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MEMORANDUM

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Productivity Interpretations of the Farrell Efficiency Measures and the Malmquist Index and its Decomposition

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Productivity Interpretations of the Farrell Efficiency Measures and the Malmquist Index and its Decomposition

by

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Abstract: The ratio definition of efficiency has the form of a productivity measure. But the weights are endogenous variables and they do not function as weights in a productivity index proper. It is shown that extended Farrell measures of efficiency can all be given an interpretation as productivity measures as observed productivity relative to productivity at the various projection points on the frontier.

The Malmquist productivity index is the efficiency score for a unit in a period relative to the efficiency score in a previous period, thus based on a maximal common expansion factor for outputs or common contraction factor for inputs not involving any individual weighting of outputs or inputs, as is the case if a Törnqvist or ideal Fisher index is used. The multiplicative decomposition of the Malmquist productivity index into an efficiency part and a frontier shift part should not be taken to imply causality.

Keywords: Farrell efficiency measures; Technically optimal scale; Malmquist productivity index; Decomposition of the Malmquist productivity index

JEL classification: C61, D24

1. Introduction

Measuring productive efficiency has been developing the last decades to become an important research strand within the fields of economics, management science and operations research. Two seminal contributions are Farrell (1957) and Charnes et al (1978). Although the latter paper uses the efficiency definition of the former the approaches for calculation the measure differ in the two papers. Farrell started out defining a frontier production function as the relevant comparison for measuring productive efficiency for observations of production units and introduced radial measures for the case of constant returns to scale. Charnes et al (1978) set up a ratio of weighted outputs on weighted inputs. This approach brought into the efficiency story the concept of productivity. However, the weights are endogenous variables found when estimating the efficiency measure and the ratio form does not represent a productivity index proper.

A purpose of the paper is to elaborate upon the productivity interpretation for the generalised Farrell efficiency measures covering the case of variable returns to scale. We then have technical efficiency measures, scale efficiency measures and a technical measure of productivity, the last two types of measures building upon the old concept of technically optimal scale in production theory. We will also have a closer look at the Malmquist productivity index because it is defined as the ratio of Farrell technical efficiency measures for a unit for two different time periods.

The paper is organised as follows. The Charnes et al (1978) ratio measure and five Farrell efficiency measures are defined in Section 2¹ and the productivity interpretations of the latter measures discussed for the case of a single output and input, and then generalised to multiple outputs and inputs. The importance of (local) constant returns to scale for productivity measurement is brought out using the elasticity of scale. In Section 3 the Malmquist index is introduced and some basic properties of the index and their consequences for choice of efficiency measures are discussed. The decomposition of productivity change into efficiency

¹ Section 2 is based on Førsund (2015), Section 4.

change and frontier shift introduced in Nishimizu and Page (1982) is discussed and compared with the decomposition done in Färe et al (1994). Section 4 offers some conclusions.

2. Productivity interpretations of the Farrell efficiency measures

The ratio definition of the efficiency measure

Charnes et al (1978) relate the ratio idea for defining an efficiency measure to how efficiency is defined in engineering as “the ratio of the actual amount of heat liberated in a given device to the maximum amount that could be liberated by the fuel” (Charnes et al 1978, p. 430). The optimisation problem set up for deriving the efficiency measure in the case of constant returns to scale (CRS) is:

$$\text{Max } h_{j_0} = \frac{\sum_{r=1}^s u_{rj_0} y_{rj_0}}{\sum_{i=1}^m v_{ij_0} x_{ij_0}} \quad \text{subject to} \quad \frac{\sum_{r=1}^s u_{rj_0} y_{rj}}{\sum_{i=1}^m v_{ij_0} x_{ij}} \leq 1, \quad j = 1, \dots, j_0, \dots, n, \quad u_{rj_0}, v_{ij_0} \geq 0 \quad \forall r, i \quad (1)$$

Here h_{j_0} is the efficiency measure, y and x are the output and input vectors, respectively, with s outputs and m inputs, number of units is n , and u_{rj_0} , v_{ij_0} are the weights associated with outputs and inputs, respectively. These weights are endogenous variables and will be determined in the optimal solution. The constraints on a productivity form requires the “productivity” of all units to be equal or less than 1, i.e. the productivity of fully efficient units is normalized to 1.

Charnes et al (1978) show how the fractional programming problem above can be transformed to a linear programming problem, thus providing a link to the Farrell approach².

² Farrell and Fieldhouse (1962) were the first to solve the problem of calculating their efficiency measure by using linear programming.

The Farrell suite of efficiency measures

Farrell (1957) defined two technical measures of efficiency, the input-oriented measure based on scaling inputs of inefficient units with a common scalar, projecting the point radially to the frontier keeping observed output constant, and the output-oriented measure scaling outputs of inefficient units with a common scalar, projecting the point radially to the frontier keeping observed inputs constant. The measures were defined for a frontier function exhibiting constant returns to scale³. However, he also discussed variable returns to scale and studied this further in Farrell and Fieldhouse (1962), without explicitly introducing measures reflecting scale properties. This was done in Førsund and Hjalmarsson (1974) and (1979), developing a family of five efficiency measures. The latter paper illustrated the measures using a smooth variable returns to scale frontier production function exhibiting an *S*-shaped graph as typical for neoclassical production functions obeying the *Regular Ultra Passum Law*⁴ of Frisch (1965)⁵. However, the efficiency measures are valid for any type of frontier function as long as a basic requirement of the variation of the elasticity of scale is fulfilled. It is in particular valid for the generic DEA model exhibiting variable returns to scale (VRS) introduced in Banker et al (1984) that will be used in this paper.⁶

The family of Farrell efficiency measures is illustrated in Fig. 1 in the case of the frontier within a non-parametric framework being a piecewise linear function (Førsund 1992). The point of departure is the observation $P^0 = (y^0, x^0)$ that is inefficient with respect to the VRS frontier. The reference point on the frontier for the input-oriented measure E_1 with respect to the VRS frontier is $P_1^{VRS} = (y^0, x_1^{VRS})$, and the reference point on the frontier for the output-oriented measure E_2 with respect to the VRS frontier is $P_2^{VRS} = (y_2^{VRS}, x^0)$. A second envelopment is indicated by the ray from the origin being tangent to the point P^{Tops} . (We will return to the interpretation of this point below.) This frontier exhibits constant returns to scale (CRS). The reference points on the frontier are $P_1^{CRS} = (y^0, x_1^{CRS})$ and $P_2^{CRS} = (y_2^{CRS}, x^0)$. The dotted factor ray from the origin to the observation gives the productivity of the observation, and the dotted factor ray from the origin to a reference point on the VRS frontier

³ Farrell (1957) points out that the two measures in the case of constant returns to scale are equal.

⁴ The Regular Ultra Passum Law requires that the scale elasticity decreases monotonically from values greater than one, through the value one to lower values when moving along a rising curve in the input space.

⁵ This may be the reason for this way of presenting the family of efficiency measures being rather unknown in the DEA literature.

⁶ In the VRS DEA specification the scale elasticity has a monotonically decreasing value in the range of increasing returns to scale, but has a more peculiar development in the range of decreasing returns to scale as shown in Førsund et al (2009). However, there is a unique face where the scale elasticity is equal to 1 along a rising curve.

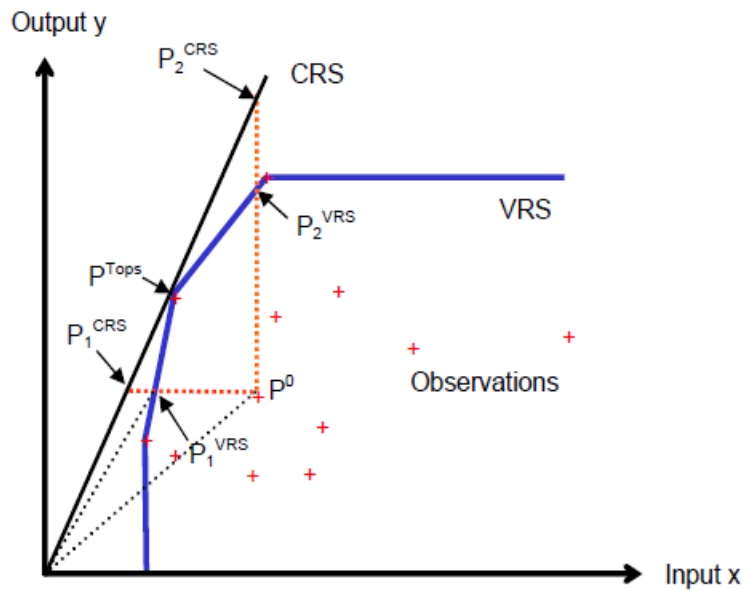


Figure 1. The Farrell efficiency measures applied to a piecewise linear frontier

gives the productivity of this reference point. As is easily seen from Fig. 1 the productivity at the CRS envelopment is the maximal productivity obtained on the VRS frontier. Comparing the observation with the reference point $P^{\text{Tops}} = (y^T, x^T)$ therefore gives the relative productivity of an observation to the maximal productivity on the VRS frontier. Continuing Farrell's numbering of measures a measure E_3 is introduced covering this measurement and is therefore termed the measure of *technical productivity*.⁷ The two remaining efficiency measures E_4 and E_5 introduced in Førsund and Hjalmarsson (1979) are the scale efficiency measures⁸ comparing the productivity of the reference points P_1^{VRS} and P_2^{VRS} , respectively, with the point P^{Tops} of maximal productivity on the frontier.

Productivity interpretations in the case of a single output and input

All Farrell measures of efficiency can be given an interpretation of relative productivity; the productivity of the observation relative to specific points on the VRS frontier marked in Fig. 1. Before showing the relative productivity interpretation in the case of a single output and a single input in a general setting, let us state the definitions of the Farrell input-and output-

⁷ In Førsund and Hjalmarsson (1979), introducing this measure, it was called the gross scale efficiency.

⁸ In Førsund and Hjalmarsson (1979) these measures were called measures of pure scale efficiency.

oriented technical efficiency measures, starting with the general definition of the production possibility set $T = \{(Y, X) : Y \geq 0 \text{ can be produced by } X \geq 0\}$. By assumption let the set T exhibit variable returns to scale (VRS) of its frontier (efficient boundary of T). The input- and output-oriented efficiency measures can be defined as⁹

$$\begin{aligned} E_1(y, x) &= \text{Min}_{\mu} \{ \mu : (\mu x, y) \in T \} \\ E_2(y, x) &= \text{Min}_{\lambda} \{ \lambda : (x, y / \lambda) \in T \} \end{aligned} \quad (2)$$

The relative productivity interpretation can be shown in the following way, starting with the input-oriented efficiency measure using the points P^0 and P_1^{VRS} in Fig. 1:

$$\frac{y^0 / x^0}{y^0 / x_1^{\text{VRS}}} = \frac{y^0 / x^0}{y^0 / E_1 x^0} = E_1 \quad (3)$$

The same productivity interpretation holds for the output-oriented efficiency measure using points P^0 and P_2^{VRS} in Fig. 1:

$$\frac{y^0 / x^0}{y_2^{\text{VRS}} / x^0} = \frac{y^0 / x^0}{(y^0 / E_2) / x^0} = E_2 \quad (4)$$

In the input-oriented case we adjust the observed input quantity so that the projection of the observation is on the frontier, and in the output-oriented case we adjust the observed output, using the symbols for adjusted input and output introduced above.

For the three remaining measures we will make a crucial use of the CRS envelopment in order to calculate the measures. The notation E_1^{CRS} and E_2^{CRS} , making explicit reference to the CRS envelopment as the frontier, together with $P^{\text{Tops}} = (y^{\text{T}}, x^{\text{T}})$, will be used. The measure of technical productivity is

$$\begin{aligned} \frac{y^0 / x^0}{y^{\text{T}} / x^{\text{T}}} &= \frac{y^0 / x^0}{y^0 / E_1^{\text{CRS}} x^0} = E_1^{\text{CRS}} = E_3 \\ \frac{y^0 / x^0}{y^{\text{T}} / x^{\text{T}}} &= \frac{y^0 / x^0}{(y^0 / E_2^{\text{CRS}}) / x^0} = E_2^{\text{CRS}} = E_3 \Rightarrow E_3 = E_1^{\text{CRS}} = E_2^{\text{CRS}} \end{aligned} \quad (5)$$

The first expression in each of the two lines of the equations is the definition of the measure of technical productivity using the productivity at the point P^{Tops} as a reference. The second expressions, input-orientation or output-orientation, respectively, show the most convenient way of calculating the productivity measure. The outputs and inputs differ between the

⁹ The Farrell efficiency measure functions correspond to the concept of distance functions introduced in Shephard (1970). Shephard's input distance function is the inverse of Farrell's input-oriented efficiency measure, and Shephard's output distance function is identical to Farrell's output-oriented efficiency measure.

observation P^0 and the P^{Top} points. But using the CRS envelopment the maximal productivity for the VRS technology is the same along the entire ray from the origin going through the point P^{Top} . The productivity measure E_3 is equal to both the input-oriented measure and the output-oriented measure using the CRS envelopment as the frontier. It is easy to see geometrically that in the case of using the CRS envelopment the two orientated efficiency measures must be identical, as pointed out by Farrell (1957).

Measures for scale efficiency are also defined using a relative productivity comparison. The input-oriented scale efficiency E_4 (keeping output fixed) and the output-oriented scale efficiency E_5 (keeping input fixed) are:

$$\begin{aligned} \frac{y^0 / x_1^{\text{VRS}}}{y^T / x^T} &= \frac{y^0 / E_1 x^0}{y^0 / E_1^{\text{CRS}} x^0} = \frac{E_1^{\text{CRS}}}{E_1} = \frac{E_3}{E_1} = E_4 \\ \frac{y_1^{\text{VRS}} / x^0}{y^T / x^T} &= \frac{(y^0 / E_2) / x^0}{y^0 / E_2^{\text{CRS}} x^0} = \frac{E_2^{\text{CRS}}}{E_2} = \frac{E_3}{E_2} = E_5 \end{aligned} \quad (6)$$

The relative productivity comparison for input-oriented scale efficiency in Fig. 1 is between the observed output on the efficiency-corrected input on the VRS frontier and the maximal productivity at the P^{Top} - point (y^T, x^T) . For output-oriented scale efficiency we have an analogous construction. The calculations of the scale efficiency measures can either be based on the ratios between the efficiency scores for input-oriented efficiency relative to the VRS frontier and the CRS envelopment, or expressed as deflating the technical productivity measure with the relevant efficiency measures relative to the VRS frontier.

The concepts of elasticity of scale and technically optimal scale

Before generalising the relative productivity interpretation to multiple outputs and inputs we need to introduce the concept of elasticity of scale. The definition of scale elasticity for a frontier production function is the same whether it is of the neoclassical differential type $F(y, x) = 0$ or if the production possibility set has a faceted envelopment border like in the DEA case. We are looking at the maximal proportional expansion β of outputs for a given proportional expansion α of inputs, i.e. looking at $F(\beta y, \alpha x) = 0$. The scale elasticity is

defined as the derivative of the output expansion factor w.r.t. the input expansion factor on the average value of the ratio of the output factor on the input factor¹⁰:

$$\varepsilon(x, y) = \frac{\partial \beta(x, y, \alpha)}{\partial \alpha} \frac{\alpha}{\beta} = \frac{\partial \beta(\alpha, x, y)}{\partial \alpha} \Big|_{\alpha=\beta=1} \quad (7)$$

The scale elasticity is evaluated, without loss of generality, for $\alpha = \beta = 1$. In the DEA case with non-differentiable points (vertex points or points on edges) the expression above is substituted with the right-hand derivative or the left-hand derivative, respectively, at such points (Krivonozhko et al 2004; Førsund and Hjalmarsson 2004b; Førsund et al 2007; Podinovski et al 2009; Podinovski and Førsund 2010).

Returns to scale is defined by the value of the scale elasticity; increasing returns to scale is defined as $\varepsilon > 1$, constant returns to scale as $\varepsilon = 1$ and decreasing returns to scale as $\varepsilon < 1$.

For a production function with variable returns to scale there is a connection between the input- and output- oriented measures via the scale elasticity. Following Førsund and Hjalmarsson (1979) in the case of a frontier function for a single output and multiple inputs we have

$$E_2 = E_1^{\bar{\varepsilon}} \Rightarrow E_1 \begin{matrix} > \\ < \end{matrix} E_2 \text{ for } \bar{\varepsilon} \begin{matrix} > \\ < \end{matrix} 1 \quad (8)$$

where the variable $\bar{\varepsilon}$ is the average elasticity of scale along the frontier function from the evaluation point for the input-saving measure to the output-increasing measure. In Førsund (1996) this result was generalised for multiple outputs and inputs in the case of a differentiable transformation relation $F(y, x) = 0$ as the frontier function, using the *Beam [Ray] variation equations* of Frisch (1965). This result holds for points of evaluation being projection points in the relative interior of faces. The path between the points will be continuous although not differentiable at vertex point or points located at edges.

We must distinguish between scale elasticity and scale efficiency (Førsund 1996). Formalising the illustration in Fig. 1 the reference for the latter is the concept of *technically optimal scale* of a frontier function (Frisch 1965). The set of points $TOPS^T$ having maximal productivities for the (efficient) border of the production possibility set

¹⁰ See Hanoch (1970); Panzar and Willig (1977); Starrett (1977).

$T = \{(Y, X) : Y \geq 0 \text{ can be produced by } X \geq 0\}$ with the frontier exhibiting VRS, can be defined as (Førsund and Hjalmarsson 2004a)

$$TOPS^T = \{(X, Y) : \varepsilon(X, Y) = 1, (X, Y) \in T\} \quad (9)$$

It must be assumed that such points exist and that for outward movements in the input space the scale elasticity cannot reach the value of 1 more than once for a smooth neoclassical frontier. However, it can in the DEA case be equal to 1 for points on the same face. The point (y^T, x^T) used above is now replaced by vectors y^T and x^T belonging to the set $TOPS^T$. From production theory we know that in general a point having maximal productivity must have a scale elasticity of 1. In a long-run competitive equilibrium the production units will realise the technically optimal scale with the scale elasticity of 1 implying zero profit.

The productivity interpretation of the efficiency measures in the general case

The interpretation of the five Farrell measures as measures of relative productivity can straightforwardly be generalised to multiple outputs and inputs. Introducing general aggregation functions¹¹ $g_y(y_1, y_2, \dots, y_M)$ and $g_x(x_1, x_2, \dots, x_N)$ for outputs and inputs, respectively, increasing in the arguments and being homogeneous of degree 1 in outputs and inputs, respectively (y and x are now interpreted as vectors and y_1, x_1 , etc., as elements of the respective vectors), we have, starting with the definition of relative productivity in the input-oriented case for an observation (y^0, x^0) :

$$\underbrace{\frac{g_y(y^0) / g_x(x^0)}{g_y(y_1^{VRS}) / g_x(x_1^{VRS})}}_{\text{Relative productivity}} = \underbrace{\frac{g_y(y^0) / g_x(x^0)}{g_y(y^0) / g_x(E_1 x^0)}}_{\text{Substituting for frontier input}} = \underbrace{\frac{g_y(y^0) / g_x(x^0)}{g_y(y^0) / E_1 g_x(x^0)}}_{\text{Using homogeneity of index function}} = E_1 \quad (10)$$

In the first expression relative productivity is defined in the input-oriented case using the observed vectors y^0, x^0 and the vectors y_1^{VRS}, x_1^{VRS} for the projection onto the VRS frontier analogous to the point P_1^{VRS} in Fig. 1 in the two-dimensional case. In the second expression the vectors for y_1^{VRS} and x_1^{VRS} are inserted, keeping the observed output levels y^0 and contracting the observed input vector using the input-oriented efficiency E_1 to project the inputs x^0 to the VRS frontier. In the third expression the homogeneity property of the input index function is used.

¹¹ There is no time index on the functions because our variables are from the same period.

In the case of output orientation of the efficiency measure E_2 we get in the multiple output – multiple input case following the procedure above:

$$\frac{g_y(y^0)/g_x(x^0)}{g_y(y_2^{VRS})/g_x(x_2^{VRS})} = \frac{g_y(y^0)/g_x(x^0)}{g_y(y^0/E_2)/g_x(x^0)} = \frac{g_y(y^0)/g_x(x^0)}{(g_y(y^0)/E_2)/g_x(x^0)} = E_2 \quad (11)$$

Using the general aggregation functions $g_y(y)$, $g_x(x)$ the measure of technical productivity can be derived using input- or output-orientation:

$$\begin{aligned} \frac{g_y(y^0)/g_x(x^0)}{g_y(y^T)/g_x(x^T)} &= \frac{g_y(y^0)/g_x(x^0)}{g_y(y^0)/g_x(E_1^{CRS}x^0)} = \frac{g_y(y^0)/g_x(x^0)}{g_y(y^0)/E_1^{CRS}g_x(x^0)} = E_1^{CRS} = E_3 \\ \frac{g_y(y^0)/g_x(x^0)}{g_y(y^T)/g_x(x^T)} &= \frac{g_y(y^0)/g_x(x^0)}{g_y(y^0/E_2^{CRS})/g_x(x^0)} = \frac{g_y(y^0)/g_x(x^0)}{(g_y(y^0)/E_2^{CRS})/g_x(x^0)} = E_2^{CRS} = E_3 \\ &\Rightarrow E_1^{CRS} = E_2^{CRS} = E_3 \end{aligned} \quad (12)$$

We obtain the same relationship between the technical productivity measure and the oriented measures with the CRS envelopment as in the simple case illustrated in Fig.1.

The case of multi-output and -input is done in the same way for the scale efficiency measures as for the other measures utilising the homogeneity properties of the aggregation functions:

$$\begin{aligned} \frac{g_y(y^0)/g_x(x_1^{VRS})}{g_y(y^T)/g_x(x^T)} &= \frac{g_y(y^0)/g_x(E_1x^0)}{g_y(y^0)/E_1^{CRS}g_x(x^0)} = \frac{g_y(y^0)/E_1g_x(x^0)}{g_y(y^0)/E_1^{CRS}g_x(x^0)} = \\ &= \frac{E_1^{CRS}}{E_1} = \frac{E_3}{E_1} = E_4 \\ \frac{g_y(y_2^{VRS})/g_x(x^0)}{g_y(y^T)/g_x(x^T)} &= \frac{g_y(y^0/E_2)/g_x(x^0)}{(g_y(y^0)/E_2^{CRS})/g_x(x^0)} = \frac{(g_y(y^0)/E_2)/g_x(x^0)}{(g_y(y^0)/E_2^{CRS})/g_x(x^0)} = \\ &= \frac{E_2^{CRS}}{E_2} = \frac{E_3}{E_2} = E_5 \end{aligned} \quad (13)$$

Again, we obtain the same relationship between the technical productivity measure and the oriented measures defining scale efficiency as in the simple case illustrated in Fig.1. The calculations of the scale efficiency measures can either be based on the ratios between the efficiency scores for input-oriented efficiency relative to the VRS frontier and the CRS envelopment or expressed as deflating the technical productivity measure with the relevant efficiency measures relative to the VRS frontier.

3. The Malmquist productivity index

The interpretation of the productivity index

Caves et al (1982) introduced the bilateral Malmquist productivity index developed for discrete time based on the ratio of distance functions (here efficiency functions) measured for two units, e.g. the same unit i measured for two different time periods u and v , relative to the same benchmark frontier production function indexed by b :

$$M_i^b(u, v) = \frac{E_j^b(x_{iv}, y_{iv})}{E_j^b(x_{iu}, y_{iu})}, \quad j = 1, 2, \quad i = 1, \dots, N, \quad u, v = 1, \dots, T, \quad u < v, \quad b = u, v \quad (14a)$$

The benchmark technology indexed by b is in many applications either the technology for period u or v , and changing over time according to the technologies of the two periods involved. It is also usual to take a geometric mean following the seminal papers of Färe et al (1992); (1994) on how to estimate the Malmquist productivity index. The reason given is that either the technologies from u or from v may be used as benchmark (Färe et al 2008). However, the time periods may be seen to impose a natural choice of the first period as a base in accordance with a “Laspeyre” view of using period u technology for the change from u to v .

It is well known in the literature how to set up LP problems to estimate the distance (or efficiency) functions involved in (14a) so we do not find it necessary to do this here (see e.g. Cooper et al 2000; Fried et al 2008).

The efficiency functions in (14a) show the maximal proportional expansion (outputs) or contraction (inputs), and the measures are called technical efficiency measures because prices are not involved. The Malmquist productivity index should then be called a technical productivity index. The productivity results may be different from the results one would get using prices for aggregating outputs and inputs. Weighting with revenue and cost shares as in the Törnqvist index means that the (real) price structure will have an influence. In the National accounts all industries that sell outputs and buy inputs in markets have information on values. All firms report values (by law in Norway) to be used in National accounts.

Another source of difference is that one or more of the weights (or shadow prices) (see Eq. (1)) of outputs and inputs may be zero, thus excluding variables from explicit influence on the efficiency scores in (14a) in order to maximise (minimise) the scaling factors in Eq. (2).¹² This may bias the Malmquist index in both directions compared with a standard Törnqvist index where all variables have strictly positive weights.

Another feature of the Malmquist productivity index that may give different results than other indices is that the efficiency functions in (14a) are based on frontier functions. In the case of capital vintage effects it is a dynamic investment process taking place in order to improve the technology of a firm, so a frontier based on the best technology may give a too optimistic view of the potential for efficiency improvements in the short run (Førsund 2010). The estimation of the frontier using DEA will also be distorted if observations picked to represent best practice by the method may in fact not be best practice, but picked due to biased technical change (Belu 2015).

Thus, there is a question about the usefulness of the information a Malmquist productivity index gives compared with indices using available price information. Public sector production activities not selling outputs in markets seem to be the best type of activities for application of the Malmquist productivity index.

In Section (2) we introduced general aggregator functions g_y and g_x for outputs and inputs. These functions may now be period-specific. However, because we do not know these or do not have data to estimate them, the Malmquist index will be estimated using the non-parametric DEA models giving us the efficiency measures in the numerator and denominator in (14a) (Färe et al 2008).

When applying the Malmquist productivity index attention should be paid to desirable properties. In the literature this is more often than not glossed over. We will therefore explain in more detail the choice of our specification. Productivity as measured by the Malmquist index (14a) may be influenced by changes in the scale of the operation, but two units that have the same ratio of outputs to inputs should be viewed as equally productive, regardless of the scale of production (Grifell-Tatjé and Lovell 1995). Doubling all inputs and outputs, keeping input and output mixes constant, should not change productivity. Therefore the benchmark envelopment of data, if we want to measure total factor productivity (TFP), is one

¹² To the best of our knowledge the occurrence of zero weights in Malmquist productivity index estimations has never been reported in the literature.

that is homogenous of degree 1 in the input and output vectors, and thus the linear-homogenous set that fits closest to the technology. The homogenous envelopment is based on the concept of technically optimal scale termed TOPS in Section 2, and is illustrated in Figure 2 at the point P_v^{tops} for a variable-returns-to-scale frontier (VRS). As pointed out in Section 2

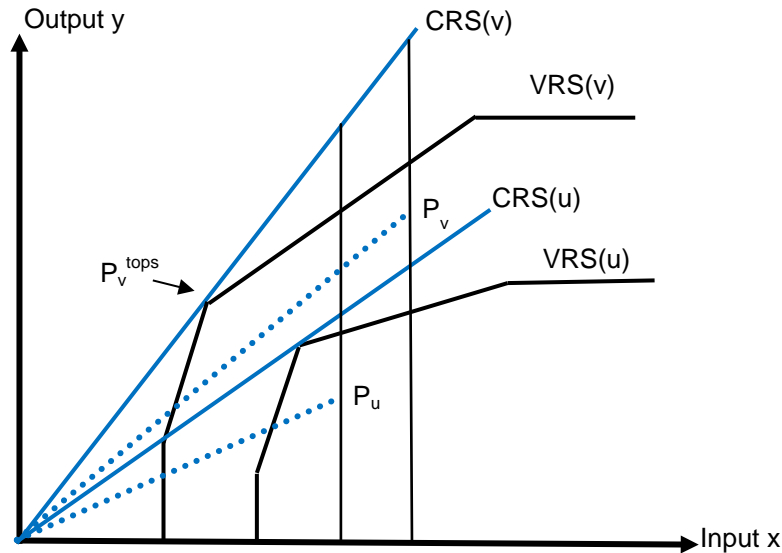


Figure 2. *The Malmquist productivity index. Productivity change for a unit measured relative to the benchmark CRS envelopment with maximal productivity*

the productivity is maximal at optimal scale where returns to scale is one, thus the CRS contemporary benchmark envelopments (assuming that the contemporaneous frontiers are VRS) are natural references for productivity changes over time. Observations of the same unit for the two periods u and v are indicated by P_u and P_v . The two corresponding VRS frontiers are drawn showing an outward shift indicating technological progress. The TOPS point for period v is labelled P_v^{tops} . Just as the productivity should be unchanged if the input-output vector is proportionally scaled, a measure of productivity should double if outputs are doubled and inputs are kept constant, and increase by half if inputs double, but outputs are constant. The desirable homogeneity properties of a TFP index is therefore to be homogenous of degree 1 in outputs in the second period and of degree (-1) in inputs of the second period, and homogenous of degree (-1) in outputs of the first period and homogenous of degree 1 in inputs of the first period. Using CRS to envelope the data is thus one way of obtaining all the required homogeneity properties of a Malmquist productivity index. Notice that in the

illustration in Fig. 2 the relative technology gap between the CRS benchmark technologies (blue lines) for observations in period v and u are identical, thus making the use of geometric mean of the Malmquist index in (14a) superfluous.¹³

The frontier technology level “jumps” from period to period from the start of one period to the start of the consecutive one. Outputs are produced and inputs consumed during the periods. This set-up is of course somewhat artificial compared with when real changes take place. The dynamic problems of adapting new technology and phasing it in are neglected.

Another property of a productivity index held to be important (Swamy and Samuelson 1974) is the *circularity* of the index (Berg et al 1992) (see Gini (1931) for an interesting exposition). The implied transitivity of the index means that the productivity change between two non-adjacent periods can be found by multiplying all the pairwise productivity changes of adjacent periods between the two periods in question. We will transitive the Malmquist index by using a single reference frontier enveloping the pooled data. In Tulkens and van den Eeckaut (1995) this type of frontier was termed the *intertemporal frontier*.

Using the same CRS reference envelopment for all units means that we have made sure that efficiency for all units and time periods refer to the same envelopment. The observations are either below the benchmark or on it in the case of the units from the pooled dataset spanning the envelopment. (In Fig. 2 imagining that the CRS envelopment for period v is the benchmark, all observations are below or on it.) Specifying CRS only is not sufficient to ensure that a specific data point occurring at different time periods get the same efficiency evaluation, because both input- and output isoquants may differ in shape over time if the technology is allowed to change over time as in Färe et al (2008).

Using a linear homogeneous envelopment implies that the orientation of the distance function does not matter. The Malmquist index for a unit i , that should be used according to the properties outlined above is then:

$$M_i^s(u, v) = \frac{E_v^s(x_{iv}, y_{iv})}{E_u^s(x_{iu}, y_{iu})}, \quad i = 1, \dots, J, \quad u, v = 1, \dots, T, \quad u < v \quad (14b)$$

¹³ However, most illustrations of the Malmquist indices using geometric mean are in fact using CRS frontiers and single output and input. But in the multi-output and -input case the distances may vary due to different mixes of outputs and inputs.

where superscript s symbolises that all data are used for estimating the technology reference set. The productivity change is the change in the productivities of the observations relative to the benchmark maximal productivity.

The decomposition of the Malmquist productivity index

Nishimizu and Page (1982) introduced the decomposition of the productivity index into efficiency change and technical change in continuous time and then applied the decomposition in discrete time¹⁴. They were aware of the problems with interpretation in the discrete case:

“Clearly, technological progress and technical efficiency change are not neatly separable either in theory or in practice. In our methodological approach ... we define technological progress as the movement of the best practice or frontier production over time. We then refer to all other productivity change as technical efficiency change. The distinction which we have adopted is therefore somewhat artificial,...” (Nishimizu and Page (1982), pp. 932-933).

Their approach is set out in Fig. 3 (the original Fig. 1, p.924). All variables are measured in

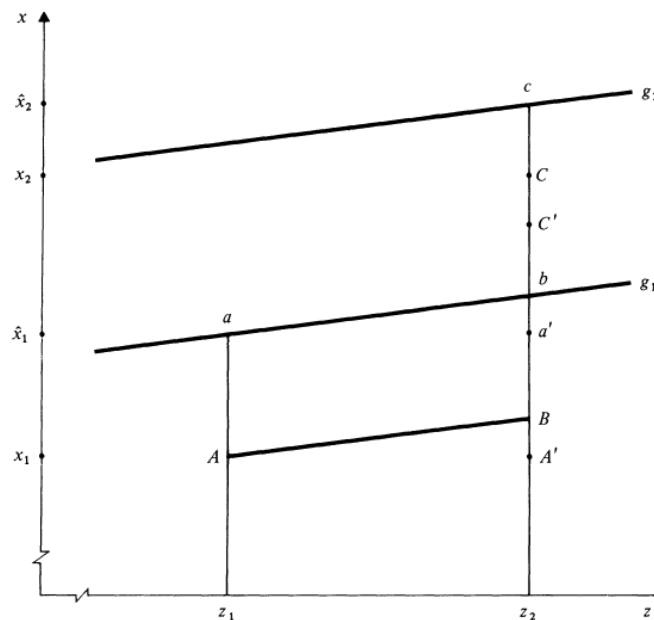


Fig. 1. Alternative interpretations of productivity change.

Figure 3. *The Nishimizu and Page (1982) decomposition*
Source: *The Economic Journal*

¹⁴ Nishimizu and Page (1982) were the first to refer to a working paper (Caves et al 1981) that was published as Caves et al (1982). However, they did not use the term Malmquist productivity index.

logarithms, and the frontier functions are linear C-D functions with Hicks-neutral technical change from period 1 to period 2. Production is x and input z . The observation A has a production function with the same parameter as the frontiers g_1 and g_2 , but with a different constant term. It is then the case that if unit A in period 1 had had the input of period 2, its production level would be at point B . From this point the frontier gap bc is added ending in point C' , so $BC' = bc$. Now, the observation in period 2 is found at C greater than C' . Nishimizu and Page then assume that the full potential frontier shift is realised in period 2, but in addition there is a positive efficiency change equal to $C'C$. So, measured in logarithms the productivity change is the sum of the efficiency gap $C'C$ and the frontier gap BC' ($= bc$).

Fig. 4 provides an explanation of their approach in the usual setting of quantities of variables

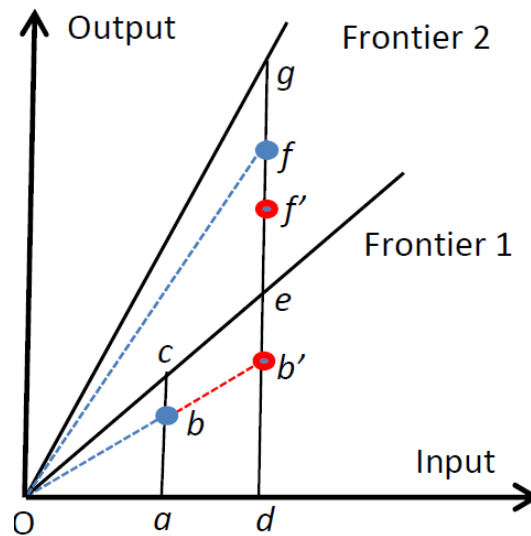


Figure 4. *The decomposition of the Malmquist index*

in the simple case of single output and input and the frontiers being CRS. We will now show that the Nishimizu and Page decomposition is the same as the decomposition introduced in Färe et al (1992); (1994).

A unit is observed at b in period 1 and at f in period 2. Using the frontier 1 as the benchmark technology instead of the pooled data for all years for simplicity the Malmquist productivity index (14b) for a unit i for change between period 1 and 2 and its decomposition are:

$$M_i^1(1,2) = \frac{E_i^1(y_i^2, x_i^2)}{E_i^1(y_i^1, x_i^1)} = \frac{E_i^2(y_i^2, x_i^2)}{E_i^1(y_i^1, x_i^1)} \cdot \frac{E_i^1(y_i^2, x_i^2)}{E_i^2(y_i^2, x_i^2)} = MC_i \cdot MF_i, \quad (15)$$

$$\frac{df/de}{ab/ac} = \frac{df/dg}{ab/ac} \cdot \frac{df/de}{df/dg}, \quad MF = \frac{df/de}{df/dg} = \frac{dg}{de}$$

The general definition of the Malmquist productivity-change index after the first equality sign is the ratio of the period efficiency measures against the same frontier technology, here for period 1. The expression after the second equality sign shows the multiplicative decomposition into a catching-up measure MC and a frontier shift measure MF . The second line relates the observations b and f in Fig. 4 to the decomposition in the case of a single output and input. Notice that to do decomposition we need a frontier for each period as illustrated in Fig. 2. To obtain the correct homogeneity properties we have to use period frontiers that exhibit CRS.

The MF -measure represents the relative gap between technologies and is thus the *potential* maximal contribution to productivity change, while the MC -measure is not the efficiency contribution to productivity change *per se*, but illustrates the actual catching-up that is also influenced by the technology shift. The components MC and MF cannot be estimated independently of the Malmquist index M .

The split into efficiency change and frontier shift that Nishimizu and Page propose is to assume that the full productivity potential of the frontier shift is actually realised. If both observations had been on their respective frontiers it is obvious that the Malmquist productivity change will be measured only by the frontier shift. If both observations are inefficient with respect to their period frontiers then the efficiency contribution is measured by changing (expanding in the figure) the input in period 1 to that of period 2, but using the actual production function in use in period 1 to predict the hypothetical output level. However, we do not operate with any production function for an inefficient observation as Nishimizu and Page did (a CRS C–D function with the same form as the frontier functions), but we will equivalently assume that the efficiency level stays constant getting the inputs of period 2 in period 1. The unit then moves from point b to point b' . The problem is now to predict where observation b' in period 2 will be if the whole potential shift is realised as productivity change. Nishimizu and Page operated with logarithms of the variables and could more conveniently illustrate this, as shown in Fig. 3 above. In our Fig. 4 this means that the predicted output at point f' must obey $df'/db' = dg/de$. Then we actually get the same measure for efficiency “contribution” as Nishimizu and Page equal to the ratio of the two period

efficiency measures. This decomposition is the same as the decomposition introduced in Färe et al (1992); (1994). We can demonstrate this in Fig. 4 by identifying the efficiency gap as df/df' and the frontier gap df'/db' building on Fig. 1 in Nishimizu and Page (Fig. 3 here), and using $df'/db' = dg/de$ and $db'/de = ab/ac$:

$$\frac{df}{df'} \cdot \frac{df'}{db'} = \frac{df}{db'} \cdot \frac{dg}{de} = \frac{df}{ab} \cdot \frac{dg}{ac} = \frac{df}{de} = M \quad (16)$$

However, note that the decomposition does not mean that there is a causation; we cannot distinguish between productivity change due to increase in efficiency and due to shift in technology using the general components in (15), as often seems to be believed in the literature (Johnes 2005; Worthington et al 2008). The actual productivity change that we estimate using the Malmquist productivity index is from the observation in one period to an observation in another period (from b to f in Fig. 4). The causation is another question related to the dynamics of technical change and how this potential is utilized. As expressed in Nishimizu and Page (1982) after identifying technological progress as the change in the best practice production frontier:

“We then refer to all other productivity change – for example learning by doing, diffusion of new knowledge, improved managerial practice as well as short run adjustment to shocks external to the enterprise – as technical efficiency change.”

Nishimizu and Page (1982, p. 921).

Nishimizu and Page consider that dynamic factors influence efficiency change, but do not consider the same for realising the new technology.

We cannot decompose efficiency effects and frontier shift effects without making assumptions, according to Nishimizu and Page. “Catching up” seems to be the best descriptive term for the efficiency component. The decomposition can then be described as the relative potential contribution from technical change multiplied by an efficiency correction factor.

Circularity and decomposition

Maintaining circularity for both components MC and Mf in the decomposition implies that the technology shift term MF will be more involved. Decomposing the index in Eq. (14b) we get:

$$M_i^s(u, v) = \underbrace{\frac{E_v^v(x_{iv}, y_{iv})}{E_u^u(x_{iu}, y_{iu})}}_{MC} \cdot \underbrace{\frac{E_v^s(x_{iv}, y_{iv}) / E_v^v(x_{iv}, y_{iv})}{E_u^s(x_{iu}, y_{iu}) / E_u^u(x_{iu}, y_{iu})}}_{MF}, \quad i = 1, \dots, N, u, v = 1, \dots, T, u < v \quad (17)$$

The frontier shift is now more involved to maintain circularity (note that taking the geometric mean of the Malmquist index (14b) is not compatible with circularity). The *MF* measure of technology shift is calculated as a ‘double’ relative measure where both period efficiency measures are relative to the benchmark efficiency measures (Berg et al 1992). It is easy to see that the decomposition is in fact the Malmquist index (14b) by shortening elements.

It can be illustrated in the case of one output and one input that the frontier shift component still measure the gap between the two benchmark technologies 1 and 2 in Figs. 2 and 4. Introducing the intertemporal benchmark *s* in Fig. 4 we can express the Malmquist index and its components in Fig. 5. The observations in period 1 and 2 are marked with blue circles at *b*

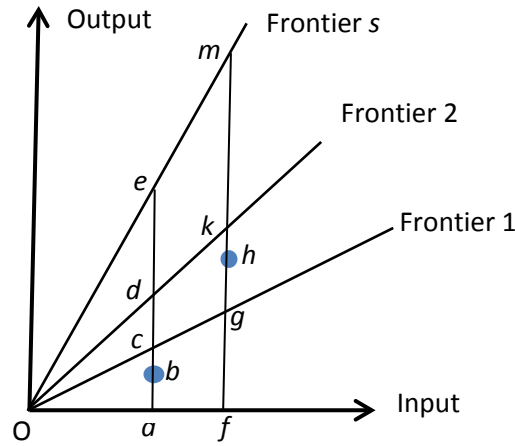


Figure 5. *The decomposition of the Malmquist index imposing circularity*

and *h*. The relative frontier gap between frontier 1 and 2 measured using the observation for period 2 is fk/fg . We shall see if the decomposition in (17) gives the same measure using the notation in Fig. 5:

$$M = \frac{fh / fm}{ab / ae} = \underbrace{\frac{fh / fk}{ab / ac}}_{MC} \cdot \underbrace{\frac{(fh / fm) / (fh / fk)}{(ab / ae) / (ab / ac)}}_{MF} \quad (18)$$

The *MF* component can be developed as follows:

$$MF = \frac{(fh / fm) / (fh / fk)}{(ab / ae) / (ab / ac)} = \frac{fk / fm}{ac / ae} \quad (19)$$

The last expression is the gap between frontier 2 and benchmark s in the numerator and the gap between frontier 1 and the benchmark in the denominator, both expressed as the inverse of the definition of the gap as expressed in the last equation in (15). But using the property of like triangles we have $ac / ae = fg / fm$. The last expression in (19) can then be written:

$$\frac{fk / fm}{ac / ae} = \frac{fk / fm}{fg / fm} = \frac{fk}{fg} \quad (20)$$

This is the relative gap between frontier 2 and 1 using the input for period 2 as the base for calculating the gap. As mentioned above the gap is influenced by where it is measured in the general case of multiple outputs and inputs. But this is a general feature of the Malmquist index and it is not avoided by taking the geometric mean of gaps using one point from each period.

4. Conclusions

Charnes et al (1978) introduced the ratio form of productivity measures for estimating the efficiency scores via estimating the weights in a linear aggregation of outputs and inputs, used to measure the productivity of a unit, and then maximising this productivity subject to no productivity ratio using these weights for all units being greater than one (as a normalisation). This ratio measure is said to be inspired by how efficiency is defined in the engineering literature. However, this way of defining efficiency measures using expressions formally equal to productivity, is not as satisfactory for economists as the Farrell approach, introducing explicitly a frontier production function as a reference for efficiency measure definitions and calculations.

The original Farrell measures developed for constant returns to scale (CRS) can be extended to five efficiency measures for a frontier production function exhibiting variable returns to scale (VRS); input- and output technical efficiency, input- and output scale efficiency, and the

technical productivity measure. The relationship between the two measures of technical efficiency involves the average scale elasticity value between the two frontier projection points along the frontier surface. The technical productivity measure and the two scale efficiency measures are developed based on the Frisch (1965) concept of technically optimal scale, predating the use of the concept most productive scale size in the DEA literature with almost 20 years.

It does not seem to be recognised in the DEA literature that in the general case of multiple outputs and inputs the Farrell efficiency measures can all be given productivity interpretations in a more satisfactory way than the ratio form of Charnes et al. (1978). Using quite general theoretical aggregation functions for outputs and inputs with standard properties, it has been shown that all five Farrell efficiency measures can be given a productivity interpretation employing a proper definition of productivity. Each of the two technical efficiency measures and the technical productivity measure can be interpreted as the ratio of the productivity of an inefficient observation and the productivity of its projection point on the frontier, using the general aggregation equations. Of course, we have not estimated any productivity index as such, this remains unknown, but that was not the motivation of the exercise in the first place.

The Malmquist productivity index for bilateral comparisons, applied to discrete volume data and no prices, utilises Farrell efficiency measures directly. In order to have the required index property of proportionality it is necessary to have as a benchmark an envelopment that exhibits global constant returns to scale, although the underlying contemporaneous production frontiers may have variable returns to scale. The proportionality properties are obtained by the benchmark envelopment being based on the technically optimal scale of the underlying frontiers. If circularity is wanted then this may be done by using envelopment for a single year, or pooling all data and using an intertemporal benchmark as is followed in this paper. There is some interest in the literature for decomposing the Malmquist productivity index multiplicatively into a component expressing efficiency change and a component showing the frontier shift impact on productivity. However, a warning of not attaching any causality to the decomposition is in place. The decomposition is based on assuming that the full potential of productivity change due to new technology is actually realised, and then the efficiency component is determined residually, but neatly expressed as the relative catching-up to the last period frontier compared with the relative distance to the frontier in the previous period.

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