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Combinatorial Optimisation Problems of the Assignment Type and a Partitioning Approach

Andreas Klose, Andreas Drexl

Abstract

Assignment type problems consist in optimally assigning or allocating a given set of "activities" to a given set of "resources". Optimisation problems of the assignment type have numerous applications in production planning and logistics. A popular approach to solve such problems or to compute lower bounds on the optimal solution value (in case of a minimisation problem) is to employ column generation. By means of considering subsets of "activities" which can feasibly be assigned to a single resource, the problem is reformulated as some kind of set-partitioning problem. Column generation is then used in order to solve the linear relaxation of the reformulation. The lower bound obtainable from this approach may, however, be improved by partitioning the set of resources into subsets and by considering subsets of activities which can feasibly be assigned to subsets of resources. This paper outlines the application of this partitioning method to a number of important combinatorial optimisation problems of the assignment type.

1 Introduction

An optimisation problem of the assignment type is to find a feasible least-cost assignment of a given set I of "activities" i to a given set J of "resources" j. Since a seminal work of Koopmans and Beckmann (1957) on linear and quadratic assignment problems, optimisation problems of the assignment type play an important role in Economics and Operations Research. Assignment type problems have numerous applications in logistics and production planning, and many mathematical models of decision problems in logistics contain an assignment type problem as a subproblem.

Formally, an assignment or allocation of activities $i \in I$ to resources $j \in J$ is a mapping $x : I \times J \longrightarrow [0,1]$. An assignment problem then consists in determining a feasible assignment $x \in X$ which minimises a given objective function $g(x)$. Assuming that the objective function $g(x)$ as well as the set X of feasible assignments x is decomposable, that is $g(x) = \sum_{j\in J} g_j(x)$ and $X = \bigcup_{j\in J} X_j$, the problem can be formulated as follows:

$$\min \sum_{j\in J} g_j(x_j) \tag{1a}$$

$$\text{s.t.:} \sum_{j\in J} x_{ij} = 1 \quad \forall i \in I \tag{1b}$$

$$x_j \in X_j \subseteq [0,1]^{|I|} \quad \forall j \in J\,. \tag{1c}$$

In addition, it is assumed that either X is a nonempty finite set or that each X_j is a polyhedron and that each $g_j(x_j)$ is concave.

Assignment problems often involve indivisibilities. Indivisibilities occur

- if an activity has to be assigned to exactly one resource and/or
- a fixed-charge is imposed on the use of a resource.

Indivisibilities often lead to strong $\mathcal{NP}$-hard combinatorial optimisation problems. In order to solve such problems to optimality, the computation of sharp lower bounds on the optimal solution value is required. One possible way to accomplish this, is to reformulate the problem. A popular reformulation is to consider all subsets of activities feasibly assignable to each single resource j. The problem then consists in choosing subsets of activities in such a way that each activity is covered by one of the selected subsets and total costs are minimised. The lower bound resulting from the linear relaxation of this reformulation may, however, be improved by means of partitioning the set of resources into (small) subsets and considering subsets of activities feasibly assignable to subsets of resources. In this paper, this partitioning approach is outlined

for some $\mathcal{NP}$-hard optimisation problems of the assignment type. The next section describes some important $\mathcal{NP}$-hard assignment problems. Sect. 3 discusses the standard way of applying column generation to these problems. Sect. 4 introduces the partitioning approach and outlines column generation procedures based on this approach for the generalised assignment problem (GAP) and the capacitated facility location problem (CFLP). Furthermore, some computational results obtained for the CFLP are given. Finally, the findings and some directions for future research are summarised in Sect. 5.

2 Optimisation Problems of the Assignment Type

Formulation (1) of a generic assignment problem excludes optimisation problems with interactions between assigned objects like the quadratic assignment problem. Nevertheless, formulation (1) still covers a large number of assignment type problems ranging from polynomial solvable cases like the matching and transportation problem to strong $\mathcal{NP}$-hard optimisation problems like the generalised assignment problem (GAP), bin-packing problem (BPP), fixed-charge transportation problem (FCTP), and discrete location problems. Furthermore, many combinatorial assignment problems are difficult in the sense that no polynomial-time approximation scheme (PTAS) exists, unless the class $\mathcal{P}$ of decision problems solvable in polynomial time equals the set $\mathcal{NP}$ of decision problems solvable in nondeterministic polynomial time. Optimisation problems of this type are called MAX-$\mathcal{SNP}$-hard. An example of a MAX-$\mathcal{SNP}$-hard assignment type problem is the uncapacitated facility location problem (see Arora and Lund (1997) for a proof). Moreover, some assignment type problems are $\mathcal{APX}$-complete, that is the existence of a PTAS for such an optimisation problem implies the existence of a PTAS for any other $\mathcal{NP}$-hard optimisation problem. An example of an $\mathcal{APX}$-complete optimisation problem is the GAP. In the case of the GAP, even the question if a feasible solution exists, is an $\mathcal{NP}$-complete problem. For an overview on complexity theory of optimisation problems see Crescenzi and Kann (1998) and Hochbaum (1997). The following discussion is restricted to "difficult" assignment problems. Some of these problems and areas of applications are described below.

2.1 Generalised Assignment Problem

The GAP is to optimally assign a set $I = \{1, \ldots, m\}$ of tasks to a set $J = \{1, \ldots, n\}$ of agents. Mathematically, the problem can be formulated as follows

$$\min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \tag{2a}$$

$$\text{s.t.:} \sum_{j \in J} x_{ij} = 1 \quad \forall i \in I \tag{2b}$$

$$\sum_{i \in I} d_{ij} x_{ij} \leq s_j \quad \forall j \in J \tag{2c}$$

$$x_{ij} \in \{0, 1\} \quad \forall i \in I, j \in J, \tag{2d}$$

where c_{ij} is the cost of assigning task i to agent j, s_j is agent's j capacity, and d_{ij} denotes the amount of resources required by agent j to perform task i. The binary variable x_{ij} is equal to 1 if agent j performs task i and 0 otherwise. In case that processing tasks by agents requires more than one type of resource, the problem is known as the multi-resource GAP.

A large number of solution methods for the GAP have been proposed in the literature. Fisher et al. (1986) and Guignard and Rosenwein (1989) use Lagrangian ascent methods, that is dual-based procedures employing Lagrangian relaxation of the semi-assignment constraints (2b). Cattrysse et al. (1994) reformulate the GAP as a set-partitioning problem and apply a heuristic

procedure for its solution. Savelsbergh (1997) solves the linear relaxation of this reformulation by means of column generation, and proposes a branch-and-price algorithm for computing optimal solutions. Cattrysse et al. (1998) develop a linear programming heuristic and a branch-and-cut approach based on employing liftet cover inequalitities. Heuristic methods based on local search, tabu search and simulated annealing are described in Osman (1995). For more comprehensive surveys of solution methods for the GAP, we refer to Martello and Toth (1990a), Cattrysse and Van Wassenhove (1992), and Romero Morales (2000).

The GAP is often a subproblem of a larger model of a decision problem in logistics. Examples of such application areas are vehicle routing and distribution system design. Fisher and Jaikumar (1981) describe a "cluster first–route second" vehicle routing heuristic. In a first phase, customer clusters are obtained by means of selecting "seed customers" and solving a GAP in order to feasibly assign the other customers to the selected seeds. Within distribution system design, a GAP has to be solved in order to evaluate selected locations of depots and to optimally assign customer demands to depots if single-sourceing of customers is required. Campell and Langevin (1995) use a mathematical model which is a 2-resource GAP in order to find a low cost assignment of snow removal sectors to snow disposal sites for the City of Montreal. The GAP is a static model of demand allocation. Romero Morales (2000) describes a dynamic model for demand allocation. The model allows time-varying demand patterns and incorporates inventory decisions. The resulting "multi-period single-sourceing problem" is reformulated as a GAP with a convex objective function; greedy heuristics resembling greedy heuristics for the GAP are developed and a branch-and-price approach is proposed.

Another application area of the GAP is tactical and operational planning of flexible manufacturing systems (FMS). A FMS is an automated production system consisting of a set of numerically controlled machines interconnected by an automated transportation system. Each machine is equipped with a tool magazine which can be armed with different tools. Each tool occupies a given number of slots. Tools available at the local tool magazine can be changed by means of an automatic tool changer. Therefore, the FMS can perform different sets of operations and produce different parts in any order, if the FMS is equipped with an approapriate set of tools. Lee and Kim (1998) formulate the order selection problem as a 2-resource GAP. Each order is specified by the due date and the number of parts to be produced. A set of orders to be produced during the planning horizon is given. It has to be decided, which order to produce in which period. Total costs consist of earliness and tardiness costs as well as subcontracting costs. Earliness and tardiness costs are incurred if an order is finished before and after the due date, respectively. Subcontracting costs, however, are incurred if an order is not selected within the planning horizon. Capacity constraints to be taken into account in each period are the total tool magazine capacity as well as total machine processing time capacity. Kuhn (1995) considers the loading problem in FMSs. For a given set of part types that have been selected to be produced simultaneously, the problem is to decide on the assignment of operations to machines. The assignment of operations determines the assignment of tools to machines. The problem is formulated as an integer program with the objective of minimising the largest workload in such a way that the tool magazine capacity for each machine cannot be violated. In order to solve the problem, Kuhn proposes a heuristic algorithm based on repeatedly solving a GAP. A feasible solution to the GAP assigns operations to machines such that the largest workload of machines in a known feasible solution to the loading problem is reduced. The objective function of the GAP approximates the use of tools slots required by the operations assigned to the machines.

In telecommunication or computer networks, terminals are often connected to the access point of a "backbone network" via so-called concentrators. The terminal layout problem addresses the question of how to interconnect terminals to their associated concentrors. Due to the complexity of the telecommunication network design problem, the concentrator location and the terminal layout problem are usually treated independently. For given locations of the

concentrators, the problem of assigning terminals to concentrators can be formulated as a GAP (see e. g. Mirzaian (1985) and Chardaire (1999)). Other applications of the GAP in the area of telecommunication and computer networks concern e. g. the assignment of tasks in a network of functionally similar computers (Balachandran (1976)) or the assignment of user nodes to processing sites with the objective of minimising telecommunication costs subject to capacity constraints on processors (Pirkul (1986)).

2.2 Bin-Packing Problem

Given a set $I = \{1, \ldots, m\}$ of items with weight $w_i > 0$, the bin-packing problem (BPP) is to fit the items into bins of capacity c in such a way that the number of bins used is a minimum. Introducing binary variables x_{ij}, which equal 1 if item i is assigned to bin $j \in J$, a mathematical formulation of the problem is

$$\begin{aligned}
&\min \sum_{j \in J} g(x_j) \\
&\sum_{j \in J} x_{ij} = 1 \quad \forall i \in I \\
&\sum_{i \in I} w_i x_{ij} \leq c \quad \forall j \in J \\
&x_{ij} \in \{0,1\} \quad \forall i \in I, j \in J,
\end{aligned}$$

where $g(x_j) = 1$ if $\sum_{i \in I} x_{ij} > 0$ and 0 otherwise. Defining $y_j = 1$ if $g(x_j) = 1$ and 0 otherwise, the linear formulation

$$\min \sum_{j \in J} y_j \tag{3a}$$

$$\sum_{j \in J} x_{ij} = 1 \quad \forall i \in I \tag{3b}$$

$$\sum_{i \in I} w_i x_{ij} \leq c y_j \quad \forall j \in J \tag{3c}$$

$$x_{ij},\, y_j \in \{0,1\} \quad \forall i \in I, j \in J, \tag{3d}$$

is obtained. For the BPP, simple approximation algorithms with a constant, asymptotic worst-case performance ratio are known (see Martello and Toth (1990a)). The linear relaxation of problem (3) can be solved by inspection. The optimal objective function value of the LP-relaxation is given by $r = \sum_{i \in I} w_i / c$. Thus, $\lceil r \rceil$ is a simple lower bound on the mininum number of bins to be used, where $\lceil r \rceil$ is the smallest integer greater or equal than r. Martello and Toth (1990b) improve this bound by partitioning the set I of items into three subsets in such a way that items of the first two subsets require separate bins and that no item of the third subset can be assigned to a bin containing an item from the first subset. Furthermore, they provide a reduction algorithm which checks, if item subsets of cardinality less than 4 have to be packed into the same bin in an optimal solution. Martello and Toth (1990a) describe a branch-and-bound algorithm for the BPP which makes use of this bounding procedure and reduction algorithm.

Heuristic vehicle routing algorithms, as e. g. parallel route building procedures (see e. g. Kontoravdis and Bard (1995)), make use of lower bounds on the minimum number of vehicles required. Such bounds can be obtained by solving bin-packing problems. In this case, the items and their weights are given by the customers and their demands; the bin size is the vehicle capacity. Kontoravdis and Bard (1995) also make use of a maximum route duration constraint

in order to obtain bounds on the number of vehicles by means of solving a bin-packing problem. In case of a multiple use of vehicles, the problem of assigning vehicles to tours can also be formulated as a bin-packing problem (see e. g. Fleischmann (1990) and Taillard et al. (1996)).

Another application of the BPP is e. g. the problem of assigning jobs with a given due date to identical machines in such a way that the number of machines used is a minimum and all jobs can be finished before the due date.

2.3 Fixed-Charge Transportation Problem

The fixed-charge transportation problem (FCTP) is obtained from the classical transportation problem by imposing a fixed cost on each transportation link if there is a positive flow on this link. Let $I = \{1, \ldots, m\}$ denote the set of destinations with demands d_i, and let $J = \{1, \ldots, n\}$ denote the set of origins with supplies s_j. The FCTP then consists in solving the mathematical program

$$\min \sum_{i \in I} \sum_{j \in J} g_{ij}(x_{ij}) \tag{4a}$$

$$\text{s.t.:} \sum_{j \in J} x_{ij} = 1 \quad \forall i \in I \tag{4b}$$

$$\sum_{i \in I} d_i x_{ij} \leq s_j \quad \forall j \in J \tag{4c}$$

$$0 \leq x_{ij} \leq u_{ij} \quad \forall i \in I, j \in J\,, \tag{4d}$$

where x_{ij} is the portion of destination i' s demand supplied from origin j, $u_{ij} d_i \leq d_i$ is an upper bound on the amount which can be shipped on link (j, i), and $g_{ij}(x_{ij})$ is the cost of shipping the amount $x_{ij} d_i$ on link (j, i). Assuming that the variable transportation costs are proportional to the amount shipped, the costs $g_{ij}(x_{ij})$ are given by $g_{ij}(x_{ij}) = c_{ij} x_{ij} + f_{ij}$ if $x_{ij} > 0$ and 0 otherwise. By means of introducing binary variables y_{ij} which equal 1 if and only if $x_{ij} > 0$, the problem is easily transformed to the linear mixed-integer program

$$\min \sum_{i \in I} \sum_{j \in J} (c_{ij} x_{ij} + f_{ij} y_{ij}) \tag{5a}$$

$$\text{s.t.:} \sum_{j \in J} x_{ij} = 1 \quad \forall i \in I \tag{5b}$$

$$\sum_{i \in I} d_i x_{ij} \leq s_j \quad \forall j \in J \tag{5c}$$

$$0 \leq x_{ij} \leq u_{ij} y_{ij} \quad \forall i \in I, j \in J \tag{5d}$$

$$y_{ij} \in \{0, 1\} \quad \forall i \in I, j \in J\,. \tag{5e}$$

A number of branch-and-bound algorithms have been proposed to solve the FCTP (see Kennington and Unger (1976), Cabot and Erenguc (1984, 1986), Palekar et al. (1990)). Most of these algorithms make use of penalties in order to fix binary variables and to be able to prematurely prune nodes of the enumeration tree. Since the FCTP is a special case of fixed-charge network flow problems, polyhedral cuts developed for this problem (see e. g. Nemhauser and Wolsey (1988) and Bienstock and Günlük (1996)) may also be used to solve the FCTP. Wright and Haehling von Lanzenauer (1989) develop a Lagrangian heuristic for the FCTP which is based on relaxing the variable upper bound constraints $x_{ij} \leq u_{ij} y_{ij}$. It can be proved that an optimal solution to the FCTP is an extreme point of the convex region defined by constraints (4b)–(4d) (see Hirsch and Dantzig (1968)). Basic feasible solutions to the system (4b)–(4d) define spanning trees of the transportation network. Sun et al. (1998) propose a tabu search

procedure for the FCTP which replaces a single link of the current spanning tree by a link not contained in the tree such that a new basic feasible solution is obtained. Göthe-Lundgren and Larsson (1994) consider the pure FCTP. In this case the objective function coefficients c_{ij} of the continuous variables x_{ij} are all equal to zero. The Benders' reformulation of the pure FCTP consists, therefore, only of feasibility cuts. A feasibility cut excludes an integer solution y which does not allow a feasible flow from the set of origins to the set of destinations. The large set of all feasibility cuts has the structure of a set-covering problem. Thus, by means of applying Benders' reformulation principle, the FCTP is transformed to a set-covering problem. In order to solve the reformulated problem to ϵ-optimality, Göthe-Lundgren and Larsson (1994) apply Benders' decomposition, where violated feasibility cuts are determined by means of solving a maximum flow problem. Furthermore, Göthe-Lundgren and Larsson (1994) propose a Lagrangian relaxation approach which relaxes a large number of the feasibility cuts. This way, a lower bound as well as a feasible solution to the pure FCTP is computed. Hultberg and Cardoso (1997) describe the "teacher assignment problem", a pure FCTP in which all fixed charges f_{ij} are equal to one. Thus, the problem is to find the most degenerate basic feasible solution to constraints (4b)–(4d). Hultberg and Cardoso (1997) show that this problem is equivalent to a maximum cardinality partition problem and provide a branch-and-bound procedure for computing optimal solutions. Herer et al. (1996) study the FCTP with a single destination. They develop two simple greedy heuristics as well as an implicit enumeration scheme which also makes use of domination rules and lower bounds for accelerating the search.

Herer et al. (1996) present three applications of the FCTP. A first application is to select suppliers and to determine periodic shipment quantities for an item to be procured in such a way that total periodic costs are minimised, periodic demand is met and a supplier's capacity is not exceeded. The total costs consist of purchasing, fixed ordering, inventory carrying and fixed supplier management costs. Periodic inventory costs (ording and inventory carrying) are computed based on Economic Order Quantity logic. As a second application of the (single-sink) FCTP, Herer et al. (1996) mention the selection of trucks for meeting a firm's delivery needs such that the sum of variable transportation and fixed vehicle costs is as low as possible. As a final application they mention process selection. A pre-specified amount of a number of products can be made using several different processes, each of which has a given capacity and fixed set-up cost. The problem is then to determine which processes to use to what extend so as to minimise costs. Moore et al. (1991) describe an integer transportation model which closely resembles the FCTP and which is used by the central dispatch control center of a metal company in the U.S. for assigning shipments to carriers. An extended version of this model as well as a simulation study were also used for the purposes of core carrier selection.

2.4 Discrete Location Problems

Discrete location problems form a large subfamily of assignment type optimisation problems. Applications of discrete location problems include location and distribution planning (see e. g. Geoffrion and Graves (1974), Gelders et al. (1987), Tüshaus and Wittmann (1998), Engeler et al. (1999), Bruns et al. (2000)), lotsizing in production planning (Pochet and Wolsey (1988)), telecommunication and computer network design (Mirzaian (1985), Boffey (1989), Chardaire (1999)), vendor selection (Current and Weber (1994)), and physical database design (Caprara and Salazar (1999)). For comprehensive surveys of discrete location problems, we refer the reader to Aikens (1985), Mirchandani and Francis (1990), Daskin (1995), Revelle and Laporte (1996), Domschke and Krispin (1997) and Owen and Daskin (1998).

As one example of a discrete location problem, we describe the capacitated facility location problem (CFLP). It consists in deciding which depots to open from a given set $J = \{1, \ldots, n\}$ of potential depot locations and how to assign a set $I = \{1, \ldots, m\}$ of customers with given

demands d_i to those depots. The objective is to minimise total fixed and shipping costs. Constraints are that each customer's demand must be satisfied and that each depot j cannot supply more than its capacity s_j if it is open. Let f_j denote the fixed cost of operating facility j and let c_{ij} denote the cost of supplying all of customer i's demand from location j. The CFLP can then be stated as follows:

$$\begin{aligned}
\min\ & \sum_{j\in J} g_j(x_j) \\
\text{s.t.:}\ & \sum_{j\in J} x_{ij} = 1 \quad \forall i \in I \\
& \sum_{i\in I} d_i x_{ij} \le s_j \quad \forall j \in J \\
& x_{ij} \ge 0 \quad \forall i \in I, j \in J,
\end{aligned}$$

where x_{ij} is the fraction of customer i's demand met from facility j, and $g_j(x_j) = f_j + \sum_{i\in I} c_{ij}x_{ij}$ if $\sum_{i\in I} x_{ij} > 0$ and 0 otherwise. A linear formulation is obtained by introducing binary variables y_j indicating if a facility j is open or not:

$$\min \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{j\in J} f_j y_j \tag{6a}$$

$$\text{s.t.:}\ \sum_{j\in J} x_{ij} = 1 \quad \forall i \in I \tag{6b}$$

$$\sum_{i\in I} d_i x_{ij} \le s_j y_j \quad \forall j \in J \tag{6c}$$

$$\sum_{j\in J} s_j y_j \ge \sum_{i\in I} d_i \tag{6d}$$

$$x_{ij} - y_j \le 0 \quad \forall i \in I, j \in J \tag{6e}$$

$$x_{ij} \ge 0 \quad \forall i \in I, j \in J \tag{6f}$$

$$y_j \in \{0,1\} \quad \forall j \in J. \tag{6g}$$

In the above formulation, constraints (6e) and (6d) are redundant; however, these constraints help to strengthen certain relaxations of the CFLP.

Numerous heuristic and exact algorithms for the CFLP have been proposed in the literature. These methods include greedy heuristics (Khumawala (1974), Jacobsen (1983), Korupolu et al. (1998)), linear programming based rounding and filtering techniques (Shmoys et al. (1997)), Benders' decomposition (Wentges (1996)), branch-and-cut methods based on polyhedral cuts (Aardal et al. (1995), Aardal (1998)), and a number of Lagrangian relaxation approaches used in heuristics or exact branch-and-bound schemes (Nauss (1978), Christofides and Beasley (1983), Beasley (1988), Ryu and Guignard (1992)).

3 Column Generation Applied to Assignment Problems

Consider the generic assignment problem (1) and assume that $X = \bigcup_{j\in J} X_j$ and thereby each X_j is a finite set, that is $X_j = \{x_j^t : t \in T_j\}$. The problem may then be rewritten as

$$\min \sum_{j\in J}\sum_{t\in T_j} g_j(x_j^t)\alpha_{jt} \tag{7a}$$

$$\text{s.t.:}\ \sum_{t\in T_j} \alpha_{jt} = 1 \quad \forall j \in J \tag{7b}$$

$$\sum_{j\in J}\sum_{t\in T_j} x_{ij}^t \alpha_{jt} = 1 \quad \forall i \in I \tag{7c}$$

$$\alpha_{tj} \in \{0,1\} \quad \forall j \in J,\, t \in T_j\,, \tag{7d}$$

which is a set-partioning problem if the x_j^t are binary. A lower bound on the optimal solution value of problem (1) can be obtained by relaxing the integrality requirements (7d) and solving the resulting linear program. In case, that each function g_j is convex, it is straightforward to show that this bound is at least as strong as a lower bound obtained by replacing in (1) the set X by a convex set $\overline{X}$ containing X (see e. g. Romero Morales (2000) for a proof).

The number of variables in the reformulation (7) grows exponentially with problem size. Thus, column generation has to be applied in order to solve its linear relaxation. The linear programming dual of (7) is given by

$$\begin{aligned} &\max \sum_{j\in J} \nu_j + \sum_{i\in I} \eta_i \\ &\text{s.t.: } \nu_j + \sum_{i\in I} x_{ij}^t \eta_i \le g_j(x_j^t) \quad \forall j \in J,\, t \in T_j\,. \end{aligned} \tag{8}$$

For each $j \in J$ let $\{x_j^t : t \in \overline{T}_j\}$ denote known subsets of columns $t \in \overline{T}_j \subseteq T_j$. The restricted linear master problem is obtained from the linear relaxation of (7) by replacing each T_j with $\overline{T}_j$. Assume that the restricted linear master is feasible (otherwise put e. g. an artificial box around the dual variables) and let $\overline{\alpha}$ and $(\overline{\nu}, \overline{\eta})$ denote an optimal basic solution of the restricted linear master and its dual. The basic solution $\overline{\alpha}$ is an optimal solution to the linear relaxation of (7) if it is also dual feasible, that is

$$\overline{\nu}_j + \sum_{i\in I} x_{ij}^t \overline{\eta}_i \le g(x_j^t) \quad \forall j \in J,\, t \in T_j\,.$$

Therefore, in order to detect a violated dual constraint and a column $t \in T_j$ for some $j \in J$ with negative reduced costs, the following pricing problem has to be solved for all or at least some $j \in J$:

$$\min\Big\{g_j(x_j) - \sum_{i\in I} \overline{\eta}_i x_{ij} : x_j \in X_j\Big\}. \tag{9}$$

If x_j^t, $t \in T_j$, is an optimal solution to (9) with $g_j(x_j^t) - \sum_i \overline{\eta}_i x_{ij}^t < \overline{\nu}_j$ new columns with negative reduced costs are found. These columns are added to the restricted linear master, and the master is reoptimised. The process continues until no column with negative reduced costs exists.

It is straightforward to see that this approach of computing a lower bound for the assignment problem (1) is equivalent to a Lagrangian relaxation of the semi-assignment constraints (1b). Dualising constraints (1b) with dual variables η_i gives the Lagrangian subproblem

$$\begin{aligned} L_1(\eta) &= \sum_{i\in I} \eta_i + \min\Big\{\sum_{j\in J}\Big(g_j(x_j) - \sum_{i\in I} \eta_i x_{ij}\Big) : x_j \in X_j\ \forall j \in J\Big\} \\ &= \sum_{i\in I} \eta_i + \sum_{j\in J} \min\Big\{g_j(x_j) - \sum_{i\in I} \eta_i x_{ij} : x_j \in X_j\Big\} \\ &= \sum_{i\in I} \eta_i + \sum_{j\in J} \min_{t\in T_j}\Big\{g_j(x_j^t) - \sum_{i\in I} \eta_i x_{ij}^t\Big\}, \end{aligned} \tag{10}$$

where the last inequality follows from the finiteness of the sets X_j. Setting $\nu_j \equiv \min_{t\in T_j}\{g_j(x_j^t) - \sum_i \eta_i x_{ij}^t\}$, the problem of maximising the lower bound $L_1(\eta)$ leads then to the linear programming dual (8) of the reformulation (7).

The function $L_1(\eta)$ is piecewise linear and concave. A number of methods having their origins in nondifferential optimisation is, therefore, applicable for the purposes of solving the linear relaxation of problem (7). Examples of such methods are mixtures of Dantzig-Wolfe decomposition and subgradient optimisation (Guignard and Zhu (1994), Klose and Drexl (2001)), bundle methods (Lemaréchal (1989)), and interior point methods (Goffin et al. (1992)). All these methods usually show a better convergence behaviour than the standard column generation procedure which is also known as Dantzig-Wolfe decomposition (Dantzig and Wolfe (1960)) or Kelley's cutting plane algorithm (Kelley (1960)). The suitability of a method for maximising the function $L_1(\eta)$, however, strongly depends on the specific problem and the hardness of the restricted master problem and pricing subproblem, respectively.

Now consider the second case in which X is not finite, but each X_j is a polyhedron and each function $g_j(x_j)$ is concave. For reasons of simplicity assume in addition, that each set X_j is bounded. Let $\{x_j^t : t \in T_j\}$ denote the set of vertices of X_j. Problem (1) may then be rewritten as follows:

$$\begin{aligned}
\min \quad & \sum_{j\in J} g\Big(\sum_{t\in T_j} \alpha_{jt} x_j^t\Big) \\
\text{s.t.:} \quad & \sum_{t\in T_j} \alpha_{jt} = 1 \quad \forall j \in J \\
& \sum_{j\in J}\sum_{t\in T_j} x_{ij}^t \alpha_{jt} = 1 \quad \forall i \in I \\
& \alpha_{tj} \geq 0 \quad \forall j \in J,\, t \in T_j\,.
\end{aligned}$$

Since g_j is concave, we have

$$g_j\Big(\sum_{t\in T_j} \alpha_{jt} x_j^t\Big) \geq \sum_{t\in T_j} \alpha_{jt} g_j(x_j^t)\,.$$

The linear program

$$\begin{aligned}
\min \quad & \sum_{j\in J}\sum_{t\in T_j} \alpha_{jt} g(x_j^t) \\
\text{s.t.:} \quad & \sum_{t\in T_j} \alpha_{jt} = 1 \quad \forall j \in J \\
& \sum_{j\in J}\sum_{t\in T_j} x_{ij}^t \alpha_{jt} = 1 \quad \forall i \in I \\
& \alpha_{tj} \geq 0 \quad \forall j \in J,\, t \in T_j
\end{aligned}$$

is, therefore, a relaxation of the original problem and its optimal objective function value is a lower bound on the optimal solution value of (1). In order to solve the above relaxation, column generation can be applied in the same way as already described. Obviously, the same relaxation results if the semi-assignment constraints (1b) are relaxed in a Lagrangian manner; due to the concavity of the functions g_j, optimal solutions to the Lagrangian subproblem are vertices x_j^t of the set X_j.

The following examples further illustrate the principle of the reformulation (7).

Example 1 Consider the GAP (2). For each agent $j \in J$ let $\{x_j^t : t \in T_j\}$ denote the set of feasible assignments of tasks $i \in I$ to agent j, that is

$$\{x_j^t : t \in T_j\} = \Big\{x_j \in \{0,1\}^{|I|} : \sum_{i\in I} d_{ij} x_{ij} \leq s_j\Big\}\,.$$

Furthermore, let $C_{jt} = \sum_{i \in I} c_{ij}^t$ be the cost of such an assignment. The GAP may then be formulated as the set-partitioning problem

$$\min \sum_{j \in J} \sum_{t \in T_j} C_{jt} \alpha_{jt} \tag{11a}$$

$$\text{s.t.:} \sum_{t \in T_j} \alpha_{jt} = 1 \quad \forall j \in J \tag{11b}$$

$$\sum_{j \in J} \sum_{t \in T_j} x_{ij}^t \alpha_{tj} = 1 \quad \forall i \in I \tag{11c}$$

$$\alpha_{jt} \in \{0,1\} \quad \forall j \in J,\, t \in T_j\,. \tag{11d}$$

For each $j \in J$ the pricing subproblem is given by the binary knapsack problem

$$\min\Big\{\sum_{i \in I} (c_{ij} - \overline{\eta}_i) x_{ij} : \sum_{i \in I} d_{ij} x_{ij} \leq s_j,\; x_{ij} \in \{0,1\}\; \forall i \in I\Big\}, \tag{12}$$

where $(\overline{\nu}, \overline{\eta})$ is an optimal dual solution to the restricted linear master problem. New columns x_j^t price out if $\overline{\nu}_j > \sum_{i \in I} (c_{ij} - \overline{\eta}_i) x_{ij}^t$.

Example 2 Consider the BPP (3). Let $I_j^t \subseteq I$ be a nonempty subset of items which fit into bin j, and let $\{I_j^t : t \in T_j\}$ denote the set of all such subsets. Since all bins have the same capacity c, we have $T_j = T$ and $I_j^t = I^t$ for all $j \in J$. Furthermore, the evaluation $g(I_j^t) = 1$ of a nonempty subset I_j^t does not depend on j. The BPP may, therefore, be rewritten as follows:

$$\begin{aligned}
&\min \sum_{t \in T} \alpha_t \\
&\text{s.t.:} \sum_{t \in T} \alpha_t \leq n \\
&\quad \sum_{t \in T} x_i^t \alpha_t = 1 \quad \forall i \in I \\
&\quad \alpha_t \in \{0,1\} \quad \forall t \in T\,,
\end{aligned}$$

where $x_i^t = 1$ if $i \in I^t$ and 0 otherwise. Since it is assumed that the BPP (3) has a feasible solution, the constraint $\sum_{t \in T} \alpha_t \leq n$ can be dropped and the BPP is reformulated as a pure set-partitioning problem. The pricing subproblem is to solve the single binary knapsack problem

$$\max\Big\{\sum_{i \in I} \overline{\eta}_i x_i : \sum_{i \in I} w_i \leq c,\; x_i \in \{0,1\}\; \forall i \in I\Big\},$$

where $\overline{\eta}$ is an optimal dual solution to the restricted linear master problem. A new column x^t prices out if $1 - \sum_{i \in I} \overline{\eta}_i x_i^t < 0$.

Example 3 For the FCTP (5) let $\{(y_j^t, x_j^t) : t \in T_j\}$ denote the set of all link selections $y_j \in \{0,1\}^{|I|}$ and transportation flows $x_j \in [0,1]^{|I|}$ respecting the link capacities u_{ij} as well as source node j's capacity s_j. Exactly speaking, $\{(y_j^t, x_j^t) : t \in T_j\}$ is the vertex set of the convex hull of the set of solutions satisfying constraints

$$\sum_{i \in I} d_i x_{ij} \leq s_j\,, 0 \leq x_{ij} \leq u_{ij} y_{ij}\; \forall i \in I \text{ and } y_{ij} \in \{0,1\}\; \forall i \in I\,.$$

This leads to the reformulation

$$\begin{aligned}
\min\ & \sum_{j\in J}\sum_{t\in T_j} C_{jt}\alpha_{jt} \\
\text{s.t.:}\ & \sum_{t\in T_j} \alpha_{jt} = 1 \quad \forall j \in J \\
& \sum_{j\in J}\sum_{t\in T_j} x_{ij}^t \alpha_{jt} = 1 \quad \forall i \in I \\
& \alpha_{jt} \in \{0,1\} \quad \forall j \in J,\, t \in T_j\,,
\end{aligned}$$

where $C_{jt} = \sum_{i\in I}(c_{ij}x_{ij}^t + f_{ij}y_{ij}^t)$. The pricing subproblem consists in solving for each $j \in J$ the program

$$\min \sum_{i\in I}\big((c_{ij} - \overline{\eta}_i)x_{ij} + f_{ij}y_{ij}\big) \tag{13a}$$

$$\text{s.t.:} \sum_{i\in I} d_i x_{ij} \le s_j \tag{13b}$$

$$0 \le x_{ij} \le u_{ij}y_{ij} \quad \forall i \in I \tag{13c}$$

$$y_{ij} \in \{0,1\} \quad \forall i \in I\,, \tag{13d}$$

where $(\overline{\nu}, \overline{\eta})$ is an optimal dual solution to the restricted linear master problem. If a slack variable is added to the capacity constraint (13b) and the roles of the source node j and the sink nodes $i \in I$ is reversed, the program (13) is easily recognized as a single-sink FCTP or single-node capacitated flow problem. New columns (y_j^t, x_j^t) price out if $\overline{\nu}_j > \sum_{i\in I}\big((c_{ij} - \overline{\eta}_i)x_{ij}^t + f_{ij}y_{ij}^t\big)$.

Example 4 In case of the CFLP, assume that $\{y^t : t \in T^y\}$ is the set of all depot selections which have enough capacity to meet total demand, that is

$$\{y^t : t \in T^y\} = \Big\{y \in \{0,1\}^{|J|} : \sum_{j\in J} s_j y_j \ge \sum_{i\in I} d_i\Big\}.$$

Furthermore, let $\{x_j^t : t \in T_j^x\}$ denote the vertex set of the set of all feasible flows from depot j to the customers, that is $\sum_{i\in I} d_i x_{ij}^t \le s_j$ and $x_j^t \in [0,1]^{|I|}$. The CFLP may then be rewritten as the linear mixed-integer program:

$$\min \sum_{t\in T^y} F_t\alpha_t + \sum_{j\in J}\sum_{t\in T_j^x} C_{jt}\beta_{jt} \tag{14a}$$

$$\text{s.t.:} \sum_{t\in T^y} \alpha_t = 1 \tag{14b}$$

$$\sum_{t\in T^y} y_j^t\alpha_t - \sum_{t\in T_j^x} \beta_{jt} \ge 0 \quad \forall j \in J \tag{14c}$$

$$\sum_{j\in J}\sum_{t\in T_j^x} x_{ij}^t\beta_{jt} = 1 \quad \forall i \in I \tag{14d}$$

$$\alpha_t \in \{0,1\} \quad \forall t \in T^y \tag{14e}$$

$$\beta_{jt} \ge 0 \quad \forall j \in J,\, t \in T_j^x\,, \tag{14f}$$

where $F_t = \sum_{j\in J} f_j y_j^t$ and $C_{jt} = \sum_{i\in I} c_{ij}x_{ij}^t$. Constraint (14b) guarantees that exactly one depot set with sufficient capacity is selected; constraints (14c) state that there can be no flow from a closed facility j, and constraints (14d) guarantee that each customer's demand is met. If

$(\overline{\zeta}, \overline{\nu}, \overline{\eta})$ is an optimal dual solution to the restricted linear master problem, the pricing problem consists in solving for each $j \in J$ the continuous knapsack problems

$$\max\Big\{\sum_{i\in I}(\overline{\eta}_i - c_{ij})x_{ij} : \sum_{i\in I} d_i x_{ij} \leq s_j,\ 0 \leq x_{ij} \leq 1\ \forall i \in I\Big\}$$

and the binary knapsack problem

$$\min\Big\{\sum_{j\in J}(f_j - \overline{\nu}_j)y_j : \sum_{j\in J} s_j y_j \geq \sum_{i\in I} d_i\,,\ y_j \in \{0,1\}\ \forall j \in J\Big\}\,.$$

New columns x_j^t and y^t price out if $\overline{\nu}_j < \sum_{i\in I}(\overline{\eta}_i - c_{ij})x_{ij}^t$ and $\overline{\zeta} > \sum_{j\in J}(f_j - \overline{\nu}_j)y_j^t$, respectively.

Remark 1 As already shown when discussing the reformulation of the assignment problem (1), in all the aforementioned cases, the linear relaxation of the integer master program can be obtained by relaxing the semi-assignment constraints in a Lagrangian fashion, rewriting the Lagrangian dual as a linear program and dualising this linear program.

4 A Partitioning Approach

Consider the generic assignment problem (1) and assume that each X_j is a finite set (similar arguments apply if each X_j is a polyhedron and each g_j is a concave function). Partition the set J into subsets J_q, $q \in Q$ and $J_l \cap J_q = \emptyset$ for $l \neq q$, in such a way that at least one subset J_q has cardinality greater than one. The semi-assignment constraints (1b) imply the constraints

$$\sum_{j\in J_q} x_{ij} \leq 1 \quad \forall i \in I,\, q \in Q\,. \tag{15}$$

Thus, if constraints (1b) are relaxed in a Lagrangian manner, the addition of the constraints (15) can help to sharpen the relaxation. Dualising the semi-assignment constraints (1b) with multipliers η_i while adding the redundant constraints (15) gives the Lagrangian subproblem

$$L_2(\eta) \equiv \sum_{i\in I}\eta_i + \min \sum_{q\in Q}\sum_{j\in J_q}\Big\{g_j(x_j) - \sum_{i\in I}\eta_i x_{ij}\Big\} \tag{16a}$$

$$\text{s.t.:} \sum_{j\in J_q} x_{ij} \leq 1 \quad \forall i \in I,\, q \in Q \tag{16b}$$

$$x_j \in X_j \quad \forall j \in J\,, \tag{16c}$$

which decomposes into the $|Q|$ subproblems

$$\nu_q = \min \sum_{j\in J_q}\Big(g_j(x_j) - \sum_{i\in I}\eta_i x_{ij}\Big) \tag{17a}$$

$$\text{s.t.:} \sum_{j\in J_q} x_{ij} \leq 1 \quad \forall i \in I \tag{17b}$$

$$x_j \in X_j \quad \forall j \in J_q \tag{17c}$$

such that

$$L_2(\eta) = \sum_{i\in I}\eta_i + \sum_{q\in Q}\nu_q\,.$$

The subproblems (17) have the same structure as the original problem (1) if a dummy resource with assignment costs of zero is added; they are, however, usually much smaller than the original problem. Let $\{(x_j^h)_{j\in J_q} : h \in H_q\}$ denote the set of solutions satisfying constraints (17b) and (17c). The Lagrangian dual, which is the problem of maximising the piecewise linear and concave function $L_2(\eta)$, may then be rewritten as the linear program:

$$\max_{\eta} L_2(\eta) = \max \sum_{i\in I}\eta_i + \sum_{q\in Q}\nu_q$$
$$\text{s.t.: } \nu_q + \sum_{i\in I}\Big(\sum_{j\in J_q} x_{ij}^h\Big)\eta_i \le \sum_{j\in J_q} g_j(x_j^h) \quad \forall q \in Q, h \in H_q .$$

Dualising this linear program with dual variables μ_{qh} gives:

$$\min \sum_{q\in Q}\sum_{h\in H_q} G_{qh}\mu_{qh} \tag{18a}$$
$$\text{s.t.: } \sum_{h\in H_q}\mu_{qh} = 1 \quad \forall q \in Q \tag{18b}$$
$$\sum_{q\in Q}\sum_{h\in H_q}\Big(\sum_{j\in J_q} x_{ij}^h\Big)\mu_{qh} = 1 \quad \forall i \in I \tag{18c}$$
$$\mu_{qh} \ge 0 \quad \forall q \in Q,\, h \in H_q , \tag{18d}$$

where $G_{qh} = \sum_{j\in J_q} g_j(x_j^h)$. The linear program (18) is the linear relaxation of an equivalent reformulation of problem (1). This reformulation is obtained if the nonnegativity constraints (18d) are replaced by $\mu_{qh} \in \{0,1\}\,\forall q,h$. If the x_j^h are binary, $I_q^h = \{i \in I : \sum_{j\in J_q} x_{ij}^h = 1\}$ is a subset of activities which can feasibly be assigned to the subset J_q of resources. The reformulated problem than simply states, that the assignment problem is to select for each subset J_q of resources exactly one subset of activities I_q^h in such a way that each activity is contained in one of the selected subsets and total costs are minimised. The following two examples further illustrate the relaxation (18) and the corresponding reformulation of the assignment problem.

Example 5 Adding constraints (15) to the GAP (2) for a given partitioning $\{J_q : q \in Q\}$ of the agent set J and dualising the semi-assignment constraints (2b) with multipliers η_i gives the Lagrangian subproblem

$$L_2(\eta) = \sum_{i\in I}\eta_i + \min \sum_{q\in Q}\sum_{j\in J_q}\sum_{i\in I} c_{ij}x_{ij}$$
$$\text{s.t.: } \sum_{j\in J_q} x_{ij} \le 1 \quad \forall i \in I,\, q \in Q$$
$$\sum_{i\in I} d_{ij}x_{ij} \le s_j \quad \forall j \in J_q,\, q \in Q$$
$$x_{ij} \in \{0,1\} \quad \forall i \in I,\, j \in J .$$

The Lagrangian subproblem decomposes into the $|Q|$ subproblems

$$\nu_q = \min \sum_{i\in I}\sum_{j\in J_q} c_{ij}x_{ij} \tag{19a}$$
$$\text{s.t.: } \sum_{j\in J_q} x_{ij} \le 1 \quad \forall i \in I \tag{19b}$$
$$\sum_{i\in I} d_{ij}x_{ij} \le s_j \quad \forall j \in J_q \tag{19c}$$
$$x_{ij} \in \{0,1\} \quad \forall i \in I, j \in J_q \tag{19d}$$

such that $L_2(\eta) = \sum_{i\in I}\eta_i + \sum_{q\in Q}\nu_q$. Each of the above subproblems is easily recognized as a GAP if a dummy task 0_q with assignment costs $c_{i0_q} = 0\,\forall i$, resource requirements $d_{i0_q} = 1\,\forall i$, and capacity $s_{0_q} = |I|$ is added to the subsets J_q of agents. If $\{(x_j^h)_{j\in J_q} : h \in H_q\}$ denotes the set of feasible solutions of subproblem q, the Lagrangian dual problem $\max_\eta L_2(\eta)$ can be written as the linear program

$$\begin{aligned}
&\max \sum_{i\in I}\eta_i + \sum_{q\in Q}\nu_q \\
&\text{s.t.: } \nu_q + \sum_{i\in I}\Big(\sum_{j\in J_q} x_{ij}^h\Big)\eta_i \le C_{qh} \quad \forall q \in Q,\, h \in H_q\,,
\end{aligned}$$

where $C_{qh} = \sum_{i\in I}\sum_{j\in J_q} x_{ij}^h$. Dualising this program with dual variables μ_{qh} then gives the primal linear master problem

$$\min \sum_{q\in Q}\sum_{h\in H_q} C_{qh}\mu_{qh} \tag{20a}$$

$$\text{s.t.: } \sum_{h\in H_q}\mu_{qh} = 1 \quad \forall q \in Q \tag{20b}$$

$$\sum_{q\in Q}\sum_{h\in H_q}\Big(\sum_{j\in J_q} x_{ij}^h\Big)\mu_{qh} = 1 \quad \forall i \in I \tag{20c}$$

$$\mu_{qh} \ge 0 \quad \forall q \in Q,\, h \in H_q\,, \tag{20d}$$

which is the linear relaxation of a set-partitioning reformulation of the GAP. Columns of this reformulation correspond to subsets of tasks feasibly assignable to subsets J_q of agents.

Example 6 In case of the CFLP (6) the situation is more difficult than for the GAP (2). The aggregate capacity constraint (6d) must be dropped in order to keep the Lagrangian subproblem decomposable, if the demand constraints (6b) are relaxed and the constraints (15) are added. Doing this and dualising constraints (6b) with multipliers η_i the Lagrangian subproblem is

$$L_2(\eta) = \sum_{i\in I}\eta_i + \sum_{q\in Q}\nu_q \tag{21a}$$

where

$$\nu_q = \min \sum_{i\in I}\sum_{j\in J_q} c_{ij}x_{ij} + \sum_{j\in J_q} f_j y_j \tag{21b}$$

$$\text{s.t.: } \sum_{j\in J_q} x_{ij} \le 1 \quad \forall i \in I \tag{21c}$$

$$\sum_{i\in I} d_i x_{ij} \le s_j y_j \quad \forall j \in J_q \tag{21d}$$

$$0 \le x_{ij} \le y_j \quad \forall i \in I,\, j \in J_q \tag{21e}$$

$$y_j \in \{0,1\} \quad \forall j \in J_q\,. \tag{21f}$$

The above subproblem can be transformed to a CFLP by adding a dummy depot 0_q with capacity $s_{0_q} = \sum_{i\in I} d_i$ and costs $f_{0_q} = c_{i0_q} = 0\,\forall i$ to the subset J_q of depots. If $\{(y_j^h,\, x_j^h)_{j\in J_q}) : h \in H_q\}$ denotes the vertex set of the set of feasible solutions to subproblem q and C_{qh} is defined as

$$C_{qh} = \sum_{i\in I}\sum_{j\in J_q} x_{ij}^h + \sum_{j\in J_q} f_j y_j^h\,,$$

the Lagrangian dual problem and the primal linear master problem can be written in the same way as in the case of the GAP. In this case, the linear primal master problem is the linear relaxation of a program which restates the CFLP as a pure integer program. Columns of this reformulation correspond to feasible flows from subsets J_q of depots to the set I of customers.

Remark 2 In an analogous way, the relaxation (18) can be applied to the BPP (3) and the FCTP (5). In case of the FCTP the resulting Lagrangian subproblem (pricing subproblem) decomposes into smaller FCTPs if a dummy sink is added to each of the $|Q|$ subproblems. In case of the BPP, however, each of the $|Q|$ resulting subproblems is a "mixture" of a BPP and a multiple knapsack problem where there is a profit for each item assigned and a fixed cost of 1 for each knapsack (bin) used.

Remark 3 The discussed examples of assignment type problems include only linear constraints; the requirements $x_j \in X_j$ are capacity constraints of the form $\sum_{i\in I} d_{ij}x_{ij} \leq s_j$. An alternative way of partitioning the problem is, therefore, to decompose the set I of activities into disjoint subsets I_r, $r \in R$, adding the implied constraints $\sum_{i\in I_r} d_{ij}x_{ij} \leq s_j$ for each $j \in J$ and $r \in R$, and relaxing the capacity constraints in a Lagrangian manner. The resulting Lagrangian subproblem decomposes again into smaller subproblems of the same or a similar structure as the original problem. It is, however, not difficult to show that this relaxation is usually not as strong as the relaxation (7) or even as strong as the conventional Lagrangian relaxation of the semi-assignment constraints (1b).

An apparent question which arises, is how to partion the subset J into subsets J_q. Some hints can be derived from the following proposition:

Proposition 1 *Let $\overline{\eta}$ denote an optimal solution to $\max_\eta L_1(\eta)$, where the Lagrangian function $L_1(\eta)$ is defined in (10). Furthermore, let $\{x^t : t \in \overline{T}\}$ denote the set of optimal solutions to the Lagrangean subproblem (10) with $\eta = \overline{\eta}$, that is*

$$L_1(\overline{\eta}) = \sum_{i\in I} \overline{\eta}_i + \sum_{j\in J}\Big(g_j(x_j^t) - \sum_{i\in I} \overline{\eta}_i x_{ij}^t\Big) \quad \forall t \in \overline{T}.$$

If $\sum_{j\in J_q} x_{ij}^t \leq 1$ holds for every $q \in Q$ and $t \in \overline{T}$ then $\max_\eta L_1(\eta) = \max_\eta L_2(\eta)$.

Proof: The linear relaxation of (7) and the corresponding dual program (8) can be rewritten in aggregated form as

$$\begin{aligned} \max_\eta L_1(\eta) = \min \quad & \sum_{t\in T} g(x^t)\alpha_t \\ \text{s.t.:} \quad & \sum_{t\in T} \alpha_t = 1 \\ & \sum_{j\in J}\sum_{t\in T} x_{ij}^t \alpha_t = 1 \quad \forall i \in I \\ & \alpha_t \geq 0 \quad \forall t \in T \end{aligned} \tag{22}$$

and

$$\max_\eta L_1(\eta) = \max\Big\{\sum_{i\in I}\eta_i + \nu : \sum_{i\in I}\sum_{j\in J}\eta_i x_{ij}^t + \nu \leq g(x^t)\ \forall t \in T\Big\}, \tag{23}$$

respectively, where $\{x^t : t \in T\} = X = \bigcup_j X_j$ and $g(x^t) = \sum_{j\in J} g_j(x_j^t)$. Analogously, if $\{x^h : h \in H\}$ is the set of solutions $x \in X$ satisfying constraints (15), the aggregated version of

the linear program (18) reads:

$$\begin{aligned} \max_{\eta} L_2(\eta) = \min \; & \sum_{h \in H} g(x^h)\mu_h \\ \text{s.t.:} \; & \sum_{h \in H} \mu_h = 1 \\ & \sum_{j \in J} \sum_{h \in H} x_{ij}^h \mu_h = 1 \quad \forall i \in I \\ & \mu_h \geq 0 \quad \forall h \in H\,. \end{aligned} \tag{24}$$

Let $\overline{\alpha}$ and $(\overline{\nu}, \overline{\eta})$ denote an optimal solution to (22) and (23), respectively. From complementary slackness it follows that

$$\overline{\alpha}_t > 0 \;\Rightarrow\; g(x^t) - \sum_{i \in I} \sum_{j \in J} \overline{\eta}_i x_{ij}^t = \overline{\nu} \equiv \min_{t \in T} \Big\{ g(x^t) - \sum_{i \in I} \sum_{j \in J} \overline{\eta}_i x_{ij}^t \Big\}\,.$$

Therefore, $\{x^t : \overline{\alpha}_t > 0\}$ is a subset of the set $\overline{T}$ of optimal solutions to the Lagrangian subproblem (10) for optimal multipliers $\eta = \overline{\eta}$ (if (22) is not degenerate then $\overline{T} = \{x^t : \overline{\alpha}_t > 0\}$). Since $\sum_{j \in J_q} x_{ij}^t \leq 1$ for all $t \in \overline{T}$ and $q \in Q$, the columns $t \in \overline{T}$ are feasible for (24), that is $\overline{T} \subseteq H$. The solution $\overline{\mu}_t = \overline{\alpha}_t$ if $t \in \overline{T}$ and $\overline{\mu}_t = 0$ if $t \in H \setminus \overline{T}$ is, therefore, a feasible solution to (24) with objective function value $\sum_{h \in H} \overline{\mu}_h x^h = \sum_{t \in \overline{T}} \overline{\alpha}_t x^t = L_1(\overline{\eta}) = \max_\eta L_1(\eta)$. Since generally $L_2(\eta) \geq L_1(\eta)$, the desired result follows. □

The above proposition states, that the inequalities (15) must cut-off at least one optimal solution of the Lagrangian subproblem (10) for given optimal Lagrangian multipliers; otherwise the relaxation (18) cannot be stronger than the linear relaxation of the reformulation (7). A plausible way of determining the subsets J_q is, therefore, to first compute (approximate) optimal multipliers for the "conventional" relaxation (10), and afterwards to construct the subsets J_q in such a way that an optimal solution to the Lagrangian subproblem violates at least one of the constraints (15). How this can be done in detail depends on the specific problem on hand. Other problem-specific topics concern algorithms for computing approximate optimal multipliers for the Lagrangian relaxation (10), the implementation of Lagrangian heuristics, the use of Lagrangian probing methods to reduce problem size, the algorithms used to solve the Lagrangian subproblems (10) and (16), the column generation method used for maximising the function L_2 defined in (16), and finally branching rules in case that the relaxation (18) is used within a branch-and-price framework. In the following, we briefly sketch a possible implementation of the described partitioning approach for solving the GAP and an implemention for the CFLP proposed in Klose and Drexl (2001).

4.1 Outline of a Partitioning Procedure for the GAP

In order to possibly sharpen the linear relaxation of the reformulation (11) of the GAP (2) by means of the partitioning approach, it suffices to group the agents $j \in J$ into pairs, that is to decompose the set J of agents into subsets J_q, $q \in Q$, such that $1 \leq |J_q| \leq 2 \forall q \in Q$ and $|J_q| = 2$ for at least one $q \in Q$. For each $j \in J$ let $\overline{x}_j$ denote an optimal solution of the pricing subproblem (12) for given optimal dual prices $\overline{\eta}$ of the semi-assignment constraints in the linear relaxation of (11). From proposition 1 it follows that

$$\exists\, i \in I : \overline{x}_{ij_q} + \overline{x}_{ik_q} > 1$$

must hold for at least one set $\overline{x} = (\overline{x}_j)_{j \in J}$ of optimal solutions to the pricing subproblems (12) and at least one pair $J_q = \{k_q, j_q\}$ of agents. This suggests to determine pairs J_q of agents by

means of solving the matching problem

$$\begin{aligned} \max \quad & \sum_{k \in J} \sum_{\substack{j \in J \\ j>k}} w_{kj} z_{kj} \\ \text{s.t.:} \quad & \sum_{\substack{k \in J \\ k<j}} z_{kj} + \sum_{\substack{k \in J \\ k>j}} z_{jk} \leq 1 \quad \forall j \in J \\ & z_{kj} \in \{0,1\} \quad \forall (k,j) \in J \times J,\ k < j\,, \end{aligned} \tag{25}$$

where $w_{kj} = \sum_{i \in I} (\overline{x}_{ik} + \overline{x}_{ij})$. Alternatively, the weights w_{kj} may be defined as $w_{kj} = \sum_{i \in I} \big(\max\{\overline{x}_{ik} + \overline{x}_{ij} - 1,\ 0\}\big)$. A possible implementation of the partitioning approach for the GAP is then to perform the following steps:

1. Apply "conventional" Lagrangian relaxation of the semi-assignment constraints (2b). Compute approximate optimal multipliers $\overline{\eta}$ and a feasible solution to the GAP by means of the multiplier adjustment method (possibly followed by subgradient optimisation) in conjunction with Lagrangian heuristics (see Fisher et al. (1986), Guignard and Rosenwein (1989), Karabakal et al. (1992)).

2. Apply Lagrangian probing techniques in order to fix binary variables x_{ij} to 0 and 1 without loss of optimality (see Guignard et al. (1997)).

3. Decompose the set of agents into pairs by means of solving the matching problem (25).

4. Solve the linear primal master problem (20) by means of a stabilised column generation method. During the column generation use Lagrangian heuristics after each call to the Lagrangian subproblem (pricing subproblem) (19) in order to obtain (improved) feasible solutions to the GAP. Reapply Lagrangian probing if an improved feasible solution has been found.

5. Let μ^* denote the computed optimal solution of the linear program (20) and let x^*, where $x^*_{ij} = \sum_{h \in H_q} \mu^*_{qh} x^h_{ij}$ for $j \in J_q$, denote the corresponding (fractional) solution in terms of the orginal variables. Apply a rounding procedure and/or solve the restricted integer master problem to optimality in order to obtain an improved feasible solution to the GAP.

6. In case that a duality gap remains, use branch-and-price for computing an optimal solution of the GAP. Possible branching rules are $x_{ij} = 0$ vs. $x_{ij} = 1$ for some fractional $\overline{x}_{ij}$, or $\sum_{j \in J_q} x_{ij} = 0$ vs. $\sum_{j \in J_q} x_{ij} = 1$ for some fractional $\sum_{j \in J_q} \overline{x}_{ij}$, or even a multi-branching which generates $|Q|$ branches and forces $\sum_{j \in J_q} x_{ij}$ to one on each of these branches.

4.2 Outline of a Partitioning Procedure for the CFLP

Klose and Drexl (2001) propose the following implementation of the partitioning approach for the CFLP:

1. Apply Lagrangian relaxation of the demand constraints (6b) in formulation (6) of the CFLP. Compute approximate optimal multipliers by means of subgradient optimisation and obtain a feasible solution to the CFLP by means of Lagrangian heuristics.

2. Use Lagrangian probing in order to reduce the problem size.

3. Consider the Lagrangian relaxation of constraints (6b) without the aggregate capacity constraint (6d) added. Apply subgradient optimisation to compute approximate optimal multipliers $\overline{\eta}$ and let $(\overline{x}, \overline{y})$ denote the solution of the corresponding Lagrangian subproblem.

4. Try to find a partitioning $\{J_q : q \in Q\}$ of the depot set J such that $\sum_{j \in J_q} \overline{x}_{ij} > 1$ for at least one $i \in I$ and $q \in Q$. For this purpose apply the following steps:

   (a) Set $O = \{j \in J : \overline{y}_j = 1\}$ and $q = 0$.

   (b) Choose $i \in I$ such that $\sum_{j \in O} \overline{x}_{ij} > 1$. If there is no such $i \in I$, go to step (d).

   (c) Set $q := q + 1$, $J_q = \{j \in O : \overline{x}_{ij} > 0\}$, and $O := O \setminus J_q$. Go to step (b).

   (d) Assign each $j \in J$ with $\overline{y}_j = 0$ to the set J_q which minimises $\min_{l \in J_q} \sum_{i \in I} |c_{ij} - c_{il}|$.

5. Solve the primal linear master problem (20) by means of column generation. After each call to the Lagrangian subproblem (pricing subproblem) (21) solve the transportation problem if the set of open depots has enough capacity to meet total demand. Reapply Lagrangian probing if an improved feasible solution to the CFLP has been found.

6. Let μ^* denote the computed optimal solution of the linear program (20) and let (x^*, y^*), where $(y_j^*, x_{ij}^*) = \sum_{h \in H_q} \mu_{qh}^* (y_j^h, x_{ij}^h)$ for $j \in J_q$, denote the corresponding (fractional) solution in terms of the orginal variables. Round a fractional solution y^* and solve the resulting transportation probem in order to possibly obtain an improved feasible solution to the CFLP. Afterwards apply an interchange procedure to the best feasible solution obtained so far; however, do not allow the procedure to open (close) depots j if y_j^* is small (large).

A branch-and-price procedure has not been implemented. Possible branching rules are, however, to branch on a single variable y_j if y_j^* is fractional, or to impose the branching constraints $\sum_{j \in S} y_j = 0$ vs. $\sum_{j \in S} y_j \geq 1$ if $\sum_{j \in S} y_j^*$ is fractional. In order to perform the column generation, Klose and Drexl (2001) use the analytic center cutting plane method (Goffin et al. (1992)). The above procedure has been tested on 75 test problems ranging in size from 100 potential depot sites and 100 customers to 200 potential depot sites and 500 customers. The test problems differ in the ratio $r = \sum_j s_j / \sum_i d_i$ of total capacity to total demand. For each problem size and ratio $r \in \{3, 5, 10\}$ five problem instances have been generated according to a proposal of Cornuejols et al. (1991). Furthermore, the bound obtained by means of the partitioning approach has been compared with Lagrangian bounds based on relaxing the demand constraints and the capacity constraints in model (6), respectively. These last two bounds were also compute by means of stabilised column generation procedures. Since the partitioning approach usually only make sense for relatively large problem instance, we reproduce here the results obtained for the largest test problems with 200 depot sites and 500 customers (Table 1). The results shown are averages over the five instances of each problem type. In Table 1, *LB* % is the percentage deviation of the lower bound from optimality; *UB* % is the percentage deviation of the computed feasible solution from an optimal one; It_{LR} and It_M denote the number of Lagrangian subproblems and restricted linear master problems solved, respectively; T_{LR}, T_H, and T_M are the computation times spent on solving Lagrangian subproblems, computing heuristic solutions, and solving master problems; T_{Tot} is the total computation in seconds of CPU time on a Sun Ultra (300 MHz).

As can be seen from Table 1, the partitioning method produces strong lower bounds on the optimal solution value. For a number of large test problems this bound even improves the very strong bound based on relaxing the capacity constraints (6c). The computational effort required to solve the relaxation based on partitioning the depot set is, however, quite large; also the observed variation in the times spent on computing this bound was substantial. This is due to the complexity of the subproblem which itself decomposes into (smaller) CFLPs. A counterintuitive result is that the best lower bounds computed by means of the partitioning approach have been obtained for the test problems with smallest capacity tightness index r, although this relaxation method does not make use of the aggregate capacity constraint (6d). This indicates that the heuristic used for decomposing the depot set does not work well for

Table 1: Computational results

r	$LB\,\%$	$UB\,\%$	It_{LR}	It_M	T_{LR}	T_H	T_M	T_{Tot}
Partitioning approach								
3	0.09	0.02	100	100	2514.2	9.6	154.8	2680.3
5	0.26	0.34	116	115	5657.0	7.1	217.9	5884.1
10	0.30	0.29	157	157	59076.4	3.5	1647.1	60727.9
"Conventional" relaxation of demand constraints								
3	0.16	0.02	300	24	8.9	17.0	9.9	36.7
5	0.40	0.37	631	68	25.7	21.2	39.6	87.6
10	0.47	1.09	901	91	42.1	35.5	172.3	251.7
Relaxation of capacity constraints								
3	0.15	0.00	290	290	1273.2	34.0	66.3	1373.5
5	0.33	0.04	267	267	11487.0	22.6	51.1	11560.9
10	0.29	0.02	190	190	9853.1	10.0	30.6	9893.9

Table 2: Results for 3 single large-scale test problems

r	Gap%	It_{LR}	It_M	T_{LR}	T_H	T_M	T_{Tot}
3	0.06	129	129	21769.8	168.7	768.7	22735.1
5	0.10	167	166	43286.1	66.4	1392.1	44754.3
10	0.40	234	233	59007.7	49.4	2927.3	61997.1

this type of test problems and should be improved for problems with relatively loose capacity constraints. Nevertheless, due to the quality of the lower bounds, the method has some potential to solve large problems to optimality or at least very near to optimality. This is also shown by the results of an experiment with three single very large problem instances with 1000 customers and 500 potential depot sites (see Table 2). Since no optimal solution is known for these large instances, Table 2 only shows the percentage deviation GAP % between the upper and lower bound computed by means of the partitioning method.

5 Conclusions

In this paper, we discussed the application of a partitioning method to a number of optimisation problems of the assignment type, and compared this approach to the standard way of applying column generation to assignment problems. It has been shown, that the conventional way of transforming assignment type problems into problems of the set-partitioning type and solving the linear relaxation of this reformulation is equivalent to a Lagrangian relaxation of the semi-assignment constraints. This relaxation can be improved by imposing the constraints that no "activity" may be assigned more than once to a given subset of resources. Applying Lagrangian relaxation of the semi-assignment constraints while adding these implied constraints leads to a Lagrangian (pricing) subproblem which decomposes into smaller optimisation problems of the same type as the original optimisation problem. A necessary condition for obtaining this way an improved lower bound is, that the added implied constraints are "Lagrangian cuts" (see proposition 1). Computational results obtained with this approach for the CFLP indicate that the method is capable to solve large problem instances to optimality or very near to optimality. The partitioning principle is generally applicable to assignment problems which are decomposable if the semi-assignment constraints are relaxed. However, the method has in any case to be fine-tuned to the specific problem on hand. This raises a number of research questions concerning algorithmic design. Topics which have to be addressed are the design of heuristics for suitably

decomposing the "resource" set J, the implementation of Lagrangian heuristics as well as primal heuristics using the information of a fractional solution to the primal master problem, the design of effective algorithms for solving the subproblems (which are of the same type as the original problem), the implementation of Lagrangian probing techniques for reducing the problem size, and finally the design of fine-tuned methods for stabilising the column generation.

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References

Aardal, K. (1998): Capacitated facility location: Separation algorithm and computational experience. *Mathematical Programming*, 81:149–175.

Aardal, K. / Pochet, Y. / Wolsey, L. A. (1995): Capacitated facility location: Valid inequalities and facets. *Mathematics of Operations Research*, 20:552–582.

Aikens, C. H. (1985): Facility location models for distribution planning. *European Journal of Operational Research*, 22:263–279.

Arora, S. / Lund, C. (1997): Hardness of approximations. In: Hochbaum (1997), pp. 399–346.

Balachandran, V. (1976): An integer generalized transportation model for optimal job assignment in computer networks. *Operations Research*, 24:742–749.

Beasley, J. E. (1988): An algorithm for solving large capacitated warehouse location problems. *European Journal of Operational Research*, 33:314–325.

Bienstock, D. / Günlük, O. (1996): Capacitated network design – polyhedral structure and computation. *INFORMS Journal on Computing*, 8:243–259.

Boffey, T. B. (1989): Location problems arising in computer networks. *The Journal of the Operational Research Society*, 40:347–354.

Bruns, A. D. / Klose, A. / Stähly, P. (2000): Restructuring of Swiss parcel delivery services. *OR-Spektrum*, 22:285–302.

Cabot, A. V. / Erenguc, S. S. (1984): Some branch-and-bound procedures for fixed-cost transportation problems. *Naval Research Logistics*, 31:145–154.

Cabot, A. V. / Erenguc, S. S. (1986): Improved penalties for fixed cost linear programs using Lagrangean relaxation. *Management Science*, 32:856–869.

Campell, J. F. / Langevin, A. (1995): The snow disposal assignment problem. *The Journal of the Operational Research Society*, 46:919–929.

Caprara, A. / Salazar, J. J. G. (1999): Separating lifted odd-hole inequalities to solve the index selection problem. *Discrete Applied Mathematics*, 92:111–134.

Cattrysse, D. / Degraeve, Z. / Tistaert, J. (1998): Solving the generalised assignment problem using polyhedral results. *European Journal of Operational Research*, 108:618–628.

Cattrysse, D. / Salomon, M. / Van Wassenhove, L. N. (1994): A set-partitioning heuristic for the generalized assignment problem. *European Journal of Operational Research*, 72:167–174.

Cattrysse, D. / Van Wassenhove, L. N. (1992): A survey of algorithms for the generalized assignment problem. *European Journal of Operational Research*, 60:260–272.

Chardaire, P. (1999): Hierarchical two level location problems. In: Sansò, B. / Soriano, P. (eds.), *Telecommunications Network Planning*, chap. 3, pp. 33–54. Kluwer Academic Publishers Group, London, Dordrecht, Boston.

Christofides, N. / Beasley, J. E. (1983): Extensions to a Lagrangean relaxation approach for the capacitated warehouse location problem. *European Journal of Operational Research*, 12:19–28.

Cornuejols, G. / Sridharan, R. / Thizy, J.-M. (1991): A comparison of heuristics and relaxations for the capacitated plant location problem. *European Journal of Operational Research*, 50:280–297.

Crescenzi, P. / Kann, V. (1998): A compendium of NP optimization problems. In: Ausiello, G. / Crescenzi, P. / Gambosi, G. / Kann, V. / Spaccamela, A. M. C. / Protosi, M. (eds.), *Approximate Solution of NP-hard Optimization Problems*. Springer-Verlag, `http://www.nada.kth.se/~viggo/problemlist/compendium.html`.

Current, J. / Weber, C. (1994): Application of facility location modeling constructs to vendor selection problems. *European Journal of Operational Research*, 76:387–392.

Dantzig, G. B. / Wolfe, P. (1960): Decomposition principle for linear programs. *Operations Research*, 8:101–111.

Daskin, M. S. (1995): *Network and Discrete Location: Models, Algorithms, and Applications.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Chichester, New York, Brisbane, Toronto, Singapore.

Domschke, W. / Krispin, G. (1997): Location and layout planning: A survey. *OR-Spektrum*, 19:181–194.

Engeler, K. / Klose, A. / Stähly, P. (1999): A depot location-allocation problem of a food producer with an outsourcing option. In: Speranza, M. G./ Stähly, P. (eds.), *New Trends in Distribution Logistics*, vol 480 of *Lecture Notes in Economics and Mathematical Systems*, chap. 1, pp. 95–109. Springer-Verlag, Berlin, Heidelberg, New York.

Fisher, M. L. / Jaikumar, R. (1981): A generalized assignment heuristic for vehicle routing. *Networks*, 11:109–124.

Fisher, M. L. / Jaikumar, R. / van Wassenhove, L. N. (1986): A multiplier adjustment method for the generalized assignment problem. *Management Science*, pp. 1095–1103.

Fleischmann, B. (1990): The vehicle routing problem with multiple use of the vehicles. Working paper, Fachbereich Wirtschafts- und Organisationswissenschaften, Universität der Bundeswehr Hamburg, Hamburg.

Gelders, L. F. / Pintelon, L. M. / van Wassenhove, L. N. (1987): A location-allocation problem in a large Belgian brewery. *European Journal of Operational Research*, 28:196–206.

Geoffrion, A. M. / Graves, G. W. (1974): Multicommodity distribution system design by Benders decomposition. *Management Science*, 20:822–844.

Goffin, J.-L. / Haurie, A. / Vial, J.-P. (1992): Decomposition and nondifferentiable optimization with the projective algorithm. *Management Science*, 38:284–302.

Göthe-Lundgren, M. / Larsson, T. (1994): A set covering reformulation of the pure fixed charge transportation problem. *Discrete Applied Mathematics*, 48:245–259.

Guignard, M. / Kim, S. / Wang, X. (1997): Lagrangean probing in branch-and-bound. Technical Report 97-03-20, Operations and Information Management Department, The Wharton School, University of Pennsylvania.

Guignard, M. / Rosenwein, M. B. (1989): An improved dual based algorithm for the generalized assignment problem. *Operations Research*, 37:658–663.

Guignard, M. / Zhu, S. (1994): A two-phase dual algorithm for solving Lagrangean duals in mixed integer programming. Report 94-10-03, Operations and Information Management Department, University of Pennsylvania, The Wharton School.

Herer, Y. T. / Rosenblatt, M. J. / Hefter, I. (1996): Fast algorithms for single-sink fixed charge transportation problems with applications to manufacturing and transportation. *Transportation Science*, 30:276–290.

Hirsch, W. M. / Dantzig, G. B. (1968): The fixed charge transportation problem. *Naval Research Logistics*, 15:413–425.

Hochbaum, D. S. (ed.) (1997): *Approximation Algorithms for NP-Hard Problems*. PWS Publishing Company, Boston.

Hultberg, T. H. / Cardoso, D. M. (1997): The teacher assignment problem: A special case of the fixed charge transportation problem. *European Journal of Operational Research*, 101:463–473.

Jacobsen, S. K. (1983): Heuristics for the capacitated plant location model. *European Journal of Operational Research*, 12:253–261.

Karabakal, N. / Bean, J. / Lohmann, J. R. (1992): A steepest descent multiplier adjustment method for the generalized assignment method. Technical report 92-11, Department of Industrial and Operations Engineering, University of Michigan.

Kelley, J. E. (1960): The cutting-plane method for solving convex programs. *Journal of the SIAM*, 8:703–712.

Kennington, J. L. / Unger, E. (1976): A new branch-and-bound algorithm for the fixed charge transportation problems. *Management Science*, 22:1116–1126.

Khumawala, B. M. (1974): An efficient heuristic procedure for the capacitated warehouse location problem. *Naval Research Logistics Quarterly*, 21:609–623.

Klose, A. / Drexl, A. (2001): Lower bounds for the capacitated facility location problem based on column generation. Working paper, University of St. Gallen, Switzerland.

Kontoravdis, G. / Bard, J. F. (1995): A GRASP for the vehicle routing problem with time windows. *ORSA Journal on Computing*, 7:10–23.

Koopmans, T. C. / Beckmann, M. J. (1957): Assignment problems and the location of economic activities. *Econometrica*, 25:53–76.

Korupolu, M. R. / Plaxton, C. G. / Rajaraman, R. (1998): Analysis of a local search heuristic for facility location problems. Technical Report 98-30, DIMACS, Rutgers University.

Kuhn, H. (1995): A heuristic algorithm for the loading problem in flexible manufacturing systems. *International Journal of Flexible Manufacturing Systems*, 7:229–254.

Lee, D.-H. / Kim, Y.-D. (1998): A multi-period order selection problem in flexible manufacturing systems. *The Journal of the Operational Research Society*, 49:278–286.

Lemaréchal, C. (1989): Nondifferentiable optimization. In: Nemhauser, G. L. / Rinnooy Kan, A. H. G. / Todd, M. J. (eds.), *Optimization*, vol 1 of *Handbooks in Operations Research and Management Science*, pp. 529–572. North-Holland, Amsterdam.

Martello, S. / Toth, P. (eds.) (1990a): *Knapsack Problems: Algorithms and Computer Implementations*. John Wiley & Sons, Chichester, New York, Brisbane, Toronto, Singapore.

Martello, S. / Toth, P. (1990b): Lower bounds and reduction procedures for the bin packing problem. *Discrete Applied Mathematics*, 28:59–70.

Mirchandani, P. B. / Francis, R. L. (eds.) (1990): *Discrete Location Theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Chichester, New York, Brisbane, Toronto, Singapore.

Mirzaian, A. (1985): Lagrangian relaxation for the star-star concentrator location problem: Approximation algorithm and bounds. *Networks*, 15:1–20.

Moore, E. W. / Warmke, J. M. / Gorban, L. R. (1991): The indispensible role of management science in centralizing freight operations at Reynolds metals company. *Interfaces*, 21:107–129.

Nauss, R. M. (1978): An improved algorithm for the capacitated facility location problem. *The Journal of the Operational Research Society*, 29:1195–1201.

Nemhauser, G. L. / Wolsey, L. A. (1988): *Integer and Combinatorial Optimization*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Chichester, New York, Brisbane, Toronto, Singapore.

Osman, I. H. (1995): Heuristics for the generalized assignment problem: Simulated annealing and tabu search approaches. *OR-Spektrum*, 17:211–225.

Owen, S. H. / Daskin, M. S. (1998): Strategic facility location: A review. *European Journal of Operational Research*, 111:423–447.

Palekar, U. S. / Karwan, M. K. / Zionts, S. (1990): A branch-and-bound method for the fixed charge transportation problem. *Management Science*, 36:1092–1105.

Pirkul, H. (1986): An integer programming model for the allocation of databases in a distributed computer system. *European Journal of Operational Research*, 26:842–861.

Pochet, Y. / Wolsey, L. A. (1988): Lot-size models with backlogging: Strong reformulations and cutting planes. *Mathematical Programming*, 40:317–335.

Revelle, C. S. / Laporte, G. (1996): The plant location problem: New models and research prospects. *Operations Research*, 44:864–874.

Romero Morales, M. D. (2000): *Optimization Problems in Supply Chain Management*. Ph.D. thesis, Erasmus University, Rotterdam.

Ryu, C. / Guignard, M. (1992): An efficient algorithm for the capacitated plant location problem. Working Paper 92-11-02, Decision Sciences Department, University of Pennsylvania, The Wharton School.

Savelsbergh, M. W. (1997): A branch-and-price algorithm for the generalized assignment problem. *Operations Research*, 45:831–841.

Shmoys, D. B. / Tardos, E. / Aardal, K. (1997): Approximation algorithms for facility location problems. In: *Proceedings of the 29th Annual ACM Symposium on Theory of Computing*, pp. 265–274.

Sun, M. / Aronson, J. E. / McKeown, P. G. / Drinka, D. (1998): A tabu search heuristic procedure for the fixed charge transportation problem. *European Journal of Operational Research*, 106:441–456.

Taillard, É. D. / Laporte, G. / Gendreau, M. (1996): Vehicle routeing with multiple use of vehicles. *The Journal of the Operational Research Society*, 47:1065–1070.

Tüshaus, U. / Wittmann, S. (1998): Strategic logistic planning by means of simple plant location: A case study. In: Fleischmann, B. / van Nunen, J. A. E. E. / Speranza, M. G. / Stähly, P. (eds.), *Advances in Distribution Logistics*, vol 460 of *Lecture Notes in Economics and Mathematical Systems*, chap. 2, pp. 241–263. Springer-Verlag, Berlin, Heidelberg, New York.

Wentges, P. (1996): Accelerating Benders' decomposition for the capacitated facility location problem. *Mathematical Methods of Operations Research*, 44:267–290.

Wright, D. / Haehling von Lanzenauer, C. (1989): Solving the fixed charge problem with Lagrangian relaxation and cost allocation heuristics. *European Journal of Operational Research*, 42:305–312.