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A Note on capacitated lot sizing with setup carry-over

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Abstract

The "Capacitated lot sizing problem with setup carry-over" is based on the well known "Capacitated lot sizing problem" and incorporates the possibility of preserving a setup-state from a period to the following. Sox and Gao [2] decompose their formulation of the first one (GCLP1) by Lagrangian relaxation. For obtaining a heuristic solution of the GCLP1 they use subgradient optimization and a dynamic programming algorithm to solve the corresponding subproblems optimally.
The present paper elucidates that this algorithm does not necessarily provide the optimal solution of the subproblem. Additionally, two improved approaches are presented. The first is an extension of the one proposed by [2] while the second is supplementary based on a slight modification of the underlying model.

1 Introduction

Lot sizing is one of the central planning activities in production management. One of the basic lot-sizing models, the "Capacitated lot-sizing problem", assumes a setup of the resource at each production's start of an arbitrary item. The "Capacitated lot sizing problem with setup carry-over" contains the possibility of conservation of a single setup-state between two consecutive periods. In order to represent this aspect, further restrictions and a new class of binary variables are introduced. Thus, complexity increases and, analogously, the need for heuristic solution methods. Sox and Gao propose one which employs Lagrangian relaxation, subgradient optimization and a method to solve the resulting subproblems optimally. However, not every aspect of the subproblems is regarded which might result in suboptimal solutions.
This paper proceeds as follows: Section 2 provides a short introduction into the model formulation and the formalisms of [2], in section 3 we present characteristics of the aspects ignored, and in section 4 improved approaches are proposed and commented.

2 The model

In order to keep this paper self-contained, we shortly introduce the model, notation and heuristic method of [2]:

GCLP1

$$\min \sum_{i=1}^{N} \sum_{t=1}^{T} (K_i z_{i,t} + p_{i,t} x_{i,t} + h_{i,t} I_{i,t}) \tag{1}$$

subject to:

$$\begin{aligned}
I_{i,t-1} + x_{i,t} - d_{i,t} &= I_{i,t} && \forall\, i,t && (2)\\
\sum_{i=1}^{N} a_i x_{i,t} &\leq C_t && \forall\, t && (3)\\
x_{i,t} &\leq M_{i,t} \cdot (z_{i,t} + \zeta_{i,t}) && \forall\, i,t && (4)\\
\sum_{i=1}^{N} \zeta_{i,t} &= 1 && \forall\, t \geq 2 && (5)\\
\zeta_{i,t} - z_{i,t-1} &\leq 0 && \forall\, i, t \geq 2 && (6)\\
I_{i,t},\ x_{i,t} &\geq 0 && \forall\, i,t && (7)\\
z_{i,t},\ \zeta_{i,t} &\in \{0,1\} && \forall\, i,t && (8)
\end{aligned}$$

a_i	consumption of capacity per unit of item i
C_t	capacity in period t
$d_{i,t}$	demand of item i in period t
$h_{i,t}$	inventory cost per unit of item i at the end of period t
$I_{i,t}$	inventory of item i at the end of period t
K_i	setup-cost for item i
$M_{i,t}$	big number ($M_{i,t} = \sum_{s=t}^{T} d_{i,s}$)
$p_{i,t}$	variable production cost per unit of item i in period t
$x_{i,t}$	amount of production of item i in period t
$z_{i,t}$	$\begin{cases} 1 \text{ if the resource is set up for item } i \text{ in period } t \\ 0 \text{ otherwise} \end{cases}$
$\zeta_{i,t}$	$\begin{cases} 1 \text{ if the setup-state for item } i \text{ is preserved into period } t \\ 0 \text{ otherwise} \end{cases}$

The objective to minimize total costs is represented by (1). The solution space is defined by the inventory balance equation (2), the capacity-restriction (3), the setup-state being a condition for production (4), the limitation to exactly one carry-over per period (5), the relation of setup and carry-over (6) and the variable domains in (7) and (8).
Sox and Gao [2] use Lagrangian relaxation in order to eliminate (3) and (5) and to gain N independent single item problems, where π_t and $\lambda_{i,t}$ are Lagrangian multipliers:

GCLP1$_i(\pi, \lambda)$

$$\min \sum_{t=1}^{T} \left(K_i z_{i,t} + (p_{i,t} + a_i \pi_t) x_{i,t} + h_{i,t} I_{i,t} + \lambda_t \zeta_{i,t}\right) \tag{9}$$

subject to:

$$(2), (4), (6), (7), (8)$$

These single-item-models are very similar to the single-item-problem of Wagner and Whitin [4]. As pointed out in [2], the only difference is the possibility to preserve a setup-state from one period to the following. According to this Sox and Gao propose a basic dynamic programming (BDP) approach, which takes carry-over into account:

BDP

$$f_i(t) = \min\Bigg\{$$

$$K_i + a_i\pi_{t-1}d_{i,t-1} + \lambda_t + a_i\pi_t d_{i,t} + f_i(t-2), \tag{10}$$

$$K_i + a_i\pi_t d_{i,t} + f_i(t-1), \tag{11}$$

$$\min_{s<t-1}\Bigg\{K_i + \sum_{k=s+1}^{t-1}\sum_{j=s}^{k-1} h_{i,j}\cdot d_{i,k} + a_i\pi_s \sum_{k=s}^{t} d_{i,k} + f_i(s-1)\Bigg\} \tag{12}$$

$$\Bigg\}$$

In order to determine an optimal lot sizing schedule with t periods all lot sizing schedules with $s < t$ periods are considered and supplemented in [2]. However, BDP does not regard every possible lot sizing schedule with t periods, so its result is not necessarily optimal. Accordingly, solving the subproblems does not guarantee finding a lower bound which is substantial for subgradient optimization.

3 Analysis of the basic dynamic programming approach

The variable production costs in (10), (11) and (12) are not correct according to the objective function (9) of the underlying model, because the variable production costs of the GCLP1 $p_{i,t}$ are ignored. The dynamic programming approach is modified retaining the basic idea at first. A further, slight modification changes the sum at the first appearance of index k in (12) from $\sum_{k=s+1}^{t-1}$ to $\sum_{k=s+1}^{t}$. This has to be done, because otherwise it would not be possible to calculate inventory-cost for the demand in period t. With these two replacements we get:

BDP1

$$f_i(t) = \min\Bigg\{$$

$$K_i + (p_{i,t-1} + a_i\pi_{t-1})d_{i,t-1} + \lambda_t + (p_{i,t} + a_i\pi_t)d_{i,t} + f_i(t-2), \tag{13}$$

$$K_i + (p_{i,t} + a_i\pi_t)d_{i,t} + f_i(t-1), \tag{14}$$

$$\min_{s<t-1}\Bigg\{K_i + \sum_{k=s+1}^{t}\sum_{j=s}^{k-1} h_{i,j}\cdot d_{i,k} + (p_{i,s} + a_i\pi_s)\sum_{k=s}^{t} d_{i,k} + f_i(s-1)\Bigg\} \tag{15}$$

$$\Bigg\}$$

Inspecting these recurrence terms, three classes of lot sizing schedules can be identified which are not taken into account when determining the optimal one.

3.1 Two-periods-production after setup

In order to find an optimal lot sizing schedule with t periods, the dynamic programming approach does not regard a schedule which consists of the optimal schedule with $s = t-2$ periods, a setup in period $s+1$ and production in $s+1$ amounting to demand of periods $s+1$ to t. All cases $s < t \wedge s \neq t-2$ are covered by terms (14) and (15). An example for an instance of problem $GCLP1_i(\pi, \lambda)$ which has an optimal solution as described above is given in the following:

$$\begin{aligned}(d_{1,t}) &= (\ 10,\ 10,\ 10)\\ (p_{1,t} + a_1\pi_t) &= (\ \ 1,\ \ 1,\ 10)\\ (\lambda_{1,t}) &= (100,100,100)\\ (h_{1,t}) &= (100,\ \ 1,\ \ 1)\\ K_1 &= \ \ 50.\end{aligned}$$

One can summarize all possible lot sizing schedules of [2] in three classes as follows:

- full production in period 1 means inventory-cost of 3000
- setup and production in every single period results in setup-cost of 150
- two setups and one carry-over leads to cost of 200

So costs sum up to at least 150. Nevertheless, there exists a lot sizing schedule which means cost of 140 and therefore is cheaper than every schedule considered by [2].

$$\begin{aligned}(x_{1,t}) &= (10,20,0)\\ (I_{i,t}) &= (\ 0,10,0)\\ (\zeta_{1,t}) &= (\ 0,\ 0,0)\\ (z_{1,t}) &= (\ 1,\ 1,0)\end{aligned}$$

Since it contains a two-periods-production after setup in period two, it is not considered in [2].

3.2 Several-periods-production after carry-over

In each lot sizing schedule considered in [2] it is not possible to produce the demand of periods s to t after the setup-state is carried over into s, except for the case $s = t$. The latter case is covered by term (13).
An example for an instance of problem $GCLP1_i(\pi, \lambda)$ which is not optimally solved by [2] is the following.

$$\begin{aligned}(d_{1,t}) &= (\ 10{,}10,\ 10)\\ (p_{1,t} + a_1\pi_t) &= (\ \ 1,\ 1{,}100)\\ (\lambda_{1,t}) &= (\ \ 1,\ 1{,}100)\\ (h_{1,t}) &= (100,\ 1{,}100)\\ K_1 &= \ \ 100\end{aligned}$$

We can summarize all possible lot sizing schedules of [2] as shown in section 2.1.
This leads to an amount of cost of at least 200. A solution with a "several-periods-production after carry-over" has costs of 141.

$$\begin{aligned}(x_{1,t}) &= (10{,}20{,}0)\\ (I_{i,t}) &= (\ 0{,}10{,}0)\\ (\zeta_{1,t}) &= (\ 0,\ 1{,}0)\\ (z_{1,t}) &= (\ 1,\ 0{,}0)\end{aligned}$$

Because of its structure the approach in [2] can not find it.

3.3 "Useless carry-over"

The costs for preserving a setup-state to the following period are given by the Lagrangian multipliers λ_t. They are used to relax the constraints of the original problem which set the number of carry-overs per period to one and therefore are equality-constraints. Thus, the multipliers λ_t are not restricted to nonnegative values. So there might be a "benefit" from a carry-over and the optimal solution may contain a carry-over into a period s with $\lambda_s < 0$ even if there is no need for the setup-state in s. Even more it is possible to setup a resource in t just for carrying over the setup-state into $t+1$ without production in both periods. Examples for both cases are outlined below:
First we regard an example for an "useless carry-over" from a period in which production takes place.

$$\begin{aligned}(d_{1,t}) &= (\ 10,\ 10,\ 10)\\ (p_{1,t} + a_1\pi_t) &= (\ \ 1{,}100{,}100)\\ (\lambda_{1,t}) &= (-1, -1, -1)\\ (h_{1,t}) &= (\ \ 1,\ \ 1,\ \ 1)\\ K_1 &= \ \ 100\end{aligned}$$

Obviously, producing the whole demand in period one is part of the optimal solution. Furthermore, preserving the setup-state lowers the objective value by one, so in result

the optimal solution is:

$$
\begin{aligned}
(x_{1,t}) &= (30,\ 0,0) \\
(I_{i,t}) &= (20,10,0) \\
(\zeta_{1,t}) &= (\ 0,\ 1,0) \\
(z_{1,t}) &= (\ 1,\ 0,0)
\end{aligned}
$$

The optimal solution of the following example contains a setup just for the purpose of preserving the setup-state into the following period.

$$
\begin{aligned}
(d_{1,t}) &= (\ 10,0,\ \ \ 0) \\
(p_{1,t} + a_1\pi_t) &= (\ \ 1,1,\ \ \ 1) \\
(\lambda_{1,t}) &= (\ \ 1,1,-101) \\
(h_{1,t}) &= (\ \ 1,1,\ \ \ 1) \\
K_1 &= \ 100
\end{aligned}
$$

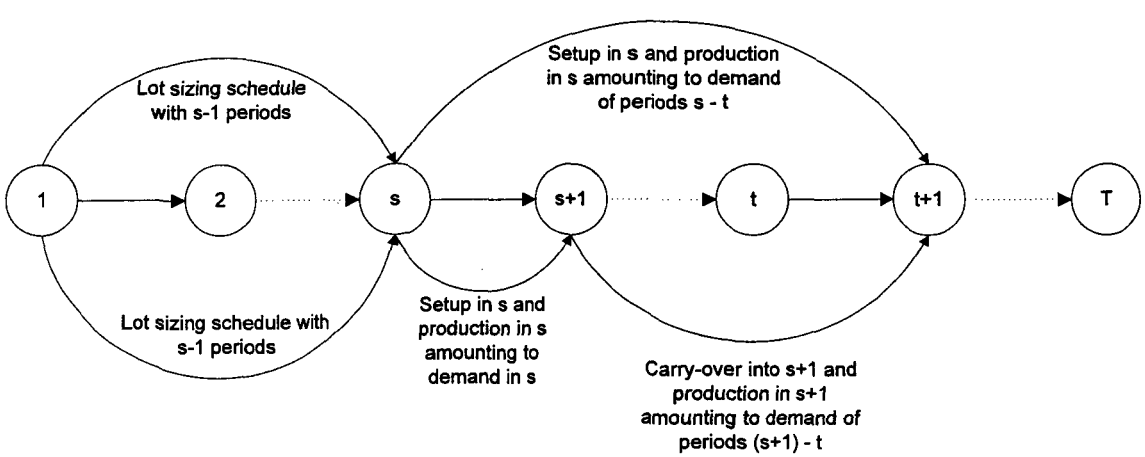
Trivially, all production takes place in period one, but there is a further setup and a carry-over.

$$
\begin{aligned}
(x_{1,t}) &= (10,0,0) \\
(I_{i,t}) &= (\ 0,0,0) \\
(\zeta_{1,t}) &= (\ 0,0,1) \\
(z_{1,t}) &= (\ 1,1,0)
\end{aligned}
$$

Because of the solution structures the approach by [2] can not find it.

4 Improved dynamic programming approach

According to the subsections of section 2 we present a dynamic programming approach which considers the missing lot sizing schedules in the following three subsections. Since Sox and Gao complement a shorter lot sizing schedule either by setup or carry-over, the basic idea of 3.1 and 3.2 is to complement it both ways.



Dynamic programming approach with two-way-complementation

In the figure the nodes represent the periods of $GCLP1_{\pi,\lambda}$. An arc between two periods, say from q to r, represents a production in q meeting the demand of periods q to r. The arcs above the nodes stand for the strategy to complement a known lot sizing schedule by setup and production as known from [4]. The arcs below the nodes illustrate the idea of [2] to enlarge a known lot sizing schedule by setup, production, carry-over and further production.

4.1 Two-periods-production after setup

For taking into account the lot sizing schedules mentioned in section 2.1 we propose a generalization of (14) and (15):

$$\min_{s\leq t}\left\{K_i + \sum_{k=s+1}^{t}\sum_{j=s}^{k-1} h_{i,j}d_{i,k} + (p_{i,s} + a_i\pi_s)\sum_{k=s}^{t} d_{i,k} + f_i(s-1)\right\} \tag{16}$$

Setting $s = t$ and $s < t-1$ in (16), respectively, we get terms (14) and (15). Additionally, $s = t-1$ covers the "two-periods-production after setup".

4.2 Several-periods-production after carry-over

In order to include the "several-periods-production after carry-over" term (13) is generalized:

$$\min_{s\leq t-1}\left\{K_i + (p_{i,s} + a_i\pi_s)d_{i,s} + \lambda_{s+1} + \sum_{k=s+2}^{t}\sum_{j=s+1}^{k-1} h_{i,j}d_{i,k} + (p_{i,s+1} + a_i\pi_{s+1})\sum_{k=s+1}^{t} d_{i,k} + f_i(s-1)\right\} \tag{17}$$

If $s = t-1$ holds, then (17) covers (13). Additionally, it includes the cases mentioned in section 2.2 for $s < t-1$.

4.3 "Useless carry-over"

In the following two possibilities to take care of the "useless carry-over"-phenomenon are presented.

4.3.1 Extension of the dynamic programming approach

For considering the lot sizing schedules mentioned in section 2.3, terms (16) and (17) have to be extended:

$$\min_{s\le t}\Bigg\{K_i + \sum_{k=s+1}^{t}\sum_{j=s}^{k-1} h_{i,j}d_{i,k} + (p_{i,s}+a_i\pi_s)\sum_{k=s}^{t} d_{i,k} + \underbrace{\min(0,\lambda_{s+1})}_{\text{iff } s+1\le t} + \sum_{r=s+2}^{t}\min(0,K_i+\lambda_r) + f_i(s-1)\Bigg\} \tag{18}$$

Obviously, (18) is an extension of (16). It is possible to preserve the setup-state into period $s+1$ if and only if $s+1 \le t$ holds. Furthermore, one can setup the resource in every single following period if the "benefit" for preserving it into the next one is higher than the setup-cost.

$$\min_{s\le t-1}\Bigg\{K_i + (p_{i,s}+a_i\pi_s)d_{i,s} + \lambda_{s+1} + \sum_{k=s+2}^{t}\sum_{j=s+1}^{k-1} h_{i,j}d_{i,k} + (p_{i,s+1}+a_i\pi_{s+1})\sum_{k=s+1}^{t} d_{i,k} + \sum_{r=s+2}^{t}(\min(0,K_i+\lambda_r)) + f_i(s-1)\Bigg\} \tag{19}$$

The latter is based on the same idea as (18) and corresponds to (17).

4.3.2 Modification of the underlying model

"Useless carry-overs" do not make any sense according to the objective function (9) if their costs are nonnegative.
If we relax the underlying model "GCLP1" by allowing less or equal to one carry-over per period instead of exactly one, the corresponding multipliers λ_t are restricted to nonnegative values. In every other aspect the relaxed model is exactly the same as the Lagrangian relaxation in [2]. Therefore, carry-overs only have to be taken into account if production takes place in the current period. So it is sufficient if the dynamic programming approach considers the lot sizing schedules mentioned in sections 2.1 and 2.2.

4.4 Complete recurrence terms

The results of sections 3.1 to 3.3 can be summarized as two formulations of the dynamic programming approach. The first one, DP1, considers "useless carry-overs" according to section 3.3.1:

DP1

$$f_i(t) = \min\Bigg\{ \min_{s \le t}\Bigg\{ K_i + \sum_{k=s+1}^{t} \sum_{j=s}^{k-1} h_{i,j} d_{i,k} + (p_{i,s} + a_i \pi_s) \sum_{k=s}^{t} d_{i,k} + \underbrace{\min(0, \lambda_{s+1})}_{\text{iff } s+1 \le t} + \sum_{r=s+2}^{t} (\min(0, K_i + \lambda_r)) + f_i(s-1) \Bigg\},$$

$$\min_{s \le t-1}\Bigg\{ K_i + (p_{i,s} + a_i \pi_s) d_{i,s} + \lambda_{s+1} + \sum_{k=s+2}^{t} \sum_{j=s+1}^{k-1} h_{i,j} d_{i,k} + (p_{i,s+1} + a_i \pi_{s+1}) \sum_{k=s+1}^{t} d_{i,k} + \sum_{r=s+2}^{t} (\min(0, K_i + \lambda_r)) + f_i(s-1) \Bigg\}$$

$$\Bigg\}$$

The second one, DP2, is based on the model reformulation mentioned in section 3.3.2 and can be stated as follows:

DP2

$$f_i(t) = \min\Bigg\{ \min_{s \le t}\Bigg\{ K_i + \sum_{k=s+1}^{t} \sum_{j=s}^{k-1} h_{i,j} d_{i,k} + (p_{i,s} + a_i \pi_s) \sum_{k=s}^{t} d_{i,k} + f_i(s-1) \Bigg\},$$

$$\min_{s \le t-1}\Bigg\{ K_i + (p_{i,s} + a_i \pi_s) d_{i,s} + \lambda_{s+1} + \sum_{k=s+2}^{t} \sum_{j=s+1}^{k-1} h_{i,j} d_{i,k} + (p_{i,s+1} + a_i \pi_{s+1}) \sum_{k=s+1}^{t} d_{i,k} + f_i(s-1) \Bigg\}$$

$$\Bigg\}$$

Obviously, taking into account a larger solution space will most likely result in longer runtime requirements. But this increase is not significant, because only twice as much lot sizing schedules have to be evaluated. The runtime requirement of DP2 is greater than the one of DP1, because it checks for "profitable" setups additionally.
At last the reformulation of the lot sizing model, upon which DP2 is based, is to be commented. By enlarging the solution space it does not exclude feasible solutions of [2]. Moreover, restricting the carry-overs per period to at most one seems to be more realistic than a restriction to exactly one. There are several scenarios where no carry-over into a specific period is needed, e.g. if there is no demand at all. Equivalent restrictions can be found in several formulations of the "Capacitated lot sizing problem with setup carry-over", for example in [1] and [3].

References

[1] Haase, K., *Lotsizing and scheduling for production planning*, Springer Verlag, Berlin, 1994.

[2] Sox, C. and Gao, Y., *The capacitated lot sizing problem with setup carry-over*, IIE Transactions **31** (1999), 173–181.

[3] Suerie, C. and Stadtler, H., *The capacitated lot-sizing problem with linked lot sizes*, Management Science **49** (2003), no. 8, 1039–1054.

[4] Wagner, H. and Whitin, T., *Dynamic version of the economic lot size model*, Management Science **5** (1958), 89–96.