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A Lagrangian Approach for Minimum Cost Tournaments

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Abstract

Single round robin tournaments are a well known class of sports leagues schedules. We consider leagues with a set T of n teams where n is even. Costs are associated to each possible match. Since matches are carried out at one of both opponents venues matches may be forbidden in certain periods due to unavailability of stadiums. The goal is to find the minimum cost tournament among those having no forbidden match. We employ a Lagrangian relaxation approach in order to obtain tight lower bounds. Moreover, we develop a cost-oriented repair mechanism yielding a feasible tournament schedule to each solution of the relaxed problem.

Keywords: Sports league scheduling, Lagrangian approach, round robin tournaments, stadium availability, forbidden matches

1 Introduction

Single round robin tournaments (RRT) cover a huge variety of different types of sports league schedules arising in practice. A set of teams T, $|T| = n$ even, competes such that each team plays exactly once against each other team, either at home or away. Such a match is specified by the team i playing at home, the team j playing away, and the period p it is carried out in. In the following a match is identified by triple (i, j, p), $i \in T$, $j \in T$, $j \neq i$, and $p \in P$. Furthermore, each team $i \in T$ has to play exactly once in each period and, hence, we have a set P of $n - 1$ periods altogether.

An illustrative example for a single RRT where $n = 6$ is given in Table 1. Here i-j denotes that team i plays at home against team j.

	1	2	3	4	5
match 1	3-4	4-5	2-4	4-6	6-5
match 2	5-2	1-3	5-1	3-5	4-1
match 3	6-1	2-6	6-3	1-2	2-3

Table 1: Single RRT for $n = 6$

A double RRT is a tournament where each team plays against each other team exactly once at home. As in single RRTs each team plays exactly once per period. Thus, $|P| = 2n - 2$. A mirrored double RRT has the additional property that team i plays at home against team j in period p, $p \leq n - 1$, if and only if team i plays at j's home against team j in period $p + n - 1$. Note that a single RRT can represent a mirrored double RRT. Hence, it is sufficient to schedule a single RRT after projecting all requirements involving periods n to $2n - 2$ on periods 1 to $n - 1$.

Since each match has to be carried out at one of both opponents' venues we have to take stadium availability into consideration. A stadium can be unavailable in a specific period for various reasons. There may be other events such as sports events or concerts. Moreover, the stadium may be closed due to a construction site. Even if it is not closed there may be a strong intention to have no match during construction because seating capacity may be reduced.

Furthermore, it may be suitable to forbid certain matches. The league's planners might be interested in having a lot of thrilling matches at the end of the season. Then, it would be appropriate to forbid matches between top level teams for the first periods. Moreover, a team can prefer to have a match against a top level team or against a traditional rival in specific periods. This, again, can be due to changing seating capacity or can be due to an other festival taking place in the the same area. Whatever the reason is, of course the set of possible matches can be reduced by forbidding all those matches that are not favored.

Models for sports league scheduling have been the topic of extensive research, see Briskorn and Drexl [6] for example. A whole stream of papers is based on the analogy between sports league scheduling and edge coloring of complete graphs. Examples are de Werra [11, 12, 13, 14, 15], de Werra et al. [16], and Drexl and Knust [17]. Brucker and Knust [8] and Drexl and Knust [17] analyze the relationship between sports league scheduling and multi-mode resource constrained project scheduling. Briskorn et al. [7] line out the similarity of structures of single RRTs and planar three index assignments. Bartsch [2], Bartsch et al. [3], and Schreuder [26, 27] examine particular formulations.

Extensive overviews of literature on sports leagues scheduling in the context of operations research are provided by Knust [24] and Rasmussen and Trick [25].

The remainder of this paper is organized as follows. In Section 2 we define a sports league scheduling problem and represent it by means of IP models. In Section 3 we develop a Lagrangian approach to obtain lower bounds for the sports league scheduling problem defined in Section 2. In Section 4 we line out computational results and, finally, Section 5 contains conclusions.

2 The models

We associate costs $c_{i,j,p}$ with the match of team i at home against team j in period p. Cost $c_{i,j,p}$ of a specific match are rather abstract here. However, there are several applications of $c_{i,j,p}$ with practical relevance, see Briskorn and Drexl [6]:

- Teams may have preferences to play at home in certain periods as outlined in Section 1. We can easily express preferences through $c_{i,j,p}$. Let $pr_{i,p} \in \mathbb{R}$ be team i's preference to play at home ($pr_{i,p} > 0$) or to play away ($pr_{i,p} < 0$), respectively, in period p. A preference $pr_{i,p}$ is stronger than a preference $pr_{i',p'}$ if $|pr_{i,p}| > |pr_{i',p'}|$. Then, costs can be defined as $c_{i,j,p} = -pr_{i,p} + pr_{j,p}$, for example. Here, cost $c_{i,j,p}$ represents neglected preferences of i and j in p decreased by fulfilled ones if this specific match is carried out in period p.

- Naturally, maximizing the overall attendance is a major objective of the organizers. We can represent the economic value of the estimated attendance by $c_{i,j,p}$. Let estimated attendances $ea_{i,j,p}$ be given for each match of team i at home against team j in period p. For a given pair of teams i, j estimated attendances might be time-dependent, that is $ea_{i,j,p} \neq ea_{i,j,p'}$ might hold for $p \neq p'$. This is due to other events in the same region or the current season, for example. We can define costs as $c_{i,j,p} = -ea_{i,j,p}$ and obtain the objective to maximize total tournament's attendance. Equivalently, $c_{i,j,p}$ can be defined as the number of seats remaining empty in the stadium of i if i plays at home against j in p.

- Often, a stadium is owned by some public agency and teams have to pay a fee for each match taking place in that particular stadium. This fee might depend on season, day of the week, and time when the match takes place as well as competing events. We can represent it by $c_{i,j,p}$ and obtain the objective to minimize the sum of fees to be paid.

We define the cost of a tournament as the sum of arranged matches' costs and consider the problem to find a minimum cost single RRT considering stadium availabilities and forbidden matches. Let $F \subset (T \times T \times P)$ be the set of forbidden matches such that $(i,j,p) \in F$ if (i,j,p) is forbidden. Furthermore, let H_p and A_p be the subset of teams that cannot play away and at home, respectively, in period p. Note that a team cannot play away in period $p \leq n-1$ if it cannot play at home in period $p+n-1$ in a mirrored double RRT. We represent the problem at hand by an IP model that is referred to as RRT in the following.

RRT

$$z = \min \sum_{i \in T} \sum_{j \in T \setminus \{i\}} \sum_{p \in P} c_{i,j,p} x_{i,j,p} \tag{1}$$

$$\text{s.t.} \sum_{p \in P} (x_{i,j,p} + x_{j,i,p}) = 1 \qquad \forall\, i,j \in T, i < j \tag{2}$$

$$\sum_{j \in T \setminus \{i\}} (x_{i,j,p} + x_{j,i,p}) = 1 \qquad \forall\, i \in T, p \in P \tag{3}$$

$$x_{i,j,p} = 0 \qquad \forall\, (i,j,p) \in F \tag{4}$$

$$x_{i,j,p} = 0 \qquad \forall\, i \in A_p, p \in P \tag{5}$$

$$x_{i,j,p} = 0 \qquad \forall\, j \in H_p, p \in P \tag{6}$$

$$x_{i,j,p} \in \{0,1\} \qquad \forall\, i,j \in T, i \neq j, p \in P \tag{7}$$

We employ binary variables $x_{i,j,p}$, $i \in T$, $j \in T$, $j \neq i$, $p \in P$, equaling 1 if and only if team i plays at home against team j in period p. Then, objective function (1) represents the goal of cost minimization. Constraints (2) and (3) assure the single RRT structure. Note that we can formulate (2) as lower-equal or greater-equal constraint without changing the solution space of RRT. Restriction (4) prevents forbidden matches from being carried out. Constraints (5) and (6) ensure matches are arranged only if the corresponding stadium is available.

A major issue when solving the problem specified by RRT in a branch-and-bound scheme is the weak lower bound $\bar{z}$ provided by the LP relaxation, see Briskorn [5] and Briskorn and Drexl [6]. Model RRT exhibits a structure such that Lagrangian relaxation is appropriate to be applied. Therefore, it is suitable to obtain tighter lower bounds by using a Lagrangian relaxation approach. Note that both constraints, (2) and (3), are candidate for relaxation. In the following we discuss both alternative relaxations and outline why relaxing (2) is more promising.

First, we employ Lagrangian multipliers $\lambda_{i,j}$, $i,j \in T$, $i < j$ in order to relax constraint (2) yields the following model RRT_λ.

It is obvious from (10) and (3) to (7) that RRT_λ can be decomposed into $n-1$ isolated models RRT_λ^p, $p \in P$.

Then, we represent the Lagrangian dual of RRT as RRT_{LD}.

RRT_λ

$$\min \sum_{i\in T}\sum_{j\in T\setminus\{i\}}\sum_{p\in P} c_{i,j,p}x_{i,j,p} + \sum_{i\in T}\sum_{j\in T,i<j}\lambda_{i,j}\left(1-\sum_{p\in P}(x_{i,j,p}+x_{j,i,p})\right) \tag{8}$$

$$= \min \sum_{i\in T}\sum_{j\in T,i<j}\left(\lambda_{i,j} + \sum_{p\in P}\left((c_{i,j,p}-\lambda_{i,j})\,x_{i,j,p} + (c_{j,i,p}-\lambda_{i,j})\,x_{j,i,p}\right)\right) \tag{9}$$

$$= \sum_{i\in T}\sum_{j\in T,i<j}\lambda_{i,j} + \min \sum_{p\in P}\sum_{i\in T}\sum_{j\in T,i<j}\left((c_{i,j,p}-\lambda_{i,j})\,x_{i,j,p} + (c_{j,i,p}-\lambda_{i,j})\,x_{j,i,p}\right) \tag{10}$$

s.t. (3), (4), (5), (6), (7)

$RRT_\lambda^{p_0}$

$$z_\lambda^{p_0} = \min \sum_{i\in T}\sum_{j\in T,i<j}\left((c_{i,j,p_0}-\lambda_{i,j})\,x_{i,j,p_0} + (c_{j,i,p_0}-\lambda_{i,j})\,x_{j,i,p_0}\right) \tag{11}$$

$$\text{s.t.} \sum_{j\in T\setminus\{i\}}(x_{i,j,p_0}+x_{j,i,p_0}) = 1 \qquad \forall\, i\in T \tag{12}$$

$$x_{i,j,p_0} = 0 \qquad \forall\,(i,j,p_0)\in F \tag{13}$$

$$x_{i,j,p_0} = 0 \qquad \forall\, i\in A_{p_0} \tag{14}$$

$$x_{i,j,p_0} = 0 \qquad \forall\, j\in H_{p_0} \tag{15}$$

$$x_{i,j,p_0} \in \{0,1\} \qquad \forall\, i,j\in T, i\neq j \tag{16}$$

It is well known that

$$\bar{z} \le z_{LD} \le z.$$

Note that $RRT_\lambda^{p_0}$ is a minimum weight perfect matching problem (details for the reduction can be found in Section 3.1.2) that is solvable in polynomial time. Furthermore, it is well known that is does not have the integrality property and, therefore, depending on the cost function c the provided lower bound is stronger than the one of the LP relaxation, that is $\bar{z} < z_{LD}$.

So far we did not specify the domain of multipliers $\lambda_{i,j}$, $i<j$. The domain depends on the formulation of constraint (2). If we formulate (2) as equation or lower-equal or greater-equal constraint, then we have $\lambda_{i,j}\in\mathbb{R}$, $\lambda_{i,j}\le 0$, or $\lambda_{i,j}\ge 0$, respectively, for all $i<j$. While all three formulations are equivalent in RRT narrowing down the domain of the multiplier may influence the convergence of a subgradient optimization process which we employ in order to solve RRT_{LD}.

Instead of (2) we can relax (3) employing Lagrangian multipliers $\mu_{i,p}$, $i\in T, p\in P$ and obtain model RRT_μ. As we can see, RRT_μ decomposes into an isolated optimization problem for each pair i and j, $i<j$, of teams. Then, the problem is to choose period $p_{i,j}$ for each pair of teams, where

RRT_{LD}

$$z_{LD} = \max_{\lambda} \left(\sum_{i \in T} \sum_{j \in T, i<j} \lambda_{i,j} + \sum_{p \in P} z_{\lambda}^{p_0} \right) \tag{17}$$

$$p_{i,j} = \arg\min_{p \in P} \quad (\{c_{i,j,p} - \mu_{i,p} - \mu_{j,p} \mid (i,j,p) \notin F, i \notin A_p, j \notin H_p\} \cup \{c_{j,i,p} - \mu_{i,p} - \mu_{j,p} \mid (j,i,p) \notin F, j \notin A_p, i \notin H_p\}).$$

Although this subproblem is trivial to solve in linear time there is no advantage over the bound provided by the LP relaxation since RRT_{μ}, obviously, has the integrality property, see Fischer [19] for further details.

RRT_{μ}

$$z = \min \sum_{i \in T} \sum_{j \in T \setminus \{i\}} \sum_{p \in P} c_{i,j,p} x_{i,j,p} + \sum_{i \in T} \sum_{p \in P} \mu_{i,p} \left(1 - \sum_{j \in T \setminus \{i\}} (x_{i,j,p} + x_{j,i,p}) \right) \tag{18}$$

$$= \sum_{i \in T} \sum_{p \in P} \mu_{i,p} + \min \sum_{i \in T} \sum_{j \in T \setminus \{i\}} \sum_{p \in P} (c_{i,j,p} - \mu_{i,p} - \mu_{j,p}) x_{i,j,p} \tag{19}$$

$$\text{s.t.} \quad (2), (4), (5), (6), (7)$$

After discussing both possible relaxations we choose to relax constraint (2) and develop an approach yielding lower and upper bounds for model RRT_{LD} in Section 3.

3 The Lagrangian Approach

The Lagrangian approach to solve RRT_{LD} follows the classic subgradient method, see Geoffrion [20] and Held et al. [21] for details, and can be sketched as follows. First, we initialize the Lagrangian multipliers $\lambda_{i,j}$. Then, we repeat the following steps until a stopping criterion is fulfilled. We solve the relaxed model RRT_{λ} by solving RRT_{λ}^{p} for each $p \in P$ yielding a lower bound of the optimal objective value to RRT. According to the obtained solution we modify the Lagrangian multipliers. Details of these steps are described in Section 3.1. Furthermore, we employ a heuristic procedure that is described in Section 3.2 to obtain feasible solutions and, thus, upper bounds. We abort the procedure if we reach an optimal solution or the update of multipliers does not have any effect anymore (see Section 3.1 for details). The scheme is sketched as algorithm "Subgradient Optimization".

3.1 Subgradient Optimization

3.1.1 Multiplier Update

The easiest way – and still a very common approach – to initialize the Lagrangian multipliers is to simply let $\lambda_{i,j} = 0$ for each $i < j$. We employ a procedure to level cost $c_{i,j,p} - \lambda_{i,j}$

Algorithm 1 Subgradient Optimization

Input: an instance I of RRT

Output: a lower bound LB, an upper bound UB and a feasible solution $bestSol$ for I

```
initialize λ_{i,j}
while STOPPING_CRITERION not reached do
  construct RRT_λ
  for all p ∈ P do
    solve RRT_λ^p
  end for
  construct currentSol
  if solution is feasible then
    update LB, UB, bestSol
    STOP
  else
    currentSol = feasibleSolution(currentSol)
    update LB, UB, λ_{i,j}, bestSol
    if UB = LB then
      STOP
    end if
  end if
end while
```

somewhat. By setting

$$\lambda_{i,j} = \min_{p \in P} c_{i,j,p} \qquad \forall i < j$$

we obtain at least one period for each pair of teams where the resulting cost $c_{i,j,p} - \lambda_{i,j}$ equals zero while $c_{i,j,p} - \lambda_{i,j} \geq 0$ in general. It turns out that the later approach slightly accelerates convergence of subgradient optimization and, thus, we employ it for computational evaluation. However, as stated in Beasley [4] the choice of initial multipliers is not expected to have a major influence on convergence and, thus, we restrict ourselves to this rather straightforward method.

For the multiplier update in each iteration we adapt the approach proposed by Baker and Sheasby [1] and Crowder [10] to our model formulation. We consider the best upper bound $\hat{z}$ obtained so far and the objective value z^k of the relaxed model RRT_{λ^k} corresponding to the current Lagrangian multipliers in iteration k. Let $x^k_{i,j,p}$ be the value of $x_{i,j,p}$ in the solution to RRT_{λ^k}. We then compute the step size t^k for the multiplier update in iteration k as

$$t^k = s^k \frac{\hat{z} - z^k}{\sum_{i \in T} \sum_{j \in T, i<j} \left(1 - \sum_{p \in P} \left(x^k_{i,j,p} + x^k_{j,i,p}\right)\right)^2},$$

where s^k is obtained by setting $s^0 = 2$ and $s^k = s^{k-1}/2$ when no improvement in the lower bound has been reached in the last 10 iterations and $s^k = s^{k-1}$ otherwise.

We employ exponential smoothing to the direction of the multiplier update, that is

$$d_{i,j}^k = (1-\alpha)d_{i,j}^{k-1} + \alpha\left(1 - \left(x_{i,j,p}^k + x_{j,i,p}^k\right)\right),$$

where $0 < \alpha \leq 1$. Note that $\alpha = 1$ is equivalent to no usage of smoothing. The actual update of the Lagrangian multiplier is done according to

$$\lambda_{i,j}^{k+1} = \lambda_{i,j}^k + t^k d_{i,j}^k$$

for each $i < j$.

One stopping condition for the subgradient optimization depends on the update function. We stop the procedure when s^k drops below 10^{-6} since then the multipliers are almost not changed at all.

3.1.2 Solving the Subproblem

As mentioned in Section 2 the Lagrangian subproblem RRT_λ decomposes into $n-1$ isolated problems RRT_λ^p, $p \in P$. Problem RRT_λ^p can be described as finding a minimum cost matchday. Here, a matchday is specified as a set of $n/2$ matches where each team contributes exactly once. We can further reduce the number of variables by fixing variables to zero such that there is no more than one free variable for each pair of teams i and j, $i < j$. If both, (i,j,p) and (j,i,p), are allowed regarding (13), (14), and (15), then we set $x_{i,j,p} = 0$ if $c_{i,j,p} > c_{j,i,p}$ or $c_{i,j,p} = c_{j,i,p}$ and $i < j$. In the following we refer to variables that are not fixed to zero (due to stadium unavailability, forbidden matches, or the consideration of costs as seen above) as free variables.

In the following we consider a directed graph $G^p = (V, A^p)$ on $|V| = n$ nodes for each period p. We have $A^p = \{(i,j) \mid x_{j,i,p}$ is a free variable$\}$. Nodes and arcs correspond to teams and matches, respectively. An arc from i to j represents a match between i and j at j's home in period p.

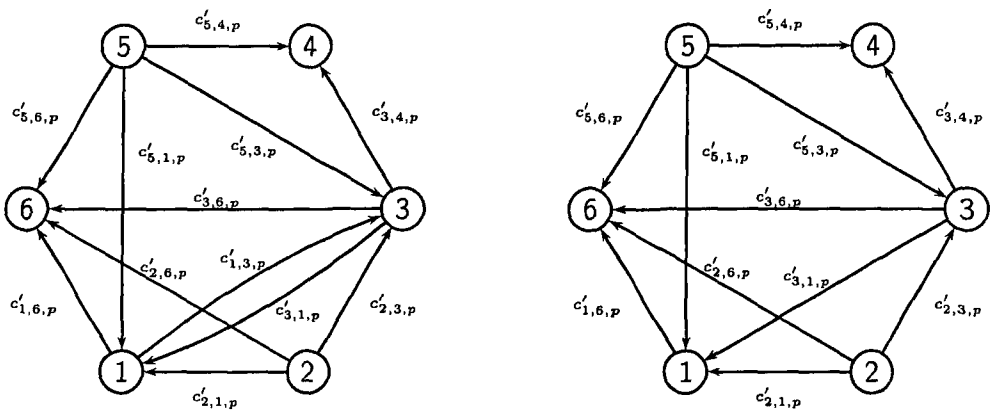

Figure 1: Example Graph for $n = 6$

Figure 1 illustrates a graph representing the structure of an examplary Lagrangian subproblem RRT_λ^p for $n = 6$ teams and period p on the left hand side. We have $H_p = \{4,6\}$, $A_p = \{2,5\}$, and $\{(1,4,p),(2,4,p)\} \subseteq F$. The set of arcs in the graph on the left hand side specifies the set of possible matches in period p (considering stadium unavailability and forbidden matches). Furthermore, we have arc weights $c'_{i,j,p} = c_{i,j,p} - \lambda_{\min(i,j),\max(i,j)}$ On the right hand side of

Figure 1 we see the reduced set of possible matches corresponding to the free variables. Teams 1 and 3 can meet in both venues. Since both matches are equivalent regarding feasibility of a solution we can restrict ourselves to the cheaper one of both matches. Assuming that $c'_{3,1,p} < c'_{1,3,p}$ we obtain the graph on the right hand side of Figure 1.

Note that the set of free variables does not depend on the Lagrangian multipliers since

$$c'_{i,j,p} < c'_{j,i,p} \quad \Leftrightarrow \quad c_{i,j,p} - \lambda_{\min(i,j),\max(i,j)} < c_{j,i,p} - \lambda_{\min(i,j),\max(i,j)} \quad \Leftrightarrow \quad c_{i,j,p} < c_{j,i,p}.$$

In all what follows we denote the cost related to teams i and j and period p as $\bar{c}_{i,j,p}$ and $\bar{c}_{j,i,p}$. If there is no free variable related to teams i and j and period p, then we set $\bar{c}_{i,j,p} = \bar{c}_{j,i,p} = \infty$. Consequently, the reduced cost in the Lagrangian subproblem is denoted by $\bar{c}'_{i,j,p}$ and $\bar{c}'_{j,i,p}$.

After reducing the number of free variables finding a minimum cost matchday reduces to finding a minimum cost perfect matching in this graph. A perfect matching A' is a subset of arcs such that each node is incident to exactly one arc in A' and corresponds to a matchday. We can solve the minimum cost perfect matching problem by the famous blossom algorithm by Edmonds [18]. For an survey on efficient implementations we refer to Cook and Rohe [9]. However, since efficiency of this algorithm is not a critical issue for problem sizes we consider here we employ the classic blossom algorithm.

3.2 Upper Bounds

In each iteration of the subgradient optimization procedure we obtain a set of $n-1$ perfect matchings (or 1-factors). Note that a feasible solution to our problem RRT corresponds to a ordered 1-factorization of the complete graph on n nodes. A 1-factorization is a partition of the edges of a graph into 1-factors as a single RRT is a partition of all matches between teams of T into matchdays. An ordered 1-factorization is a 1-factorization where the 1-factors are given in a specific order which corresponds to the assignment of matchdays to periods. Hence, in order to generate a feasible solution based on the optimal solution of the Lagrangian subproblem we have to transform an ordered set of $n-1$ 1-factors (matchdays) into an ordered 1-factorization (a single RRT).

Hilton and Johnson [22] develop a technique to transform a set of factors of the complete graph K_n into a factorization of the same graph. They assume they have a number of $t \leq n-1$ subgraphs and each subgraph G_g, $1 \leq g \leq t$, is a l_g-edge-connected and k_g-regular where $k_g \geq l_g$. Hilton and Johnson [22] provide a procedure to deterministically transform these subgraphs into a factorization $\mathcal{F}$ of K_n such that $|\mathcal{F}| = t$ and each factor F_i, $1 \leq g \leq t$, is l_g-edge-connected and k_g-regular. Note that this exactly covers the requirements for a construction of feasible solution: in our case we have $k_g = 1$, $l_g = 0$, and, therefore, $t = n-1$.

In the following we develop a procedure that is based on the mechanism of Hilton and Johnson [22] and is extended for cost orientation and consideration of runtime requirements. First, we translate the basic foundings of Hilton and Johnson [22] into the terms of our problem formulation in order to keep the paper self-contained. Moreover, we can simplify the notation since we consider a special case of the setting in Hilton and Johnson [22], here.

Having a solution to the Lagrangian subproblem we denote the number of matches between two teams i and j, $i < j$, by $x_{j,i} = x_{i,j} = \sum_{p \in P} (x_{i,j,p} + x_{j,i,p})$. Suppose we have $x_{i,j} \neq 1$ for at

least one pair i and j (otherwise, we have a feasible solution) and let $f = \sum_i \sum_{j>i} |x_{i,j} - 1|$ be the total violation of constraint (2). Since the number of matches of each team i is $\sum_j x_{j,i} = n - 1$ (due to the perfect matching structure) we can always find three distinct teams i, j, and k such that $x_{i,k} > 1$ and $x_{j,k} = 0$. Then, we can find a sequence of distinct teams $(k_0, \ldots, k_m)$, $1 \leq m \leq n-1$, such that

- $k_0 = k$
- $x_{i,k_g,p_g} = 1$ and $x_{j,k_{g+1},p_g} = 1$ for each $0 \leq g \leq m-1$ and $p_g \in P$ and
- $x_{i,k_m} = 0$ or $x_{j,k_m} > 1$.

In plain english, each team and its right neighbour in the sequence are the opponents of i and j, respectively, in an arbitrary but fixed period. Additionally, the last team in the sequence is a team that does not play against i at all or does play against j at least twice (possible both). The essential step of the method is to exchange the opponents in each period p_g, $0 \leq g \leq m-1$. Note that $p_g \neq p_{g'}$ if $g \neq g'$ since the sequence of teams is distinct. It is obvious that we obtain an other set of $n-1$ matchdays. Moreover, Hilton and Johnson [22] prove that $f' \leq f - 2$ if f' is the total violation of constraint (2) after the step. More accurately, if $x_{i,k_m} = 0$ and $x_{j,k_m} > 1$, then $f' = f - 4$ and if either $x_{i,k_m} = 0$ or $x_{j,k_m} > 1$, then $f' = f - 2$. Note that $f \leq n(n-2)$ and, therefore, we obtain a single RRT after no more than $n(n-2)/2$ steps.

Now, we illustrate the procedure described above on a small example. We consider $n = 6$ teams and let us suppose we have obtained the matchdays as given in the set labelled "Initial" (in increasing order of periods) in Figure 2. Here, we have $x_{2,3} = x_{3,5} = x_{4,6} = 2$, $x_{2,5} = x_{3,4} = x_{3,6} = 0$ and $x_{i,j} = 1$ for each other $i < j$ and, therefore, $f = 6$. Moreover, $i = 3$, $j = 2$, and $k = 5$ fulfills the requirements described earlier. We can find the sequence of teams $(5, 1, 6)$ that obviously has the properties stated above. Note that $x_{3,6} = 0$ but $x_{2,6} = 1$. For this sequence we have $p_0 = 5$ and $p_1 = 4$. Accordingly, the first step is to exchange opponents of teams $i = 3$ and $j = 2$ in both, periods $p_0 = 5$ and $p_1 = 4$. We obtain the set of matchdays labelled "Step 1". Now, $x_{2,3} = x_{4,6} = 2$, $x_{2,6} = x_{3,4} = 0$ and $x_{i,j} = 1$ for each other $i < j$ and, therefore, $f' = 4$. The total number of violation has dropped by 2 which corresponds to the fact that either $x_{i,k_m} = 0$ or $x_{j,k_m} > 1$ for the sequence found. Since $f' > 0$ we have to iterate the procedure at least once more. We can find $i = 2$, $j = 4$, and $k = 3$ now. We find sequence $(3, 6)$ where $p_0 = 2$. Here, $x_{2,6} = 0$ and $x_{4,6} = 2$. The second step is to exchange the opponents of $i = 2$ and $j = 4$ in period $p_0 = 2$. We obtain the set of matchdays labelled "Step 2" forming a single RRT. This means the total number of violations has dropped by 4 which corresponds to the fact that $x_{i,k_m} = 0$ or $x_{j,k_m} > 1$ for the second sequence.

Hilton and Johnson [22] prove that a sequence as described above exists for each pair of teams i and j such that a third team k exists with $x_{i,k} > 1$ and $x_{j,k} = 0$. In the following we propose an approach to find a cost efficient sequence $(k_0, \ldots, k_m)$ in terms of the costs implied by switching opponents in periods $p_0, \ldots, p_{m-1}$. First, we consider the case where we already chose a pair of teams i and j and consider them fixed in the following. Note that we can express the cost related to each pair of teams k_g and k_{g+1}, $0 \leq g \leq n-2$ in the sequence as

$$c^{i,j}_{k_g,k_{g+1}} = \min_{p \in P^{i,j}_{k_g,k_{g+1}}} \left\{ \bar{c}_{\min\{j,k_g\},\max\{j,k_g\},p} + \bar{c}_{\min\{i,k_{g+1}\},\max\{i,k_{g+1}\},p} - \left(\bar{c}_{\min\{i,k_g\},\max\{i,k_g\},p} - \bar{c}_{\min\{j,k_{g+1}\},\max\{j,k_{g+1}\},p} \right) \right\}, \tag{20}$$

where $P^{i,j}_{k_g,k_{g+1}}$ denotes the set of periods where the opponents of i and j are k_g and k_{g+1}, respectively. The minimization term in (20) is due to the fact that are k_g and k_{g+1}, $0 \leq g \leq m-1$, may be opponents of i and j, respectively, in more than one period. Since we can arbitrarily choose one of those periods as far as correctness of step is concerned, we choose the period where the switch of opponents implies minimum cost. Note that the expression that is to be minimized describes the change in costs if the opponents are exchanged.

We can see that $c_{k_g,k_{g+1}}$ is independent from $c_{k_{g'},k_{g'+1}}$ for $1 \leq g < g' \leq n-2$. This is why we can reduce the problem to find the minimum cost step for a given pair i and j of teams to the shortest path problem. We define a directed weighted graph $G^{i,j} = (V^{i,j}, A^{i,j}, w^{i,j})$ as follows:

$$
\begin{aligned}
V^{i,j} &= \left\{so^{i,j}\right\} \cup \left\{si^{i,j}\right\} \cup \left\{k^{i,j} \mid k \in T\right\} \\
A^{i,j} &= A^{i,j}_{so} \cup A^{i,j}_{si} \cup A^{i,j}_{T} \\
A^{i,j}_{so} &= \left\{(so^{i,j}, k^{i,j}) \mid x_{i,k^{i,j}} > 1 \wedge x_{j,k^{i,j}} = 0\right\} \\
A^{i,j}_{si} &= \left\{(i, si^{i,j}) \mid x_{i,k^{i,j}} = 0 \vee x_{j,k^{i,j}} > 1\right\} \\
A^{i,j}_{T} &= \left\{(k^{i,j}, l^{i,j}) \mid \exists p \in P, x_{i,k^{i,j},p} = x_{j,l^{i,j},p} = 1\right\} \\
w^{i,j}_{a} &= 0 \quad \forall a \in A^{i,j}_{so} \\
w^{i,j}_{a} &= -\delta \quad \forall a = (k^{i,j}, si^{i,j}) \in A^{i,j}_{si}, x_{i,k^{i,j}} = 0 \wedge x_{j,k^{i,j}} > 1 \\
w^{i,j}_{a} &= 0 \quad \forall a = (k^{i,j}, si^{i,j}) \in A^{i,j}_{si}, x_{i,k^{i,j}} = 0 \oplus x_{j,k^{i,j}} > 1 \\
w^{i,j}_{a} &= c^{i,j}_{k,l} \quad \forall a = (k^{i,j}, l^{i,j}) \in A^{i,j}_{T}
\end{aligned}
$$

There is an artificial source so and an artificial sink si in the graph. Each other node corresponds to a team. There is an arc $(k^{i,j}, l^{i,j})$ if and only if k and l are opponents of i and j, respectively, in at least one period and therefore can be consecutive elements in a sequence for i and j. There is an arc $(so^{i,j}, k^{i,j})$ and an arc $(k^{i,j}, si^{i,j})$ if $k^{i,j}$ is allowed to be the first and the last, respectively, element in a sequence for i and j. Then, a path from $so^{i,j}$ to

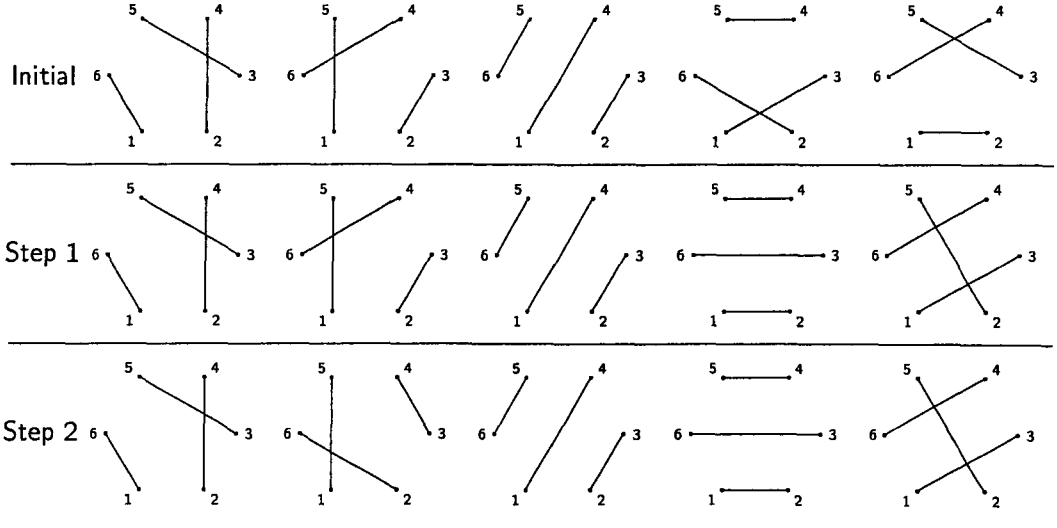

Figure 2: Example for Repair Scheme

$si^{i,j}$ corresponds to a sequence for i and j and the shortest path corresponds to the sequence causing the lowest increasement of cost in the solution to the Lagrangian subproblem.

Arcs in $A_T^{i,j}$ correspond to switches of opponents and therefore the weight of $a = (k^{i,j}, l^{i,j}) \in A_T^{i,j}$ corresponds to change $c_{k,l}^{i,j}$ of total cost in the current solution caused by the switch represented by a. Arcs in $A_{so}^{i,j}$ do not influence the repair step as far as costs are concerned. It depends on the last element of the sequence whether the total violation of constraints is reduced by 2 or by 4. We prefer those step reducing the total violation by 4 and, therefore, give the corresponding arcs a negative weight $-\delta$. Of course, we can prevent steps reducing the total violation by 2 by setting $\delta = M$. Then, a sequence reducing the total violation by 2 is chosen only if there is no sequence reducing total violation by 4. On the other hand, by setting $\delta = 0$ we do not distinguish between sequences depending on the reduction of total violation.

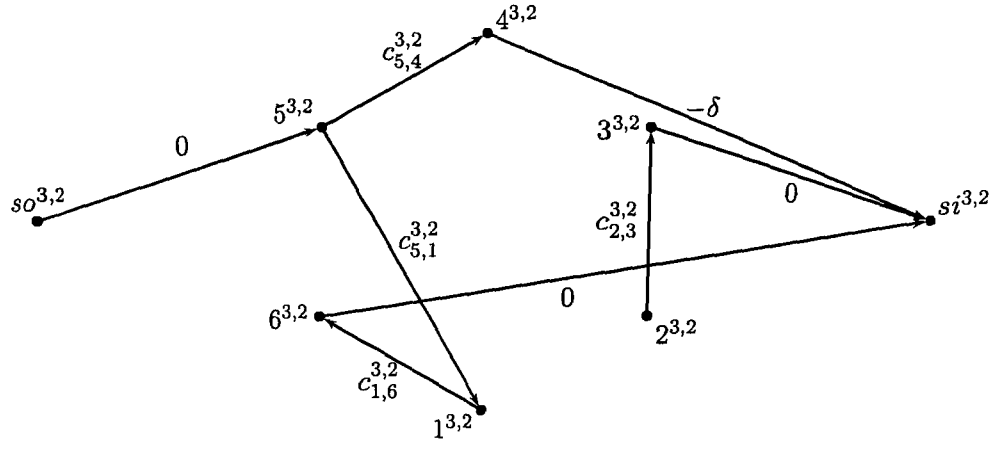

Figure 3: Example for Shortest Path Graph

Figure 3 illustrates graph $G^{i,j}$ corresponding to teams $i = 3$ and $j = 2$ and to the set of matchdays labelled "Initial" in Figure 2. There are two paths from $so^{3,2}$ to $si^{3,2}$, these are $so^{3,2} \rightarrow 5^{3,2} \rightarrow 1^{3,2} \rightarrow 6^{3,2} \rightarrow si^{3,2}$ and $so^{3,2} \rightarrow 5^{3,2} \rightarrow 4^{3,2} \rightarrow si^{3,2}$. Since the second path leads to a sequence reducing the total number of violations by 4 (which does only depend on the last element $4^{3,2}$) the arc leading to $si^{3,2}$ has weight $-\delta$.

Graph $G^{i,j}$ serves to find the minimum cost sequence if teams i and j are fixed. However, ususally there is more than one pair of teams i and j as described above. Next, we define a graph enabling us to employ a shortest path algorithm to find among all pairs i and j the corresponding sequence implying the cheapest modification of the solution. Let IJ be the set of all possible pairs (i, j) of teams i and j with $i < j$. We define a graph $G = (V, A, w)$ as follows:

$$
\begin{aligned}
V &= \{so\} \cup \{si\} \cup \bigcup_{(i,j)\in IJ} V^{i,j} \\
A &= A_{so} \cup A_{si} \cup \bigcup_{(i,j)\in IJ} A^{i,j} \\
A_{so} &= \left\{(so, so^{i,j}) \mid (i,j) \in IJ\right\} \\
A_{si} &= \left\{(si^{i,j}, si) \mid (i,j) \in IJ\right\} \\
w_a &= w_a^{i,j} \quad \forall a \in A^{i,j}, (i,j) \in IJ \\
w_a &= 0 \quad \forall a \in A_{so} \cup A_{si}
\end{aligned}
$$

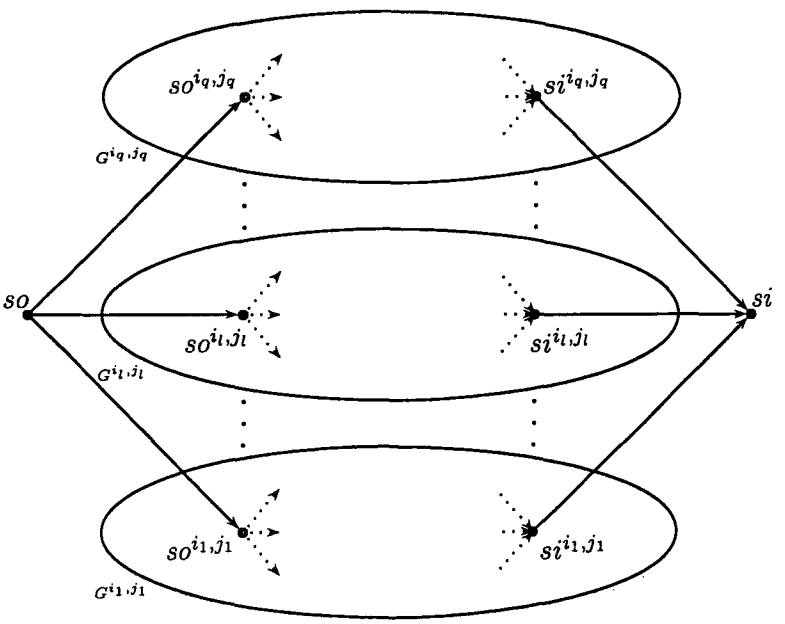

Figure 4: Example for Shortest Path Graph for all Pairs (i,j)

Figure 4 illustrates graph G. We have a unique artificial source so and a unique artificial sink si. For each pair $(i,j) \in IJ$ we have a subgraph $G^{i,j}$. In Figure 4 we have $|IJ| = q$ and $1 \leq l \leq q$. Now a path from so to si corresponds to the choice of a pair $(i,j) \in IJ$ by choosing one of the subgraphs. Note that the subgraphs are not connected. Furthermore, a sequence is chosen by finding a path within the chosen subgraph.

Graph G enables us to find the minimum cost step to reduce the total violation of constraints unless there is negative length circle in one of the subgraphs. This might happen since the solution of the Lagrangian subproblem is determined based on reduced cost $\bar{c}_{i,j,p}$ while we aim at finding a feasible solution to RRT minimizing overall costs based on $c_{i,j,p}$.

Note that a negative length circle can be interpreted in the same way as a sequence. Let $k_0, \ldots, k_m$ be a negative circle that is $k_0 = k_m$ where k_0 is arbitrarily chosen as the first element. Then, both, i and j, play against each team in $k_0, \ldots, k_{m-1}$ and switching the opponents in corresponding periods $p_0, \ldots, p_{m-2}$ will lead to a solution with identical $x_{i',j'}$, $i' \neq j'$. Hence, we do not increase the total violation of constraints but do reduce the overall cost. Consequently, we apply this step. Note, that in the graph G of the new solution the

negative circle just applied cannot occur again. Moreover, since we reach a cost reduction each time we detect a negative circle and the cost reduction cannot be arbitrarily small we can only find a negative circle a finite times before obtaining a graph G without any negative circle.

We propose two alternative strategies to apply a low-cost step. First, we can construct graph G, find the minimum cost step and apply it. Second, in order to save runtime we find an arbitrary pair (i, j), construct graph $G^{i,j}$, find a minimum cost step for pair (i, j), and apply it. We refer to the first and the second strategy as REPAIR and REPAIR(i, j), respectively, in the following. In both cases we apply the Bellmann-Ford algorithm to solve the shortest path problem. Note that we can identify a negative circle if there is still a violated triangle inequality after $|V|$ ($|V^{i,j}|$) steps of the Bellmann-Ford algorithm. Summarizing, our repair scheme to find a feasible solution can be sketched as repeating algorithm "Repair Step" until the resulting solution is feasible.

Algorithm 2 Repair Step

Input: set of $n-1$ matchdays having total violation f, cost $c_{i,j,p}$
Output: set of $n-1$ matchdays having total violation $f' \leq f-2$

```
if REPAIR then
  construct G
else
  if REPAIR(i, j) then
    find an arbitrary pair (i, j)
    construct G^{i,j}
  end if
end if
found=FALSE
while found = FALSE do
  apply Bellmann-Ford algorithm
  if there is a negative circle then
    apply circle
  else
    apply sequence
    found=TRUE
  end if
end while
```

In preliminary tests strategy REPAIR(i, j) has proved to be more efficient and, therefore, we restrict everything what follows to REPAIR(i, j).

4 Computational Results

In the section at hand we outline results of a computational study carried out to evaluate the efficiency of the approach proposed in Section 3. The study has been carried out on a 3.8 GHz Pentium 4 machine with 3 GBs of RAM running Microsoft Windows Server 2003. Naturally, our focus is on run times and lower bound quality of our approach. Moreover, we

consider the quality of solutions obtained by our repair scheme. We compare the solutions and lower bounds of our approach to optimal solutions and lower bounds, respectively, obtained by employing Cplex.

For each number of teams between 12 and 30 we have 16 classes of 20 instances each. The classes are specified by two parameters with four possible values each. In the following we detail how we generate instances of each class.

1. Parameter $P^f \in \{0.0, 0.1, 0.2, 0.3\}$ specifies the probability of each match (i, j, t) to be forbidden. Given P^f, we randomly decide whether (i, j, t) is forbidden according to P^f. Note that $F = \emptyset$ if $P^f = 0$.

2. Parameter $P^s \in \{0.0, 0.1, 0.2, 0.3\}$ specifies the probability for each team i and each period p that i's venue is restricted in p. Given P^s, we randomly decide whether i's venue is restricted in p according to P^s. If the venue is not restricted, then i can play at home and away, respectively, in p. If it is restricted, then we randomly decide whether i must play at home, that is $i \in H_p$, or away, that is $i \in A_p$.

3. In each class costs are randomly drawn from integer values between 0 and 20.

In the following, we refer to the class of instances having n teams and being generated using P^f and P^s as $\mathcal{I}(n, P^f, P^s)$. Table 2 gives an overview of the results of our computational study. For all classes of instances we outline run time (r.t.) of our approach in seconds and the percentage by which the best found lower bound exceeds the LP relaxation lower bound (l.b.). For $\mathcal{I}(n, P^f, P^s)$ where $n \leq 16$ we additionally provide the percentage by which the best found upper bound exceeds the optimal objective value (u.b.).

We can clearly observe that the lower bound yielded by our approach is larger than the LP relaxation lower bound. For common numbers of teams in real world tournaments, that is for $n \leq 20$, we obtain a lower bound that is at least larger by 0.9% than the LP relaxation lower bound. However, quality of our lower bounds seems to decrease in n which may be due to the quality of upper bounds which is decreasing as well and will be discussed later on.

While P^f seems not to influence the quality of our lower bound we can see that the lower bound gets slightly worse (in comparison to the LP relaxation lower bound) in increasing P^s. An intuitive reason for this can be given as follows. As mentioned before the LP relaxation lower bound coincides with the optimal objective value of the Langrangian dual if the subproblem has the integrality property. Now, consider a problem instance out of $\mathcal{I}(n, P^f, 1.0)$, that is for each team the venue in each period is fixed. Then, RRT^p_λ is equivalent to the bipartite minimum cost matching problem which is well known to have the integrality property. Hence, by increasing P^s we "transform" the minimum cost perfect matching problem which does not have the integrality property step by step to a problem having the integrality property. Therefore, the gap between LP relaxation lower bound and optimal solution to the Lagrangian dual narrows down when we increase P^s.

As we can see the upper bounds obtained by our approach are far from optimality. Still, in comparison to basic construction schemes like the circle method, see Kirkman [23], our results are reasonably good. We could find feasible solutions to all instances while the circle method fails in many cases when $P^f > 0$ or $P^s > 0$. If the circle method found a feasible solution the solution obtained by our approach had lower cost. However, the upper bounds obtained using our approach are not competitive to those obtained from sophisticated optimization methods.

n	P^s	0.0				0.1				0.2				0.3			
	P^f	0.0	0.1	0.2	0.3	0.0	0.1	0.2	0.3	0.0	0.1	0.2	0.3	0.0	0.1	0.2	0.3
12	l. b.	1.7	1.2	1.7	1.4	1.3	1.7	1.2	1.6	1.4	1.3	1.3	1.1	1.3	1.2	1.6	1.4
	r. t.	0.42	0.43	0.40	0.44	0.39	0.41	0.45	0.43	0.40	0.42	0.41	0.45	0.42	0.43	0.44	0.47
	u. b.	18	19	17	20	15	19	17	19	15	15	14	16	16	15	14	21
14	l. b.	1.5	1.6	1.8	1.6	1.4	1.6	1.6	1.6	1.5	1.2	1.6	1.1	1.2	1.3	1.1	0.9
	r. t.	0.68	0.79	0.73	0.75	0.75	0.70	0.72	0.74	0.69	0.75	0.74	0.80	0.74	0.74	0.79	0.69
	u. b.	42	48	43	47	34	35	40	46	32	38	40	47	34	38	52	34
16	l. b.	1.5	1.3	1.1	1.2	1.1	1.2	1.2	1.2	1.3	1.5	1.3	1.4	1.3	1.3	1.1	1.0
	r. t.	1.29	1.22	1.26	1.26	1.22	1.26	1.21	1.26	1.20	1.25	1.22	1.12	1.25	1.24	1.28	1.07
	u. b.	79	75	72	78	68	65	68	81	62	65	71	99	59	67	85	99
18	l. b.	1.3	1,4	1.4	1.6	1.3	1.0	1.1	1.6	1.1	1.0	1.2	1.1	1.1	1.1	1.1	1.2
	r. t.	2.04	2.00	2.00	2.11	2.04	2.05	1.99	2.01	1.91	2.08	2.00	1.73	1.96	2.17	1.93	1.69
20	l. b.	1.4	1.2	1.4	1.2	1.3	1.1	1.1	0.9	1.0	1.0	1.0	1.0	1.1	0.9	1.1	0.9
	r. t.	3.09	3.16	3.11	3.13	3.20	3.24	3.23	2.85	3.20	3.06	3.15	2.70	3.18	3.36	2.74	2.65
22	l. b.	1.3	1.0	1.2	1.1	1.1	1.1	1.1	1.1	1.1	1.0	0.9	0.9	0.9	1.0	0.9	0.9
	r. t.	4.51	4.45	4.49	4.89	4.88	4.51	4.57	3.85	4.64	4.64	4.48	3.98	4.54	4.57	3.99	3.80
24	l. b.	1.1	1.1	0.9	0.9	1.0	1.0	1.0	0.8	1.0	0.9	0.9	0.9	0.8	0.8	0.8	0.7
	r. t.	6.69	6.69	6.56	6.83	6.81	6.64	6.67	5.87	6.33	6.77	5.82	5.64	6.58	6.41	6.01	5.82
26	l. b.	1.1	0.9	1.0	0.8	1.0	1.0	1.0	0.7	0.8	0.8	0.8	0.7	0.8	0.7	0.7	0.7
	r. t.	9.00	9.35	8.91	8.29	9.10	9.04	9.48	7.73	8.97	9.25	7.78	7.98	9.24	8.91	8.03	7.98
28	l. b.	1.0	1.0	1.0	0.9	1.8	0.9	0.8	0.8	0.8	0.8	0.8	0.7	0.8	0.8	0.7	0.7
	r. t.	12.97	13.16	12.68	10.86	12.24	12.71	11.63	11.04	12.08	13.28	10.66	10.38	12.70	10.97	10.74	11.12
30	l. b.	1.2	1.0	0.8	0.8	0.9	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.8	0.6	0.6	0.6
	r. t.	17.60	17.91	17.48	15.00	16.55	16.88	16.09	15.71	17.54	18.23	14.66	15.88	18.02	14.78	14.68	14.85

Table 2: Computational results

Naturally, run times increase in n. Values of P^f and P^s do not have a significant influence on run times. The steep increasement of run times is due to the solution procedure for the shortest path problems. Hence, finding feasible solutions bears a relatively high computational burden.

5 Conclusions and Outlook

In the paper at hand, we presented a Lagrangian approach to obtain lower bounds for the problem to find a mimimum cost tournaments subject to stadium unavailability and forbidden matches. We discussed different relaxation approaches and gave details for the implementation of a subgradient optimization approach. Moreover, we propose a cost oriented repair scheme which transforms a solution to the Lagrangian relaxation into a feasible solution to the underlying problem. For the unconstrained problem, that is when we have $P^f = P^s = 0$, this approach finds a feasible solution for an arbitrary solution to the relaxation. If $P^f > 0$ or $P^s > 0$, then the repair scheme may fail. However, we were able to find feasible solutions to each single test instance. Finally, we discussed the quality of lower bounds and solutions found by our approach.

We believe an obvious way to proceed is to use the lower bound obtained by the Lagrangian approach in a branching framework. Naturally, branching can be done by fixing and forbidding, respectively, specific matches. Fixed matches are considered in our relaxation model and, hence, it seems promising to combine these two approaches. However, the challenging part may be to derive solutions to the Lagrangian dual from the solutions found in father nodes in order to save run time. Another promising field of work is the use of the repair scheme in a neighbourhood search framework. We may slightly modify the current solution such that we obtain a set of $n - 1$ 1-factors which is not a 1-factorization. Afterwards, we can repair it afterwards to a new feasible solution using our repair scheme.

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