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## Constructing fair sports leagues schedules

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No. 635

**Constructing fair sports leagues schedules**

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## Abstract

A single round robin tournament (SRRT) consists of  $n$  teams and a set of periods. Matches between the teams have to be scheduled such that each team plays against each other team exactly once and each team plays at most once per period. In order to establish fairness among teams we consider a partition of the teams into strength groups. Then, the goal is to avoid that a team plays against extremely weak or extremely strong teams in consecutive periods. In this paper we pick up two concepts ensuring different degrees of fairness and address several questions which remained open so far.

**Keywords:** sports scheduling, round robin tournaments, fairness, strength groups, graphs

## 1 Introduction

In a **single round robin tournament (SRRT)** a set of  $n$  teams is given where each team has to play against each other team exactly once. All these matches have to be scheduled in a set of periods (rounds)  $P$  where  $|P| = n - 1$  if  $n$  is even and  $|P| = n$  if  $n$  is odd. Furthermore, each team is supposed to play at most one match per period. If  $n$  is even, this means that each team plays exactly once per period; if  $n$  is odd, in each period exactly one team has a "bye", i.e. does not play. The situation with an odd number of teams is usually reduced to the case with an even number of teams by adding a dummy team  $n + 1$ . Then in each period the team playing against  $n + 1$  has a bye. It is well known that feasible schedules for SRRTs can be constructed for every number  $n$  (even and odd) from 1-factorizations (near-1-factorizations) of the complete graph  $K_n$  (see e.g. de Werra [4]).

In practice for sports tournaments often additional restrictions have to be satisfied. As outlined in Bartsch et al. [2], in real world sports leagues fairness issues are quite important. One possibility of measuring fairness is the concept of so-called carry-over effects (Russell [7]), where it is not favourable if teams have the same sequence of opponents.

Another requirement is that a team should not play against extremely weak or extremely strong teams in consecutive periods. In order to model such a constraint, as in Bartsch [1] and Briskorn [3] we partition the set of all teams into  $g \geq 2$  strength groups  $S_0, \dots, S_{g-1}$ . Note that in the following all indices are 0-based, i.e. we always use 0 as starting index. The index of the group containing team  $i \in \{0, \dots, n - 1\}$  is denoted by  $g(i)$ .

Unless mentioned otherwise, we restrict ourselves to the situation that all groups have the same size  $|S_s| = \frac{n}{g} > 1$  for  $s = 0, \dots, g - 1$ . Without loss of generality we may assume that

the teams are numbered such that  $S_s = \left\{ s \frac{n}{g} + k \mid k = 0, \dots, \frac{n}{g} - 1 \right\}$  for  $s = 0, \dots, g - 1$ .

In Briskorn [3] two concepts are introduced assuring a certain degree of fairness with respect to groups. A schedule for a SRRT is called

- **group-changing**, if no team plays against teams of the same group in two consecutive matches,
- **group-balanced**, if no team plays more than once against teams of the same group within  $g$  consecutive matches.

It is easy to see that the second property implies the first one, since each group-balanced schedule (where each team plays against each group exactly once within  $g \geq 2$  consecutive matches) is also group-changing.

As an example consider a tournament with  $n = 6$  teams and  $g = 3$  groups  $S_0 = \{0, 1\}$ ,  $S_1 = \{2, 3\}$ ,  $S_2 = \{4, 5\}$ .

- If team 0 plays against the opponents 1, 2, 3, 4, 5 in this order, the corresponding schedule is not group-changing since 2 and 3 (as well as 4 and 5) belong to the same group.
- If team 0 plays against the opponents 1, 2, 4, 3, 5 in this order, the corresponding schedule is group-changing but not group-balanced since in the subsequences 2, 4, 3 and 4, 3, 5 one group appears twice.
- If team 0 plays against the opponents 2, 4, 1, 3, 5 in this order, the corresponding schedule is group-balanced according to team 0.

An interesting question is whether for given parameters  $n$  and  $g$  a schedule for a tournament with (at least) one of these properties exists. In Briskorn [3] it has been proved that for even  $n$  a group-balanced schedule for a SRRT exists if and only if  $g$  and  $\frac{n}{g}$  are even. Furthermore, it was shown that for certain combinations of the parameters  $g$  and  $\frac{n}{g}$  group-changing schedules can be constructed and that for  $g = 2$  and odd  $\frac{n}{g}$  no group-changing schedule exists. All known results are summarized in Table 1, where “–” indicates that this combination is not possible.

	$n$ even		$n$ odd
	$n/g$ even	$n/g > 1$ odd	$n/g$ odd
$g = 2$	group-balanced [3]	no group-changing [3]	–
$g = 3$	open, Section 2	–	open, Section 3
$g \geq 4$ even	group-balanced [3]	group-changing [3]	–
$g \geq 5$ odd	group-changing [3]	–	open, Section 3

Table 1: Existence results for group-balanced and group-changing schedules

From the table it can be concluded that all problems with an odd number  $n$  of teams as well as problems with  $g = 3$  groups are open. In the remainder of this paper we consider these situations. While in Section 2 problems with even  $n$  and  $g = 3$  groups are studied, in Section 3 we consider problems with odd  $n$ . Furthermore, in Section 4 we relax the restriction that all groups contain the same number of teams and present some results for arbitrary group sizes to allow group-changing schedules. Finally, some conclusions and topics for further research can be found in Section 5.

## 2 Problems with even $n$ and $g = 3$

In this section we consider problems with an even number  $n$  of teams and  $g = 3$  groups. According to Briskorn [3] it is already known that in this case (since  $g$  is odd) group-balanced schedules cannot exist. On the other hand, for SRRTs with even  $n$  and odd  $g \geq 5$  in [3] group-changing schedules have been constructed. Unfortunately, this construction scheme fails if  $g = 3$  holds. However,  $g = 3$  is an important case because in practice often strong, weak and middle leveled teams are distinguished.

Since  $n$  is even and each of the  $g = 3$  groups contains the same number  $\frac{n}{g}$  of teams, we must have  $n \bmod 6 = 0$  and  $\frac{n}{3}$  must be even. By complete enumeration (using an integer programming formulation) it can be shown that for  $n = 6$  no group-changing schedule exists. On the other hand, we give a construction scheme for group-changing schedules for all  $n \geq 12$  with  $\frac{n}{3} \notin \{26, 34, 58, 74\}$ .

The basic idea is to arrange all matches between teams of the same group in the equidistant periods  $2, 5, 8, \dots, n - 4$  (with two periods in between). These  $\frac{n}{3} - 1$  periods  $P^{int} = \{3k - 1 \mid k = 1, \dots, \frac{n}{3} - 1\}$  are called **internal periods** in the following. Matches between teams of two different groups will be scheduled in the periods in between. Two consecutive periods  $p, p + 1 \in P \setminus P^{int}$  are called **block** in the following.

We start with constructing a schedule for all matches between teams of the same group. For each group (containing an even number  $\frac{n}{3}$  of teams) we consider the complete graph  $K_{\frac{n}{3}}$  where all teams of the group are considered as nodes and the edges correspond to the matches between these teams. Recall that a 1-factorization of a graph  $G = (V, E)$  is a partition of the edges  $E$  into disjoint 1-factors, where a 1-factor is a set of edges such that each node is incident to exactly one of them. It is well-known that for each even  $k \in \mathbb{N}$  a 1-factorization  $F = \{F_0, \dots, F_{k-2}\}$  of the complete graph  $K_k$  exists (see e.g. Wallis [8]). Since each team is supposed to play exactly one game per period, every 1-factor induces a schedule for a single period. Thus, an arbitrary 1-factorization of the complete graph  $K_{\frac{n}{3}}$  can be used for scheduling all matches between teams belonging to the same group since the number of internal periods equals  $\frac{n}{3} - 1$ .

It remains to schedule all other matches between teams belonging to different groups. All these matches can be represented by the complete tripartite graph  $K(\frac{n}{3}; 3)$  with  $\frac{n}{3}$  nodes in each partite set. While the nodes in each partite set correspond to the teams of one group, the edges model the matches between teams of different groups. Figure 1 illustrates the complete tripartite graph  $K(4; 3)$  representing  $n = 12$  teams and the matches between teams belonging to different groups. Solid lines represent matches between teams of  $S_0$  and  $S_1$ , dashed lines represent matches between teams of  $S_0$  and  $S_2$ , and dotted lines represent matches between teams of  $S_1$  and  $S_2$ .

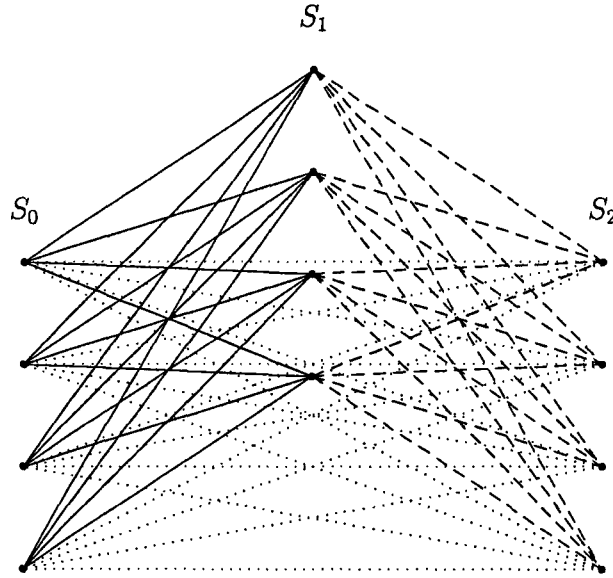


Figure 1: Complete tripartite graph  $K(4; 3)$

In the following we define the problem of finding a group-changing schedule for all matches between teams of different groups in the periods  $P \setminus P^{int}$  in terms of graph theory. Clearly, all matches of a block (containing two periods) induce a 2-factor of  $K(\frac{n}{3}; 3)$  since each team plays exactly once in each period. Recall that a 2-factor of a graph  $G = (V, E)$  is a subset

$E'$  of edges such that each node  $v \in V$  is incident to exactly two edges  $e_0, e_1 \in E'$ ,  $e_0 \neq e_1$ . Note that each 2-factor consists of a collection of node-disjoint cycles.

A 2-factor of  $K(\frac{n}{3}; 3)$  has to meet two requirements to qualify as a subset of matches which can be scheduled in a block of a group-changing schedule:

- (i) Each cycle contained in the 2-factor must have even length since otherwise the corresponding matches cannot be distributed to the two periods of a block such that each team plays exactly once in each period.
- (ii) In the 2-factor each team  $i \in S_s$ ,  $s \in \{0, 1, 2\}$ , must be incident to exactly one match against a team of  $S_{(s-1) \bmod 3}$  and exactly one match against a team of  $S_{(s+1) \bmod 3}$  since otherwise the corresponding schedule is not group-changing.

We define a **circular even 2-factor** (ce2f) as a 2-factor of  $K(\frac{n}{3}; 3)$  fulfilling requirements (i) and (ii). Accordingly, a **circular even 2-factorization** (ce2F) is a decomposition of all edges into disjoint ce2fs.

**Theorem 1.** *If a ce2F of the complete tripartite graph  $K(\frac{n}{3}; 3)$  for even  $n$  exists, a group-changing schedule for  $n$  teams and  $g = 3$  groups can be constructed by*

- *scheduling the matches between teams of the same group in the internal periods according to an 1-factorization of  $K_{\frac{n}{3}}$ , and*
- *scheduling all matches between teams of different groups according to a ce2F of  $K(\frac{n}{3}; 3)$  in the blocks in between.*

*Proof.* First, we show that by the described construction a SRRT is obtained. Obviously, each match is scheduled exactly once. Moreover, each team plays exactly once in each period  $p \in P^{int}$ . According to (i) each cycle  $c$  contained in a 2-factor of a ce2F is 2-colorable. A 2-coloring of  $c$  induces a partition of the matches (edges) in  $c$  into two subsets such that each team (node) covered by  $c$  has exactly one match in each color class. Consequently, a ce2f induces a subset of matches decomposable into two subsets such that each team plays exactly once per subset. This means that each team plays in both periods of a block exactly once. Finally, we show that the resulting schedule is group-changing. According to property (ii), each team  $i$  plays against teams of different groups in the two periods of a block. Furthermore, it plays against teams of its own group  $S_{g(i)}$  in the period immediately preceding the block (if it is not the first block) and the period immediately following the block (if it is not the last block). Thus, the schedule is group-changing.  $\square$

It is easy to see that no ce2F of  $K(\frac{n}{3}; 3)$  exists if  $\frac{n}{3}$  is odd. Since then also the number of nodes  $n$  is odd, there must be at least one cycle having odd length in each 2-factor. Therefore, we restrict ourselves to  $\frac{n}{3}$  being even in the following. In the next two subsections we construct ce2Fs of the graph  $K(\frac{n}{3}; 3)$  for all even  $\frac{n}{3}$  with  $\frac{n}{3} \notin \{2, 26, 34, 58, 74\}$ . While Subsection 2.1 deals with the case  $\frac{n}{3} = 4k$  for  $k \in \mathbb{N}$ , in Subsection 2.2 a construction scheme for all even  $\frac{n}{3} \notin \{2, 26, 34, 58, 74\}$  is given.

## 2.1 Case $\frac{n}{3} = 4k$ with $k \in \mathbb{N}$

In this subsection we provide a construction scheme for a ce2F of the graph  $K(\frac{n}{3}; 3)$  with  $\frac{n}{3} = 4k$  for all  $k \in \mathbb{N}$ . Note that the subgraph induced by two of the three groups is

the complete bipartite graph  $K_{\frac{n}{3}, \frac{n}{3}}$  with  $\frac{2n}{3}$  nodes. It is well-known that for this graph 1-factorizations exist, e.g. the bipartite 1-factorization  $F^{bip} = \{F_0^{bip}, \dots, F_{\frac{n}{3}-1}^{bip}\}$  with

$$F_l^{bip} = \left\{ \left[ m, \frac{n}{3} + (m + l) \bmod \frac{n}{3} \right] \mid m = 0, \dots, \frac{n}{3} - 1 \right\} \text{ for } l = 0, \dots, \frac{n}{3} - 1. \quad (1)$$

As an example, in Figure 2 the 1-factorization  $F^{bip}$  for  $K_{\frac{n}{3}, \frac{n}{3}}$  with  $\frac{n}{3} = 4$  is illustrated.

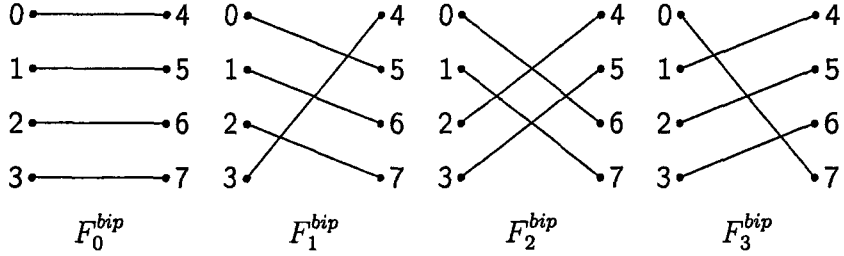


Figure 2: 1-factorization  $F^{bip}$  of  $K_{4,4}$

The basic idea of our construction scheme is to combine certain 1-factors of  $F^{bip}$  between the groups in each block. First, we address the question how to combine 1-factors of  $F^{bip}$  in order to obtain a ce2f of  $K(\frac{n}{3}; 3)$ . For two different groups  $S_s, S_{s'}$  and  $l \in \{0, \dots, \frac{n}{3} - 1\}$  we denote by  $F_l^{bip}(S_s, S_{s'})$  the 1-factor  $F_l^{bip}$  between the groups  $S_s$  and  $S_{s'}$ .

**Lemma 1.** *Three 1-factors  $F_{l_0}^{bip}(S_0, S_1)$ ,  $F_{l_1}^{bip}(S_1, S_2)$ , and  $F_{l_2}^{bip}(S_2, S_0)$  with  $l_0, l_1, l_2 \in \{0, \dots, \frac{n}{3} - 1\}$  form a ce2f of  $K(\frac{n}{3}; 3)$  if  $l_0 + l_1 + l_2$  is odd.*

*Proof.* Obviously,  $F_{l_0}^{bip}(S_0, S_1)$ ,  $F_{l_1}^{bip}(S_1, S_2)$ , and  $F_{l_2}^{bip}(S_2, S_0)$  form together a 2-factor of  $K(\frac{n}{3}; 3)$  since exactly two 1-factors are chosen for each group. Moreover, the 2-factor fulfills condition (ii), i.e. it is circular. Let us consider an arbitrary path from a node  $i \in S_s$  to another node  $i' \in S_s$  induced by  $F_{l_0}^{bip}$ ,  $F_{l_1}^{bip}$ , and  $F_{l_2}^{bip}$ . Obviously, the length of this path is a multiple of 3. Now consider an arbitrary path having length 3. Note that there are exactly two such paths for each  $i \in S_s$  and both connect  $i$  with another node  $i' \in S_{g(i)}$ . Let  $i$  and  $i'$  be the start and end node of such a path. Since  $\frac{n}{3}$  is even and  $l_0 + l_1 + l_2$  is odd,  $|i - i'|$  is odd, as well. This implies that an even number of paths of length 3 must be concatenated to construct a circle and, hence, circles have even length.  $\square$

In the following we will show that a ce2F of the graph  $K(\frac{n}{3}; 3)$  with  $\frac{n}{3} = 4k$  for all  $k \in \mathbb{N}$  exists. According to Lemma 1 combining 1-factors  $F_{l_0}^{bip}(S_0, S_1)$ ,  $F_{l_1}^{bip}(S_1, S_2)$ , and  $F_{l_2}^{bip}(S_2, S_0)$  with indices  $l_0, l_1, l_2 \in \{0, \dots, \frac{n}{3} - 1\}$  yields a ce2f if

- exactly one index is odd, and the other two are even, or
- all three indices are odd.

According to Lemma 1 the specific combination of 1-factors can be chosen arbitrarily. However, we focus on a specific arrangement in the following.

At first we combine the odd 1-factors  $F_l^{bip}$  for  $l = 4\lambda + 1$  ( $\lambda = 0, \dots, \frac{n}{12} - 1$ ) with the two even 1-factors  $F_{l-1}^{bip}$  and  $F_{l+1}^{bip}$ . As an example Figure 3 illustrates the case  $l = 1$  for the graph  $K(4; 3)$ , where the 1-factors  $F_0^{bip}(S_0, S_1)$ ,  $F_1^{bip}(S_1, S_2)$  and  $F_2^{bip}(S_2, S_0)$  are shown. Here, the

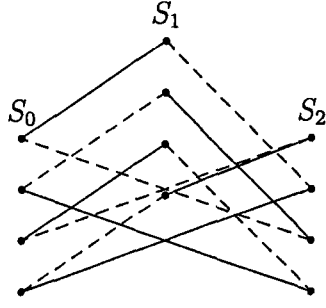


Figure 3:  $F_0^{bip}(S_0, S_1)$ ,  $F_1^{bip}(S_1, S_2)$ , and  $F_2^{bip}(S_2, S_0)$  for  $K(4; 3)$

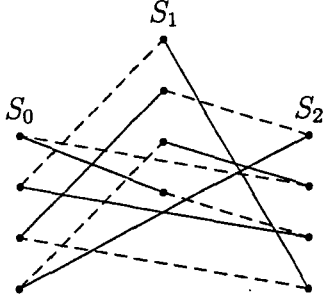


Figure 4:  $F_3^{bip}(S_0, S_1)$ ,  $F_3^{bip}(S_1, S_2)$ , and  $F_3^{bip}(S_2, S_0)$  for  $K(4; 3)$

node on top of the column representing a group has the smallest index, solid and dashed lines symbolize the two color classes.

We apply each combination three times by rotating the specific 1-factors such that each 1-factor is applied to each pair of groups exactly once. This means that each combination is arranged in three ce2fs:  $F_{l-1}^{bip}(S_0, S_1)$ ,  $F_l^{bip}(S_1, S_2)$ ,  $F_{l+1}^{bip}(S_2, S_0)$  in the first ce2f,  $F_{l-1}^{bip}(S_1, S_2)$ ,  $F_l^{bip}(S_2, S_0)$ ,  $F_{l+1}^{bip}(S_0, S_1)$  in the second ce2f, and  $F_{l-1}^{bip}(S_2, S_0)$ ,  $F_l^{bip}(S_0, S_1)$ ,  $F_{l+1}^{bip}(S_1, S_2)$  in the third ce2f.

By arranging this configuration for all  $\lambda = 0, \dots, \frac{n}{12} - 1$  we arrange  $\frac{3n}{12}$  ce2fs and cover  $F_l^{bip}$  for each even index  $l \in \{0, \dots, \frac{n}{3} - 1\}$ . It remains to cover  $F_l^{bip}$  for all odd  $l = 4\mu + 3$  ( $\mu = 0, \dots, \frac{n}{12} - 1$ ) in the  $\frac{n}{12}$  remaining ce2fs. This can be done by arranging  $F_l^{bip}$  between each pair of groups, i.e. by choosing  $F_l^{bip}(S_0, S_1)$ ,  $F_l^{bip}(S_1, S_2)$ , and  $F_l^{bip}(S_2, S_0)$ . Clearly, the prerequisite of Lemma 1 is fulfilled and, hence, we obtain a ce2f in each block.

As an example Figure 4 illustrates the case  $l = 3$  for the graph  $K(4; 3)$ , where the 1-factors  $F_3^{bip}(S_0, S_1)$ ,  $F_3^{bip}(S_1, S_2)$  and  $F_3^{bip}(S_2, S_0)$  are shown.

The preceding construction implies:

**Theorem 2.** *For the graph  $K(\frac{n}{3}; 3)$  with  $\frac{n}{3} = 4k$  a ce2F exists for all  $k \in \mathbb{N}$ .*

## 2.2 Case $\frac{n}{3} \notin \{26, 34, 58, 74\}$

In this subsection we provide a construction scheme for a ce2F of the graph  $K(\frac{n}{3}; 3)$  for all even  $\frac{n}{3} \notin \{2, 26, 34, 58, 74\}$ . If  $\frac{n}{3}$  is a multiple of 4, the construction method from the previous subsection can be applied. By using an integer programming formulation it can be shown that for  $\frac{n}{3} = 2$  no ce2F exists. On the other hand, using Cplex we found ce2Fs  $F^6$  and  $F^{10}$  for  $K(6; 3)$  and  $K(10; 3)$ . These ce2Fs are given in the appendix.



At first we will show that from ce2Fs for a graph  $K(l; 3)$  with  $l \in \mathbb{N}$  ce2Fs for the larger graphs  $K(kl; 3)$  can be constructed for all  $k \in \mathbb{N}$ .

**Lemma 2.** *If a ce2F of  $K(l; 3)$  for some integer  $l \in \mathbb{N}$  exists, then also a ce2F of  $K(kl; 3)$  for all  $k \in \mathbb{N}$  exists.*

*Proof.* The basic idea of this constructive proof is to cluster nodes and to use the structure known for the graph induced by the cluster. This technique has been applied in several works concerning decomposition of graphs, see e.g. Laskar [5] or Laskar and Auerbach [6].

Suppose we have a ce2F  $F$  of  $K(l; 3)$  for some  $l \in \mathbb{N}$ . Then we construct a ce2F  $F'$  of  $K(kl; 3)$  for all  $k \in \mathbb{N}$  as follows:

1. In the graph  $K(kl; 3)$  partition the nodes of each group  $S_s$ ,  $s \in \{0, 1, 2\}$ , into  $l$  subgroups  $S_{s,m}$  for  $m = 0, \dots, l-1$ , having size  $k$  each.
2. Consider  $K(l; 3)$  as the complete tripartite graph induced by the  $3l$  subgroups. Each 2-factor in the 2-factorization  $F$  of  $K(l; 3)$  is expanded to  $k$  2-factors in the 2-factorization  $F'$  of  $K(kl; 3)$ . Note that for each pair of subgroups (belonging to different groups) all edges between the nodes in these subgroups can be represented by the complete bipartite graph  $K_{k,k}$ . We may employ  $F^{bip}$  to decompose them into  $k$  1-factors. For this purpose we assign an arbitrary orientation to each circle from the 2-factors in  $F$ . Then, we combine the 1-factors  $F_\lambda^{bip}$  for  $\lambda = 0, \dots, k-1$  between each pair of subgroups being consecutive in a circle.

It is easy to see that  $F'$  is a 2-factorization of  $K(kl; 3)$ . Moreover, we have a circular structure due to the circular structure of  $F$ . It is obvious that circles of even length are constructed since each underlying circle in  $F$  is of even length  $q$  and circles in  $F'$  can only have lengths being multiples of  $q$ .  $\square$

Thus, according to Lemma 2, we can construct ce2Fs of  $K(6k; 3)$  and  $K(10k; 3)$  for all  $k \in \mathbb{N}$  using the ce2Fs  $F^6$  and  $F^{10}$  as building blocks.

In a similar way the following lemma provides a basis for a more powerful recursive construction.

**Lemma 3.** *If a ce2F of  $K(l; 3)$  for some  $l \in \mathbb{N}$  exists, then also a ce2F of  $K(kl + 4m; 3)$  for each pair of integers  $k, m$  with  $4m \geq kl \geq 3m$  exists.*

*Proof.* The basic idea of this constructive proof is to divide the nodes into two parts inducing subgraphs  $G^{kl} := K(kl; 3)$  and  $G^{4m} := K(4m; 3)$ . Suppose we have a ce2F  $F$  of  $K(l; 3)$  for some  $l \in \mathbb{N}$ . Then we construct a ce2F of  $K(kl + 4m; 3)$  using structural elements of  $F$  and known ce2Fs of  $G^{4m}$  (cf. Subsection 2.1) as follows:

1. According to Lemma 2 from  $F$  we can construct a ce2F  $F'$  of  $G^{kl}$  for all  $k \in \mathbb{N}$ . We arrange  $F'$  of  $G^{kl}$  as part of the first  $kl$  ce2fs of  $K(kl + 4m; 3)$ .
2. We complete the first  $kl$  ce2fs of  $K(kl + 4m; 3)$  by ce2fs of  $G^{4m}$ . These ce2fs of  $G^{4m}$  are arranged by combining 1-factors according to  $F^{bip}$  of  $K_{4m,4m}$  as in the construction in Section 2.1. More specifically, we choose all  $2m$  even 1-factors  $F_{2\lambda}^{bip}$  for  $\lambda = 0, \dots, 2m-1$  and the  $kl - 2m$  odd 1-factors  $F_{4m-kl+1+2\lambda}^{bip}$  for  $\lambda = 0, \dots, kl - 2m - 1$ . Since  $kl \geq 3m$ , the number of used odd 1-factors is at least  $m$  and, therefore, at least half the number of used even 1-factors. Regarding Lemma 1, we can arrange  $3m$  ce2fs of  $G^{4m}$  by combining two even 1-factors with one odd 1-factor in each ce2f. Furthermore, we can arrange  $kl - 3m$  ce2fs of  $G^{4m}$  using the remaining odd 1-factors (only a single 1-factor in each ce2f, cf. Section 2.1).

Up to now, we have  $kl$  complete ce2fs of  $K(kl + 4m; 3)$ . It remains to arrange all edges of  $G^{4m}$  according to  $F_{2\lambda+1}^{bip}$  and  $F_{4m-1-2\lambda}^{bip}$  for all  $\lambda = 0, \dots, \frac{4m-kl}{2} - 1$  (these edges have been left out in Step 2) and all edges between  $G^{kl}$  and  $G^{4m}$  in the remaining  $4m$  ce2fs of  $K(kl + 4m; 3)$ . In the following we represent node  $i$  by  $(g(i), i')$  where  $g(i)$  again denotes the group of  $i$  and  $i' := i \bmod \frac{n}{3}$  is the index of  $i$  within its group. W.l.o.g. we assume that the nodes of  $G^{4m}$  are numbered such that they satisfy  $0 \leq i' \leq 4m - 1$ .

3. For  $\lambda = 0, \dots, 4m - 1$  in the ce2f with index  $kl + \lambda$  for  $q = 0, 1, 2$  and  $i' = 4m, \dots, kl + 4m - 1$  we arrange edges

$$[(q, i'), ((q + 1) \bmod 3, (i' - 4m + \lambda) \bmod 4m)] \quad \text{and} \\ [(q, i'), ((q - 1) \bmod 3, (i' - 4m + \lambda) \bmod 4m)].$$

In Step 3 we exclusively arrange circles of length 6. In the ce2f with index  $\mu \in \{kl, \dots, kl + 4m - 1\}$  of  $K(kl + 4m; 3)$  the nodes  $(q, (\mu + i') \bmod 4m)$  for  $q = 0, 1, 2$  and  $i' = 0, \dots, 4m - kl - 1$  are not involved yet.

4. For  $\mu = kl, \dots, kl + 4m - 1$  in the ce2f with index  $\mu$  for  $q = 0, 1, 2$  and  $i' = 0, \dots, 4m - kl - 1$  we arrange edges

$$[(q, (\mu + i') \bmod 4m), ((q + 1) \bmod 3, (\mu + 4m - kl - 1 - i') \bmod 4m)] \quad \text{and} \\ [(q, (\mu + i') \bmod 4m), ((q - 1) \bmod 3, (\mu + 4m - kl - 1 - i') \bmod 4m)].$$

In Step 4 we, again, exclusively arrange circles of length 6. Edges originating from Step 4 form ce2fs corresponding to  $F_{2\lambda+1}^{bip}$  and  $F_{4m-1-2\lambda}^{bip}$  for  $\lambda = 0, \dots, \frac{4m-kl}{2} - 1$ . Summarizing, we obtain a ce2F of  $K(kl + 4m; 3)$ .  $\square$

Figure 5 illustrates the ce2f with index  $\mu = kl + 2 = 8$  of  $K(14; 3)$  where  $k = 1$ ,  $l = 6$ , and  $m = 2$ . Nodes of  $G^{4m} = G^8$  and  $G^{kl} = G^6$  are drawn as solid and empty dots, respectively. Edges arranged in Step 4 are incident to the both upper nodes of each column.

In the following we want to apply Lemma 3 with  $k = 1$ . Thus, we study which even numbers  $N \in \mathbb{N}$  can be written in the form  $N = l + 4m$  with  $l, m \in \mathbb{N}$  and  $4m \geq l \geq 3m$ . We get

$$(l, m) = \begin{cases} (\frac{N}{2}, \frac{N}{8}) & \text{for } N \bmod 8 = 0, \quad N \geq 8, \\ (\frac{N-6}{2}, \frac{N+6}{8}) & \text{for } N \bmod 8 = 2, \quad N \geq 42, \\ (\frac{N-4}{2}, \frac{N+4}{8}) & \text{for } N \bmod 8 = 4, \quad N \geq 28, \\ (\frac{N-2}{2}, \frac{N+2}{8}) & \text{for } N \bmod 8 = 6, \quad N \geq 14. \end{cases} \quad (2)$$

Thus, each even number  $\frac{n}{3} \geq 36$  can be written in the form  $\frac{n}{3} = l + 4m$  with  $l, m \in \mathbb{N}$  and  $4m \geq l \geq 3m$ .

**Theorem 3.** For the graph  $K(\frac{n}{3}; 3)$  a ce2F exists for all even  $\frac{n}{3} \notin \{2, 26, 34, 58, 74\}$ .

*Proof.* Due to Theorem 2 we already know a ce2F of  $K(\frac{n}{3}; 3)$  for all  $\frac{n}{3}$  which are a multiple of 4. For all other even  $\frac{n}{3} < 36$  in Table 2 we summarize the results whether a ce2F is known for the graph  $K(\frac{n}{3}; 3)$  or not. Only for  $\frac{n}{3} \in \{2, 26, 34\}$  no ce2F is known. Note that for  $\frac{n}{3} = 26, 34$  no numbers  $k, l, m$  exist which satisfy the conditions of Lemma 3.

Due to (2) each even  $\frac{n}{3} \geq 36$  can be represented as  $\frac{n}{3} = l + 4m$  with  $4m \geq l \geq 3m$ . Thus, if a ce2F for the specific graph  $K(l; 3)$  is known, according to Lemma 3 also a ce2F of  $K(\frac{n}{3}; 3)$  exists.

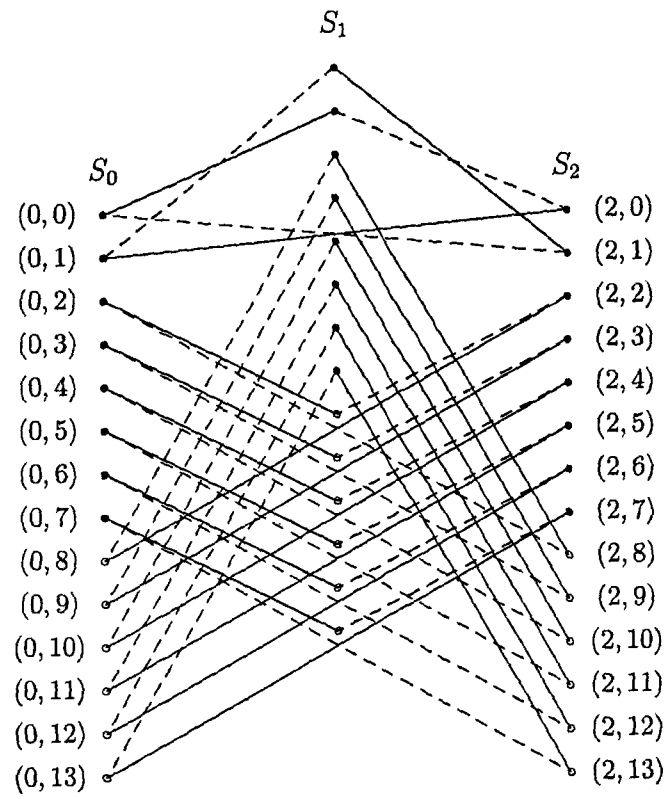


Figure 5: ce2f with index  $\mu = 8$  of  $K(14; 3)$

$\frac{n}{3}$	existence	argument
2	no	IP
6	yes	IP
10	yes	IP
14	yes	Lemma 3 ( $k = 1, l = 6, m = 2$ )
18	yes	Lemma 2 ( $k = 3, l = 6$ )
22	yes	Lemma 3 ( $k = 1, l = 10, m = 3$ )
26	?	—
30	yes	Lemma 2 ( $k = 5, l = 6$ )
34	?	—

Table 2: Existence results for a ce2F of  $K(\frac{n}{3}; 3)$

This construction covers all values  $\frac{n}{3}$  which do not depend on a (direct or indirect) representation with  $l = 26$  or  $l = 34$ . The first part of Table 3 lists all numbers which directly depend on  $l = 26$  or  $l = 34$ . The second column gives the feasible domain  $D_m$  consisting of all values  $m \in \mathbb{N}$  with  $4m \geq l \geq 3m$  for the specific value of  $l$ . For each element of  $D_m$  the third column contains the value  $l + 4m$ . If another argument not based on the specific value of  $l$  for the existence of a ce2F of  $K(l + 4m; 3)$  is known, it is stated in the fourth column.

The only numbers which directly depend on  $l_1 = 26$  or  $l_2 = 34$  and for which no other argument is known are  $l_1 + 4 \cdot 8 = 58$  and  $l_2 + 4 \cdot 10 = 74$ . These numbers are treated in the same way in the second part of Table 3. It can be seen that all numbers which directly depend on  $l_3 = 58$  or  $l_4 = 74$  can be covered by other arguments. Thus, our construction only fails for all even  $\frac{n}{3} \in \{2, 26, 34, 58, 74\}$ . While for  $\frac{n}{3} = 2$  it is proved that no ce2F exists, the other four cases remain open.  $\square$

$l$	$D_m$	$m$	$l + 4m$	argument
26	{7,8}	7	54	Lemma 2 ( $k = 9, l = 6$ )
		8	58	—
34	{9,10,11}	9	70	Lemma 3 ( $k = 1, l = 30, m = 10$ )
		10	74	—
		11	78	Lemma 2 ( $k = 13, l = 6$ )
58	{15,16,17,18,19}	15	118	Lemma 3 ( $k = 1, l = 54, m = 16$ )
		16	122	Lemma 3 ( $k = 1, l = 54, m = 17$ )
		17	126	Lemma 3 ( $k = 1, l = 54, m = 18$ )
		18	130	Lemma 3 ( $k = 1, l = 62, m = 17$ )
		19	134	Lemma 3 ( $k = 1, l = 62, m = 18$ )
74	{19,20,21,22,23,24}	19	150	Lemma 3 ( $k = 1, l = 70, m = 20$ )
		20	154	Lemma 3 ( $k = 1, l = 70, m = 21$ )
		21	158	Lemma 3 ( $k = 1, l = 70, m = 22$ )
		22	162	Lemma 3 ( $k = 1, l = 70, m = 23$ )
		23	166	Lemma 3 ( $k = 1, l = 78, m = 22$ )
		24	170	Lemma 3 ( $k = 1, l = 78, m = 23$ )

Table 3: Numbers  $l + 4m$  depending on  $l \in \{26, 34, 58, 74\}$

### 3 Problems with odd $n$

In this section we consider problems with an odd number  $n$  of teams. Note that in this case  $g$  and  $\frac{n}{g}$  must be odd as well. As mentioned in the introduction, in each period exactly one team has a bye. Usually, tournaments with an odd number of teams are reduced to the situation with an even number of teams by adding a dummy team. This construction is not possible in our situation, since the dummy team had to be assigned to one group which would lead to different group sizes and additionally would destroy the concept of group changes. For this reason we must consider the case of odd  $n$  individually.

A schedule for a SRRT with odd  $n$  corresponds to a near-1-factorization of the complete graph  $K_n$ . Recall that a near-1-factorization of a graph  $G = (V, E)$  is a partition of  $E$  into near-1-factors, where a near-1-factor is a set of edges such that each but one node is incident to exactly one edge. This node is incident to no edge. It is well-known that for each odd  $n \in \mathbb{N}$  a near-1-factorization  $F = \{F_0, \dots, F_{n-1}\}$  of the complete graph  $K_n$  exists.

In the following we will show that a group-balanced schedule exists for each odd  $n$  and each odd  $g$  with  $\frac{n}{g} \in \mathbb{N}$  by providing an appropriate construction scheme. At first we construct pairs of groups according to a near-1-factorization of  $K_g$ . We pair each group with each other group exactly once in the first  $g$  periods. Then each group is not paired in exactly one period and in each period there is exactly one group which is not paired. Furthermore, we copy this structure  $\frac{n}{g} - 1$  times in the remaining  $n - g$  periods. An example for  $n = 15$  and  $g = 5$  is shown in Figure 6, where rows correspond to groups and columns to periods. The entry for row  $S_s$  ( $s = 0, \dots, g - 1$ ) and column  $p \in \{0, \dots, n - 1\}$  gives the opponent group of group  $S_s$  in period  $p$ .

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$S_0$	—	$S_2$	$S_4$	$S_1$	$S_3$	—	$S_2$	$S_4$	$S_1$	$S_3$	—	$S_2$	$S_4$	$S_1$	$S_3$
$S_1$	$S_4$	—	$S_3$	$S_0$	$S_2$	$S_4$	—	$S_3$	$S_0$	$S_2$	$S_4$	—	$S_3$	$S_0$	$S_2$
$S_2$	$S_3$	$S_0$	—	$S_4$	$S_1$	$S_3$	$S_0$	—	$S_4$	$S_1$	$S_3$	$S_0$	—	$S_4$	$S_1$
$S_3$	$S_2$	$S_4$	$S_1$	—	$S_0$	$S_2$	$S_4$	$S_1$	—	$S_0$	$S_2$	$S_4$	$S_1$	—	$S_0$
$S_4$	$S_1$	$S_3$	$S_0$	$S_2$	—	$S_1$	$S_3$	$S_0$	$S_2$	—	$S_1$	$S_3$	$S_0$	$S_2$	—

Figure 6: Pairs of groups for  $n = 15$  and  $g = 5$

Afterwards, we arrange matches according to these pairs of groups as follows. If two different groups  $S_\lambda$  and  $S_\mu$  are paired in period  $p$ , then each team in group  $S_\lambda$  plays against a team from group  $S_\mu$  in  $p$ . If  $S_\lambda$  is not paired in period  $p$ , then the teams in  $S_\lambda$  play against each other. These internal matches can be arranged according to a near-1-factorization of the complete graph  $K_r$  with  $r := \frac{n}{g} \in \mathbb{N}$ . Note that each group is not paired in exactly  $r$  periods (with distance  $g$ ), and  $r$  equals the number of near-1-factors of  $K_r$ . Thus, in each of these periods exactly one team does not play and each team does not play exactly once.

The other matches between teams of different groups may be arranged according to an arbitrary 1-factorization of the complete bipartite graph  $K_{r,r}$ , e.g. using  $F^{bip}$  as defined in (1).

According to such a 1-factorization we can arrange all matches between teams of different groups  $S_\lambda$  and  $S_\mu$  in  $r$  periods, which is exactly the amount of times  $S_\lambda$  and  $S_\mu$  are paired. Thus, as a result we obtain a feasible SRRT schedule. From the construction of pairs of groups it follows that this schedule is group-balanced.

As an example in Figure 7 the resulting schedule for  $n = 15$  and  $g = 5$  is shown. In order to see the group changes, the teams  $0, \dots, 14$  are additionally labeled with the corresponding group indices  $0, \dots, 4$ .

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0 <sub>0</sub>	–	6 <sub>2</sub>	12 <sub>4</sub>	3 <sub>1</sub>	9 <sub>3</sub>	2 <sub>0</sub>	7 <sub>2</sub>	13 <sub>4</sub>	4 <sub>1</sub>	10 <sub>3</sub>	1 <sub>0</sub>	8 <sub>2</sub>	14 <sub>4</sub>	5 <sub>1</sub>	11 <sub>3</sub>
1 <sub>0</sub>	2 <sub>0</sub>	7 <sub>2</sub>	13 <sub>4</sub>	4 <sub>1</sub>	10 <sub>3</sub>	–	8 <sub>2</sub>	14 <sub>4</sub>	5 <sub>1</sub>	11 <sub>3</sub>	0 <sub>0</sub>	6 <sub>2</sub>	12 <sub>4</sub>	3 <sub>1</sub>	9 <sub>3</sub>
2 <sub>0</sub>	1 <sub>0</sub>	8 <sub>2</sub>	14 <sub>4</sub>	5 <sub>1</sub>	11 <sub>3</sub>	0 <sub>0</sub>	6 <sub>2</sub>	12 <sub>4</sub>	3 <sub>1</sub>	9 <sub>3</sub>	–	7 <sub>2</sub>	13 <sub>4</sub>	4 <sub>1</sub>	10 <sub>3</sub>
3 <sub>1</sub>	12 <sub>4</sub>	–	9 <sub>3</sub>	0 <sub>0</sub>	6 <sub>2</sub>	13 <sub>4</sub>	5 <sub>1</sub>	10 <sub>3</sub>	2 <sub>0</sub>	7 <sub>2</sub>	14 <sub>4</sub>	4 <sub>1</sub>	11 <sub>3</sub>	1 <sub>0</sub>	8 <sub>2</sub>
4 <sub>1</sub>	13 <sub>4</sub>	5 <sub>1</sub>	10 <sub>3</sub>	1 <sub>0</sub>	7 <sub>2</sub>	14 <sub>4</sub>	–	11 <sub>3</sub>	0 <sub>0</sub>	8 <sub>2</sub>	12 <sub>4</sub>	3 <sub>1</sub>	9 <sub>3</sub>	2 <sub>0</sub>	6 <sub>2</sub>
5 <sub>1</sub>	14 <sub>4</sub>	4 <sub>1</sub>	11 <sub>3</sub>	2 <sub>0</sub>	8 <sub>2</sub>	12 <sub>4</sub>	3 <sub>1</sub>	9 <sub>3</sub>	1 <sub>0</sub>	6 <sub>2</sub>	13 <sub>4</sub>	–	10 <sub>3</sub>	0 <sub>0</sub>	7 <sub>2</sub>
6 <sub>2</sub>	9 <sub>3</sub>	0 <sub>0</sub>	–	12 <sub>4</sub>	3 <sub>1</sub>	11 <sub>3</sub>	2 <sub>0</sub>	8 <sub>2</sub>	13 <sub>4</sub>	5 <sub>1</sub>	10 <sub>3</sub>	1 <sub>0</sub>	7 <sub>2</sub>	14 <sub>4</sub>	4 <sub>1</sub>
7 <sub>2</sub>	10 <sub>3</sub>	1 <sub>0</sub>	8 <sub>2</sub>	13 <sub>4</sub>	4 <sub>1</sub>	9 <sub>3</sub>	0 <sub>0</sub>	–	14 <sub>4</sub>	3 <sub>1</sub>	11 <sub>3</sub>	2 <sub>0</sub>	6 <sub>2</sub>	12 <sub>4</sub>	5 <sub>1</sub>
8 <sub>2</sub>	11 <sub>3</sub>	2 <sub>0</sub>	7 <sub>2</sub>	14 <sub>4</sub>	5 <sub>1</sub>	10 <sub>3</sub>	1 <sub>0</sub>	6 <sub>2</sub>	12 <sub>4</sub>	4 <sub>1</sub>	9 <sub>3</sub>	0 <sub>0</sub>	–	13 <sub>4</sub>	3 <sub>1</sub>
9 <sub>3</sub>	6 <sub>2</sub>	12 <sub>4</sub>	3 <sub>1</sub>	–	0 <sub>0</sub>	7 <sub>2</sub>	13 <sub>4</sub>	5 <sub>1</sub>	11 <sub>3</sub>	2 <sub>0</sub>	8 <sub>2</sub>	14 <sub>4</sub>	4 <sub>1</sub>	10 <sub>3</sub>	1 <sub>0</sub>
10 <sub>3</sub>	7 <sub>2</sub>	13 <sub>4</sub>	4 <sub>1</sub>	11 <sub>3</sub>	1 <sub>0</sub>	8 <sub>2</sub>	14 <sub>4</sub>	3 <sub>1</sub>	–	0 <sub>0</sub>	6 <sub>2</sub>	12 <sub>4</sub>	5 <sub>1</sub>	9 <sub>3</sub>	2 <sub>0</sub>
11 <sub>3</sub>	8 <sub>2</sub>	14 <sub>4</sub>	5 <sub>1</sub>	10 <sub>3</sub>	2 <sub>0</sub>	6 <sub>2</sub>	12 <sub>4</sub>	4 <sub>1</sub>	9 <sub>3</sub>	1 <sub>0</sub>	7 <sub>2</sub>	13 <sub>4</sub>	3 <sub>1</sub>	–	0 <sub>0</sub>
12 <sub>4</sub>	3 <sub>1</sub>	9 <sub>3</sub>	0 <sub>0</sub>	6 <sub>2</sub>	13 <sub>4</sub>	5 <sub>1</sub>	11 <sub>3</sub>	2 <sub>0</sub>	8 <sub>2</sub>	–	4 <sub>1</sub>	10 <sub>3</sub>	1 <sub>0</sub>	7 <sub>2</sub>	14 <sub>4</sub>
13 <sub>4</sub>	4 <sub>1</sub>	10 <sub>3</sub>	1 <sub>0</sub>	7 <sub>2</sub>	12 <sub>4</sub>	3 <sub>1</sub>	9 <sub>3</sub>	0 <sub>0</sub>	6 <sub>2</sub>	14 <sub>4</sub>	5 <sub>1</sub>	11 <sub>3</sub>	2 <sub>0</sub>	8 <sub>2</sub>	–
14 <sub>4</sub>	5 <sub>1</sub>	11 <sub>3</sub>	2 <sub>0</sub>	8 <sub>2</sub>	–	4 <sub>1</sub>	10 <sub>3</sub>	1 <sub>0</sub>	7 <sub>2</sub>	13 <sub>4</sub>	3 <sub>1</sub>	9 <sub>3</sub>	0 <sub>0</sub>	6 <sub>2</sub>	12 <sub>4</sub>

Figure 7: Group-balanced SRRT for  $n = 15$  and  $g = 5$

Obviously, the construction scheme given above can be applied for every odd number  $n$  and each odd  $g > 1$  with  $n = rg$ ,  $r \in \mathbb{N}$ . Thus, we have

**Theorem 4.** *For every odd number  $n$  of teams and each odd number  $g > 1$  of groups with  $\frac{n}{g} \in \mathbb{N}$  a group-balanced SRRT exists.*

## 4 Problems with arbitrary group sizes

Up to now (in this paper and Briskorn [3]) only groups with identical sizes have been studied. Clearly, a generalization to arbitrary group sizes is interesting. For example, in practice often sports leagues have few excellent teams, few weak teams, and mostly middle leveled teams. While the concept of group-balanced schedules is not applicable in such a situation, the concept of group-changing schedules can be transferred. In the following we give first insights into the problem of arranging group-changing SRRTs with different group-sizes.

In the following we assume that the  $n$  teams are partitioned into  $g$  groups  $S_0, \dots, S_{g-1}$  with  $|S_0| \geq |S_1| \geq \dots \geq |S_{g-1}|$ . Obviously, necessary for the existence of a group-changing schedule is the condition

$$\sum_{\lambda=1}^{g-1} |S_\lambda| \geq |S_0| \quad (3)$$

since otherwise it cannot be avoided that teams  $i \notin S_0$  play at least two consecutive matches against teams from the largest group  $S_0$ . This condition implies  $|S_0| \leq \lfloor \frac{n}{2} \rfloor$ . For even  $n$  where  $\frac{n}{2}$  is odd, we may strengthen this condition even to  $|S_0| < \frac{n}{2}$  for the following reason. Assume that  $|S_0| = \frac{n}{2}$  is odd and let  $\bar{S}_0 := S_1 \cup \dots \cup S_{g-1}$  be the complement of  $S_0$ . Then also  $|\bar{S}_0| = \frac{n}{2}$  and all matches of teams in  $\bar{S}_0$  against teams from  $S_0$  must take place in the  $\frac{n}{2}$  periods  $0, 2, 4, \dots, n-2$ . Thus, for the internal matches in  $\bar{S}_0$  only  $n-1-\frac{n}{2} = \frac{n}{2}-1$  periods remain. Since  $|\bar{S}_0| = \frac{n}{2}$  is odd,  $\frac{n}{2}-1$  periods are not sufficient for a complete tournament in  $\bar{S}_0$ .

Thus, we have the following necessary condition for the existence of group-changing schedules in the case of an even number  $n$  of teams:

$$|S_0| \leq \frac{n}{2} \text{ if } \frac{n}{2} \text{ is even, and } |S_0| < \frac{n}{2} \text{ if } \frac{n}{2} \text{ is odd.} \quad (4)$$

On the other hand, in the following we derive a sufficient condition for the existence of group-changing schedules by joining groups to larger "supergroups". For a partitioning of the index set  $\{0, 1, \dots, g-1\}$  into  $k \leq g$  subsets  $A_0, \dots, A_{k-1} \subset \{0, \dots, g-1\}$  with  $\bigcup_{\lambda=0}^{k-1} A_\lambda = \{0, \dots, g-1\}$  and  $A_\lambda \cap A_\mu = \emptyset$  for all  $\lambda \neq \mu$  we consider induced "supergroups"  $\tilde{S}_\lambda := \bigcup_{\mu \in A_\lambda} S_\mu$  for  $\lambda = 0, \dots, k-1$ . Then we have

**Lemma 4.** *If for supergroups  $\tilde{S}_0, \dots, \tilde{S}_{k-1}$  induced by a partitioning of  $g \geq k$  groups  $S_0, \dots, S_{g-1}$  a group-changing schedule exists, this schedule is also group-changing for the original groups.*

*Proof.* Obviously, if for the supergroups  $\bigcup_{\mu \in A_0} S_\mu, \dots, \bigcup_{\mu \in A_{k-1}} S_\mu$  a group-changing schedule exists, this schedule is also group-changing according to the groups  $S_0, \dots, S_{g-1}$  since the original groups are a refinement of the supergroups.  $\square$

Using Lemma 4 we can state the following result:

**Lemma 5.** *If for an even number  $n$  of teams and a fixed number  $g \geq 3$  of groups a group-changing schedule exists for all possible group configurations  $\mathcal{S} = \{S_0, S_1, \dots, S_{g-1}\}$  with  $|\mathcal{S}| = g$  satisfying (4), then also a group-changing schedule exists for all possible group configurations  $\mathcal{S}' = \{S'_0, S'_1, \dots, S'_{g'}\}$  with  $|\mathcal{S}'| = g' > g$  satisfying (4).*

*Proof.* We prove the statement by induction on the number  $g$  of groups and distinguish whether  $\frac{n}{2}$  is even or odd. At first we consider the case that  $\frac{n}{2}$  is even. Suppose that a group-changing schedule for each configuration  $\mathcal{S}$  with  $|\mathcal{S}| = g$  and  $|S_0| \leq \frac{n}{2}$  exists. Let  $\mathcal{S}'$  be an arbitrary configuration with  $|\mathcal{S}'| = g+1$  and  $|S'_0| \leq \frac{n}{2}$ .

We claim that  $|S'_{g-1}| + |S'_g| \leq \frac{n}{2}$ . Suppose to the contrary that  $|S'_{g-1}| + |S'_g| > \frac{n}{2}$ . Since  $|S'_0| \geq \dots \geq |S'_g|$ , we have  $|S'_0| + |S'_1| \geq |S'_{g-1}| + |S'_g|$ . Due to  $g \geq 3$  we obtain  $\sum_{s=0}^g |S'_s| > n$ , which contradicts the fact that all groups together contain  $n$  elements. Thus, we must have  $|S'_{g-1}| + |S'_g| \leq \frac{n}{2}$ .

We consider "supergroups" of  $\mathcal{S}'$  defined by  $\mathcal{S}'' := \{S'_0, \dots, S'_{g-2}, S'_{g-1} \cup S'_g\}$ . Since  $|\mathcal{S}''| = g$  and  $\max\{|S'_0|, |S'_{g-1} \cup S'_g|\} \leq \frac{n}{2}$ , a group-changing schedule for  $\mathcal{S}''$  exists by assumption. Due to Lemma 4 such a schedule is also group-changing for  $\mathcal{S}'$ .

In order to prove the case of odd  $\frac{n}{2}$  with similar arguments, we must show that  $\max\{|S'_0|, |S'_{g-1} \cup S'_g|\} < \frac{n}{2}$ . Since  $|S'_0| < \frac{n}{2}$  is satisfied due to (4), it suffices to show  $|S'_{g-1} \cup S'_g| < \frac{n}{2}$ .

We claim that  $|S'_0| > |S'_g|$  or  $g > 3$  holds. Assume  $g = 3$ . Since  $\frac{n}{2}$  is odd,  $n$  is not a multiple of 4. Thus,  $n$  teams cannot be partitioned into  $g+1 = 4$  groups of identical sizes  $|S'_0| = |S'_1| = \dots = |S'_g|$  in  $\mathcal{S}'$  and, hence,  $|S'_0| > |S'_g|$  must hold.

If  $|S'_0| > |S'_g|$ , then  $|S'_0| + |S'_1| > |S'_{g-1}| + |S'_g|$  and due to  $g \geq 3$  by similar arguments as above  $|S'_{g-1}| + |S'_g| < \frac{n}{2}$  is implied.

On the other hand, if  $g > 3$ , then  $|\mathcal{S}'| = g+1 \geq 5$  and we obtain  $|S'_{g-1}| + |S'_g| \leq \frac{2n}{5} < \frac{n}{2}$  concluding the proof.  $\square$

In the following we show that Lemma 4 only provides a sufficient but not necessary condition for the existence of group-changing schedules. Figure 8 provides an example for a group-changing schedule with  $n = 8$  teams and  $g = 3$  groups  $S_0 = \{0, 1, 2\}$ ,  $S_1 = \{3, 4, 5\}$ ,  $S_2 = \{6, 7\}$ . All partitionings of  $\mathcal{S} = \{S_0, S_1, S_2\}$  into supergroups satisfy  $|S^0| > \frac{n}{2} = 4$ , i.e. for them no group-changing schedule exists.

	0	1	2	3	4	5	6
0 <sub>0</sub>	5 <sub>1</sub>	2 <sub>0</sub>	4 <sub>1</sub>	6 <sub>2</sub>	3 <sub>1</sub>	1 <sub>0</sub>	7 <sub>2</sub>
1 <sub>0</sub>	4 <sub>1</sub>	6 <sub>2</sub>	3 <sub>1</sub>	2 <sub>0</sub>	7 <sub>2</sub>	0 <sub>0</sub>	5 <sub>1</sub>
2 <sub>0</sub>	7 <sub>2</sub>	0 <sub>0</sub>	5 <sub>1</sub>	1 <sub>0</sub>	4 <sub>1</sub>	6 <sub>2</sub>	3 <sub>1</sub>
3 <sub>1</sub>	6 <sub>2</sub>	4 <sub>1</sub>	1 <sub>0</sub>	5 <sub>1</sub>	0 <sub>0</sub>	7 <sub>2</sub>	2 <sub>0</sub>
4 <sub>1</sub>	1 <sub>0</sub>	3 <sub>1</sub>	0 <sub>0</sub>	7 <sub>2</sub>	2 <sub>0</sub>	5 <sub>1</sub>	6 <sub>2</sub>
5 <sub>1</sub>	0 <sub>0</sub>	7 <sub>2</sub>	2 <sub>0</sub>	3 <sub>1</sub>	6 <sub>2</sub>	4 <sub>1</sub>	1 <sub>0</sub>
6 <sub>2</sub>	3 <sub>1</sub>	1 <sub>0</sub>	7 <sub>2</sub>	0 <sub>0</sub>	5 <sub>1</sub>	2 <sub>0</sub>	4 <sub>1</sub>
7 <sub>2</sub>	2 <sub>0</sub>	5 <sub>1</sub>	6 <sub>2</sub>	4 <sub>1</sub>	1 <sub>0</sub>	3 <sub>1</sub>	0 <sub>0</sub>

Figure 8: Group-changing schedule for groups  $S_0 = \{0, 1, 2\}$ ,  $S_1 = \{3, 4, 5\}$ , and  $S_2 = \{6, 7\}$

The preceding example can be generalized leading to the following construction scheme:

**Lemma 6.** *For all even  $n \geq 4$  which are a multiple of 4 and group sizes  $|S_0| = |S_1| = \frac{n-2}{2}$ ,  $|S_2| = 2$  group-changing schedules exist.*

*Proof.* W.l.o.g. we assume that the teams are numbered such that  $S_2 = \{n-2, n-1\}$ . Let  $S'_0 := S_0 \cup \{n-2\}$ ,  $S'_1 := S_1 \cup \{n-1\}$  and  $k := |S'_0| = |S'_1| = \frac{n}{2}$ . Since  $n$  is a multiple of 4,  $k$  is even.

1. For the matches between the two groups  $S'_0, S'_1$  we consider the bipartite graph  $K_{k,k}$ . We choose an arbitrary 1-factorization of  $K_{k,k}$  (e.g.  $F^{bip}$  according to (1)). The corresponding 1-factors are assigned to the periods  $0, 2, 4, \dots, n-2$  (in an arbitrary order).
2. In the remaining periods  $1, 3, 5, \dots, n-3$  we schedule the "internal" matches among teams in  $S'_0$  and  $S'_1$ , respectively. In the following we show that 1-factorizations for the graph  $K_k$  with even  $k$  can be chosen such that the schedule is also group-changing w.r.t. group  $S_2$ .

We consider the group  $S'_0$ , the other group can be treated symmetrically. For each period  $t \in \{1, 3, \dots, n-3\} = \{2\mu+1 \mid \mu = 0, \dots, |S_0|-1\}$  let  $V_t$  be the set of all teams in  $S_0$  which are opponents of team  $n-1 \in S'_1$  in period  $t-1$  or  $t+1$  according to the schedule from Step 1. In the 1-factor corresponding to period  $t$  these teams may not be paired with team  $n-2 \in S'_0$  since teams  $n-1, n-2$  belong to the same group.

Each set  $V_t$  contains at most two teams, furthermore, each team  $i \in S_0$  appears in at most two sets  $V_t$ . Thus, for each  $t$  we can find a team  $i_t \in S_0 \setminus V_t$  such that

$$\bigcup_{t \in \{1, 3, \dots, n-3\}} \{i_t\} = S_0. \text{ Let } j_\mu := i_{2\mu+1} \text{ for } \mu = 0, \dots, |S_0|-1.$$

According to the canonical 1-factorization (cf. de Werra [4]) for  $\mu = 0, \dots, |S_0|-1$  we may set

$$F_{2\mu+1} = \{[n-2, j_\mu]\} \cup \left\{ [j_{(\mu+\lambda) \bmod |S_0|}, j_{(\mu-\lambda) \bmod |S_0|}] \mid \lambda = 1, \dots, \frac{|S_0|-1}{2} \right\} \quad (5)$$



and schedule the matches in period  $2\mu + 1$  according to  $F_{2\mu+1}$ . The same construction is done for the set  $S'_1$  resulting in a group-changing schedule.  $\square$

An example for this construction with  $n = 12$  teams can be found in Figure 9. After scheduling the matches in periods  $0, 2, \dots, n - 2$  according to  $F^{bip}$ , for the group  $S'_0 = \{0, 1, 2, 3, 4, 10\}$  we get  $V_1 = \{4\}$ ,  $V_3 = \{3, 4\}$ ,  $V_5 = \{2, 3\}$ ,  $V_7 = \{1, 2\}$ , and  $V_9 = \{0, 1\}$ . Then, team  $n - 2 = 10$  can be paired with team  $j_0 = i_1 = 3$  in period 1, team  $j_1 = i_3 = 2$  in period 3, team  $j_2 = i_5 = 1$  in period 5, team  $j_3 = i_7 = 0$  in period 7 and team  $j_4 = i_9 = 4$  in period 9. According to (5) the remaining internal matches are scheduled.

	0	1	2	3	4	5	6	7	8	9	10
0	5	1	6	4	7	2	8	10	9	3	11
1	6	0	7	3	8	10	9	4	11	2	5
2	7	4	8	10	9	0	11	3	5	1	6
3	8	10	9	1	11	4	5	2	6	0	7
4	9	2	11	0	5	3	6	1	7	10	8
10	11	3	5	2	6	1	7	0	8	4	9
5	0	7	10	9	4	6	3	8	2	11	1
6	1	11	0	8	10	5	4	7	3	9	2
7	2	5	1	11	0	9	10	6	4	8	3
8	3	9	2	6	1	11	0	5	10	7	4
9	4	8	3	5	2	7	1	11	0	6	10
11	10	6	4	7	3	8	2	9	1	5	0

Figure 9: Group-changing schedule for groups  $S_0 = \{0, 1, 2, 3, 4\}$ ,  $S_1 = \{5, 6, 7, 8, 9\}$ , and  $S_2 = \{10, 11\}$

Due to the arguments given above and some computational results in connection with integer programming formulations for all even  $n \leq 14$  and all possible group sizes it is known whether a group-changing schedule exists or not. These results are summarized in Table 4.

Note that in this table Lemma 4 is also used to provide negative results. If for some configuration no group-changing schedule exists, then also for all "supergroup configurations" no group-changing schedule exists. For  $n = 8, 12, 14$  we use Lemma 5 in order to state that for all  $g \geq 3$  group-changing schedules exist. On the other hand, for  $n = 10$  one configuration with  $g = 4$  does not allow a group-changing schedule. For  $n \geq 12$  it seems that group-changing schedules always exist whenever the necessary condition (4) is satisfied.

## 5 Conclusions and outlook

We picked up a common idea to guarantee a certain degree of fairness in a SRRT with teams belonging to different groups. The contribution of this paper is threefold:

- We considered the case where the number of groups  $g$  equals 3 and all groups have identical sizes. No group-changing schedule exists for  $n = 6$  teams. Moreover, we gave construction schemes for  $\frac{n}{3} \notin \{2, 26, 34, 58, 74\}$ . Although we conjecture that group-changing schedules also for  $\frac{n}{3} \in \{26, 34, 58, 74\}$  exist, these cases remain open.
- For the case where  $n$  is odd and group sizes are identical we proposed a construction scheme for all possible  $g$  (i.e. with  $\frac{n}{g} \in \mathbb{N}$ ).

$n$	group sizes	existence	reason
4	$ S_0  > 2$	no	$ S_0  > \frac{n}{2}$ violates (4)
	2+2	yes	2 even, construction in [3]
	$ S_0  = 2$	yes	Lemma 4, supergroups with sizes 2+2
6	$ S_0  \geq 3$	no	$ S_0  \geq \frac{n}{2}$ violates (4)
	2+2+1+1	no	IP
	2+2+2	no	Lemma 4, supergroups of 2+2+1+1
	2+1+1+1+1	yes	IP
8	$ S_0  > 4$	no	$ S_0  > \frac{n}{2}$ violates (4)
	4+4	yes	4 even, construction in [3]
	$ S_0  = 4$	yes	Lemma 4, supergroups with sizes 4+4
	3+3+2	yes	construction of Lemma 6
	all other combinations	yes	by Lemma 5 with $g = 3$
10	$ S_0  \geq 5$	no	$ S_0  \geq \frac{n}{2}$ violates (4)
	4+3+3	no	IP
	4+4+1+1	no	IP
	4+4+2	no	Lemma 4, supergroups of 4+4+1+1
	4+3+2+1	yes	IP
	4+2+2+2	yes	IP
	3+3+3+1	yes	IP
	3+3+2+2	yes	IP
	all other combinations	yes	supergroups with sizes 4+3+2+1 or 3+3+2+2
12	$ S_0  > 6$	no	$ S_0  > \frac{n}{2}$ violates (4)
	6+6	yes	6 even, construction in [3]
	$ S_0  = 6$	yes	supergroups with sizes 6+6
	5+5+2	yes	construction of Lemma 6
	5+4+3	yes	IP
	4+4+4	yes	construction of Section 2.1
	all other combinations	yes	by Lemma 5 with $g = 3$
14	$ S_0  \geq 7$	no	$ S_0  \geq \frac{n}{2}$ violates (4)
	6+6+2	yes	IP
	6+5+3	yes	IP
	6+4+4	yes	IP
	5+5+4	yes	IP
	all other combinations	yes	by Lemma 5 with $g = 3$

Table 4: Existence of group-changing schedules for even  $n \leq 14$  and arbitrary group sizes

- We provided several results for cases where group sizes are not identical. Besides stating necessary conditions as well as sufficient conditions we gave a construction scheme for a special case. However, several questions remain open. Especially, for  $n \geq 16$  it is not known whether for all configurations satisfying the necessary condition (4) group-changing schedules exist.

In order to visualize the construction methods of this paper and from Briskorn [3] we implemented an applet which can be found at

<http://www.informatik.uos.de/knust/sportssched/webapp/group.html>.

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## A Ce2Fs of $K(6; 3)$ and $K(10; 3)$

In the following we represent the nodes  $i = 0, \dots, n-1$  of the graph  $K(\frac{n}{3}; 3)$  by pairs  $(g(i), i')$  where  $i' = i \bmod \frac{n}{3}$ . In the following a ce2F  $F^6 = \{F_f^6 \mid 0 \leq f \leq 5\}$  of  $K(6; 3)$  and a ce2F  $F^{10} = \{F_f^{10} \mid 0 \leq f \leq 9\}$  of  $K(10; 3)$  are shown. Each ce2f is given as collection of circles.

$$\begin{aligned}
 F_0^6 &= \{[(0, 0), (1, 3), (2, 0), (0, 3), (1, 0), (2, 3)], \\
 &\quad [(0, 1), (2, 4), (1, 1), (0, 4), (2, 1), (1, 4)], \\
 &\quad [(0, 2), (1, 5), (2, 2), (0, 5), (1, 2), (2, 5)]] \\
 F_1^6 &= \{[(0, 0), (1, 1), (2, 2), (0, 3), (1, 4), (2, 5)], \\
 &\quad [(0, 1), (1, 2), (2, 3), (0, 4), (1, 5), (2, 0)], \\
 &\quad [(0, 2), (1, 3), (2, 4), (0, 5), (1, 0), (2, 1)]] \\
 F_2^6 &= \{[(0, 0), (2, 2), (1, 4), (0, 2), (2, 0), (1, 2), \\
 &\quad (0, 3), (2, 1), (1, 1), (0, 5), (2, 3), (1, 3), \\
 &\quad (0, 4), (2, 4), (1, 5), (0, 1), (2, 5), (1, 0)], \} \\
 F_3^6 &= \{[(0, 0), (1, 4), (2, 0), (0, 4), (1, 2), (2, 1)], \\
 &\quad [(0, 1), (2, 2), (1, 3), (0, 5), (2, 5), (1, 5), \\
 &\quad (0, 3), (2, 3), (1, 1), (0, 2), (2, 4), (1, 0)]] \\
 F_4^6 &= \{[(0, 0), (2, 0), (1, 1), (0, 3), (2, 4), (1, 2), \\
 &\quad (0, 2), (2, 2), (1, 0), (0, 4), (2, 5), (1, 3), \\
 &\quad (0, 1), (2, 3), (1, 4), (0, 5), (2, 1), (1, 5)]] \\
 F_5^6 &= \{[(0, 0), (1, 2), (2, 2), (0, 4), (1, 4), (2, 4)], \\
 &\quad [(0, 1), (1, 1), (2, 5), (0, 3), (1, 3), (2, 1)], \\
 &\quad [(0, 2), (2, 3), (1, 5), (0, 5), (2, 0), (1, 0)]]
 \end{aligned}$$

$$\begin{aligned}
F_0^{10} &= \{[(0, 0), (1, 7), (2, 3), (0, 4), (1, 3), (2, 1), (0, 9), (1, 9), (2, 4), (0, 3), (1, 8), (2, 0)], \\
&\quad [(0, 1), (1, 6), (2, 7), (0, 7), (1, 0), (2, 8), (0, 5), (1, 4), (2, 6), (0, 8), (1, 5), (2, 2), \\
&\quad (0, 2), (1, 2), (2, 9), (0, 6), (1, 1), (2, 5)]\} \\
F_1^{10} &= \{[(0, 0), (1, 3), (2, 0), (0, 6), (1, 5), (2, 5), (0, 4), (1, 2), (2, 4), (0, 8), \\
&\quad (1, 0), (2, 6), (0, 7), (1, 7), (2, 9), (0, 5), (1, 9), (2, 3), (0, 1), (1, 4), \\
&\quad (2, 7), (0, 3), (1, 1), (2, 1), (0, 2), (1, 8), (2, 2), (0, 9), (1, 6), (2, 8)]\} \\
F_2^{10} &= \{[(0, 0), (1, 8), (2, 7), (0, 9), (1, 1), (2, 3)], [(0, 3), (1, 2), (2, 1), (0, 6), (1, 7), (2, 5)], \\
&\quad [(0, 1), (1, 3), (2, 8), (0, 2), (1, 5), (2, 9), (0, 7), (1, 4), (2, 0), (0, 8), (1, 9), (2, 6)], \\
&\quad [(0, 4), (1, 0), (2, 4), (0, 5), (1, 6), (2, 2)]\} \\
F_3^{10} &= \{[(0, 0), (1, 6), (2, 9), (0, 2), (1, 0), (2, 0), (0, 5), (1, 2), (2, 7), (0, 4), \\
&\quad (1, 9), (2, 1), (0, 3), (1, 3), (2, 5), (0, 7), (1, 1), (2, 2), (0, 6), (1, 4), \\
&\quad (2, 4), (0, 1), (1, 7), (2, 8), (0, 9), (1, 5), (2, 3), (0, 8), (1, 8), (2, 6)]\} \\
F_4^{10} &= \{[(0, 0), (1, 0), (2, 7), (0, 1), (1, 9), (2, 5), (0, 6), (1, 8), (2, 1), (0, 8), \\
&\quad (1, 6), (2, 3), (0, 5), (1, 5), (2, 8), (0, 4), (1, 1), (2, 4), (0, 2), (1, 7), \\
&\quad (2, 0), (0, 7), (1, 3), (2, 2)], [(0, 3), (1, 4), (2, 9), (0, 9), (1, 2), (2, 6)]\} \\
F_5^{10} &= \{[(0, 0), (1, 2), (2, 2), (0, 3), (1, 5), (2, 0), (0, 9), (1, 4), (2, 8), (0, 1), \\
&\quad (1, 8), (2, 4), (0, 6), (1, 3), (2, 6), (0, 4), (1, 6), (2, 5)], \\
&\quad [(0, 2), (1, 1), (2, 7), (0, 5), (1, 0), (2, 3), (0, 2), (1, 1), (2, 7), (0, 5), (1, 0), (2, 3)]\} \\
F_6^{10} &= \{[(0, 0), (1, 1), (2, 8), (0, 3), (1, 0), (2, 9), (0, 1), (1, 5), (2, 7), (0, 6), \\
&\quad (1, 9), (2, 2), (0, 8), (1, 2), (2, 0), (0, 4), (1, 7), (2, 6), (0, 2), (1, 4), \\
&\quad (2, 1), (0, 5), (1, 8), (2, 5), (0, 9), (1, 3), (2, 3), (0, 7), (1, 6), (2, 4)]\} \\
F_7^{10} &= \{[(0, 0), (1, 5), (2, 1), (0, 1), (1, 0), (2, 5), (0, 5), (1, 7), (2, 4), (0, 4), \\
&\quad (1, 4), (2, 2), (0, 7), (1, 2), (2, 8), (0, 8), (1, 1), (2, 6), (0, 9), (1, 8), \\
&\quad (2, 3), (0, 6), (1, 6), (2, 0), (0, 2), (1, 3), (2, 9), (0, 3), (1, 9), (2, 7)]\} \\
F_8^{10} &= \{[(0, 0), (1, 4), (2, 5), (0, 8), (1, 3), (2, 4), (0, 9), (1, 7), (2, 7), (0, 2), \\
&\quad (1, 9), (2, 0), (0, 1), (1, 2), (2, 3), (0, 3), (1, 6), (2, 1), (0, 4), (1, 8), \\
&\quad (2, 8), (0, 7), (1, 5), (2, 6), (0, 6), (1, 0), (2, 2), (0, 5), (1, 1), (2, 9)]\} \\
F_9^{10} &= \{[(0, 0), (1, 9), (2, 8), (0, 6), (1, 2), (2, 5), (0, 2), (1, 6), (2, 6), (0, 5), \\
&\quad (1, 3), (2, 7), (0, 8), (1, 4), (2, 3), (0, 9), (1, 0), (2, 1)], \\
&\quad [(0, 1), (1, 1), (2, 0), (0, 3), (1, 7), (2, 2)], [(0, 4), (1, 5), (2, 4), (0, 7), (1, 8), (2, 9)]\}
\end{aligned}$$