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Sales Force Deployment by Mathematical Programming

Knut Haase

Abstract: The sales force deployment problem is considered which arises in many selling organizations. As a solution a novel mixed-integer formulation is introduced which is specifically characterized by an infinite number of variables. A column generation approach is proposed to obtain upper and lower bounds. The associated subproblem is solved analytically. A computational study shows that the approach outperforms recently introduced approaches.
Keywords: Marketing models, sales force sizing, salesperson location, sales territory alignment, sales resource allocation, nonlinear mixed–integer programming, column generation

1 Introduction

In many selling organizations, sales force deployment is an important instrument by which sales management can improve profit. It involves the concurrent resolution of four interrelated subproblems: sizing of the sales force, salespersons locations, sales territory alignment, and sales resource allocation. All these subproblems have to be solved simultaneously so that the total profit contribution will be maximized (see Haase and Drexl, 1999).

The subproblems can be briefly described as follows: Consider a large geographical market area which is partitioned to a set of so-called sales coverages units (SCUs). *Sizing of the sales force* advocates selecting the appropriate number of salespersons required to penetrate the total market area. A sales territory and a location will be assigned to each salesperson. Thus by sales force sizing we also decide on the number of sales territories and locations. The *salesperson location* aspect of the problem involves determining the location of each salesperson in one of the SCUs. *Sales territory alignment* may be viewed as the partitioning problem of grouping the SCUs around the locations of the salespersons into larger geographic areas called sales territories. *Sales resource allocation* refers to the problem of allocating salesperson time to the assigned SCUs.

Note, Sales force deployment is an aggregate planning problem with, typically, a planning horizon of one year so that decisions as to how often (e.g. every week, once a month), on which days and at which time a customer has to be visited are the subject of subsequent planning stages.

Related Work. Recently, a comprehensive review of related work is given by Drexl and Haase (1999). Therefore, for we are satisfied with a description of the most important contributions regarding this paper:

Skiera and Albers (1998) formulated a model which addresses both the sales territory alignment and the sales resource allocation problems simultaneously. For the solution a so-called backward deletion procedure is considered. If desired, the algorithm can also construct connected sales territories.

In Drexl and Haase (1999) all subproblems are covered. For the solution a kind of tournament selection approach is proposed, which solves large instances within reasonable time.

Haase *et al.*. (1999) address the location and alignment problem for salespersons selling motor-cars. The sales force sizing and the sales resource allocation problem were not relevant for their practical application. Considering a linear profit contribution function problems with 1219 SCUs have been optimally solved in under 11 minutes applying the standard software package CPLEX (see Bixby and Boyd, 1996).

Contribution. An innovative approach to sales force deployment is introduced which outperforms all previously proposed approaches. The approach is based on a new mathematical model of the sales force deployment problem. The model is characterized by an infinite number of variables the linear relaxation of which can be optimally solved using column generation. The optimal linear relaxation solution supplies an upper bound. An efficient linear programming-based heuristic is presented to derive lower bounds also. Based on a computational study it can be stated that the gap between the lower and upper bound is very small and thus a high solution quality is provided by the approaches. Moreover, it is shown that instances arising in practice can be solved efficiently within reasonable time.

Overview. The organization of this paper is as follows: Section 2 describes the problem setting. In particular, we define important terms and explain assumptions which are implicitly involved in these definitions. Section 3 introduces a new mathematical model for sales force deployment. Section 4 provides an approach to this formulation in which column generation is applied to derive lower and upper bounds. Section 5 covers the results of a computational study. To stress the practical relevance we consider two applications in Section 6. A summary and conclusions are provided in Section 7.

2 Problem Description

In the following we at first explain and discuss some terms which are important for understanding the addressed problem and then we provide a problem statement.

Sales Coverage Unit (SCU). A sales coverage unit is considered here to be a relatively small geographical area where the choice and thus the size of the SCUs depends upon the specific application and on whether the required data can be obtained (at a reasonable cost). Counties, zip codes, and company trading areas are some examples of SCUs (see for instance Zoltners and Sinha, 1983, and Churchill *et al.*, 1993).

Sales Territory. A sales territory is a connected geographic area which consists of a set of SCUs for which a salesperson is responsible. Connected means that a salesperson can travel from her location to each SCU in her sales territory without entering the sales territory of another salesperson. Clearly, this implies that the location of the salesperson is within his sales territory. An SCU which belongs to an island is adjacent to an SCU from the mainland if a bridge or a ship connection between both SCUs exists. Thus islands can also be assigned to a sales territory with a location on the mainland. Usually, the connectivity requirement is demanded by the management of a company for some organizational reasons. Note, a solution in which one or more territories are not connected may provide a better objective function value than the optimal solution in which all territories are connected. Moreover, allowing an SCU be further partitioned between two or more salespersons may also improve the solution quality. But this is not the subject here. To be responsible for a sales territory means that the salesperson has to provide service for all (potential) customers located in his sales territory. For example, a salesperson who is working for a company which provides materials concerning dental surgery will be responsible for all dentists practicing in her sales territory. She is only allowed to sell goods in SCUs which belong to her sales territory.

Travel Time, Calling Time, Selling Time. In order to sell a product a salesperson has to do some time consuming activities. The time required to travel from a location to a customer, from a customer to another customer, and back to the location is called travel time. Presenting a product and performing contract negotiations are examples of calling activities. The associated required time is denoted as calling time. Now, selling time is defined as the sum of calling time and travel time. For each salesperson the total selling time is restricted. We assume that the calling time is a constant fraction of the selling time. Clearly, this may be a more or less a rough average consideration. However, in Skiera and Albers (1998) an example is provided in which the travel time was assumed to be underestimated by 20%: The result was that only 0.1% of the selling time was used for calling when it was actually used for traveling.

Sales. The sales in a specific SCU depends on the calling time a salesperson is allocating to the SCU. Now, as the calling time is a constant fraction of the selling time and a restriction is defined on the selling time, we are going to consider the relationship between sales and selling time instead of sales and calling time. The relation between selling time and sales can be described by a sales response function (see Skiera and Albers, 1998). In applications such a function has to be specified by an econometric analysis. An example is provided in Haase *et al.* (1999).

Fixed Location Costs, Travel Costs. For each potential location fixed costs are incurred. These costs arise due to the fixed salary of a salesperson, the rent of an office, car insurance of a car, and so on. Some of these fixed costs may depend on the SCU in which a location will be set up. This may be reasoned by the fact that rents or site costs can be different across SCUs. Travel costs are proportional to travel time.

Profit Contribution, Objective. Each unit of sales provides a certain amount of profit contribution. The profit contribution obtained by one salesperson is the sum of the profit contributions resulting from the realized sales across the SCUs of his sales territory minus the incurred costs. We consider travel costs and fixed location costs. The sum of profit contributions obtained by the sales force has to be maximized. It may be noteworthy to mention that, as in applications the realized sales is stochastic so the expected profit contribution will be maximized. Typically, in order to describe the relation between selling time and sales a concave sales response function is considered. Assuming a concave sales response function more selling time is necessary for the last sold product unit than for the first one. Now, more selling time indicates more travel time and thus more travel cost. So the travel cost incurred for the last sold product is larger than for the first one. Thus, in general the average profit contribution per sales unit in

which travel cost is taken into account decreases as the number of sold products increases. Therefore, to maximize the profit contribution obtained by a salesperson accurately a (travel) cost function has to be taken explicitly into account.

Note, in applications travel cost is not significant.

Problem Statement. Determine the appropriate sales force size, construct a sales territory for each salesperson, and allocate the total available selling time of each salesperson over her sales territory such that the sum of profit contributions obtained by the salespersons will be maximized.

3 Mathematical Formulation

First, we formalize the profit contribution function needed to calculate objective function coefficients. Then we present an innovative mathematical model for the sales force deployment problem under concern.

3.1 Profit Contribution Function

Let

α per unit profit contribution of sales,

and consider the salesperson located in SCU i allocating selling time of t to SCU j then the profit contribution which will be obtained by the salesperson is derived from the profit contribution function

$$p_{ij}(t) = \alpha \cdot s_{ij}(t) - k_{ij}(t) \tag{1}$$

where

$s_{ij}(t)$ is the sales response function,

$k_{ij}(t)$ is the selling cost function

of the salesperson, respectively.

In the literature several types of sales response functions are considered (see e.g. Skiera and Albers, 1998). Very often a concave function is proposed.

Example. To formulate an exemplary profit contribution function we introduce some more symbols:

b calling time elasticity $(0 < b < 1)$

β_{ij} travel time fraction of the selling time of the salesperson located in SCU i allocating selling time to SCU j $(0 < \beta_{ij} < 1)$

η scaling parameter $(\eta > 0)$

h cost per travel time unit $(h \geq 0)$

Now, by

$$s_{ij}(t) = \eta\,((1 - \beta_{ij})\,t)^b \tag{2}$$

we define a concave sales response function and by

$$k_{ij}(t) = h \cdot \beta_{ij}\, t \tag{3}$$

a cost function and then we derive the profit contribution function

$$p_{ij}(t) = c_{ij}\, t^b - o_{ij}\, t \tag{4}$$

where

$$c_{ij} = \alpha \cdot \eta\,(1 - \beta_{ij})^b \tag{5}$$

and

$$o_{ij} = h \cdot \beta_{ij} \tag{6}$$

It should be noted here, that the following solution approach is also suited for considering a calling time elasticity, say b_{ij}, depending both on salesperson i and SCU j.

3.2 Model

We define the parameters

J	set of SCUs, indexed by j,
I	set of SCUs ($I \subseteq J$) for locating salespersons, indexed by i,
$\mathcal{N}_j$	set of SCUs which are adjacent to SCU j, and
T	total selling time available per period of a salesperson,
f_i	fixed cost per incurred per period for locating a salesperson in SCU i ($f_i \geq 0$)

introduce the decision variables

x_{ijt} =1, if salesperson located in SCU i is allocating a selling time of t in SCU j ($x_{ijt} = 0$, otherwise),

and then formulate an optimization model for sales force sizing, salesperson location, sales territory alignment, and sales resource allocation as follows:

$$\text{Maximize} \sum_{i \in I} \sum_{j \in J} \sum_{t \in [0,T]} p_{ij}(t)\, x_{ijt} - \sum_{i \in I} \sum_{t \in [0,T]} f_i\, x_{iit} \tag{7}$$

subject to

$$\sum_{i \in I} \sum_{t \in [0,T]} x_{ijt} = 1 \qquad j \in J \tag{8}$$

$$\sum_{j \in J} \sum_{t \in [0,T]} t \cdot x_{ijt} - \sum_{t \in [0,T]} T\, x_{iit} \leq 0 \qquad i \in I \tag{9}$$

$$\sum_{j \in \bigcup_{v \in V} \mathcal{N}_v - V} \sum_{t \in [0,T]} x_{ijt} - \sum_{k \in V} \sum_{t \in [0,T]} x_{ikt} \geq 1 - |V| \qquad i \in I, \quad V \subseteq J - \mathcal{N}_i - i \tag{10}$$

$$x_{ijt} \in \{0,1\} \qquad i \in I,\ j \in J,\ t \in [0,T] \tag{11}$$

The objective (7) maximizes the total profit contribution obtained by all salespersons. Equations (8) in combination with the binary conditions (11) assign each SCU to exactly one of the salespersons. By (9) selling time of a salesperson is allocated over the assigned SCUs. Note that, due to travel cost it may be suboptimal to allocate the available selling time completely. However, this is only a theoretical consideration. Relations (10) guarantee that SCUs assigned to one sales territory are connected. Note that these relations work similarly to constraints destroying short cycles in traveling salesperson model formulations (an example can be found in Haase (1997)). Clearly, it would be sufficient to take care of connected subsets $V \subseteq J - \mathcal{N}_i - i, i \in I$, of SCUs only. Finally, binary requirements on the decision variables are given by (11).

Note, $x_{iit} = 1$ means that SCU i is assigned to the salesperson located in sales territory i. In other words, $x_{iit} = 1$ does not only tell us where to locate a salesperson and how much working time the salesperson has to allocate to SCU i, it also defines how to align SCU i. Moreover, we assume by definition of the binary alignment variables $x_{ijt} \in \{0,1\}$ that accounts are exclusively assigned to individual salesperson. Note, this is an assumption in marketing science and marketing management, made for several appealing reasons.

Remark 3.1 The objective function (7) is for given t linear, although the function $p_{ij}(t)$ may be a nonlinear function. We prefer a linear formulation as, in general, much more powerful algorithms are available for linear optimization problems than for nonlinear ones. However, the "price we have to pay" for this helpful transformation is that we have to solve a linear formulation with an infinite number of variables.

4 Solution Approach

Due to the infinite number of variables it seems to be very hard to derive an optimal solution for the sales force deployment problem (7)–(11). Now, it is well known that for a given sales territory and some specific types of sales response function the optimal resource allocation can be derived very easily by solving a simple analytical problem (see e.g. Skiera and Albers, 1998). This refers to the development of a branch and bound procedure in which branching and bounding are done on the location and assignment decision variables $x_{ij} = \sum_{t\in[0,T]} x_{ijt}$. For example, if we consider the branch $x_{ij} = 1$ we have to set the variables $x_{kjt} = 0$ for all $k \in I \mid k \neq i$ and $t \in [0,T]$. Clearly, we have then to branch on a finite number of variables.

However, to derive a fast approach we require efficient methods of obtaining tight upper and lower bounds within reasonable time. This is the subject of the next two sections. In the next section we present a column generation procedure to compute upper bounds by solving the linear programming relaxation of (7)–(9) and (11). Then, in Section 4.2, we introduce an approach to obtain lower bounds in which column generation is applied.

4.1 Upper Bounds

In this section we present a column generation approach to derive an upper bound for the problem (7)–(11). The associated master- and subproblem of the column generation procedure will be defined in the following.

Master problem. Let $T_{ij} \subset [0,T]$, then

$$\text{Maximize} \sum_{i\in I}\sum_{j\in J}\sum_{t\in T_{ij}} p_{ij}(t)\, x_{ijt} \tag{12}$$

subject to

$$\sum_{i\in I}\sum_{t\in T_{ij}} x_{ijt} \leq 1 \qquad j \in J \tag{13}$$

$$\sum_{\substack{j\in J\\ j\neq i}}\sum_{t\in T_{ij}} t\cdot x_{ijt} + \sum_{t\in T_{ij}} (t-T)\, x_{iit} \leq 0 \qquad i \in I \tag{14}$$

$$x_{ijt} \geq 0 \qquad i \in I,\ j \in J,\ t \in T_{ij} \tag{15}$$

the master problem associated with the sales force deployment problem (7)–(11) is defined, i.e. the connectivity restrictions (10) and the integer requirements (11) are relaxed. Note, the equality of (8) is now changed to inequality. This may provide more stability to the simplex algorithm when solving the master problem (see Merle *et al.*, 1998).

Reduced Cost Function. Let

π_j be the dual variable associated with the j-th constraint of (13) ($\pi_j \geq 0$) and

σ_i be the dual variable associated with the i-th restriction of (14) ($\sigma \geq 0$)

then the reduced cost function $\bar{p}_{ij}(t)$ associated with the salesperson located in SCU i who allocates selling time t in SCU j is defined by

$$\bar{p}_{ij}(t) = \begin{cases} p_{ij}(t) - \pi_j - (t-T)\,\sigma_i & : \ i = j \\ p_{ij}(t) - \pi_j - t\cdot\sigma_i & : \ \text{otherwise} \end{cases} \tag{16}$$

The reduced cost function $\bar{p}_{ij}(t)$ computes the reduced cost of the variable x_{ijt}.

Subproblem. A solution of (12)–(15) provides an upper bound for (7)–(9), if

$$\bar{p}_{ij}(t) \leq 0 \qquad i \in I, j \in J, t \in [0,T] \tag{17}$$

is satisfied. Checking this condition is denoted as the subproblem.

Solution of the Subproblem. Let t'_{ij} be the selling time which maximizes $\overline{p}_{ij}(t)$. Considering $t \in [0, T]$, we determine the optimal feasible selling time by

$$t^*_{ij} = \begin{cases} t'_{ij} & : \ 0 \leq t'_{ij} \leq T \\ T & : \ t'_{ij} > T \\ 0 & : \ \text{otherwise} \end{cases} \tag{18}$$

Now, if

$$\overline{p}_{ij}(t^*_{ij}) \leq 0 \qquad i \in I, j \in J \tag{19}$$

then (17) is satisfied.

Note, we have to consider only a finite number of selling time values to check (17).

Example. Consider the profit contribution function (4) with the sales response function (2) and the cost function (3). Then we have to maximize

$$\overline{p}_{ij}(t) = \begin{cases} c_{ij}\, t^b - o_{ij}\, t - \pi_j - (t - T)\, \sigma_i & : \ i = j \\ c_{ij}\, t^b - o_{ij}\, t - \pi_j - t \cdot \sigma_i & : \ \text{otherwise} \end{cases} \tag{20}$$

for all $i \in I$ and $j \in J$. Differentiating (20) with respect to t and setting the derivative equal to zero leads to:

$$b\, c_{ij} \cdot t^{b-1} - o_{ij} - \sigma_i = 0 \tag{21}$$

Rearranging and applying (18) leads to:

$$t^*_{ij} = \begin{cases} \left(\frac{o_{ij} + \sigma_i}{b\, c_{ij}}\right)^{\frac{1}{(b-1)}} & : \ o_{ij} + \sigma_i > 0 \\ T & : \ \text{otherwise} \end{cases} \tag{22}$$

Thus to check whether a basic solution of the master problem provides an upper bound we have to compute the expression (22) and then $\overline{p}_{ij}(t^*_{ij})$ for each combination of $i \in I$ and $j \in J$. Now, if for at least one combination $i \in I$ and $j \in J$ $\overline{p}_{ij}(t^*_{ij}) > 0$ then the basic solution can be improved by appending the column corresponding to the variable x_{ijt^*} where $t^* = t^*_{ij}$ to the master problem.

Column Generation Procedure. The column generation approach to computing an upper bound of the addressed problem (7)–(11) is outlined in Table 1. In *step 0* we define an initial solution in which every potential location is assigned to a salesperson. Note, the initial master problem contains only the variables x_{iiT}, i.e. $x_{ijt} = 0$ for all $i, j, t \mid i \neq j$. It follows that the SCUs which are not suited to the location of a salesperson are not assigned in *step 0*. In *step 1* we solve the current master problem (with the simplex algorithm). From the optimal basic solution we compute the associated dual variables in *step 2*. Solving (18) we derive the optimal selling times in *step 3*. Then in *step 4* we extend the master problem only by those variables which have positive reduced cost (multiple pricing). This process is repeated as long as at least one variable (column) is appended to the master problem in *step 3* (see *step 5* and *step 6*).

Table 1: Upper Bounds – Sales Force Deployment

step	*0:*	$\forall i, j : T_{ij} = \begin{cases} T & : \ i = j \\ \emptyset & : \ \text{otherwise} \end{cases}$
step	*1:*	solve (12)–(14);
step	*2:*	derive $\pi_j \ \forall j$ and $\sigma_i \ \forall i$ from current solution;
step	*3:*	$\forall i, j$: determine t^*_{ij};
step	*4:*	$\forall i, j$: if $\overline{p}_{ij}(t^*_{ij}) > 0$ then $T_{ij} := T_{ij} \cup t^*_{ij}$;
step	*5:*	if $\exists\, \overline{p}_{ij}(t^*_{ij}) > 0$ goto *step 1*;
step	*6:*	stop;

4.2 Lower Bounds

In this subsection we are now going to describe a heuristic solution approach to the problem (7)–(11). It is based on the upper bound solution approach introduced in the preceding subsection. At first, we focus on determining the locations which should be set up. Clearly when determining the locations to be set up we also derive the sales force size. For each set of locations – which may provide a "good" lower bound – we obtain the sales territories and compute the associated selling time allocations.

In the following we present some procedures and functions which are used by the lower bound method.

Location Deletion. Given a solution of (12)–(15) compute for each potential location $i \in I$ the sum of associated decision variables

$$X_i = \sum_{t \in T_{ii}} x_{iit}.$$

Then calculate

$$\tilde{X}_k = \min\{X_i \mid i \in I; X_i > 0\}$$

and remove k from I, i.e. remove all decision variables associated with the salesperson located in SCU k from the master problem (12)–(15).

Initial Alignment. We assume $0 \notin I$. Define $y_j \in I \cup \{0\}$ where $y_j = i$ means that the SCU j is assigned to the salesperson located in SCU i and $y_j = 0$ that SCU j is assigned to none of the salespersons. Initialize $y_j = 0$ for all $j \in J$. Then set $y_i = i$ for all $i \in I$. Repeat as long as $(y_j = 0)$ exists: $\forall k \in J$ and $\forall i \in N_k$ if $(y_k > 0)$ and $(y_i = 0)$ then set $y_i = y_k$.

Note, using a first-in-first-out list the initial alignment can be done in linear time.

Resource Allocation. For a given sales territory the resource allocation can be solved by applying the column generation procedure proposed in the preceding subsection whereby the total master problem can be decomposed in $|I|$ independent small master problems in order to accelerate the solution process. Now, if we consider the special case of (4) with $o_{ij} = 0$ then an optimal resource allocation is provided by

$$t^*_{ij} = \frac{(c_{ij})^a}{\sum_{k \in J_i} (c_{ik})^a} T \qquad i \in I,\ j \in J_i \tag{23}$$

where $a = 1/(1-b)$ and J_i denotes the set of SCUs assigned to the salesperson located in SCU i (see Einbu, 1981, and Skiera and Albers, 1998).

Insertion Feasibility. An SCU j can only be assigned to the sales territory of the salesperson who is located in SCU i, if $k \in N_j$ with $y_k = i$ exists. With respect to this requirement the sales territory of the salesperson located in SCU i remains connected.

Removement Feasibility. An SCU can only be removed from a sales territory if the sales territory remains connected. Let m be the number of SCUs which are assigned to a specific sales territory. The sales territory is connected if m is equal to the number of SCUs, say n, which can be reached from a location without the necessity of going through another sales territory. n can be computed in linear time as follows: Consider the sales territory of the salesperson located in SCU i. Define $z_j = 1$ indicating that SCU j is already considered, and $z_j = 0$ otherwise. Let F be a subset of SCUs which can be reached from SCU i. We initialize $z_j = 0$ for all $j \in J \mid j \neq i$, $z_i = 1$, $F = \{i\}$, and $n = 1$. Now, for all $k \in F$ first remove k from F and secondly consider all $j \in N_k$: if $(z_j = 0)$ and $(y_j = i)$ then let $n = n+1$, $F = F \cup j$, and $z_j = 1$. This procedure terminates as long as F is empty. Note, each $j \in J$ is at most an element of F once. Clearly, if $(n < m)$ then the sales territory of the salesperson located in SCU i is not connected.

Improving Sales Territory Alignment. A given sales territory alignment may be improved by removing an SCU from a sales territory and inserting the SCU into another sales territory. If such an exchange is feasible, i.e. checking insertion and removement feasibilty, we perform a new resource allocation (see Skiera and Albers, 1994). Now, if this does not result in an increased lower bound value we will revoke the exchange. In general, the resource allocation may require much computation time. Therefore, we propose to develop a specific criteria to decide whether an SCU exchange probably results in an improvement or not. For example, consider the special case which is assumed for (23) and we want to evaluate the exchange of SCU j from the salesperson located in SCU i who is allocating a selling time

of t in SCU j to the salesperson located in SCU k. We propose to compare $p_{ij}(t)$ and $p_{kj}(t^*)$ where t^* is derived from (23) with $J_k = J_k \cup j$. Now, if

$$p_{ij}(t) < p_{kj}(t^*) \tag{24}$$

then an exchange improves the current alignment.

Lower Bound Method. An outline of the proposed heuristic approach is given in Table 2: In *step 0* we initialize the best lower bound value LB^* and the set of SCUs I^* in which a salesperson should be located. In *step 1* we apply the upper bound procedure proposed in the preceding subsection which now terminates if for all $i \in I$ and $j \in J$ the reduced cost $\overline{p}_{ij}(t)$ is less than $\epsilon_1 \geq 0$. Note, the larger ϵ_1 the earlier the upper bound procedure terminates. Thus by ϵ_1 we influence the required total computation time. In *step 2* the UB takes the objective function value, denoted as ofv, of the basic solution derived in *step 1*. Now, if this value is less than the objective function value of the best known lower bound solution LB^* we proceed with *step 14*. This means that we conclude that the lower bound cannot be improved further by reducing the number of locations. As described above, in *step 3* we remove one SCU, indexed by k, from the set of SCUs corresponding to the set of potential locations. Clearly, all variables associated with this location, i.e. x_{kjt} for all $j \in J$ and $t \in [0, T]$, will be removed from the master problem. Now, if the associated value of $\tilde{X}_k$ is less than $\epsilon_2 < 1$, i.e. if $\tilde{X}_k$ is relatively small we suppose that the current set I does not provide a competitive feasible solution (see *step 5*). From *step 5* to *step 8* we determine a feasible solution. The corresponding statements are described above. Note, in *step 8* the improvement procedure terminates depending on ϵ_3, i.e. the larger ϵ_3 the earlier the termination criterion is satisfied. Thus by the choice of ϵ_3 we influence the total computation time. The current lower bound value is LB (see *step 9*). Now, if this current lower bound is less than the best known lower bound we conclude that no better lower bound can be obtained by reducing the number of locations (see *step 10*). In *step 10* and *step 11* we update the variables for taking the actual best lower bound value and the corresponding set of locations. In *step 14* we try to improve the current best solution. This is apporpriate due to *step 8*.

Note, in the following SFDCG, sales force deployment by column generation, denotes the lower bound method as described in Table 2.

Table 2: Lower Bound – Sales Force Deployment – SFDCG

step	*0*:	initialize $LB^* = -\infty$ and $I^* = I$;
step	*1*:	upper bound procedure with pricing out $\overline{p}_{ij}(t) > \epsilon_1$;
step	*2*:	UB = ofv of master problem solution;
step	*3*:	if $UB \leq LB^*$ then goto *step 14*;
step	*4*:	location deletion $\Rightarrow \tilde{X}_k, k$ is removed from I;
step	*5*:	if $\tilde{X}_k < \epsilon_2$ then goto *step 1*;
step	*6*:	initial alignment;
step	*7*:	resource allocation;
step	*8*:	improve sales territory alignment as long as $\exists p_{ij}(t) + \epsilon_3 < p_{kj}(t')$;
step	*9*:	LB = ofv of *step 8* solution;
step	*10*:	if $LB < LB^*$ goto *step 14*;
step	*11*:	$LB^* = LB$;
step	*12*:	$I^* = I$;
step	*13*:	goto *step 1*;
step	*14*:	improve sales territory alignment based on I^* as long as $\exists p_{ij}(t) < p_{kj}(t')$;
step	*15*:	stop.

5 Computational Experimentation

The algorithms have been coded in C and implemented on a 350 MHz Pentium machine under the operating system Linux. For solving the master problem we used the LP–solver provided by CPLEX.

To evaluate the performance of the proposed methods we take the instance generator as described in Drexl and Haase (1999). We assume that the cardinality of the set I of potential sales territory centers, the cardinality of the set J of SCUs, and the expected fixed costs $E[f_i]$ to setup a location in SCU i have a major impact on the performance of the algorithms.

The fixed cost are generated as follows:

$$f_i = U(0.75 \cdot E[f_i],\ 1.25 \cdot E[f_i]) \tag{25}$$

where $U(a, b)$ denotes the uniform distribution over the interval $[a, b]$. We assume that o_{ij} can be negleted, i.e. $o_{ij} = 0$. As such, (23) can be applied with ease to the computation of the optimal resource allocation and we have the opportunity to compare SFDCG with the approaches proposed in Haase and Drexl (1999) and in Skiera and Albers (1998).

In Table 3 the parameter settings of SFDCG which are used to derive lower bounds are given.

Table 3: Values of SFDCG Control Parameters

ϵ_1	ϵ_2	ϵ_3
5.00	0.10	0.01

Table 4 provides the results required to analyze the solution quality of SFDCG. Each entry is the average result of 10 solved instances. On the average the solution gap is less than 1%, i.e. the upper and lower bounds are close to the optimum objective function value. Increasing the expected fixed cost the solution gap increases slightly. The same result can be observed when the number of potential locations is increased. However, the results indicate – at least in case of a concave objective function – that very large instances can now be solved very close to optimality.

Table 4: Bounds - Generated Instances

$\|J\|$	$\|I\|$	$E[f_i]$	UB	LB	% gap
500	100	1,000	138,320	137,590	0.53
500	100	2,500	95,933	95,080	0.89
1,000	100	1,000	272,642	271,435	0.44
1,000	100	2,500	195,771	195,001	0.40
1,000	250	1,000	292,148	290,861	0.44
1,000	250	2,500	203,389	202,417	0.48
5,000	250	5,000	738,070	736,185	0.26
5,000	500	5,000	757,005	754,709	0.30
5,000	250	10,000	554,457	549,161	0.95
5,000	500	10,000	564,743	558,096	1.18

The computation times in seconds required to derive the lower and upper bounds are given in Table 5. We see that for extremely large instances upper and lower bounds can be obtained within reasonable time. Note, in Drexl and Haase (1999) an upper bound procedure is presented by which instances with up to 500 SCUs are solved. This upper bound procedure was not suited to solving much larger instances. Thus the upper bound method proposed here is much more efficient. Now, let us consider some details of Table 5: The larger the instance size or the expected fixed cost the more computation time is required, whereby the increase is moderate. Note, the larger the fixed costs the smaller the sales force will be, i.e. the more often we have to perform *step 4* to *step 9* of SFDCG.

Table 6 provides some information about the solution process. The results emphasize the interpretation of the computation times in Table 5, i.e. the number of generated columns required to compute an upper and a lower bound increases when increasing the instance size or the expected fixed cost.

Table 7 provides a comparison between the approach of Skiera and Albers (1998), denoted as COSTA, and SFDCG. In COSTA the location aspect and sales force sizing is not considered, i.e. for a given set of locations the sales territories are designed and the associated sales resource allocations are computed. For the 19 instances refered to in Table 7 the average solution gap of SFDCG was 0.099%. The algorithm

Table 5: CPU-sec. – Generated Instances

$\|J\|$	$\|I\|$	$E[f_i]$	UB	LB	TOTAL
500	100	1,000	16	37	53
500	100	2,500	22	50	71
1,000	100	1,000	42	56	98
1,000	100	2,500	48	127	175
1,000	250	1,000	90	361	452
1,000	250	2,500	161	527	688
5,000	250	5,000	2,408	8,170	10,578
5,000	500	5,000	4,410	19,246	23,656
5,000	250	10,000	2,195	7,061	9,256
5,000	500	10,000	3,527	16,902	20,428

Table 6: Results – No. of Generated Columns and $|I^*|$ – Generated Instances

$\|J\|$	$\|I\|$	$E[f_i]$	UB	LB	$\|I^*\|$
500	100	1,000	7,825	14,630	60
500	100	2,500	10,507	20,425	20
1,000	100	1,000	11,812	15,772	90
1,000	100	2,500	13,964	25,616	37
1,000	250	1,000	24,441	57,981	127
1,000	250	2,500	31,295	77,440	43
5,000	250	5,000	79,374	157,683	72
5,000	500	5,000	152,361	359,112	80
5,000	250	10,000	87,870	177,919	31
5,000	500	10,000	153,200	378,263	33

of Skiera and Albers (1998) provides a solution gap of 0.195% (see Skiera and Albers, 1998). That is, the more general approach SFDCG provides a higher solution quality than COSTA. Furthermore, also when dividing the computation times of COSTA by 4 to take into account the different machine powers SFDCG is significantly faster than COSTA. Thus, SFDCG outperforms COSTA.

Table 7: Comparison with Skiera and Albers (1998) – Generated Instances ($E[f_i] = 0$, (10) relaxed)

		SFDCG			COSTA
$\|J\|$	$\|I\|$	LB	CPU	% gap	CPU[1]
50	10	18,883	0.31	0.00	1
100	10	35,808	0.85	0.06	4
200	10	57,732	2.40	0.00	15
400	10	92,152	6.21	0.00	49
75	15	33,978	0.70	0.37	3
150	15	53,585	1.42	0.02	10
300	15	87,576	4.49	0.06	41
600	15	139,531	12.99	0.01	141
125	25	54,976	1.87	0.08	8
250	25	98,477	4.19	0.04	31
500	25	138,239	14.23	0.03	132
1,000	25	231,474	38.84	0.01	502
250	50	122,331	6.63	0.25	40
500	50	175,947	21.90	0.12	161
1,000	50	289,913	60.77	0.02	933
2,000	50	473,985	210.06	0.02	3,093
500	100	220,727	34.17	0.45	290
1,000	100	360,316	83.01	0.25	1,359
2,000	100	587,120	289.61	0.09	5,951

[1] Computed on a 133 MHz Pentium machine

Now, we will compare SFDCG with CONIMP which denotes the heuristic approach for sales force deployment introduced in Drexl and Haase (1999). We perform the same computational study as in Drexl and Haase (1999) to explore the solution quality of CONIMP. The results are given in Table 8. We see that for all instance sizes SFDCG provides on average a significantly smaller solution gap than CONIMP. Moreover, SFDCG computes a solution much faster that CONIMP.

Table 8: Comparison with Drexl and Haase (1999) – Generated Instances

		SFDCG			CONIMP		
$\|J\|$	$\|I\|$	LB	CPU	% gap	LB	CPU	% gap
50	10	11,712	0	1.70	11,509	0	3.39
50	25	12,408	1	2.14	12,217	0	3.65
50	50	12,919	2	1.93	12,736	0	3.30
100	10	25,680	1	0.41	25,611	0	0.68
100	25	27,858	2	0.83	27,619	1	1.69
100	50	29,040	4	1.13	28,465	4	3.04
250	50	70,139	11	0.57	69,775	27	1.09
500	50	131,289	25	0.54	130,962	107	0.79

6 Applications

In this section we consider two applications published in the literature. The first one is presented by Skiera and Albers (1998) and the other one by Drexl and Haase (1999).

For the computation of the solutions we have again used a 350 MHz Pentium machine under the operating system Linux and the parameter settings as defined in Tabel 3.

Application 1. In Skiera and Albers (1998) the results of an application of COSTA in a mid-sized German company are presented. The company had hired 10 salespersons and adopted the 95 two-digit postal areas of Germany as SCUs. The following sales response function was considered:

$$s_{ij}(t) = 1350\ \mathrm{POT}_i^{0.625}\ (1+\tilde{q}_{ij})^{-0.375}\ t_{ij}^{0.375} \tag{26}$$

where

POT_i is the number of potential accounts and

$\tilde{q}_{ij}$ is the ratio of travel time to calling regarding salesperson i in SCU j.

Now, defining

$$c_{ij} = 1350\ \mathrm{POT}_i^{0.625}\ (1+q_{ij})^{-0.375} \tag{27}$$

we derive

$$s_{ij}(t) = c_{ij} \cdot t_{ij}^{0.375} \tag{28}$$

Thus, (23) can be applied for resource allocation. The total available selling time is $T = 1,300$ time units for each salesperson. Each salesperson has a fixed location, i.e. fixed costs have not to be taken into account in the setup of a location. Moreover, as no significant travel costs are considered, we define $o_{ij} = 0$. A comparison of the solution qualities of COSTA and SFDCG is given in Table 9. Again SFDCG outperforms COSTA.

Table 9: Application 1 – Skiera and Albers (1998)

		SFDCG				COSTA
$\|J\|$	$\|I\|$	UB	CPU[1]	% gap[2]	% gap	% gap
95	10	21,075,077	0.67	0.05	0.83	1.13

[1] UB computation time (in seconds)
[2] Connectivity relaxed

Application 2. In Drexl and Haase (1999) an application in the distribution of beverages is presented. The purpose of the application was to redesign the sales force deployment of FAXE – part of the Danish distributor of beverages "Bryggerigroupen Danmark AS" – in Schleswig–Holstein (northern part of Germany with 2.7 million inhabitants).

The following sales response function was considered[1]:

$$s_{ij}(t) = 0.205\ H_j \cdot\ (4.6+d_{ij})^{-0.285}\ t_{ij}^{0.285} \tag{29}$$

where

H_j is the number of inhabitants of SCU j and

d_{ij} is the time to drive from SCU i to SCU j.

Note, again (23) can be applied for resource allocation.

Here, as demanded by the management we had considered a modification of this instance. In the city Kiel the company runs a central distribution store. To reduce the total location cost and for some other reasons the management decided that one location of a salesperson has to be setup in the central distribution store. This requirement was easily modeled by assigning fixed location costs of zero regarding the SCU in which the store is located. The results of our solution approach are summarized in Table 10

[1] Note, some of the data are biased, because FAXE views our work with them as proprietary. They do not want their competitors to know all the details.

and Table 11, respectively. The solution was obtained in less than 10 minutes with a gap of 2.59% (see Table 10). The gap is small. However, the gap is larger than one may expect when taking into account the results of the preceding section. This may be reasoned as follows: The values of c_{ij} are now not as uniformly distributed across the SCUs as in the randomly generated instances. Furthermore, the fixed costs are larger than the expected fixed costs of the generated instances.

The proposed locations (see Table 11) and sales territories are illustrated in Figure 1. It is worth noting, that the results have been very well accepted by the management.

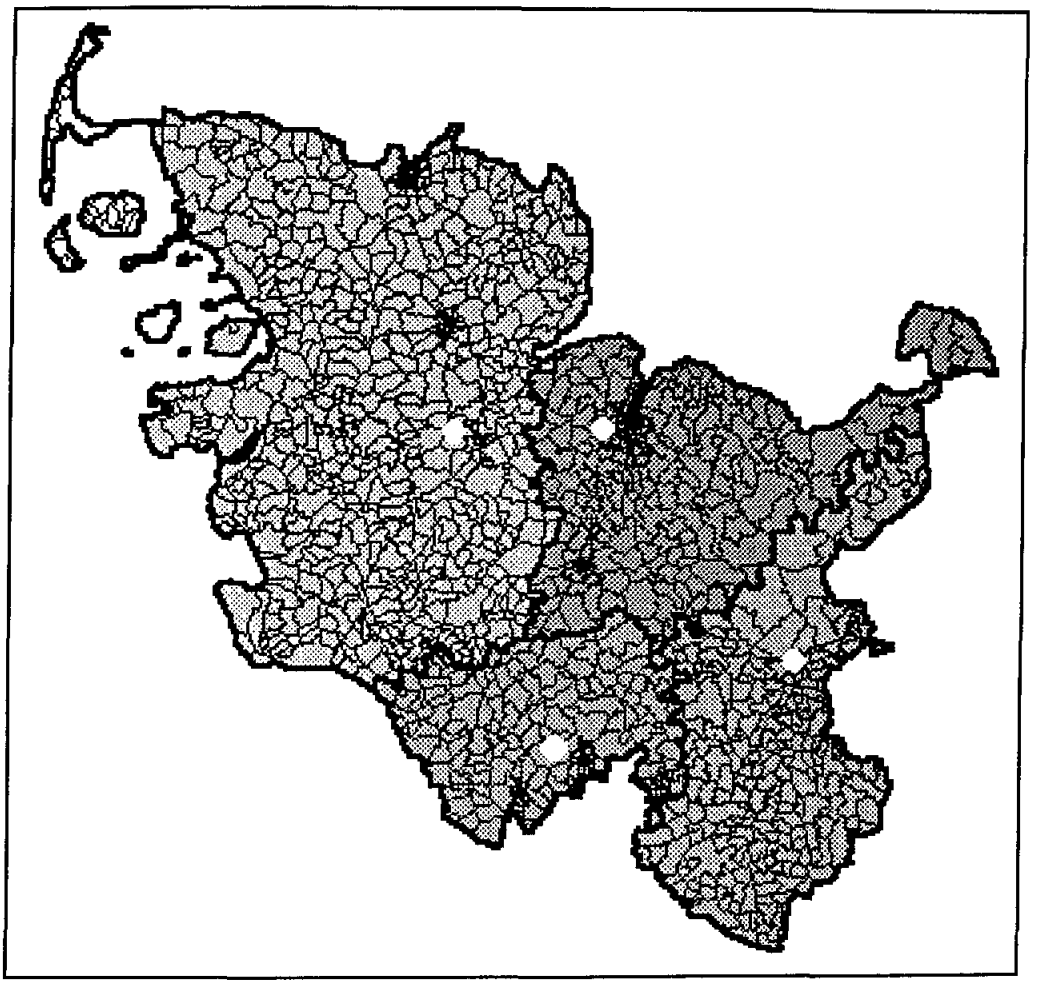

Figure 1: Redesigned Sales Territory for Schleswig-Holstein

Table 10: Performance of Application 2 – Drexl and Haase (1999)

$\|J\|$	$\|I\|$	UB	LB	CPU-TOTAL	% gap
1219	125	1,326,183.67	1,292,197.33	574.33	2.56

7 Conclusions

In this paper it is shown how four interrelated sales force deployment subproblems can be modeled and solved simultaneously. These subproblems are: sizing the sales force, salesperson location, sales territory alignment, and sales resource allocation. More specifically an integrated mixed-integer programming

Table 11: Solution of Application 2 – Drexl and Haase (1999)

GEOCODE	Name	Inhabitants
1002000250	KI-Mettenhof	18,689
1003000603	HL-Dornbreite	7,086
1056041000	Quickborn	18,008
1058054000	Fockbek	5,180

model is formulated. For the solution of the model we present a newly developed effective and efficient column generation approach.

The methods are evaluated on two sets of instances. The first one stems from a systematic generation of a representative set of problem instances covering all problem parameters at hand. The second consists of two applications. Benchmarking the results with the help of upper bounds shows that the methods allow very fast solution of large–scale instances close to optimality.

The methods provide lower bounds and upper bounds for the optimal objective function. On average the solution gap, i.e. the difference between upper and lower bound, is roughly 1%. Moreover, previously proposed solution approaches are outperformed.

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