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Discussion Paper #1583

“Monotone Persuasion”

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Northwestern University

July 9, 2015

# Monotone Persuasion

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## PRELIMINARY AND INCOMPLETE

### Abstract

We explore when it is optimal for senders to commit to signal structures which induce the receiver to take higher actions when the underlying state is higher and the preferences of the receiver satisfy strategic complementarity conditions. Building on the literature on monotone comparative statics, we provide sufficient conditions for the sender's optimal signal structure consists of a monotone partition of the state space, and characterize the boundary conditions. When the action space is binary, it is optimal to use a monotone partition if the sender's preferences are supermodular in the action and the state. In the case of a continuum of actions, though, one must take into account the additional effect that altering the receiver's posteriors also affect her choice. We provide a new single-crossing condition that takes account of this effect, and guarantees monotonicity given appropriate conditions on the cost of implementing the signal structure. If it is costless to provide information, it will be optimal for the sender to reveal all information. Applications are provided to preference disagreement with biases, as well as to expected revenue maximization.

## 1 Introduction

A prominent problem in economic literature is that of signalling one's quality: one party, referred to as the "sender," transmits some information to another party, the "receiver," who then implements some action based on this information (using Bayesian reasoning) which affects the sender. One angle from which to analyze this problem is found in the literature on persuasion, where the sender is given commitment power in designing the signal structure,

although it may be costly to do so. In their seminal paper, Kamenica and Gentzkow (2011) (henceforth KG) showed that, when payoffs from signals are convex in the possible posterior beliefs and signal design is costless, it is optimal to reveal all information to the receiver; this result was extended by Kamenica and Gentzkow (2014) to environments where signals' costs are increasing in their informativeness. More generally, the optimal signal structure generates a payoff to the sender from the concave closure of the possible payoffs for any given prior.

One can think of many environments that fit into the persuasion framework in which the structure of signals is monotone, meaning that a higher realization of the underlying state will lead to a “higher” message being sent in the sense that it induces the receiver to take a higher action. For instance, credit rating agencies formulate a set of criteria by which different ratings are assigned to various financial assets, and then apply those criteria once an asset is issued: intuitively, a safer asset receives a higher score. Another such environment is that of recommendations: a better candidate for a job receives a better review, in the sense that it leads to the better candidate getting better offers. One may also consider a school administrator placing students in courses of the optimal level of difficulty, where teachers (once the students are placed) set the difficulty level based on their own criteria. We formalize this intuition through sufficient conditions that guarantee that the optimal signal structure will be monotone.

While KG demonstrate the existence of an optimal signal, their approach does not address the issue of whether the resultant structure will be monotone. Indeed, by their techniques, it is quite possible that one wish to pool types that are not within a connected interval, or to send multiple possible messages with positive probability for the same type. To supplement their methods, one needs an appropriate single-crossing condition for the sender's payoffs which will ensure that the sender wishes to match the higher types to higher messages.

Furthermore, while the approach of KG is relatively straightforward for low-dimensional state spaces, the problem becomes intractable when the state becomes large. Indeed, as noted by Kamenica and Gentzkow (2014), for a given state space  $\Omega$ , the dimension of possible signals is  $(\Delta(\Omega))^{|\Omega|}$ .<sup>1</sup> Additional methods are often needed in practice to simplify the problem.

In this paper, we propose to simplify the problem by deriving sufficient conditions under which the optimal persuasion technique is to partition the state space. Specifically, if  $\Omega$  is a subset of the real line, then each message generated by the signal structure will give a posterior to the receiver which simply consists of the prior restricted to a subinterval of  $\Omega$ .

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<sup>1</sup>If  $\Omega$  is an interval subset of the real line, the cardinality of  $|\Omega|$  is uncountably infinite, and so it becomes difficult to find the concave closure of the sender's payoffs.

We do this by providing a single-crossing property for the signals: if the support for two messages overlap, then it will be optimal to “swap” elements of the support from one to the other to remove the overlap. This “swap” will consist of a Gâteaux derivative in the direction of the change of the conditional distribution. This method is most similar to that used in Bergemann and Pesendorfer (2007), who simultaneously consider the problem of providing information and designing an optimal auction for bidders with independent private values; they show that the optimal information structure is generated by partitioning each player’s value space into a finite set of intervals by showing that for any non-monotone partition, there exists a local improvement via swapping for some type. Thus our results can be viewed as a generalization of theirs by precisely formulating a single-crossing condition to generate local improvements.

Interestingly, the single-crossing conditions required when the sender can commit to a signal structure differ from those when the sender cannot commit, on which the previous literature has focused. In these papers, the sender first chooses an action to signal his type, while the receiver chooses another action in response. The seminal papers of Spence (1973) about costly job-market signalling, and Crawford and Sobel (1982) modelling costless communication, merely involve a single-crossing condition for the sender’s payoffs between the underlying state on one hand, and the respective actions chosen by the sender and the receiver. More recent papers by Kartik (2009) and Mensch (2015) extend the conditions under which there exists an equilibrium in which the sender’s signalling action will be increasing in the underlying state. When there is no commitment, in equilibrium, the receiver’s beliefs are specified for any action that the sender may take. The sender must therefore take the receiver’s beliefs and hence her response as given when considering a potential deviation. So, a deviation by the sender does not change the action that the receiver will take, and so the single-crossing conditions for the sender will be straightforward.

When the receiver has two actions available, this intuition remains correct to a certain extent, although the reasoning is slightly different: it turns out that supermodularity of the receiver’s and sender’s preferences in the action and state are sufficient to guarantee that the induced action will be higher when the state is higher. Consider a pair of posteriors in which, conditional on one of them, one takes the higher of the two possible actions, and one takes the lower of the two conditional on the other. If we increase the weight of a high state in the former posterior, the receiver is even more inclined to take a high action; similarly, by decreasing the weight of a high state conditional on the latter posterior, the receiver will be even more inclined to take a low action. Rather than look at the change in sender’s payoff from shifting an individual state, though (as in the case of no commitment), one must look at the change from this method of changing the weights of states in the posteriors: it must

be that the payoff from pairing a high action with a high state, along with a low action with a low state, is an improvement over pairing the high action with the low state and vice versa. A weaker single-crossing condition will therefore not suffice to guarantee monotonicity.

In the case where the receiver has more than two actions available, even supermodularity of the sender's payoffs will no longer be sufficient. This is because, when the sender can commit to a certain signal structure, the sender can no longer take the beliefs of the receiver as given when optimizing: a change in the signal structure changes the posteriors of the receiver. Perturbing a posterior by placing more weight on a certain state will therefore change what action the receiver takes for that posterior. The naive intuition that only complementarities between the resulting action and the underlying state matter is thus incorrect: one must also consider the effect that changing a signal will have on the action chosen by the receiver given that her beliefs have been perturbed.

The resulting structure will also depend on the costs required to generate the signals. If the costs to generate a certain posterior only depend on the probability that it is generated, regardless of the resultant distribution over states, then it is optimal to partition the state space into a non-degenerate set of intervals. Under weaker complementarity conditions, it is also optimal to partition in the same way if the cost only depends on the number of messages that the signal structure can generate. However, if providing more precise signals is costless, then it turns out that the single-crossing condition will also imply that it is optimal to reveal all information. While KG showed that this is true when the sender's payoffs are convex in the resultant posteriors, the single-crossing condition provides an alternative method of analyzing the problem. It also suggests that optimal monotone structures are degenerate, since the conditions that would allow for improvement over non-monotone structures would also make it optimal to completely separate.

When our single-crossing conditions are satisfied, it becomes relatively straightforward to find the cutoffs for each realization in the partition. As the conditions will automatically hold for all states that are not on the boundary of the partition conditional on their holding at the boundary, one need only check that they hold at the boundary. This ensures that, for a finite partition, one need only check a finite number of first-order conditions for any hypothetical solution, as opposed to checking for every state.

Our approach allows us to reexamine some of the classic signalling problems which did not assume commitment on the part of the sender. Specifically, we will examine the implications of our results for the model of preference disagreement with biases of Crawford and Sobel (1982) and the job market signalling model of Spence (1973). We show that the standard examples of these models satisfy the single-crossing conditions, and so will make it optimal

to partition the state space under full commitment. We also present applications to other economically relevant issues such as expected revenue maximization.

## 2 Setup

For simplicity, we mainly keep the notation consistent with that of KG;<sup>2</sup> this will allow for easy cross-reference and comparison of results. There are two players, a sender (S) and a receiver (R); for convenience, we will occasionally refer to the sender as “he” and the receiver as “she.” There is an underlying compact state space  $\Omega \subset \mathbb{R}$ , with an associated prior over the states  $\mu_0 \in \text{int}(\Delta(\Omega))$  that is common to both parties. Prior to the realization of  $\omega \in \Omega$ , the sender can choose a signal structure  $(\pi, \mathcal{S})$ , where  $\mathcal{S}$  is a compact metric space representing the set of possible realizations of the signal, and

$$\pi : [0, 1] \rightarrow \Omega \times \mathcal{S}$$

$$x \rightarrow (\pi_1(x), \pi_2(x))$$

is a measurable function defined so that the realization of  $\pi_2(x) = s \in \mathcal{S}$  is correlated with  $\pi_1(x) = \omega \in \Omega$ . We assume that  $x$  is uniformly distributed over  $[0, 1]$ . Thus the random variable  $s \in \mathcal{S}$  is generated from  $\omega$  via the probability integral transform as modelled by  $x$  through the mapping  $\pi$ .

Upon receiving signal realization  $s$ , the receiver, using Bayesian reasoning from the signal structure and the prior  $\mu_0$ , forms conditional probability assessment  $\mu(\cdot|s)$ . The receiver then chooses some action  $a$  from some compact set  $A \subset \mathbb{R}$ . The utility function for the receiver is given by  $u_R(a, \omega)$ , which is bounded and measurable in  $\omega$ , and is supermodular in  $(a, \omega)$ . The sender has utility function  $v_S(a, \omega, \pi, \mathcal{S}) = u_S(a, \omega) - c(\pi, \mathcal{S})$ , where  $u_S$  represents the payoff to the sender from the action chosen by the receiver, while  $c(\pi, \mathcal{S})$  represents the cost of implementing the signal structure. Thus  $u_S$  is the portion of the payoff dependent on a particular realization of the state and action, while  $c$  is the portion that depends on implementing the entire signal structure. Assume that  $u_S$  is bounded and measurable in  $\omega$ , and that  $c(\cdot, \cdot)$  is increasing in the informativeness of the  $(\pi, \mathcal{S})$  in the sense of Blackwell (1951).

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<sup>2</sup>Since we allow for a continuum of states, we refer the reader to their online appendix, which deals specifically with this environment.

Upon observing  $s$ , the receiver's problem is then

$$\max_{a \in A} \int u_R(a, \omega) d\mu(\omega|s)$$

Define a distribution of posteriors by  $\tau \in \Delta(\Delta(\Omega))$ , so that  $\pi$  induces  $\tau$ , i.e. if  $\pi_2(x) \in \mathcal{S}' \subset \mathcal{S}$  for some measurable set  $X \subset [0, 1]$ , then

$$\int_{\mathcal{S}'} d\tau(\mu(\cdot|s)) = \int_0^1 1[\pi_2(x) \in \mathcal{S}'] dx$$

In order to be feasible via Bayes' Theorem, we must have for all  $\omega$  that

$$\int_{\Delta(\Omega)} \mu d\tau(\mu) = \mu_0$$

The sender's problem is then, since  $(\pi, \mathcal{S})$  pins down  $\tau$ ,

$$\max_{(\pi, \mathcal{S})} \int u_S(a^*(\mu), \omega) d\tau(\mu) - c(\pi, \mathcal{S})$$

$$\text{s.t. } \int_{\Delta(\Omega)} \mu d\tau(\mu) = \mu_0$$

A perfect Bayesian equilibrium of this game will consist of a vector  $(\pi, \mathcal{S}, \mu, a^*)$  such that (i)  $\mu$  is Bayes-consistent with the signal structure  $(\pi, \mathcal{S})$ , (ii)  $a^*$  is optimal for the receiver given  $\mu$ , and (iii)  $(\pi, \mathcal{S})$  is optimal for the sender given  $a^*$ . KG showed that in such an environment, the set of perfect Bayesian equilibria is non-empty as long as  $c = 0$  for all  $(\pi, \mathcal{S})$ ; Kamenica and Gentzkow (2014) extended this result to environments in which  $\Omega$  is finite and  $c$  is increasing in the informativeness of the signal.

### 3 Monotone Signal Structures

Our central question is to analyze when the optimal signal structure is monotone, in that it partitions the state space into subintervals of  $\Omega$ . Before we can identify precisely what this will mean, we first introduce an ordering over conditional probabilities. The idea will be to rank distributions according to the actions that they induce of the sender: the higher the action taken, the higher the distribution is ranked. Suppose that  $|\mathcal{S}|$  is finite;<sup>3</sup> in this case,  $\tau(\mu(\cdot|s)) > 0$  for all  $s \in \mathcal{S}$ . Consider any two conditional distributions  $\mu, \mu' \in \Delta(\Omega)$ ,

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<sup>3</sup>By Proposition 4 in the online appendix of KG, this is guaranteed whenever  $\min\{|\mathcal{A}|, |\Omega|\} < \infty$ .



and assign them the probabilities  $\tau(\mu)$  and  $\tau(\mu')$ , respectively. Then we define an addition operation between  $\mu$  and  $\mu'$  so that  $\mu + \mu'$  yields a new conditional distribution which is generated with probability  $\tau(\mu + \mu') = \tau(\mu) + \tau(\mu')$ , where, for any measurable  $\Psi \subset \Omega$ ,

$$(\mu + \mu')(\Psi) = \frac{\mu(\Psi)\tau(\mu) + \mu'(\Psi)\tau(\mu')}{\tau(\mu) + \tau(\mu')}$$

Analogously, we define a subtraction operation which inverts the addition operation, i.e. if  $\hat{\mu} = \mu - \mu'$ , then  $\mu' + \hat{\mu} = \mu$ , assuming the posteriors from each such conditional are between 0 and 1 for all such  $\Psi$ .

In tandem with the above operations, we allow for scaling of signal realizations. Thus,  $\delta \cdot \mu + \epsilon \cdot \mu'$  is defined so that  $\tau(\delta \cdot \mu + \epsilon \cdot \mu') = \delta\tau(\mu) + \epsilon\tau(\mu')$

$$(\delta \cdot \mu + \epsilon \cdot \mu')(\Psi) = \frac{\delta\mu(\Psi)\tau(\mu) + \epsilon\mu'(\Psi)\tau(\mu')}{\delta\tau(\mu) + \epsilon\tau(\mu')}$$

In much of the ensuing analysis, we examine the effects to  $\mu$  that adding a conditional distribution that occurs with probability  $\epsilon$  which places conditional probability 1 on the state  $\omega'$ ; with a slight abuse of notation, we label this by  $\epsilon\omega'$ , so that  $\tau(\epsilon\omega') = \epsilon$ .

Next, we define an ordering over conditional distributions. This will allow for comparisons to show that a signal structure that is “more monotone” in the sense of the ordering will be a local improvement.

**Definition 1:** A binary relation  $\succeq_\sigma$  is an *ordering over conditional distributions* if the following two properties hold:

- (i) *Completeness:* For any  $\mu_1, \mu_2$ , either  $\mu_1 \succeq_\sigma \mu_2$  or  $\mu_2 \succeq_\sigma \mu_1$
- (ii) *Transitivity:* For any  $\mu_1, \mu_2, \mu_3$ , if  $\mu_1 \succeq_\sigma \mu_2$  and  $\mu_2 \succeq_\sigma \mu_3$ , then  $\mu_1 \succeq_\sigma \mu_3$ . Furthermore, if one of the orderings is strict, then  $\mu_1 \succ_\sigma \mu_3$ .

The ordering of the conditional distributions will be useful to essentially “summarize” the information contained therein. In particular, we will be interested in the case where higher distributions induce higher actions on the part of the receiver.

**Assumption 1 (Action Monotonicity):**  $\mu_1 \succeq_\sigma \mu_2 \iff a^*(\mu_1) \geq a^*(\mu_2)$ .

While this assumption may seem restrictive, one should note that for any game that fits into our model, there will always exist such an ordering  $\succeq_\sigma$ : one can simply define the ordering to be that induced by the action chosen by the receiver upon believing  $\mu$ .

To relate Assumption 1 to the underlying preferences of the receiver, we introduce the following property that posteriors may hold.

**Definition 2:** The ordering  $\succeq_\sigma$  is (strictly) *FOSD-consistent* if  $\mu_1((-\infty, \omega]) \leq \mu_2((-\infty, \omega])$  for all  $\omega \in \Omega$ , then  $\mu_1(\succ_\sigma) \succeq_\sigma \mu_2$ .

The following lemma characterizes the relationship between action monotonicity and the FOSD-consistency condition. If the induced beliefs from  $\mu_1$  first-order stochastically dominated those from  $\mu_2$ , we write this as  $\mu_1 \succeq_{FOSD} \mu_2$ . This will be useful for our later results, since there will be occasions where we will need to construct an improvement over a given signal structure through a multi-step perturbation, each step of which preserves first-order stochastic dominance. We will then be able to preserve ordering of the signals through each step.

**Lemma 1:** Suppose that there is a unique optimal action  $a^*(\mu)$ . Let  $\succeq_\sigma$  be a complete and transitive ordering. Then is FOSD-consistent if it satisfies action monotonicity (strictly so if either  $a^*(\mu_1) \in \text{int}(A)$  or  $a^*(\mu_2) \in \text{int}(A)$ ).

The proofs of all lemmas and propositions are contained in the appendix.

The above result shows that it is only possible to have the optimal action by the receiver be increasing in the conditional distribution if the ordering ranks the induced distribution higher when it first-order stochastically dominates the other. In this sense, the ordering  $\succeq_\sigma$  is a completion of the ordering induced by first-order stochastic dominance. There are many possible methods of ranking that fall into this latter category; an important one is by the conditional mean.

**Example 1:** Suppose that signal realizations are ranked by the conditional mean of the distribution, i.e.  $\mu_1 \succeq_\sigma \mu_2 \iff E_{\mu_1}[\omega] \geq E_{\mu_2}[\omega]$ . This is a complete and transitive ordering. To see that it is FOSD-consistent, note that  $E_\mu[\omega] = \int \omega d\mu(\omega)$ . Since  $f(\omega) = \omega$  is an increasing function, it follows that if  $\mu_1 \succeq_{FOSD} \mu_2$ , then  $E_{\mu_1}[\omega] \geq E_{\mu_2}[\omega]$ .  $\square$

We can see from here how changing a conditional distribution affects its ordering. Specifically, if we consider adding more weight to the highest portion of the support of a distribution, then it will increase its ranking in the ordering  $\succeq_\sigma$ . Similarly, subtracting weight to the highest portion will decrease the distribution.

Now that we have defined an ordering over conditional distributions, we can present our definition of monotone signal structures.

**Definition 3:** A signal structure  $(\pi, \mathcal{S})$  is *monotone* if, for all  $\omega_1, \omega_2 \in \Omega$ , if  $\omega_1 > \omega_2$  and  $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_2)$ , then

$$\omega_1 \in \text{supp}(\mu(\cdot|s_2)) \implies \omega_2 \notin \text{supp}(\mu(\cdot|s_1))$$

The following lemma will be useful for our results, stating that without loss of generality, it is possible to merge realizations which are ranked equally under  $\succeq_\sigma$ .<sup>4</sup> The intuition is that they result in the same action chosen by the receiver, and so the sender is indifferent to combining the two realizations into one. We can therefore assume without loss of generality that, for every  $s_1, s_2 \in \mathcal{S}$ , either  $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_2)$  or  $\mu(\cdot|s_2) \succ_\sigma \mu(\cdot|s_1)$ .

**Lemma 2:** *Suppose that the optimal signal  $(\pi, \mathcal{S})$  generates some measurable family of subsets  $\{\mathcal{S}^*(a)\} \subset \mathcal{S}$ , where  $|\mathcal{S}^*(a)| \geq 2$  and  $\int_{\cup\{\mathcal{S}^*(a)\}} d\tau(\mu(\cdot|s))$  such that, for all  $s_1, s_2 \in \mathcal{S}^*(a)$ ,  $a^*(\mu(\cdot|s_1)) = a^*(\mu(\cdot|s_2)) = a$ . Then the signal  $(\pi', \mathcal{S}')$  is also optimal, where  $\mathcal{S} = \mathcal{S}'$  except that each  $\mathcal{S}^*(a)$  is replaced by a single realization  $s'(a)$ , i.e.*

1.  $\mathcal{S} \setminus \cup\{\mathcal{S}^*(a)\} = \mathcal{S}' \setminus \{s'(a)\}$ ;
2. If  $\pi_2(x) \in \mathcal{S} \setminus \cup\{\mathcal{S}^*(a)\}$ , then  $\pi'_2(x) = \pi_2(x)$  and  $\pi_1(x) = \pi'_1(x)$ ;
3. If  $\pi_2(x) \in \mathcal{S}^*(a)$ , then  $\pi'_2(x) = s'(a)$

Using the previous lemma, it is easy to see that any monotone signal structure partitions the state space into intervals (though there may be some overlap in supports at the boundaries if there is an atom at a particular  $\omega$  in the prior  $\mu_0$ ). We formalize this in the following proposition.

**Proposition 1:** *If a signal structure is monotone, then:*

- (a) *For any two states  $\omega_1, \omega_2 \in \text{supp}(\mu(\cdot|s))$ , any  $\omega' \in \Omega \cap (\omega_1, \omega_2)$  is contained in  $\text{supp}(\mu(\cdot|s))$ .*
- (b) *If  $\omega \in \text{supp}(\mu(\cdot|s))$  and there exist  $\omega_1 < \omega$  and  $\omega_2 > \omega$  such that  $\omega_1, \omega_2 \in \text{supp}(\mu(\cdot|s))$  (i.e.  $\omega$  is in the “interior” of the support of  $s$ ), then for all  $s'$  such that either  $\mu(\cdot|s') \succ_\sigma \mu(\cdot|s)$  or  $\mu(\cdot|s) \succ_\sigma \mu(\cdot|s')$ ,  $\omega \notin \text{supp}(\mu(\cdot|s'))$ .*
- (c)  $|\text{supp}(\mu(\cdot|s)) \cap (\text{supp}(\mu(\cdot|s')))| \leq 1$ .

If the signal structure is monotone, then it is quite easy to define the posteriors  $\mu(\cdot|s)$  using Bayes' theorem: they will just be the prior  $\mu_0$  restricted to the set of types in the support of  $\mu$  given  $s$ . If there is an atom at some  $\omega$  that is not in the interior of the support of the posterior, then we just scale it by the probability that  $\omega$  generates signal realization  $s$ . If there is just a single  $\omega$  in the support of  $s$ , then  $\mu(\omega|s) = 1$ .

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<sup>4</sup>This is analogous to Proposition 1 in KG.

## 4 Two Actions

There are many relevant situations in which a sender must convince a receiver to take one of two actions. In many cases, there is complementarity between the state and the decision to be taken. For instance, in the well-known example of KG, a prosecutor wishes to convince a judge to convict a defendant. The judge is more inclined to convict the defendant if there is a greater likelihood that he is guilty, and so there is complementarity between the state (the guilt of the defendant) and the action (the verdict). As shown in their two-state example, the optimal structure is monotone: one realization has only in its support one of the states (in their case, that the defendant is innocent), while the other is supported by both of the states (that the defendant is guilty, as well as, with some probability, that he is innocent). We extend this reasoning more generally to environments with strategic complementarities.

Consider an environment in which  $A = \{a_1, a_2\}$ , with  $a_1 < a_2$ . By Lemma 2, one can restrict attention to  $|\mathcal{S}| \leq 2$ , and so  $\tau(\mu(\cdot|s)) > 0$  for each  $s \in \mathcal{S}$ .

Suppose that the preferences of both the sender and the receiver are weakly supermodular in  $(a, \omega)$ . Assuming that the induced action  $a$  does not change for sufficiently small  $\epsilon$ , the marginal change in payoff to the sender from adding  $\epsilon\mu'$  to  $\mu$  in the limit as  $\epsilon \rightarrow 0$  is

$$\begin{aligned} d(\mu; \mu') &\equiv \lim_{\epsilon \rightarrow 0} \frac{\tau(\mu) \int u_S(a^*(\mu), \omega) d\mu + \epsilon \tau(\mu') \int u_S(a^*(\mu), \omega) d\mu' - \tau(\mu) \int u_S(a^*(\mu), \omega) d\mu}{\epsilon \tau(\mu')} \\ &= \int u_S(a^*(\mu), \omega) d\mu' \end{aligned} \tag{1}$$

This is the Gâteaux derivative of the payoff of the sender from conditional distribution  $\mu$  in the direction of  $\mu'$ . If we define a posterior  $\mu'$  and assign it the value  $\tau(\mu')$ , then at the optimal signal structure  $(\pi, \{s_1, s_2\})$ , if it is feasible to subtract  $\epsilon \cdot \mu'$  (for some small enough  $\epsilon > 0$ ) from  $\mu(\cdot|s_1)$  and add it to  $\mu(\cdot|s_2)$ , then

$$d(\mu(\cdot|s_2); \mu') - d(\mu(\cdot|s_1); \mu') \leq 0 \tag{2}$$

If not, then for small enough  $\epsilon$ , there would exist an improvement from subtracting  $\epsilon \cdot \mu'$  from such a swap. This enables us to characterize a condition for monotone persuasion in the case of binary action spaces.

**Definition 4:** The *restriction* of a signal realization  $s$  to some measurable subset  $\Psi \subset \mathbb{R}$

such that  $\Psi \cap \Omega \neq \emptyset$ ,  $s_\Psi$ , assigns

$$\mu(A|s_\Psi) = \frac{\mu(A \cap \Psi|s)}{\mu(\Psi|s)}$$

and

$$\tau(s_\Psi) = \tau(s) \cdot \mu(\Psi|s)$$

**Theorem 2:** *If the sender's preferences are supermodular in  $(a, \omega)$ , then a monotone signal structure is optimal.*

The intuition for Theorem 2 is straightforward: due to the complementarity between actions and states, the sender would like to align the higher states with higher actions. If a signal structure is not monotone, he can always increase the level of alignment by swapping some of the support from the lower conditional distribution to the higher. This will not affect incentive compatibility of the receiver, as now the receiver is even more inclined to choose the higher action conditional on the higher signal realization.

One might be tempted to extend Theorem 2 to environments in which the sender's payoff is submodular to show that it is optimal to reveal no information. Similarly, one might be tempted to show that a monotone signal structure is optimal when  $|A| > 2$  as well. However, this cannot be assumed, as once the posteriors are modified, the incentive compatibility constraints no longer need hold. Thus the critical condition that the changes in the conditional distributions do not change the induced actions by the receiver will not necessarily hold, either. For instance, when  $u_S$  is submodular, transferring a high state  $\omega$  to the distribution of the lower of the two signal realizations,  $\mu(\cdot|s_1)$ , will increase the latter's conditional distribution, and so may induce the receiver to instead take action  $a_2$ . Similarly, for  $|A| > 2$ , increasing  $\mu(\cdot|s_2)$  may lead to the receiver taking an even higher action, which may or may not be beneficial to the sender. In Section 5, we will extend our analysis to an environment in which  $A$  is a continuum (specifically, an interval) in which a different but related condition will allow for monotonicity.

In tandem with the inequality (2), one can observe that the optimal cutoff state  $\omega^*$  for the signal structure, such that  $\omega > \omega^* \in \text{supp}(\mu(\cdot|s_2))$  and  $\omega > \omega^* \in \text{supp}(\mu(\cdot|s_1))$ , must satisfy the following first-order condition.

**Corollary 3:** *Let  $(\pi, \mathcal{S})$  be an optimal (monotone) signal structure such that  $\mu(\cdot|s_2) \succ_\sigma \mu(\cdot|s_1)$ . Then one of the following conditions must hold:*

$$(a) \int u_R(a_1, \omega) d\mu(\omega|s_1) = \int u_R(a_2, \omega) d\mu(\omega|s_1)$$

$$(b) \int u_R(a_1, \omega) d\mu(\omega|s_2) = \int u_R(a_2, \omega) d\mu(\omega|s_2)$$

(c) If  $\bar{\omega}_1 \equiv \sup\{\omega \in \text{supp}(\mu(\cdot|s_1))\} < \inf\{\omega \in \text{supp}(\mu(\cdot|s_2))\} \equiv \underline{\omega}_2$ , then  $u_S(a_2, \underline{\omega}_2) \geq u_S(a_1, \underline{\omega}_2)$  and  $u_S(a_2, \bar{\omega}_1) \leq u_S(a_1, \bar{\omega}_1)$ . In particular, if  $u_S$  is continuous in  $\omega$  and  $\bar{\omega}_1 = \underline{\omega}_2 \equiv \omega^*$ , then  $u_S(a_2, \omega^*) = u_S(a_1, \omega^*)$ .

Note that the first two scenarios of Corollary 3 are incentive compatibility constraints for the receiver: one must check to see if, from a small perturbation in the posterior, the receiver would jump to a different action. If they are not binding, then on the margin, the sender can shift some weight at the margin to the sender's preferred action at state  $\omega^*$ . Thus the problem to find the optimal signal structure is simple: one checks the incentive compatibility constraints, and if they are not binding, one finds the value of  $\omega^*$  are which the sender is indifferent between the actions taken.

## 5 Continuum of Actions

We now explore the case where  $A$  is an interval,  $[\underline{a}, \bar{a}]$ . As alluded to above, one may have the intuition to attempt to align the higher states with higher actions as in the environment with a binary action space. However, one must consider here an additional effect: by perturbing the posteriors, one also changes the action that the receiver takes. Thus, one must incorporate this effect as well when checking if such an alignment is an improvement for the sender.

To proceed with our analysis, we make a few additional assumptions. Assume that  $u_R$  is twice continuously differentiable with  $\frac{\partial^2 u_R}{\partial a^2} < 0$  and  $\frac{\partial^2 u_R}{\partial a \partial \omega} > 0$ . Because  $\frac{\partial^2 u_R}{\partial a^2} < 0$ , there will exist a unique optimal action for the receiver for each posterior  $\mu$ , which we define as  $a^*(\mu)$ . Assume that  $u_S$  is continuously differentiable in  $a$  and both  $u_S$  and  $\frac{\partial u_S}{\partial a}$  are continuous in  $\omega$ .

Suppose that there are two signal realizations  $s_1, s_2$  such that  $\mu(\cdot|s_1) \succeq_\sigma \mu(\cdot|s_2)$  that have overlapping distributions, so that there exists  $\omega^*$  such that  $\mu((-\infty, \omega^*]|s_1) > 0$  but  $\mu((\omega^*, \infty)|s_2) > 0$  as well. We want to show that there exists a local improvement by swapping some of the distribution above  $\omega^*$  from the support of signal realization  $s_2$  with some of the distribution below  $\omega^*$  from that of  $s_1$ . To precisely define what this means, define the marginal change in payoff from adding two conditional distributions  $\mu$  and  $\mu'$  for which  $\tau(\mu)$  and  $\tau(\mu')$  are positive by  $D(\mu, \mu')$  (if we subtract  $\mu'$  from  $\mu$ , this change will be  $D(\mu, -\mu')$ ). Thus, if one adds  $\mu$  and  $\epsilon \cdot \mu'$ ,  $D(\cdot, \cdot)$  is defined to be

$$D(\mu, \epsilon \cdot \mu') \equiv \frac{1}{\epsilon \tau(\mu')} \left\{ \tau(\mu) \left[ \int u_S(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu - \int u_S(a^*(\mu), \omega) d\mu \right] + \epsilon \tau(\mu') \cdot \int u_S(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu \right\} \quad (3)$$

This illustrates two effects of adding  $\mu'$  to  $\mu$  on the payoff of the sender. First, this will perturb  $\mu$ , and so perturb the optimal action for the receiver,  $a^*(\mu)$ . Second, there will now be an additional measure of type  $\omega'$  which will now have the action associated with  $\mu$  chosen for it as well.

For the case where we add  $\epsilon\omega'$  to  $\mu$ , the marginal change in payoff to the sender can be written as

$$D(\mu, \epsilon\omega') = \frac{1}{\epsilon} \left\{ \tau(\mu) \left[ \int u_S(a^*(\mu + \epsilon\omega'), \omega) d\mu - \int u_S(a^*(\mu), \omega) d\mu \right] + \epsilon \cdot u_S(a^*(\mu + \epsilon\omega'), \omega') \right\} \quad (4)$$

Before this can be applied directly to form the single-crossing condition, one must note that adding some weight to some state in some signal realization requires subtracting some from another. It is not clear that these effects are symmetric, i.e. subtracting a certain amount of weight from some realization results in the same marginal effects as from adding the same amount of weight. Fortunately, due to the assumptions on the preferences of the receiver, the best-reply  $a^*(\mu)$  is well-behaved. To see the intuition, note that if the receiver were completely certain of the state, then by taking the second derivative with respect to  $\omega$ , we have

$$\frac{\partial^2 u_R}{\partial a \partial \omega}(a^*(\omega), \omega) + \frac{\partial^2 u_R}{\partial a^2}(a^*(\omega), \omega) \cdot \frac{da^*(\omega)}{d\omega} = 0$$

and so the implicit function  $a^*(\omega)$  is differentiable with derivative

$$\frac{da^*(\omega)}{d\omega} = - \frac{\frac{\partial^2 u_R}{\partial a \partial \omega}(a^*(\omega), \omega)}{\frac{\partial^2 u_R}{\partial a^2}(a^*(\omega), \omega)} > 0$$

Note that the marginal effects of increasing or decreasing the underlying state are the same; since it is the receiver's reaction that affects the payoff of the sender, the marginal effect is symmetric for the sender as well. A similar idea holds in showing that the effect of shifting the conditional distribution up or down is symmetric. To do so, as in the previous section, we define

$$d(\mu; \mu') \equiv \lim_{\epsilon \rightarrow 0} D(\mu, \epsilon \cdot \mu')$$

to be the Gâteaux derivative of the payoff of the sender from conditional distribution  $\mu$  in the direction of  $\mu'$ .

**Lemma 3:** (a) For all  $\mu$  such that  $a^*(\mu) \in (\underline{a}, \bar{a})$ ,

$$d(\mu; \mu') = - \frac{\int \frac{\partial u_S}{\partial a}(a^*(\mu), \omega) d\mu \cdot \int \frac{\partial u_R}{\partial a}(a^*(\mu), \omega) d\mu'}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu), \omega) d\mu} + \int u_S(a^*(\mu), \omega) d\mu' \quad (5)$$

Otherwise, if for any  $\omega \in \Omega$ ,  $a^*(\mu) = a^*(\mu + \epsilon\omega) \in \{a, \bar{a}\}$  for small enough  $\epsilon > 0$ , then  $d(\mu; \mu') = \int u_S(a^*(\mu), \omega) d\mu'$ .

(b) Suppose that it is feasible to subtract  $\epsilon \cdot \mu'$  from  $\mu$  for small enough  $\epsilon$ . Then

$$d(\mu; -\mu') = -d(\mu; \mu')$$

Lemma 3 provides a formula for the effects of adding an infinitesimally small amount of  $\mu'$  to  $\mu$ . For the special case of adding  $\epsilon\omega'$  to  $\mu$  in the limit as  $\epsilon \rightarrow 0$ , this is given by

$$d(\mu; \omega') = u_S(a^*(\mu), \omega') - \frac{\frac{\partial u_R}{\partial a}(a^*(\mu), \omega') \int \frac{\partial u_S}{\partial a}(a^*(\mu), \omega) d\mu}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu), \omega) d\mu} \quad (6)$$

Hence we can write  $d(\mu; \mu')$  as

$$d(\mu; \mu') = \int d(s; \omega') d\mu' \quad (7)$$

Interestingly,  $d(\mu; \mu')$  does not depend on  $\tau$ , so we do not need to take account of the probability of a given posterior occurring in our ordering. This is due to the inverse proportionality of the size of the effects from perturbing the distribution: if  $\tau(\mu)$  is larger, then the posterior will change less from adding  $\epsilon \cdot \mu'$ , and so the induced action will shift less; on the other hand, there is a greater proportion of states (with respect to the prior) that are affected by this change in posterior. These two effects will exactly cancel out.

An economic interpretation of the expression for  $d(\mu; \omega')$  is in order. As mentioned above, there are two effects from changing the posterior. First, the action taken at the particular states will change, since the action taken at the old posterior is now taken at the new posterior. Second, there is an effect of changing the action taken at the old posterior, since the receiver will change his action as the posterior has changed. For marginal changes in the distribution, only these two first-order effects matter. Thus, the first term in the expression,  $u_S(a^*(\mu), \omega')$ , is the effect of having action  $a^*(\mu)$  taken when the state is  $\omega'$ , since that is how the distribution was perturbed. The second term is the marginal effect on the payoff of the sender from the receiver changing his action from that taken at posterior  $\mu$ . Thus  $\int \frac{\partial u_S}{\partial a}(a^*(\mu), \omega) d\mu$  is the marginal utility of the sender from the change in action of the receiver at posterior  $\mu$ , while  $-\frac{\frac{\partial u_R}{\partial a}(a^*(\mu), \omega')}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu), \omega) d\mu}$  is the marginal amount by which  $a^*(\mu)$  is perturbed by placing more weight on  $\omega^*$ . These two effects are combined additively for a readily interpretable expression.<sup>5</sup>

<sup>5</sup>This is similar to the expression for virtual utility found in Myerson (1981): in both cases, the principal (here, the sender) maximizes his utility subject to the constraint of incentive compatibility on the part of the agent (i.e. the receiver).



The definition of  $d$  will extend naturally to distributions  $\mu$  which do not arise as posteriors from  $(\pi, \mathcal{S})$ , as long as we assign such  $\mu$  a positive value of  $\tau$ . As seen in the previous section, if  $(\pi, \mathcal{S})$  is optimal, then for any  $s_1, s_2 \in \mathcal{S}$  for which  $\tau(\mu(\cdot|s_1)), \tau(\mu(\cdot|s_2)) > 0$  such that it is feasible to subtract  $\epsilon \cdot \mu'$  from  $\mu(\cdot|s_1)$ , it must be that

$$d(\mu(\cdot|s_2); \mu') - d(\mu(\cdot|s_1); \mu') \leq 0$$

or else there would exist an improvement by swapping  $\epsilon \cdot \mu'$  from  $\mu(\cdot|s_1)$  to  $\mu(\cdot|s_2)$ . This will greatly ease our definitions for our single-crossing conditions, which we now present.

**Definition 4:** The payoffs for the sender are *d-supermodular* if, for  $\omega' > \omega$  and  $\mu' \succ_{\sigma} \mu$ ,

$$d(\mu'; \omega') - d(\mu'; \omega) > d(\mu; \omega') - d(\mu; \omega) \quad (8)$$

Similarly, the payoffs to the sender are *d-submodular* if

$$d(\mu'; \omega') - d(\mu'; \omega) < d(\mu; \omega') - d(\mu; \omega) \quad (9)$$

**Definition 5:** The payoffs for the sender are *d-quasisupermodular* if

$$d(\mu'; \omega) - d(\mu; \omega) \geq 0 \implies d(\mu; \omega') - d(\mu; \omega) > 0 \quad (10)$$

The payoffs to the sender are *d-quasisubmodular* if

$$d(\mu'; \omega) - d(\mu; \omega) \leq 0 \implies d(\mu; \omega') - d(\mu; \omega) < 0 \quad (11)$$

The intuition here is that, if the sender could choose which action to induce once the state was realized, he would want to choose a higher action when the state was higher if his preferences were quasisupermodular; this follows from the standard analysis of Milgrom and Shannon (1994). This justifies the reasoning in signaling games where the sender cannot precommit to a signal structure (such as cheap talk), since in a perfect-Bayesian equilibrium, the sender takes the receiver's beliefs conditional on any message as given, and so the sender will not change the induced action by deviating to a particular message. In our environment, though, the sender *will* change the action taken following a particular message (i.e. when the posterior is perturbed), since a different action will now be optimal for the receiver. Thus we must combine this effect as well into the problem of the sender, and only then check for single-crossing in the posterior (and hence the action) that the sender wants to induce given the state.

We are now ready to present our main theorem.

**Theorem 4:** *Let  $c = 0$  for all  $(\pi, \mathcal{S})$ .*

*(i) If the sender's payoffs are  $d$ -quasisupermodular, then it is optimal for the sender to reveal all information.*

*(ii) Conversely, if his payoffs are  $d$ -quasisubmodular, then it is optimal to reveal no information, i.e. the posterior beliefs for the receiver will just be  $\mu_0$ .*

The intuition for the proof of Theorem 4(i) is that, if there is a posterior  $\mu$  with more than one state in the support, then we can duplicate the posterior to form  $\mu_1, \mu_2$ , and perturb them so that a little more weight is placed at the top of the support of, say,  $\mu_2$ , or a little less weight at the bottom, while the opposite is done for  $\mu_1$ . After this perturbation,  $\mu_2$  will be ranked higher than  $\mu_1$  due to FOSD-consistency, and so it will be a strict improvement to swap even more of the support. Since this initial perturbation can be arbitrarily small, it turns out that there will be a strict improvement over this initial posterior. Thus there cannot be any posteriors which have more than one state in the support, implying that full separation is optimal.

The intuition for Theorem 4(ii) is that, if there were two posteriors  $\mu_2 \succ_{\sigma} \mu_1$  that induced different actions, then by FOSD-consistency, there must be some pair of states for which a higher action  $a^*(\mu_2)$  is taken for the higher of the two states, while  $a^*(\mu_1)$  is taken for the latter. It will then be an improvement to swap either the higher state or the lower state, and so this could not be optimal. All realizations must therefore lead to the same action in the optimal signal structure, and so by Lemma 2 one can just as well reveal nothing.

A key element of the proof of Theorem 4 was the absence of costs of implementing a signal. However, in many cases of interest, this assumption is not reasonable. This may arise from physical costs of providing additional information, which may make it optimal to only partially reveal the information to the receiver. We therefore include the following result for the case where it is not optimal to reveal all information.

**Theorem 5:** *Assume that  $c(\pi, \mathcal{S})$  is increasing in  $|\mathcal{S}|$  and  $\lim_{|\mathcal{S}| \rightarrow \infty} c(\pi, \mathcal{S}) = \infty$ .*

*(i) If  $c(\pi, \mathcal{S})$  is solely a (continuous) function of the vector of probabilities  $\{\tau(\mu)\}$  (irrespective of the posteriors that they induce) and the sender's payoffs are  $d$ -supermodular, then the optimal signal structure is monotone.*

*(ii) If  $c(\pi, \mathcal{S})$  is solely a function of the number of posteriors generated (i.e.  $|\mathcal{S}|$ ) and the sender's payoffs are  $d$ -quasisupermodular, then the optimal signal structure is monotone.*

In terms of economic interpretation of Theorem 5, in addition to the obvious interpretation of

it being costly to the sender to generate the signal, one can reinterpret this result to apply to when the receiver has a fixed capacity. Recall that in Lemma 2, we saw that without loss of generality, we could merge signal realizations if they resulted in the same outcome. Suppose that the receiver will choose to implement  $N$  actions, and the sender can only influence what those  $N$  actions are. One possible instance in which this might apply is in tracked courses: a school may offer a fixed number of tracked classes in a certain subject, and teach them at a difficulty level determined by the students in the respective classes. The school principal's optimal course difficulty level may differ from that of the teachers, as the must take into account the preferences of the students' parents; however, the teachers are the ones who have control over their classrooms. The principal's problem would then be to optimize the distribution of students so as to induce the most favorable incidence of student quality and course difficulty level. Alternatively, one could view the capacity as regarding class size: the school can only fit a fixed number of students in each class, and so the sender must optimize with the constraint that the vector of class sizes remains constant. Assuming that the number of students is large, this can be viewed as fixing the proportion of students in each class. We summarize these results in the following corollary.

**Corollary 6:** *Let  $c(\pi, \mathcal{S}) = 0$ .*

*(i) Suppose that the receiver has a capacity constraint on the number of actions that it can implement, so that it is costless for the receiver to implement up to  $N$  different actions, but for any  $n > N$ , it is infinitely costly for the receiver. Then if the sender's preferences are  $d$ -quasisupermodular, the optimal signal structure is monotone, but without full disclosure.*

*(ii) Suppose that the receiver has a fixed capacity  $\hat{\tau}(n)$  for each action  $a_n \in A$ , where  $n \in \{1, \dots, N\}$  such that  $\sum_{n=1}^N \hat{\tau}(n) = 1$  (i.e. each of the  $N$  actions taken must be taken with a fixed probability, but which action to choose is up to the receiver) Then if the sender's preferences are  $d$ -supermodular, the optimal signal structure is monotone, but without full disclosure.*

**Remark:** KG also provide sufficient conditions for which it is optimal to either reveal all information, or reveal nothing: the former will be optimal when the sender's payoff is convex in the posterior, while the latter is optimal when it is concave. A simple example demonstrates that these notions are distinct. Suppose that  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ , with  $\omega_1 < \omega_2 < \omega_3 < \omega_4$  and  $\mu_0$  assigning equal probability to each state. Convexity implies that the signal structure  $(\pi, \{s_1, s_2\})$  in which  $\text{supp}(\mu(\cdot|s_1)) = \{\omega_1, \omega_3\}$  and  $\text{supp}(\mu(\cdot|s_2)) = \{\omega_2, \omega_4\}$  is at least as good for the sender as revealing no information, whereas this is not necessarily the case in the presence of either  $d$ -supermodularity or  $d$ -quasisupermodularity. On the other hand, there is some monotone signal structure  $(\pi', \{s'_1, s'_2\})$  which is preferable to  $(\pi, \{s_1, s_2\})$  under

d-quasisupermodularity, which is not necessarily the case under convexity.  $\square$

The characterization of the signal structure as monotone greatly eases the search for the optimal structure, as one must now only search over a countable number of possible cutoff points to describe the intervals over which the signals are supported. This also allows the construction of first-order conditions in looking for the optimal structure. Suppose, as in case (ii) of Theorem 5, that the cost structure only depends on the number of signal realizations. Then one cannot be better off by switching the marginal type of some interval of support to another, and vice versa. We describe these incentive conditions in the following result.

**Corollary 7:** *Suppose that  $c(\pi, \mathcal{S})$  is solely a function of  $|\mathcal{S}|$  and the sender's payoffs are d-quasisupermodular. Let  $(\pi, \mathcal{S})$  be an optimal (monotone) signal structure. Consider  $s_1, s_2 \in \mathcal{S}$  such that  $\mu(\cdot|s_1) \succ_{\sigma} \mu(\cdot|s_2)$  and there does not exist  $s_3 \in \mathcal{S} \setminus (s_1, s_2)$  such that  $\mu(\cdot|s_1) \succ_{\sigma} \mu(\cdot|s_3) \succ_{\sigma} \mu(\cdot|s_2)$ . If*

$$\bar{\omega}_2 \equiv \sup\{\omega \in \text{supp}(\mu(\cdot|s_2))\} < \inf\{\omega \in \text{supp}(\mu(\cdot|s_1))\} \equiv \underline{\omega}_1$$

*then  $d(\mu(\cdot|s_1); \underline{\omega}_1) \geq d(\mu(\cdot|s_2); \underline{\omega}_1)$  and  $d(\mu(\cdot|s_1); \bar{\omega}_2) \leq d(\mu(\cdot|s_2); \bar{\omega}_2)$ . In particular, if  $\bar{\omega}_2 = \underline{\omega}_1 \equiv \omega^*$ , then  $d(\mu(\cdot|s_1); \omega^*) = d(\mu(\cdot|s_2); \omega^*)$ .*

We now present applications of our results.

## 6 Applications

### 6.1 Preference disagreement

One of the classic models of signalling is the cheap talk model of Crawford and Sobel (1982). In their model, they show that in any perfect Bayesian equilibrium, the sender's types are partitioned, so that a higher type sends a higher message. They also show that when types are uniformly distributed, and

$$u_S(a, \omega) = -(a - \omega - b)^2$$

$$u_R(a, \omega) = -(a - \omega)^2$$

then if it were possible for the sender to commit to reveal his type, he would receive a higher payoff than the equilibrium from their model. However, they do not explore whether this would indeed be optimal overall, nor do they analyze other distributions of types.

KG explored a more general class of sender preferences in the persuasion environment, and

showed that if the sender's preferences are given by

$$u_S(a, \omega) = -(a - (b_1 + b_2\omega))^2 \quad (12)$$

for  $[\underline{a}, \bar{a}] = \Omega = [0, 1]$ , then it is optimal to reveal all information if  $b_2 > \frac{1}{2}$ , and reveal nothing if  $b_2 < \frac{1}{2}$ .

We extend the analysis here further to look at the optimal structure when full separation is not possible. When the sender's preferences are given as in (12), we find that, since  $a^*(\mu) = E_\mu[\omega]$ ,

$$\begin{aligned} d(\mu; \omega') &= -(a^*(\mu) - (b_1 + b_2\omega'))^2 + 2(a^*(\mu) - \omega') \left( \int (a^*(\mu) - (b_1 + b_2)\omega) d\mu \right) \\ &= -(a^*(\mu) - (b_1 + b_2\omega'))^2 + 2(a^*(\mu) - \omega')(a^*(\mu) - b_1 - b_2a^*(\mu)) \end{aligned}$$

To check for d-supermodularity, we check the sum of the terms involving both  $a^*(\mu)$  and  $\omega'$ , which is

$$2b_2a^*(\mu)\omega' - 2a^*(\mu)\omega' + 2b_2a^*(\mu)\omega'$$

Because  $a^*(\mu)$  is increasing in  $\mu$ , this above expression is supermodular in  $(\mu, \omega')$  if  $b_2 > \frac{1}{2}$ , and submodular if  $b_2 < \frac{1}{2}$ . Thus it is optimal to reveal all information in the former case, and reveal none in the latter, as anticipated by KG. Furthermore, if costs are as in Theorem 5, it will be optimal to have a monotone signal structure.

To give a concrete example of a finite partition, let us look at an optimal partition when  $|\mathcal{S}| = 2$ ,  $A = [0, 1]$ ,  $b_2 = 1$ , and  $\mu_0$  induces the uniform distribution over  $[0, 1]$ . Consider the optimal signal structure  $(\pi, \{s_1, s_2\})$ , with  $\mu(\cdot|s_2) \succ_\sigma \mu(\cdot|s_1)$ . Since the signal structure will be monotone, there will exist  $\omega^*$  such that  $\mu(\cdot|s_1)$  induces the uniform distribution on  $[0, \omega^*]$ , while  $\mu(\cdot|s_2)$  induces the uniform distribution on  $[\omega^*, 1]$ . Thus

$$\begin{aligned} a^*(\mu(\cdot|s_1)) &= \frac{\omega^*}{2} \\ a^*(\mu(\cdot|s_2)) &= \frac{1 + \omega^*}{2} \end{aligned}$$

and so, by Corollary 7, it must be that

$$-\left(\frac{\omega^*}{2} - (b + \omega^*)\right)^2 + 2\left(\frac{\omega^*}{2} - \omega^*\right)\left(\frac{\omega^*}{2} - b - \frac{\omega^*}{2}\right) = -\left(\frac{1 + \omega^*}{2} - (b + \omega^*)\right)^2 + 2\left(\frac{1 + \omega^*}{2} - \omega^*\right)\left(\frac{1 + \omega^*}{2} - b - \frac{1 + \omega^*}{2}\right)$$

Simplifying this expression yields

$$-\left(\frac{\omega^*}{2} - (b + \omega^*)\right)^2 - 2b\left(-\frac{\omega^*}{2}\right) = -\left(\frac{1 + \omega^*}{2} - (b + \omega^*)\right)^2 - 2b\left(\frac{1 - \omega^*}{2}\right)$$

$$b = 2b\omega^* \implies \omega^* = \frac{1}{2}$$

Thus it is optimal to divide the interval  $[0, 1]$  into two equal pieces. The intuition for this is that, were the intervals to be uneven (say, with the upper interval being larger and the bias  $b$  being low), the concavity of the sender's payoff would imply that there would be a greater loss from high states (i.e.  $\omega$  near 1) being farther from the induced action of the receiver than there would a benefit of moving the higher states of the lower interval (i.e.  $\omega$  close to  $\omega^*$ ) closer to the induced action of the receiver.

## 6.2 Expected revenue maximization

A central issue in many economic environments is the effect that additional information has on the expected revenue raised from the actions of the receivers. Milgrom and Weber (1982) showed that in many auction environments, when bidders are symmetric, releasing information raises expected revenues when bidders' signals are affiliated. Similarly, Ottaviani and Prat (2001) showed that a monopolistic seller maximizes his expected revenue by releasing information when it is affiliated with the buyer's private signal.

We focus here on the model of Spence (1972), in which potential workers signal their quality/type  $\omega$  by undertaking costly education. Without commitment, the sender must choose a sufficiently high message to successfully convince the receiver that he is indeed the high type, so much so that a lower type would not choose to send so costly a message. The receiver then offers a wage (in our model,  $a$ ) equal to the conditional expectation of the sender's type.

Using the results in the previous subsection, we examine the optimal signal structure when the sender can commit to send specific messages for any true state  $\omega$ . In our analysis, we briefly abstract from the costly nature of the message technology, and just look at the payoffs from outcomes. Suppose that, as in the case of cheap talk, the receiver's utility function is a quadratic loss function of the form  $-(a - \omega)^2$ . Then

$$-\frac{\frac{\partial u_R}{\partial a}(a^*(\mu), \omega') \int \frac{\partial u_S}{\partial a}(a^*(\mu), \omega) d\mu}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu), \omega) d\mu} = a^*(\mu) - \omega$$

Thus

$$d(\mu; \omega') = a^*(\mu)$$

This function is both weakly supermodular and submodular in  $(\mu, \omega')$ , and so we would intuitively expect that revealing all (respectively, no) information is weakly optimal. To show this more formally, suppose that we perturb the payoffs of the sender by adding a term  $v_k(a, \omega) = -\frac{1}{2^k}(a - \omega)^2$ . Importantly,

$$\frac{\partial v_k}{\partial a \partial \omega}(a, \omega) > 0$$

and

$$\limsup_{k \rightarrow \infty} \sup_{a, \omega} \left| \frac{\partial v_k}{\partial a}(a^*(s), \omega) \right| = 0$$

Let  $u_S^k(a, \omega) = u_S(a, \omega) + v_k(a, \omega)$ . Indicating the marginal change in payoff from adding  $\epsilon \omega'$  to signal  $\mu$  when the sender has payoff  $u_S^k$  by  $d_k(\mu; \omega')$ , we find that

$$\begin{aligned} d_k(\mu; \omega') &= a^*(\mu) - \frac{1}{2^k}(a^*(\mu) - \omega)^2 - (a^*(\mu) - \omega')(1 - 2 \int (a^*(\mu) - \omega) d\mu) \\ &= a^*(\mu) - \frac{1}{2^k}(a^*(\mu) - \omega)^2 - (a^*(\mu) - \omega') \end{aligned}$$

This function is supermodular in  $(s, \omega)$ , and so by Theorem 4, it is optimal to reveal all information. Now consider any optimal signal structure  $(\pi, \mathcal{S})$  with associated  $\tau$ , and compare it with the structure which reveals all information. Then

$$\begin{aligned} 0 &\leq \int \int u_S(a^*(\mu), \omega) d\mu d\tau(\mu) - \int u_S(a^*(\omega), \omega) d\mu_0(\omega) \\ &\leq \left| \int \int u_S(a^*(\mu), \omega) d\mu d\tau(\mu) - \int \int u_S^k(a^*(\mu), \omega) d\mu d\tau(\mu) \right| \\ &\quad + \int \int u_S^k(a^*(\mu), \omega) d\mu d\tau(\mu) - \int u_S^k(a^*(\omega), \omega) d\mu_0(\omega) \\ &\quad + \left| \int u_S^k(a^*(\omega), \omega) d\mu_0(\omega) - \int u_S(a^*(\omega), \omega) d\mu_0(\omega) \right| \end{aligned}$$

Note that

$$\int \int u_S^k(a^*(\mu), \omega) d\mu d\tau(\mu) - \int u_S^k(a^*(\omega), \omega) d\mu_0(\omega) \leq 0$$

and the first and last terms are bounded by  $\frac{1}{2^k} \sup_{a, \omega} (a - \omega)^2$ . Since  $\Omega$  is compact and  $[a, \bar{a}]$  is bounded, the first and last terms vanish as  $k \rightarrow \infty$ , and so

$$\int \int u_S(a^*(s), \omega) d\mu(\omega|s) d\tau(s) - \int u_S(a^*(s), \omega) d\mu_0(\omega) \leq 0$$

which shows that revealing all information is optimal. A similar method in which the sign of

$v_k$  is flipped shows that it is also optimal to reveal no information. If, in the spirit of Spence's costly signalling approach, it is costly to generate more than one signal (corresponding to the effort in his model), then it will be strictly optimal to reveal no information. The intuition is simple: since the sender only cares about the expected wage, there is nothing that the sender can do to change the global expected value of  $\omega$ , which determines the overall expected wage. Therefore, there is no incentive either to hide or reveal information.

## References

- Bergemann, D., and Pesendorfer, M. (2007): "Information structures in optimal auctions." *Journal of Economic Theory*, 137, 580-609.
- Blackwell, D. (1951): "Comparison of experiments." *Proceedings of the second Berkeley symposium on mathematical statistics and probability*, 93-102.
- Crawford, V. P., and Sobel, J. (1982): "Strategic information transmission." *Econometrica*, 50, 1431-1451.
- Edlin, A. S., and Shannon, C. (1998): "Strict monotonicity in comparative statics." *Journal of Economic Theory*, 81, 201-219.
- Kamenica, E., and Gentzkow, M. (2011): "Bayesian persuasion." *American Economic Review*, 101, 2590-2615.
- (2014): "Costly persuasion." *American Economic Review (Papers and Proceedings)*, 104, 457-462.
- Kartik, N. (2009): "Strategic communication with lying costs." *Review of Economic Studies*, 76, 1359-1395.
- Mensch, J. (2015): "On the existence of monotone pure-strategy perfect-Bayesian equilibrium in dynamic games with complementarities." Mimeo.
- Milgrom, P., and Shannon, C. (1994): "Monotone comparative statics." *Econometrica*, 62, 157-180.
- Milgrom, P. R., and Weber, R. J. (1982): "A theory of auctions and competitive bidding." *Econometrica*, 50, 1089-1122.
- Myerson, R. B. (1981): "Optimal auction design." *Mathematics of Operations Research*, 6, 58-73.
- Ottaviani, M., and Prat, A. (2001): "The value of public information in monopoly." Econo-



metrica, 69, 1673-1683.

Spence, M. (1973): "Job market signaling." *Quarterly Journal of Economics*, 87, 355-374.

## Appendix: Proofs

**Proof of Lemma 1:** Suppose that  $\mu_1 \succeq_{FOSD} \mu_2$ . Since (strict) supermodularity is preserved by first-order stochastic dominance,  $\mu_1 \succeq_{FOSD} \mu_2$  if and only if, for any  $a_1 > a_2$ ,

$$\int u_R(a_1, \omega) d\mu_1 - \int u_R(a_2, \omega) d\mu_1 \geq \int u_R(a_1, \omega) d\mu_2 - \int u_R(a_2, \omega) d\mu_2$$

where the inequality is strict if  $\mu_1 \succ_{FOSD} \mu_2$ , by Theorem 7 of Milgrom and Shannon (1994). By Theorem 5 of Milgrom and Shannon (1994), this implies that  $a^*(\mu_1) \geq a^*(\mu_2)$  (strictly so if  $\mu_1 \succ_{FOSD} \mu_2$  and either  $a^*(\mu_1)$  or  $a^*(\mu_2)$  is located in the interior of  $A$  by Theorem 1 of Edlin and Shannon (1998)).  $\square$

**Proof of Lemma 2:** Clearly the payoff from any realization  $s \notin \mathcal{S}^*(a)$  is the same, so we only need to check that the payoff from  $s'$  is equal to that from  $\bigcup \{\mathcal{S}^*(a)\}$ . The payoff from the former is

$$\int_{\bigcup \{\mathcal{S}^*(a)\}} \int u_S(a^*(\mu(\cdot|s')), \omega) d\mu(\omega|s') d\tau(\mu(\cdot|s'))$$

while from the latter, it is

$$\int_{\bigcup \{\mathcal{S}^*(a)\}} \int u_S(a^*(\mu(\cdot|s)), \omega) d\mu(\omega|s) d\tau(\mu(\cdot|s))$$

Since  $a^*(\mu(\cdot|s)) = a$  for all  $s \in \mathcal{S}^*(a)$ , and action  $a$  is feasible conditional on  $s'$  being realized, we conclude that  $a^*(\mu(\cdot|s')) = a$ , and so the two signals generate the same payoffs.  $\square$

**Proof of Proposition 1:** To show (a) and (b), by Bayes' Theorem, there must exist  $s'$  such that  $\omega' \in \text{supp}(\mu(\cdot|s'))$ . Suppose that  $s' \neq s$ . By Lemma 2, we can assume that  $\mu(\cdot|s') \approx_\sigma \mu(\cdot|s)$ . If  $\mu(\cdot|s') \succ_\sigma \mu(\cdot|s)$ , then monotonicity would be violated since  $\omega_1 \in \text{supp}(\mu(\cdot|s))$ . On the other hand, if  $\mu(\cdot|s') \prec_\sigma \mu(\cdot|s)$ , then monotonicity would be violated by  $\omega_2$ . We therefore conclude that  $s' = s$ .

For (c), if  $\omega_1, \omega_2 \in \text{supp}(\mu(\cdot|s)) \cap \text{supp}(\mu(\cdot|s'))$ , where  $\omega_1 \neq \omega_2$ , monotonicity is violated since the higher of  $\{\omega_1, \omega_2\}$  is in the support of the lower distribution, while the lower is in the support of the higher distribution.  $\square$

**Proof of Theorem 2:** In the case where the sender's preferences are supermodular, consider

any optimal signal structure  $(\pi, \mathcal{S})$ . If  $|\mathcal{S}| = 1$ , then the signal structure is trivially monotone, and we are done. Thus we can assume without loss of generality that  $|\mathcal{S}| = 2$  and that there exist  $s_1, s_2 \in \mathcal{S}$  such that  $a^*(\mu(\cdot|s_1)) = a_1$  and  $a^*(\mu(\cdot|s_2)) = a_2$ . Suppose that the  $(\pi, \mathcal{S})$  is not monotone. Then there exist  $\omega_1 \in \text{supp}(\mu(\cdot|s_1))$  and  $\omega_2 \in \text{supp}(\mu(\cdot|s_2))$  such that  $\omega_1 > \omega_2$ . Without loss of generality, let  $\omega_1 = \sup\{\omega : \omega \in \text{supp}(\mu(\cdot|s_1))\}$  and  $\omega_2 = \inf\{\omega : \omega \in \text{supp}(\mu(\cdot|s_2))\}$ . For any  $\delta > 0$ , let  $s_2^\delta$  be the restriction of  $s_2$  to  $[\omega_2, \omega_2 + \delta)$ , and  $s_1^\delta$  be the restriction of  $s_1$  to  $[\omega_1 - \delta, \omega_1]$ . Let  $\omega^* \in [\omega_2, \omega_1]$  be the maximal value of  $\omega$  at which  $\tau(\mu(\cdot|s_2^{\omega^* - \omega_2})) \leq \tau(\mu(\cdot|s_1^{\omega_1 - \omega^*}))$ . Then for some  $\nu_i \geq 0$ , we can define  $\bar{\mu}_1 = \mu(\cdot|s_1^{\omega_1 - \omega^*}) + \nu_1 \omega^*$  and  $\bar{\mu}_2 = \mu(\cdot|s_2^{\omega^* - \omega_2}) + \nu_2 \omega^*$  such that  $\tau(\bar{\mu}_1) = \tau(\bar{\mu}_2)$ .

Since  $u_S$  is supermodular in  $(a, \omega)$ , it follows that

$$d(\mu(\cdot|s_2); \bar{\mu}_1) - d(\mu(\cdot|s_2); \bar{\mu}_2) \geq d(\mu(\cdot|s_1); \bar{\mu}_1) - d(\mu(\cdot|s_1); \bar{\mu}_2) \quad (13)$$

as long as the induced action  $a^*$  does not change. Note that for all  $\eta \in (0, 1]$ ,

$$\mu(\cdot|s_2) - \eta \bar{\mu}_2 + \eta \bar{\mu}_1 \succ_{FOSD} \mu(\cdot|s_2) \quad (14)$$

and so by single-crossing this implies that

$$a^*(\mu(\cdot|s_2) - \eta \bar{\mu}_2 + \eta \bar{\mu}_1) \geq a^*(\mu(\cdot|s_2)) = a_2$$

Similarly, since

$$\mu(\cdot|s_1) \succ_{FOSD} \mu(\cdot|s_1) + \eta \bar{\mu}_2 - \eta \bar{\mu}_1 \quad (15)$$

by single-crossing,

$$a^*(\mu(\cdot|s_1) + \eta \bar{\mu}_2 - \eta \bar{\mu}_1) \leq a^*(\mu(\cdot|s_1)) = a_1$$

Moreover, by (14) and (15), (13) remains true if we replace  $\mu(\cdot|s_2)$  with  $\mu(\cdot|s_2) - \eta \bar{\mu}_2 + \eta \bar{\mu}_1$  and  $\mu(\cdot|s_1)$  with  $\mu(\cdot|s_1) + \eta \bar{\mu}_2 - \eta \bar{\mu}_1$ . By the fundamental theorem of calculus, it follows that

$$\begin{aligned} & \tau(\mu(\cdot|s_2)) \int u_S(a_2, \omega) d(\mu(\cdot|s_2) - \bar{\mu}_2 + \bar{\mu}_1) + \tau(\mu(\cdot|s_1)) \int u_S(a_1, \omega) d(\mu(\cdot|s_1) + \bar{\mu}_2 - \bar{\mu}_1) \\ &= \tau(\mu(\cdot|s_2)) \int u_S(a_2, \omega) d\mu(\cdot|s_2) + \tau(\mu(\cdot|s_1)) \int u_S(a_1, \omega) d\mu(\cdot|s_1) \\ &+ \int_0^1 [d(\mu(\cdot|s_2) - \eta \bar{\mu}_2 + \eta \bar{\mu}_1; \bar{\mu}_1) - d(\mu(\cdot|s_2) - \eta \bar{\mu}_2 + \eta \bar{\mu}_1; \bar{\mu}_2)] d\eta \\ &- \int_0^1 [d(\mu(\cdot|s_1) + \eta \bar{\mu}_2 - \eta \bar{\mu}_1; \bar{\mu}_1) - d(\mu(\cdot|s_1) + \eta \bar{\mu}_2 - \eta \bar{\mu}_1; \bar{\mu}_2)] d\eta \end{aligned}$$

$$\geq \tau(\mu(\cdot|s_2)) \int u_S(a_2, \omega) d\mu(\cdot|s_2) + \tau(\mu(\cdot|s_1)) \int u_S(a_1, \omega) d\mu(\cdot|s_1)$$

Hence adding  $\bar{\mu}_1 - \bar{\mu}_2$  to  $\mu(\cdot|s_2)$  and adding  $\bar{\mu}_2 - \bar{\mu}_1$  to  $\mu(\cdot|s_1)$  results in a weak improvement over  $(\pi, \mathcal{S})$ . Note that the resultant signal structure is now monotone, as there is no  $\omega < \omega^*$  in the support of  $\mu(\cdot|s_2) + \bar{\mu}_1 - \bar{\mu}_2$ , and no  $\omega > \omega^*$  in the support of  $\mu(\cdot|s_1) + \bar{\mu}_2 - \bar{\mu}_1$ . Since  $(\pi, \mathcal{S})$  was optimal, there also exists a monotone signal structure which is optimal.  $\square$

**Proof of Corollary 3:** Suppose that neither (a) nor (b) hold. Then for any  $\omega \in \Omega$  and for small enough  $\epsilon$ ,  $a^*(\mu(\cdot|s_i) + \epsilon\omega) = a^*(\mu(\cdot|s_i))$ . For each of the signals  $s_1, s_2$ , we define the signal realizations  $s_1^\delta, s_2^\delta$  which restrict  $s_1, s_2$  to the open interval within  $\delta$  of  $\underline{\omega}_2$  and  $\bar{\omega}_1$ , respectively. For every  $\delta > 0$ , if  $(\pi, \mathcal{S})$  is optimal, then  $d(\mu(\cdot|s_1); \mu(\cdot|s_1^\delta)) \geq d(\mu(\cdot|s_2); \mu(\cdot|s_1^\delta))$  and  $d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta)) \geq d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta))$ . Note that as  $\delta \rightarrow 0$ ,  $\mu(\cdot|s_1^\delta) \rightarrow \mu(\cdot|\bar{\omega}_1)$  in the weak-\* topology. Hence

$$\lim_{\delta \rightarrow 0} d(\mu(\cdot|s_i); \mu(\cdot|s_1^\delta)) = u_S(a_i, \bar{\omega}_1)$$

which is precisely the value of  $d(\mu(\cdot|s), \epsilon\bar{\omega}_1)$ . Similarly, we find that

$$\lim_{\delta \rightarrow 0} d(\mu(\cdot|s_i); \mu(\cdot|s_2^\delta)) = u_S(a_i, \underline{\omega}_2)$$

Suppose that  $u_S(a_2, \underline{\omega}_2) < u_S(a_1, \underline{\omega}_2)$ . Then for sufficiently small  $\delta$ ,  $d(\mu(\cdot|s_1); \mu(\cdot|s_2^\delta)) > d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta))$ . Thus there would be a feasible improvement from shifting a weight of  $\epsilon$  of  $\mu(\cdot|s_2^\delta)$  from  $\mu(\cdot|s_2)$  to  $\mu(\cdot|s_1)$ , and so  $(\pi, \mathcal{S})$  could not be optimal. Similarly, if  $u_S(a_2, \bar{\omega}_1) > u_S(a_1, \bar{\omega}_1)$ , then there exists an improvement by shifting (for small enough  $\delta$  and  $\epsilon$ ) a weight of  $\epsilon$  of  $\mu(\cdot|s_1^\delta)$  from  $\mu(\cdot|s_1)$  to  $\mu(\cdot|s_2)$ .  $\square$

**Proof of Lemma 3:** For the receiver, the change in payoff from adding  $\epsilon \cdot \mu'$  to  $\mu$ , for a given choice of  $a \in [\underline{a}, \bar{a}]$ , is

$$\int u_R(a, \omega) d(\mu + \epsilon \cdot \mu') - \int u_R(a, \omega) d\mu = \frac{\epsilon\tau(\mu')}{\tau(\mu) + \epsilon\tau(\mu')} \cdot \int u_R(a, \omega) d\mu'$$

Hence the receiver's problem is now

$$\max_a \frac{\tau(\mu)}{\tau(\mu) + \epsilon\tau(\mu')} \int u_R(a, \omega) d\mu + \frac{\epsilon\tau(\mu')}{\tau(\mu) + \epsilon\tau(\mu')} \cdot \int u_R(a, \omega) d\mu'$$

and so the first-order condition becomes

$$\frac{\tau(\mu)}{\tau(\mu) + \epsilon\tau(\mu')} \int \frac{\partial u_R}{\partial a}(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu(\omega|s) + \frac{\epsilon\tau(\mu')}{\tau(\mu) + \epsilon\tau(\mu')} \cdot \int \frac{\partial u_R}{\partial a}(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu' = 0$$

Note that the second term of the above equation vanishes as  $\epsilon \rightarrow 0$ . By Berge's maximum

theorem,  $\lim_{\epsilon \rightarrow 0} a^*(\mu + \epsilon \cdot \mu') = a^*(\mu)$ . Combining these cases and dividing by  $\epsilon\tau(\mu')$ , we have

$$\begin{aligned} \frac{\tau(\mu)}{\epsilon\tau(\mu')(\tau(\mu) + \epsilon\tau(\mu'))} \int \frac{\partial u_R}{\partial a}(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu + \frac{\epsilon\tau(\mu')}{\epsilon\tau(\mu')(\tau(\mu) + \epsilon\tau(\mu'))} \cdot \int \frac{\partial u_R}{\partial a}(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu' \\ - \frac{1}{\epsilon\tau(\mu')} \int \frac{\partial u_R}{\partial a}(a^*(\mu), \omega) d\mu = 0 \end{aligned}$$

Taking the limit as  $\epsilon \rightarrow 0$ ,

$$\begin{aligned} -\frac{1}{\tau(\mu)} \int \frac{\partial u_R}{\partial a}(a^*(\mu), \omega) d\mu' = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon\tau(\mu')} \left\{ \int \frac{\partial u_R}{\partial a}(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu - \int \frac{\partial u_R}{\partial a}(a^*(\mu), \omega) d\mu \right\} \\ = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon\tau(\mu')} \int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu), \omega) d\mu \cdot (a^*(\mu + \epsilon \cdot \mu') - a^*(\mu)) \end{aligned} \quad (16)$$

and so

$$\lim_{\epsilon \rightarrow 0} \frac{a^*(\mu + \epsilon \cdot \mu') - a^*(\mu)}{\epsilon\tau(\mu')} = \frac{-\frac{1}{\tau(\mu)} \int \frac{\partial u_R}{\partial a}(a^*(\mu), \omega) d\mu'}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu), \omega) d\mu}$$

Substituting this into the sender's problem gives

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} D(\mu, \epsilon \cdot \mu') = \\ \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon\tau(\mu')} \left\{ \tau(\mu) \left[ \int \frac{\partial u_S}{\partial a}(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu \right] \cdot [a^*(\mu + \epsilon \cdot \mu') - a^*(\mu)] + \epsilon\tau(\mu') \cdot \int u_S(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu' \right\} \\ = -\frac{\int \frac{\partial u_S}{\partial a}(a^*(\mu), \omega) d\mu \cdot \int \frac{\partial u_R}{\partial a}(a^*(\mu), \omega) d\mu'}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu), \omega) d\mu} + \int u_S(a^*(\mu), \omega) d\mu' \end{aligned}$$

Note that if  $a^*(\mu) = a^*(\mu + \epsilon\omega) \in \{a, \bar{a}\}$  for small enough  $\epsilon > 0$ , then the first term vanishes, and we are left with  $\lim_{\epsilon \rightarrow 0} D(\mu, \epsilon \cdot \mu') = \int u_S(a^*(\mu), \omega) d\mu'$ .

Since this is true regardless of the sign of  $\epsilon$ ,

$$\lim_{\epsilon \rightarrow 0^+} D(\mu, \epsilon \cdot \mu') = -\lim_{\epsilon \rightarrow 0^-} D(\mu, -\epsilon \cdot \mu')$$

whenever such a subtraction is feasible.  $\square$

**Proof of Theorem 4:** In both cases, we first proceed for the case in which  $\Omega$  is finite, and then extend to the infinite case through successive finite approximations of  $\Omega$ . By Proposition 4 in the online appendix of KG, this implies that  $|\mathcal{S}|$  is finite, and so  $\tau(\mu(\cdot|s)) > 0$  for all  $s \in \mathcal{S}$ .

(i) Suppose that  $(\pi, \mathcal{S})$  is optimal and there exists some signal  $s \in \mathcal{S}$  such that  $\mu(\omega|s) \notin \{0, 1\}$  for some  $\omega$ . We show that there exists an improvement over  $(\pi, \mathcal{S})$ . Consider the alteration

$(\pi', \mathcal{S}')$  which is identical to  $(\pi, \mathcal{S})$  except that it duplicates  $s$  into  $s_1$  and  $s_2$  such that  $\mu(\omega|s) = \mu(\omega|s_1) = \mu(\omega|s_2)$  and  $\tau(\mu(\cdot|s_1)) = \tau(\mu(\cdot|s_2)) = \frac{1}{2}\tau(\mu(\cdot|s))$ . For shorthand, we write  $\mu(\cdot|s_i) = \mu_i$ .

Suppose that  $a^*(\mu(\cdot|s)) \in (\underline{a}, \bar{a})$ . Let  $\underline{\omega} \equiv \min\{\omega : \omega \in \text{supp}(\mu(\cdot|s))\}$  and  $\bar{\omega} \equiv \max\{\omega : \omega \in \text{supp}(\mu(\cdot|s))\}$ . Then for  $\nu, \eta \geq 0$  (with at least one strictly so),

$$\mu_2 + \nu\bar{\omega} - \eta\underline{\omega} \succ_{\sigma} \mu_1 - \nu\bar{\omega} + \eta\underline{\omega} \quad (17)$$

Moreover, by d-quasisupermodularity, either

$$d(\mu_2 + \nu\bar{\omega} - \eta\underline{\omega}; \bar{\omega}) - d(\mu_1 - \nu\bar{\omega} + \eta\underline{\omega}; \bar{\omega}) > 0 \quad (18)$$

or

$$d(\mu_1 - \nu\bar{\omega} + \eta\underline{\omega}; \underline{\omega}) - d(\mu_2 + \nu\bar{\omega} - \eta\underline{\omega}; \underline{\omega}) > 0 \quad (19)$$

Thus for any  $\beta, \gamma \in (0, \frac{1}{2})$ , by the fundamental theorem of calculus, there exist  $\frac{1}{2} \geq \nu \geq \beta$  and  $\frac{1}{2} \geq \eta \geq \gamma$  (with at least one strict) such that

$$\begin{aligned} & \tau(\mu_1 - \nu\bar{\omega} + \eta\underline{\omega}) \int u_S(a^*(\mu_1 - \nu\bar{\omega} + \eta\underline{\omega}), \omega) d(\mu_1 - \nu\bar{\omega} + \eta\underline{\omega})(\omega) \\ & + \tau(\mu_2 + \nu\bar{\omega} - \eta\underline{\omega}) \int u_S(a^*(\mu_2 + \nu\bar{\omega} - \eta\underline{\omega}), \omega) d(\mu_2 + \nu\bar{\omega} - \eta\underline{\omega})(\omega) \\ & > \tau(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega}) \int u_S(a^*(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega}), \omega) d(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega})(\omega) \\ & + \tau(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega}) \int u_S(a^*(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega}), \omega) d(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega})(\omega) \end{aligned} \quad (20)$$

and so we can without loss of generality set  $\max\{\nu, \eta\} = \frac{1}{2}$ , as there will exist an improvement by incrementing the weight on either  $\bar{\omega}$  or  $\underline{\omega}$  for the cases where  $\nu < \frac{1}{2}$  and  $\eta < \frac{1}{2}$  since the payoffs will be continuous in  $(\nu, \eta)$  by Lemma 3. Note that  $(\beta, \gamma)$  were arbitrary, and that

$$\begin{aligned} & \lim_{(\beta, \gamma) \rightarrow (0, 0)} \tau(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega}) \int u_S(a^*(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega}), \omega) d(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega})(\omega) \\ & + \tau(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega}) \int u_S(a^*(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega}), \omega) d(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega})(\omega) \\ & = \tau(\mu) \int u_S(a^*(\mu), \omega) d\mu(\omega) \end{aligned} \quad (21)$$

Hence we can take a sequence of values  $\{(\beta_k, \gamma_k, \nu_k, \eta_k)\}_{k=1}^{\infty}$  such that  $(\beta_k, \gamma_k) \rightarrow (0, 0)$  and

(without loss of generality)  $\nu_k = \frac{1}{2}$  and  $\eta_k \leq \frac{1}{2}$  for all  $k$  such that each  $(\beta_k, \gamma_k, \nu_k, \eta_k)$  satisfies (19). Since the interval  $[0, \frac{1}{2}]$  is compact, there exists a convergent subsequence of  $\{\eta_k\}_{k=1}^\infty$  to some  $\eta_\infty \in [0, \frac{1}{2}]$  (without loss of generality, the sequence itself). By Lemma 3, the payoff to the sender is continuous in  $(\beta, \gamma, \nu)$ , and so in the limit, we find from (20) that

$$\begin{aligned} & \tau(\mu_1 - \frac{1}{2}\bar{\omega} + \eta_\infty\underline{\omega}) \int u_S(a^*(\mu_1 - \frac{1}{2}\bar{\omega} + \eta_\infty\underline{\omega}), \omega) d(\mu_1 - \frac{1}{2}\bar{\omega} + \eta_\infty\underline{\omega})(\omega) \\ & + \tau(\mu_2 + \frac{1}{2}\bar{\omega} - \eta_\infty\underline{\omega}) \int u_S(a^*(\mu_2 + \frac{1}{2}\bar{\omega} - \eta_\infty\underline{\omega}), \omega) d(\mu_2 + \frac{1}{2}\bar{\omega} - \eta_\infty\underline{\omega})(\omega) \\ & \geq \tau(\mu) \int u_S(a^*(\mu), \omega) d\mu(\omega) \end{aligned} \quad (22)$$

Moreover,

$$\begin{aligned} & \mu_2 + \frac{1}{2}\bar{\omega} - \eta_\infty\underline{\omega} \succ_{FOSD} \mu_2 \\ & \sim_{FOSD} \mu_1 \succ_{FOSD} \mu_1 - \frac{1}{2}\bar{\omega} + \eta_\infty\underline{\omega} \end{aligned}$$

and so  $\mu_2 + \frac{1}{2}\bar{\omega} - \eta_\infty\underline{\omega} \succ_\sigma \mu_1 - \frac{1}{2}\bar{\omega} + \eta_\infty\underline{\omega}$  by Lemma 1. There then exists a local improvement from either adding small enough weight  $\epsilon$  of  $\bar{\omega}$  from  $\mu_1 - \frac{1}{2}\bar{\omega} + \eta_\infty\underline{\omega}$  to  $\mu_2 + \frac{1}{2}\bar{\omega} - \eta_\infty\underline{\omega}$ , or small enough  $\epsilon$  of  $\underline{\omega}$  from  $\mu_2 + \frac{1}{2}\bar{\omega} - \eta_\infty\underline{\omega}$  to  $\mu_1 - \frac{1}{2}\bar{\omega} + \eta_\infty\underline{\omega}$ , due to inequalities (18) and (19), respectively. Hence there exists an improvement over  $(\pi, \mathcal{S})$ , and so it cannot be optimal.

For the case in which  $a^*(\mu(\cdot|s)) \in \{\underline{a}, \bar{a}\}$ , the above method does not quite work, since it may be that  $a^*(\mu_2 + \frac{1}{2}\bar{\omega} - \eta_\infty\underline{\omega}) = a^*(\mu_1 - \frac{1}{2}\bar{\omega} + \eta_\infty\underline{\omega})$ . To circumvent this issue, we take a closer look at the support of  $\mu(\cdot|s)$ . Suppose that  $a^*(\mu(\cdot|s)) = \bar{a}$ . If, for all  $\omega \in \text{supp}(\mu(\cdot|s))$ ,  $a^*(\omega) = \bar{a}$ , then splitting  $s$  into separate signals for each state does not change the payoff, and so it will again be optimal to reveal all information. Otherwise, by supermodularity of  $u_R$  in  $(a, \omega)$ , it must be that  $a^*(\underline{\omega}) < \bar{a}$ . Since  $a^*$  is continuous in  $\mu$ , there exists  $\beta^* \in (0, 1]$  for which  $a^*(\beta^* \cdot \mu(\cdot|s) - (1 - \beta^*)\underline{\omega}) = \bar{a}$ , but for all  $\beta \in (0, \beta^*)$ ,  $a^*(\beta \cdot \mu(\cdot|s) - (1 - \beta)\underline{\omega}) < \bar{a}$ . It will then follow that for  $\eta, \nu \geq 0$  (with one strict),

$$\beta^* \cdot \mu(\cdot|s) - (1 - \beta^*)\underline{\omega} \succ_\sigma \beta^* \cdot \mu(\cdot|s) - (1 - \beta^*)\underline{\omega} + \eta\underline{\omega} - \nu\bar{\omega}$$

Thus we can split  $s$  into two signals  $s_1, s_2$  such that, for some  $c > 0$ ,

$$\mu_1 = c\beta^* \cdot \mu(\cdot|s) - c(1 - \beta^*)\underline{\omega}$$

and

$$\mu_2 = \mu(\cdot|s) - (c\beta^* \cdot \mu(\cdot|s) - c(1 - \beta^*)\underline{\omega})$$

This will not change the payoff, as in both cases  $a^*(\mu(\cdot|s_i)) = \bar{a}$ . The argument then proceeds as above, where now  $d(\mu_1; \omega)$  is given by (5), but  $d(\mu_2; \omega) = u_S(\bar{a}_2, \omega)$ . The case where  $a^*(\mu(\cdot|s)) = \underline{a}$  is symmetric, and so is omitted.

(ii) Suppose that  $(\pi, \mathcal{S})$  is optimal, and there exist  $s_1, s_2 \in \mathcal{S}$  such that (without loss of generality)  $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_2)$ ; define these respective distributions as  $\mu_1$  and  $\mu_2$ , respectively. By Lemma 1,  $\succ_\sigma$  is FOSD-consistent, and so there must exist  $\omega_1 \in \text{supp}(\mu(\cdot|s_1))$  and  $\omega_2 \in \text{supp}(\mu(\cdot|s_2))$  such that  $\omega_1 > \omega_2$ . By d-quasisubmodularity, either

$$d(\mu_2; \omega_1) - d(\mu_1; \omega_1) > 0$$

or

$$d(\mu_1; \omega_2) - d(\mu_2; \omega_2) > 0$$

In either case, an improvement exists, and so  $(\pi, \mathcal{S})$  can only be optimal if for all  $s_1, s_2 \in \mathcal{S}$ ,  $\mu_1 \sim_\sigma \mu_2$ . By Lemma 2, merging all such realizations generates the same payoff. Hence the signal structure which generates the posterior beliefs identical to the prior is optimal.

Now consider the case where  $\Omega$  is infinite. Consider any information structure  $(\pi, \mathcal{S})$ . Let  $\bar{\omega} = \sup\{\omega \in \Omega\}$  and  $\underline{\omega} = \inf\{\omega \in \Omega\}$ . Consider a sequence of approximating state spaces  $\{\Omega_k\}$  defined by dividing the interval  $[\underline{\omega}, \bar{\omega}]$  into subintervals of length  $\frac{\bar{\omega} - \underline{\omega}}{2^k}$ , and so that all states  $\omega$  in the same interval  $I_k^n$  (indexed by  $n \in \{1, \dots, 2^k\}$ ) are mapped to the same value,  $\omega_k^n$ ; the prior distribution  $\mu_0^k$  is then defined by assigning to  $\omega_k^n$  the probability

$$\mu_0^k(\omega_k^n) = \int_{I_k^n} \mu_0(\omega)$$

where  $I_k^n = [\frac{n(\bar{\omega} - \underline{\omega}) + \underline{\omega}}{2^k}, \frac{(n+1)(\bar{\omega} - \underline{\omega}) + \underline{\omega}}{2^k})$ , with the interval closed for  $n = 2^k$ . We correspondingly approximate  $(\pi, \mathcal{S})$  by  $(\pi_k, \mathcal{S}_k)$  in which for any  $s \in \mathcal{S}$ , there is a signal realization  $s_k \in \mathcal{S}_k$  such that

$$\mu^k(\omega_k^n | s_k) = \int_{I_k^n} \mu(\omega | s)$$

and

$$d\tau(\mu^k(\cdot | s_k)) = d\tau(\mu(\cdot | s))$$

Since  $\pi$  is measurable, this construction is well-defined. Note that this may lead to some duplicates, i.e. there may be multiple values of  $s_k$  which lead to the same  $\mu^k$ ; if so, these realizations can be merged by Lemma 2 without changing the payoff from the resultant signal structure.

As we have already shown, when the sender's payoffs are d-quasisubmodular it is optimal to

reveal no information when the state space is  $\Omega_k$  with prior  $\mu_0^k$ , and so

$$\int u_S(a^*(\mu_0^k), \omega) d\mu_0^k(\omega) \geq \int \int u_S(a^*(\mu^k), \omega) d\mu^k(\omega) d\tau(\mu^k)$$

Since  $u_R$  is continuous and both  $\mu^k \rightarrow \mu$  and  $\mu_0^k \rightarrow \mu_0$  in the weak-\* topology, it follows that  $a^*(\mu^k) \rightarrow a^*(\mu)$  and  $a^*(\mu_0^k) \rightarrow a^*(\mu_0)$  by Berge's maximum theorem. By the continuity of  $u_S$ , this implies that

$$\int u_S(a^*(\mu_0), \omega) d\mu_0(\omega) \geq \int \int u_S(a^*(\mu), \omega) d\mu(\omega) d\tau(\mu)$$

and so the signal structure which reveals no information provides at least as good a payoff to the sender as  $(\pi, \mathcal{S})$ .

Similarly, when the sender's payoffs are d-quasisupermodular, it is optimal to reveal all information, and so

$$\int u_S(a^*(\omega), \omega) d\mu_0^k(\omega) \geq \int \int u_S(a^*(\mu^k), \omega) d\mu^k(\omega) d\tau(\mu^k)$$

Taking the limit as  $\mu^k \rightarrow \mu$  and  $\mu_0^k \rightarrow \mu_0$  yields

$$\int u_S(a^*(\omega), \omega) d\mu_0(\omega) \geq \int \int u_S(a^*(\mu), \omega) d\mu(\omega) d\tau(\mu)$$

and so it will be optimal to reveal all information.  $\square$

**Proof of Theorem 5:** For both (i) and (ii), we show that there is a local improvement for any  $(\pi, \mathcal{S})$  that is not monotone. Without loss of generality, by Lemma 2 we can assume that for all  $s_1, s_2 \in \mathcal{S}$ , either  $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_2)$  or  $\mu(\cdot|s_2) \succ_\sigma \mu(\cdot|s_1)$ . Now suppose that  $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_2)$ , but there exists  $\omega_2 \in \text{supp}(\mu(\cdot|s_2))$  and  $\omega_1 \in \text{supp}(\mu(\cdot|s_1))$  such that  $\omega_2 > \omega_1$ .

To show that an optimal signal structure exists, note that since  $\lim_{|\mathcal{S}| \rightarrow \infty} c(\pi, \mathcal{S}) = \infty$ , there must exist some  $N^*$  such that optimal structure must have  $|\mathcal{S}| < N^*$ . We then check for the optimal signal structure conditional on  $|\mathcal{S}| = N < N^*$ , and optimize over  $N$ . For fixed  $N$ , the function

$$\sum_{s \in \mathcal{S}} \tau(\mu(\cdot|s)) \int_{\Omega} u_S(a^*(\mu), \omega) d\mu(\omega|s) - c(\pi, \mathcal{S})$$

is continuous in  $(\mu, \tau(\mu))$ , since  $a^*(\mu)$  is continuous in  $\mu$  and  $c(\cdot, \cdot)$  is continuous in  $\{\tau(\mu)\}$ . The set of such  $\mu$  is compact under the weak-\* topology, while  $\{\tau(\mu)\}$  is an  $N$ -dimensional Euclidean vector contained in the  $N$ -dimensional simplex (a compact set, as long as we allow



duplicate signal realization that induce posterior  $\mu$ ), and so an optimal  $(\pi, \mathcal{S})$  exists.

To show that a monotone signal structure is optimal, fix  $\delta < \frac{1}{2}(\omega_2 - \omega_1)$ , and define  $s_1^\delta$  to be the restriction of  $s_1$  to  $[\omega_1 - \delta, \omega_1 + \delta]$ , and  $s_2^\delta$  to be the restriction of  $s_2$  to  $[\omega_2 - \delta, \omega_2 + \delta]$ .

(i) By d-supermodularity, for all  $\omega'_2 \in \text{supp}(\mu(\cdot|s_2^\delta))$  and  $\omega'_1 \in \text{supp}(\mu(\cdot|s_1^\delta))$ , we have that

$$d(\mu(\cdot|s_1); \omega'_2) - d(\mu(\cdot|s_1); \omega'_1) > d(\mu(\cdot|s_2); \omega'_2) - d(\mu(\cdot|s_2); \omega'_1)$$

and so

$$d(\mu(\cdot|s_1); \mu(\cdot|s_2^\delta)) - d(\mu(\cdot|s_1); \mu(\cdot|s_1^\delta)) > d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta)) - d(\mu(\cdot|s_2); \mu(\cdot|s_1^\delta))$$

Hence there exists an improvement (excluding costs) by adding  $(\epsilon \frac{\tau(\mu(\cdot|s_1^\delta))}{\tau(\mu(\cdot|s_2^\delta))}) \cdot \mu(\cdot|s_2^\delta)$  (where  $\epsilon$  is sufficiently small) to  $\mu(\cdot|s_1)$  and subtracting an equal weight from  $\mu(\cdot|s_2)$ , while subtracting  $\epsilon \cdot \mu(\cdot|s_1^\delta)$  from  $\mu(\cdot|s_1)$  and adding it to  $\mu(\cdot|s_2)$ . Moreover,

$$\tau(\mu(\cdot|s_1) + \epsilon \frac{\tau(\mu(\cdot|s_1^\delta))}{\tau(\mu(\cdot|s_2^\delta))} \cdot \mu(\cdot|s_2^\delta) - \epsilon \mu(\cdot|s_1^\delta)) = \tau(\mu(\cdot|s_1))$$

$$\tau(\mu(\cdot|s_2) + \epsilon \cdot \mu(\cdot|s_1^\delta) - \epsilon \frac{\tau(\mu(\cdot|s_1^\delta))}{\tau(\mu(\cdot|s_2^\delta))} \cdot \mu(\cdot|s_2^\delta)) = \tau(\mu(\cdot|s_2))$$

and so the vector of probabilities  $\{\tau\}$  (and hence  $c$ ) is unaffected. Therefore  $(\pi, \mathcal{S})$  is not optimal.

(ii) Similarly, if the sender's payoffs are d-quasisupermodular, either

$$d(\mu(\cdot|s_1); \omega'_2) - d(\mu(\cdot|s_2); \omega'_2) > 0$$

or

$$d(\mu(\cdot|s_2); \omega'_1) - d(\mu(\cdot|s_2); \omega'_1) > 0$$

and so either

$$d(\mu(\cdot|s_1); \mu(\cdot|s_2^\delta)) - d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta)) > 0$$

or

$$d(\mu(\cdot|s_2); \mu(\cdot|s_1^\delta)) - d(\mu(\cdot|s_2); \mu(\cdot|s_1^\delta)) > 0$$

In either case, ignoring costs, there exists a local improvement, in the former case by transferring sufficiently small  $\epsilon$  weight of  $\mu(\cdot|s_2^\delta)$  from  $\mu(\cdot|s_2)$  to  $\mu(\cdot|s_1)$ , or in the latter case from transferring  $\epsilon$  of  $\mu(\cdot|s_1^\delta)$  from  $\mu(\cdot|s_1)$  to  $\mu(\cdot|s_2)$ . Let the new signal structure be  $(\pi', \mathcal{S}')$ . Since  $|\mathcal{S}'| = |\mathcal{S}|$ , the cost  $c$  is unaffected. Hence  $(\pi, \mathcal{S})$  is not optimal.  $\square$

**Proof of Corollary 6:** We show that, in both cases, the problem reduces to that described in Theorem 3.

(i) By Lemma 2,  $|\mathcal{S}| \leq N$ ; moreover, if it is optimal to set  $|\mathcal{S}| < N$ , it is also optimal to set  $|\mathcal{S}| = N$ , as one can always construct duplicate signals with the same posterior. We can therefore rewrite the problem as one in which

$$c(\pi, \mathcal{S}) = \begin{cases} 0, & |\mathcal{S}| \leq N \\ \infty, & \text{otherwise} \end{cases}$$

(i) As shown in part (i), the signal structure must have  $|\mathcal{S}| \leq N$ . Let the optimal signal structure then be  $(\pi, \mathcal{S})$  (such a structure will exist because the set of signal structures with  $|\mathcal{S}| \leq N$  is compact). Suppose that the set of actions taken is  $\{a_n^*\}_{n=1}^N$ , and let the posterior distribution of types conditional on each  $a_n^*$  be  $\nu(\cdot|a_n^*)$  (this may be achieved through randomization by the receiver). Then the signal structure  $(\pi', \mathcal{S}')$  in which  $\mathcal{S}' = \{s_n\}_{n=1}^N$ ,  $\mu(\cdot|s_n) = \nu(\cdot|a_n^*)$ , and  $\tau(\mu(\cdot|s_n)) = \hat{\tau}(n)$  will also be optimal. To see this, note that the receiver to take the same set of actions  $\{a_n^*\}_{n=1}^N$ , as any other set of actions  $\{\hat{a}_n^*\}_{n=1}^N$  with associated posterior distributions of types  $\hat{\nu}(\cdot|\hat{a}_n^*)$  that would be optimal under  $(\pi', \mathcal{S}')$  would also be feasible under  $(\pi, \mathcal{S})$  since  $\hat{\nu}(\cdot|\hat{a}_n^*) = \sum_{n=1}^N \alpha_n \nu(\cdot|a_n^*)$  for some  $\{\alpha_n\}_{n=1}^N$  such that  $\sum_{n=1}^N \alpha_n \tau(\mu(\cdot|s_n)) = \hat{\tau}(n)$ . Thus the problem of the sender can be rewritten as one in which

$$c(\pi, \mathcal{S}) = \begin{cases} 0, & \{\tau(\mu)\} = \{\hat{\tau}(n)\} \\ \infty, & \text{otherwise} \end{cases}$$

In both cases, the result then follows from Theorem 5.  $\square$

**Proof of Corollary 7:** For each of the signals  $s_1, s_2$ , we define the signal realizations  $s_1^\delta, s_2^\delta$  which restrict  $s_1, s_2$  to within  $\delta$  of  $\underline{\omega}_1$  and  $\bar{\omega}_2$ , respectively. For every  $\delta > 0$ , if  $(\pi, \mathcal{S})$  is optimal, then  $d(\mu(\cdot|s_1); \mu(\cdot|s_1^\delta)) \geq d(\mu(\cdot|s_2); \mu(\cdot|s_1^\delta))$  and  $d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta)) \geq d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta))$ . Note that as  $\delta \rightarrow 0$ ,  $\mu(\cdot|s_1^\delta) \rightarrow \mu(\cdot|\underline{\omega}_1)$  in the weak-\* topology. Hence if we take the limit as  $\delta \rightarrow 0$ , we find that

$$\lim_{\delta \rightarrow 0} d(\mu(\cdot|s); \mu(\cdot|s_1^\delta)) = - \int \frac{\partial u_S}{\partial a}(a^*(\mu(\cdot|s)), \omega) d\mu(\omega|s) \cdot \frac{\frac{\partial u_R}{\partial a}(a^*(\mu(\cdot|s)), \underline{\omega}_1)}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu(\cdot|s)), \omega) d\mu(\omega|s)} + u_S(a^*(\mu(\cdot|s)), \underline{\omega}_1)$$

which is precisely the value of  $d(\mu(\cdot|s), \underline{\omega}_1)$ . Similarly, we find that

$$\lim_{\delta \rightarrow 0} d(\mu(\cdot|s); \mu(\cdot|s_2^\delta)) = - \int \frac{\partial u_S}{\partial a}(a^*(\mu(\cdot|s)), \omega) d\mu(\omega|s) \cdot \frac{\frac{\partial u_R}{\partial a}(a^*(\mu(\cdot|s)), \bar{\omega}_2)}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu(\cdot|s)), \omega) d\mu(\omega|s)} + u_S(a^*(\mu(\cdot|s)), \bar{\omega}_2)$$

Suppose that  $d(\mu(\cdot|s_1); \bar{\omega}_2) > d(\mu(\cdot|s_2); \bar{\omega}_2)$ . Then for sufficiently small  $\delta$ ,  $d(\mu(\cdot|s_1); \mu(\cdot|s_2^\delta)) > d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta))$ . Thus there would be a feasible improvement from shifting a weight of  $\epsilon$  of  $\mu(\cdot|s_2^\delta)$  from  $\mu(\cdot|s_2)$  to  $\mu(\cdot|s_1)$ , and so  $(\pi, \mathcal{S})$  could not be optimal. Similarly, if  $d(\mu(\cdot|s_2); \underline{\omega}_1) > d(\mu(\cdot|s_1); \underline{\omega}_1)$ , then there exists an improvement by shifting (for small enough  $\delta$  and  $\epsilon$ ) a weight of  $\epsilon$  of  $\mu(\cdot|s_1^\delta)$  from  $\mu(\cdot|s_1)$  to  $\mu(\cdot|s_2)$ .  $\square$