scheduling techniques try to find a solution (being probably non-optimal but as close as possible to an optimal one), which has the property of flexibility to changes of input data. In this section we give a brief description of the literature concerning both theoretical and practical aspects of scheduling in the robustness framework.

- [46] **Al-Fawzan, M.A. and Haouari M.** (2005). A bi-objective model for robust resource-constrained project scheduling. *International Journal of Production Economics* 96, 175 – 187.

A common problem which arises in project management is the fact that the planned (baseline) schedule is often disrupted by several uncontrollable factors. In order to take into account possible disruptions and their negative consequences at the project design stage a concept of schedule robustness based on the concept of job free slack is introduced. The robustness of schedule is defined as the total sum of the free slacks, i.e. amounts of time that every activity can slip without delaying the start of the very next activity while maintaining resource availability and precedence constraints. Hence, the robustness of a schedule is understood as its ability to be maintained in the case of some undesired events influencing the realization of particular activities. Two objectives – robustness maximization and makespan minimization – are considered. A tabu search algorithm to generate an approximate set of efficient solutions is proposed.

- [47] **Ballestin F. and Leus R.** (2006). Metaheuristics for stable scheduling on a single machine. Working Paper.

A single machine scheduling problem is considered. Two metaheuristics for solving an approximate formulation of the model that assumes that exactly one job is disrupted during schedule execution are proposed. Uncertainty is modelled for job duration and primal objective is to minimize deviation between planned and actual job starting times.

- [48] **Davenport A. and Beck J.** (2000). A survey of techniques for scheduling with uncertainty. Working Paper, Toronto.
(<http://www.eil.utoronto.ca/profiles/chris/chris.papers.html>)

This paper surveys some robust scheduling techniques that have been appeared during the last decade since the well-known survey by McKay et al. about the state-of-the-art in job-shop scheduling was written. Many new approaches are discussed: redundancy-based techniques (a reservation of extra time and resources for unexpected events), probabilistic techniques (they do not construct a robust schedule, but we have the possibility to measure the probability of uncertainty and, moreover, to construct an optimal schedule so to maximize), different on-line and off-line approaches as well as rescheduling techniques.

- [49] **Jensen M.** (2001). Robust and flexible scheduling with evolutionary computation. *Ph.D. thesis, University of Aarhus, Department of Computer Science.*

This thesis presents two fundamentally different approaches for scheduling job shops facing machine breakdowns. The first method is called neighbourhood based robustness and is based on an idea of minimizing the cost of a neighbourhood of schedules. The scheduling algorithm attempts to find a small set of schedules with an acceptable level of performance. The other method for stochastic scheduling uses the idea of co-evolution to create schedules with a guaranteed worst-case performance for a known set of scenarios. The method is demonstrated to improve worst-case performance of the schedules when compared to ordinary scheduling; it substantially reduces running time when compared to a more standard approach explicitly considering all scenarios.

- [50] **Kouvelis P., Daniels R. and Vairaktarakis G.** (2000). Robust scheduling of a two-machine flow shop with uncertain processing times. *IIE Transactions* 36, 667 – 682.

This paper is one of the first attempts to introduce the concept of robustness for scheduling problems. The authors suggest a robust schedule when processing times are uncertain, but they compute this robust schedule based on maximum absolute deviation between the robust solution and all the possible scenarios, but this requires knowledge of all possible scenarios. Moreover, the optimal solution of each scenario is supposed to be known a priori.

- [51] **Lambrechts O., Demeulemeester E. and Herroelen W.** (2006). Proactive and reactive strategies for resource-constrained project scheduling with uncertain resource availability. Working Paper.

The goal of the paper is to build a robust schedule that meets the project due date and minimizes the schedule instability cost, defined as the expected weighted sum of the absolute deviations between the planned and actually realized activity starting times during project execution. The authors describe how stochastic resource breakdowns can be modeled, which reaction is recommended when a resource infeasibility occurs due to a breakdown and how one can protect the initial schedule from the adverse effects of potential breakdowns.

[52] **Leus R.** (2003). The generation of stable project plans. *PhD thesis. Department of applied economics, Katholieke Universiteit Leuven, Belgium.*

[53] **Leus R. and Herroelen W.** (2004). Robust and reactive project scheduling: a review and classification of procedures. *International Journal of Production Research* 42, 1599 – 1620.

Predictive-reactive scheduling refers to the process where a baseline schedule is developed prior to the start of the project and updated if necessary during project execution. The objective of this paper is to review possible procedures for generation of proactive (robust) schedules, which are as good as possible protected against schedule disruptions, and for deployment of reactive scheduling procedures that may be used to revise or re-optimize the baseline schedule when unexpected events occur. Finally, the authors propose a survey of the basics of critical chain scheduling and indicate in which environments it is useful.

[54] **Leus R. and Herroelen W.** (2005). The complexity of machine scheduling for stability with a single disrupted job. *Operations Research Letters* 33, 151 – 156.

[55] **Policella N., Smith S., Cesta A. and Oddi A.** (2004). Generating robust schedules through temporal flexibility. *Proceedings 14th International Conference on Automated Planning and Scheduling, Whistler CA.*

This paper considers the problem of generating partial order schedules that retain temporal flexibility and thus provide some degree of robustness. Two different orthogonal procedures for construction a POS were proposed. The first, which the authors call the resource envelope based approach, uses computed bounds on cumulative resource usage (i.e., a resource envelope) to identify potential resource conflicts, and progressively winnows the total set of temporally feasible solutions into a smaller set of resource feasible solutions by resolving detected conflicts. The second, referred to as the earliest start time approach, instead uses conflict analysis of a specific (i.e., earliest start time) solution to generate an initial fixed-time schedule, and then expands this solution to a set of resource feasible solutions in a post-processing step.

[56] **Sevaux M. and Sörensen K.** (2004). A genetic algorithm for robust scheduling in a just in-time environment. *Technical Report LAMIH, SR-2003-1, University of Valenciennes, France.*

It is shown how a robust genetic algorithm can be applied to single machine scheduling problem when release dates are subject to small variations. This method leads to a robust solution, meaning that the value of the objective function remains high when small variations in some release dates occur.

- [57] **Sevaux M. and Zann Le Quere.** (2003). Solving a robust maintenance scheduling problem at the French railways company. *Technical Report LAMIH, SR-2003-3, University of Valenciennes, France.*

This paper gives an extension of the results derived by Sevaux and Sørensen for a different scheduling problem with a different objective. The authors focus on solving a robust scheduling problem at the French railway company, which can be modelled as a special case of a resource-constrained scheduling problem with additional constraints. The objective is to construct a robust schedule, i.e. a sequence of the tasks on each resource for which the makespan value can be predicted when the duration of the task is increased.

- [58] **Sørensen K.** (2001). Tabu search for robust solutions. In *Proceedings of the 4th Metaheuristics International Conference, Porto, Portugal*, 707 – 712.

The robust tabu search firstly introduced in this paper is a new and original technique based on ideas taken from theory of robust optimization for continuous mathematical functions.

- [59] **Tsutsui S. and Ghosh A.** (1997). Genetic algorithms with a robust solution searching scheme. *IEEE Transactions on Evolutionary Computation* 1, 201 – 208.

- [60] **Tsutsui S. and Jain J.C.** (1998). Properties of robust solution searching in multi-dimensional space with genetic algorithms. In *Proceedings of the 2nd International Conference on Knowledge-Based Electronic Systems.*

- [61] **Van de Vonder S., Demeulemeester E., Herroelen W. and Leus R.** (2005). The use of buffers in project management. The trade-off between stability and makespan. *International Journal of Production Economics* 97, 227 – 240.

The case of stochastic activity duration is considered. The concept of buffers is introduced to protect an optimal schedule from disruption.

There are also some interesting papers devoted to the concept of super solutions for constraint programming: Super solutions are a mechanism to provide robustness. They are solutions in which, if a small number of variables lose their values, one can guarantee to be able to repair the solution with only a few changes.

- [62] **Hebrard E, Hnich B. and Walsh T.** (2004). Super solutions in constraint programming. In *Proceedings of CP-AI-OR 2004*, 157 – 172.

- [63] **Holland A. and O'Sullivan B.** (2004). Weighted Super Solutions for Constraint Programs. Technical Report UCC-CS-2004-12-02.

[64] **Mehta S. and Uzsoy R.** (1998). Predictive scheduling of a job shop subject to breakdowns. *IEEE Transactions on Robotics and Automation* 14, 365 – 378.

[65] **Mehta S. and Uzsoy R.** (1998). Predictable scheduling of a single machine subject to breakdowns. *Int. J. Computer Integrated Manufacturing* 12, 15 – 38.

Finally, we would like to mention the text book [66] (written and available in German only), which contains a comprehensive description of a large variety of robustness concepts.

[66] **Scholl, A.** Robust scheduling and Optimization: Basics, Concepts and Methodology, Physics, Heidelberg, 2001 (in German: Robuste Planung und Optimierung: Grundlagen, Konzepte und Methoden, Experimentelle Untersuchungen. Physica-Verlag, Heidelberg).

Part IV. Robustness in Economics

Additionally, we give a short list of references on robustness and stability in economics, namely in portfolio optimization, supply chain management, master production scheduling, lot-sizing and monetary policy.

- [67] **Aghassi M. and Bertsimas D.** (2006). Robust game theory. *Mathematical Programming. Ser. B* 107, 231 – 273.

- [68] **Atamturk A. and Zhang M.** (2004) Two-stage robust network flow and design for demand uncertainty. *Research Report, University of California at Berkeley.*
The authors describe a two-stage robust optimization approach for solving network flow and design problems with uncertain demand. The approach is generalised to multi-commodity network flow and applications to lot-sizing and location-transportation problems are presented.

- [69] **Ben-Tal A., Margalit T. and Nemirovski A.** (2000). Robust modelling of multi-stage portfolio problems. In K. Roos, T. Terlaky, and S. Zhang, editors, *High Performance Optimization*, Kluwer Academic Publisher, Dordrecht, 303 – 328.

- [70] **Ben-Tal A., Golany B., Nemirovski A. and Vial J.-P.** (2003). Supplier-retailer flexible commitments contracts: A robust optimization approach. *Technical Report, Dept. of Management Studies, University of Geneva.*

- [71] **Bertsimas D. and Thiele A.** (2003). A robust optimization approach to supply chain management. *Technical Report, MIT.*

- [72] **Bertsimas D. and Thiele A.** (2006). A robust optimization approach to inventory theory. *Operations Research* 54, 150 – 168.

- [73] **Butler R., Ammons J. and Sokol J.** (2004). A robust optimization model for strategic production and distribution planning for a new product. *Working Paper, University of Central Florida.*

- [74] **El Ghaoui L., Oks M. and Oustry F.** (2003). Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. *Operations Research* 51, 543 – 556.

- [75] **Erdogan E. and Iyengar G.** (2004) Ambiguous chance constrained problems and robust optimization. *CORC Technical Report TR-2004-10, Columbia University.*

- [76] **Erdogan E., Goldfarb D. and Iyengar G.** (2004). Robust portfolio management. *CORC Technical Report TR-2004-11, Columbia University.*

- [77] **Giannoni M.** (2002). Does model uncertainty justify caution? Robust optimal monetary policy in a forward-looking model. *Macroeconomic Dynamics* 6, 111 – 144.
- [78] **Goldfarb D. and Iyengar G.** (2003). Robust portfolio selection problems. *Mathematics of Operations Research* 21, 1 – 38.
- [79] **Kimms, A.** (1998). Stability measures for rolling schedules with applications to capacity expansion planning, master production scheduling, and lot sizing. *OMEGA* 26, 355 – 366.
- [80] **Levin A. and Williams J.** (2003). Robust monetary policy with competing reference models. *Working Paper, Federal Reserve Bank of San Francisco.*
- [81] **Lutgens F. and Sturm J.** (2002). Robust option modelling. *Technical Report, University of Maastricht.*
- [82] **Popescu I.** (2003). Robust mean-covariance solutions for stochastic optimization. *Technical Report, INSEAD.*
- [83] **Schyder L.** (2005). Facility location under uncertainty: a review. *Technical Report, 04T-015, Department of Industrial and Systems Engineering, Lehigh University. USA*
- This paper reviews the literature on stochastic and robust facility location models. It illustrates the rich variety of approaches for optimization under uncertainty and their application to facility location problem.
- [84] **Yu C. and Li H.** (2000). A robust optimization model for stochastic logistic problems. *International Journal of Production Economics* 64, 385 – 397.
- [85] **Yu G.** (1997). Robust economic order quantity models. *European Journal of Operational Research* 100, 482 – 493.