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How to escape a liquidity trap with interest rate rules

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May 2016



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Abstract

I give necessary and sufficient conditions under which interest-rate feedback rules eliminate aggregate instability by inducing a globally unique optimal equilibrium in a canonical New Keynesian economy with a binding zero lower bound. I consider a central bank that initially keeps interest rates pegged at zero for a length of time that depends on the state of the economy and then switches to a standard Taylor rule. There are two crucial principles to achieving global uniqueness. In response to deepening deflationary expectations, the central bank must, first, sufficiently extend the initial period of zero interest rates and, afterward, follow a Taylor rule that does not obey the Taylor principle. I obtain all results assuming a passive or Ricardian fiscal policy stance, so that it is monetary policy alone that eliminates undesired equilibria. The interest rate rules that I consider do not require central banks to undergo any significant institutional change and do not rely on the Neo-Fisherian mechanism of inducing an increase in inflation by first increasing interest rates.

Key words: zero lower bound (ZLB), liquidity trap, New Keynesian model, indeterminacy, monetary policy, Taylor rule, Taylor principle, interest rate rule, forward guidance

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1 Introduction

Short-term nominal interest rates in many developed economies—including Japan, the US and Europe— have by now spent several years against their effective zero lower bound (ZLB) or very close to it. One key tool that central banks have used when constrained by the ZLB is forward guidance, whose main element is a promise to keep short-term nominal interest rates low for an extended period of time. How long should this period be? Should it be extended if inflation turns out to be lower than expected? Should it be stage-contingent at all? Is it more dangerous to over or undershoot the desired optimal time? How should the promise be communicated to the public and how should the central bank react—or threaten to react—if outcomes are not what it aimed for? What does it take to “anchor expectations” and prevent self-fulfilling deflationary traps? Can the central bank do it without the help of fiscal policy? Is the Neo-Fisherian idea that raising interest rates will cause an increase in inflation the only way to exit the ZLB?

In this paper, I answer these questions by studying interest rate rules for an economy in a liquidity trap. I use a canonical deterministic New Keynesian (NK) model in continuous time with a binding ZLB, as in [Werning \(2011\)](#). A binding ZLB arises because the exogenous natural rate of interest is negative for some finite initial period of time. Eventually, the natural rate becomes positive, so there is no fundamental reason why nominal interest rates should remain at the ZLB indefinitely or experience recurring ZLB episodes after liftoff. However, the economy is susceptible to expectational traps in which the expectation of low inflation can be self-fulfilling, pushing the economy into the ZLB irrespective of the level of the natural rate, a suboptimal outcome. The self-fulfilling expectations can also create macroeconomic instability in the form of multiple equilibria and chaos, also an undesirable result ([Benhabib, Schmitt-Grohe, and Uribe \(2001\)](#), [Benhabib, Schmitt-Grohé, and Uribe \(2002b\)](#), [Schmitt-Grohé and Uribe \(2009\)](#)). I assume a credible central bank with commitment follows a monetary policy rule that is very close to actual current practice. The central bank first promises to keep interest rates pegged at zero for some length of time that can be made contingent on past, present or forecasted future states of the economy. Once the promise expires, the central bank forever after follows a standard Taylor rule of the form $i(t) = \max\{0, r + \xi_\pi \pi(t) + \xi_x x(t)\}$, where $i(t)$ is the short-term nominal interest rate, $\pi(t)$ is inflation, $x(t)$ is the output gap and r , ξ_π and ξ_x are parameters of the rule chosen by the central bank. I provide necessary and sufficient conditions for this kind of monetary policy to produce the socially optimal “forward guidance” equilibrium characterized by [Werning \(2011\)](#); [Eggertsson and Woodford \(2003\)](#); [Jung, Teranishi, and Watanabe \(2005\)](#) as globally determinate (i.e. unique) equilibrium. My goal is not to provide an alternative to forward guidance, but rather to show how to properly communicate it.

In NK models without a ZLB, following a Taylor rule in which the short-term nominal interest rate reacts more than one-for-one with inflation —the so-called Taylor principle— is necessary and sufficient to achieve local determinacy (in a neighborhood of the economy’s steady-state). In the baseline “three equation” version of the model, the Taylor principle also delivers optimality of outcomes. In contrast, when the ZLB binds, I find that there are two crucial principles needed for determinacy.

The first is that the Taylor principle cannot hold for (off-equilibrium) paths of the economy in which private sector expectations of falling into a deflationary trap are strong enough. Following the Taylor principle is detrimental because promising to be tough on inflation outside the ZLB prevents the future boom in inflation and output necessary to arrest the deflationary expectations while at the ZLB. Delaying liftoff does not help either; it only confirms the pessimistic private sector expectations. For all other paths that do not have strong enough deflationary expectations, the key is to follow the second principle for determinacy that I explain next; for these paths, whether the Taylor principle holds is neither necessary nor sufficient for uniqueness of equilibria.

The second principle is that the date for liftoff must be postponed sufficiently far into the future when initial private sector expectations of falling into the deflationary trap become stronger. Although compact and relatively easy to communicate, this principle entails several elements. The most salient one is that the length of time for which interest rates are promised to be kept at zero must be long enough to preclude deflationary expectations to take hold. A longer span of interest rates pegged at zero guarantees, by stimulating future output and inflation, that by the time the central bank reverts to a Taylor rule, the economy will no longer be constrained by the ZLB. As expectations become more deflationary, the central bank can fight them by delaying liftoff. This strategy is effective only up to a point. When deflationary expectations are strong enough, the first principle explained in the last paragraph applies and the only way to avoid indeterminacy is to not follow the Taylor principle. Conversely, when deflationary expectations are weak enough, the liftoff time does not need to respond to changes in expectations. It is only for the intermediate range of deflationary expectations that are neither too weak nor too strong that delaying liftoff is both necessary and sufficient to achieve global determinacy.

Another important element of the second principle is that the date of liftoff must be state-dependent. Because rational expectations of falling into the trap are functions of inflation and the output gap, responding to changes in expectations entails a state-dependent liftoff. If liftoff is not state-dependent, irrespective of whether the Taylor principle holds outside the ZLB, there always are multiple equilibria. This is true even if the central bank is allowed to follow, after liftoff, a Taylor rule with path-dependent coefficients, a rule almost as flexible as an unconstrained non-linear rule. It follows that calendar-based forward guidance, in which

the central bank commits to zero interest rates until a specified future date that does not change with the state of the economy, can never anchor expectations. In addition, the date of liftoff must depend on *initial* expectations of the private sector, expectations that prevail at the beginning of the liquidity trap. The liftoff rule cannot be purely forward-looking. The rule for liftoff cannot solely depend on future expected values for inflation and the output gap. Policies such as the “threshold strategy” pursued by the Federal Reserve (sometimes called the “Evans rule”) in which interest rates are promised to be kept at zero (or very low) at least until projected inflation rises above 2.5 percent or unemployment falls below 6.5 percent, even if state-contingent, fail to anchor expectations because they are purely forward-looking.

The two principles become sharper if the interest rate rule is continuous in the state of the economy so that, as is usually the case in practice and in most monetary policy rules in the literature, small changes in the economy lead to commensurately small reactions by the central bank. With continuous rules, the first principle is now that the Taylor principle can never hold. To achieve determinacy, even if expectations of falling into the expectational trap are completely absent, the Taylor principle still cannot hold. If the Taylor principle holds for some states but not for others, continuity implies that there are some in-between states exactly at the threshold between following and not following the Taylor principle. For these states, the Taylor rule is neither tough enough on inflation to anchor expectations outside the ZLB nor stimulative enough inside the ZLB to guarantee an eventual exit from it, resulting in multiple equilibria. Because the Taylor principle cannot hold for at least some states (by the first principle without continuity), it then cannot hold for any state. With continuity of the interest rate rule, the second principle also becomes more dovish. Rather than requiring a late enough liftoff when expectations of falling into the deflationary trap are of medium strength only, the second principle now requires a late enough liftoff for both medium and weak expectations. Imagine the central bank promises that, for a particular state of the economy, liftoff will occur earlier the critical time needed for eternacy. By continuity, this means that liftoff occurs earlier than the critical time for all states, since otherwise, by continuity, there always is one state with liftoff occurring at exactly the critical time, giving a suboptimal equilibrium. But by the second principle without continuity, liftoff before the critical times for all states cannot sustain a unique equilibrium

I have expressed the first and second principles in terms of expectations of falling into the deflationary trap. But what exactly are these expectations? How should they be measured? The levels of inflation and the output gap are, on their own, not very informative about whether the economy is in a liquidity trap, at risk of converging to a deflationary steady-state or on a desirable policy path, as has been already pointed out through various arguments by Benhabib et al. (2001), Werning (2011) and others. Most strikingly, Benhabib et al.

(2001) show that the economy can be arbitrarily close to the preferred non-deflationary steady-state (the one that minimizes deviations of inflation and output from target) and yet be inexorably headed to the deflationary trap. My results show that, in the context of the basic NK model with a ZLB that I use, expectations of falling into the deflationary trap are defined by two specific linear combinations of inflation and the output gap. The first linear combination, which I refer to as the “trap factor”, measures the degree to which expectations are pushing the economy into the undesirable deflationary steady-state. Higher inflation and higher output gaps, present or future, make these expectations weaker. The second linear combination, the “exit factor”, gives expectations of moving away from the undesirable deflationary steady-state. These expectations become stronger when the output gap increases or inflation decreases. The tug of war between the two factors determines whether the economy can escape the ZLB.

While indeterminacy is an important issue in all NK models, it is especially difficult to eliminate it in the presence of a binding ZLB. [Benhabib et al. \(2001\)](#) show that indeterminacy is a robust feature of Taylor-style feedback rules when a ZLB is introduced, especially when the Taylor principle is satisfied. While the same exact results apply in the framework I use, I am able to eliminate the multiplicity of equilibria by considering a broader class of interest rate rules that allow for a state-contingent liftoff date. [Benhabib et al. \(2002b\)](#) show that a Taylor rule that is non-decreasing, respects the ZLB and follows the Taylor principle around the desired (non-deflationary) steady-state gives rise to chaotic trajectories for the economy. In contrast, I find that there can be global determinacy without chaos while using a Taylor rule. Indeed, the Taylor principle helps eliminate chaos in the model I use. Thus, whether the Taylor principle induces macroeconomic instability of this kind depends on the specific setting under consideration.

In the same framework of [Werning \(2011\)](#) that I use in this paper, [Cochrane \(2013\)](#) shows that an economy in a liquidity trap exhibits indeterminacy for *any* given (non-state contingent) path of nominal interest rates. I extend [Cochrane \(2013\)](#)’s result by showing that if the interest rate path is not state-contingent only until liftoff —instead of for the entire path— and then follows a Taylor rule, indeterminacy still obtains for any choice of Taylor rule coefficients. This result shows that a key source of indeterminacy is the zero interest rate peg during the liquidity trap. [Cochrane \(2015\)](#), building on ideas in [Werning \(2011\)](#), shows that for any given (non-stage contingent) path of nominal interest rates, there always exists a Taylor rule that can implement the given path as a locally determinate equilibrium by having a time-varying intercept or inflation target and following the Taylor principle. My results show that this strategy does not work if the goal is to implement the given interest rate path as a *globally* determinate equilibrium. On the other hand, although I only prove how to implement the optimal equilibrium as the unique global equilibrium of the economy,

the same techniques can be used to implement any other equilibrium as the unique global equilibrium. Thus, I generalize the conclusions in [Cochrane \(2015\)](#) to situations in which the ZLB binds and uniqueness is global.

In the discrete-time version of the model I use in this paper, [Eggertsson and Woodford \(2003\)](#) implement the optimal equilibrium as a locally determinate equilibrium by means of an “output-gap adjusted” price-level target and as a globally determinate equilibrium by adding an “active” or “non-Ricardian” fiscal policy commitment. They are the first to put forth a fiscal-monetary rule that produces global determinacy in NK models with a ZLB. My results are different in four ways. First, the monetary policy rule I study requires little change in the institutional arrangements of most central banks, an advantage when putting it into practice. The policy instrument is the short-term nominal interest rate, already the predominant instrument of choice. State-contingent forward guidance has already been tried, as with the Evans rule mentioned above, and Taylor rules fit the observed behavior of many central banks outside the ZLB quite well. Implementation capabilities for the rules I investigate are in place today. A switch to a price-level target may entail some initial fixed costs, such as a temporary loss of credibility, although the size and duration of these costs are difficult to assess. Perhaps a more relevant argument is that, rightly or not, central banks are not currently considering a move to price-level targets, at least not to the general public’s knowledge¹. Of course, a price-targeting regime can be implemented and explained by means of an interest rate rule and, conversely, the interest-rate rules that I study can be communicated as a price-targeting rule if the target is defined appropriately. However, the essence of the rule in [Eggertsson and Woodford \(2003\)](#) is more naturally explained as a price level targeting regime while the rules I examine are more naturally explained as interest rate rules. Since communication is a key aspect of policy rules, the distinction seems, at the very least, worth mentioning.

Second, unlike price-level targeting, the interest rate rules I consider can be made memoryless after liftoff and still guarantee global determinacy. All path-dependence can end when the promise to keep interest rates at zero ends. The Taylor rule that the central bank follows after liftoff can be the same —have the same coefficients— for any path that the economy takes. The path dependence for the rules I consider is essential before liftoff but not afterwards. In price level targeting, the closing of the price gap that opens while at the ZLB continues after interest rates become positive. The trajectory of nominal interest rates implied by the target is always backward looking for at least some time after the end of the ZLB period.

¹ Judging from meeting minutes, the Federal Open Market Committee (FOMC) discussed and rejected nominal GDP targeting in 1982, 1992 and 2011 ([Committee \(2015a\)](#), [Committee \(2015b\)](#), [Committee \(2015c\)](#)), but not without noting some of its merits. [Bernanke \(2015b\)](#) and [Bernanke \(2015a\)](#) explain why the FOMC rejected nominal GDP targeting in 2011.

Third, Eggertsson and Woodford (2003) consider an economy with shocks and perfect foresight while I have a deterministic economy without shocks. This allows Eggertsson and Woodford (2003) to show that the rule they propose can be implemented without any estimate or knowledge of the statistical process for the natural rate of interest, a very appealing feature. Since I do not consider shocks, it is not possible to evaluate whether the same would hold true for the rules I deal with. In terms of other informational requirements, both types of rule are similar and require knowledge of current inflation, the output gap and parameters of the model. One distinction is that with price-level targeting the central bank must implement a specific time-varying target, while the interest rate rules in this paper only require a “strong enough” response. While this makes no difference strictly inside the perfect-information model, in practice it may be a desirable trait of rules. For example, having to set the policy instrument above a certain bound instead of exactly equal to a single value may ease the practical burden of having to estimate the parameters in the model, even if the bound itself depends on parameters. A central bank can judge a certain set of parameters to be ex-ante reasonable and then enact a policy that works for all such parameters. All it takes is to keep interest rates pegged at zero for a length of time longer than the maximum lower bound across all parameters under consideration. The “strong enough” requirement is akin to the Taylor principle requiring that the response of interest rates to inflation is strong enough. It gives the central bank a higher margin of error; overshooting the intended length of time for which interest rates are promised to be zero does not re-introduce indeterminacy. On the other hand, undershooting could produce indeterminacy. It is in this sense that undershooting the desired optimal time is more dangerous than overshooting it. For determinacy, an ambivalent central bank should err on the side of too long, rather than too short, a ZLB period.

Fourth, and perhaps most importantly, global determinacy for the interest rate rules I contemplate is achieved with “passive” or “Ricardian” fiscal policy, in which the government always adjusts taxes or spending ex-post to validate any path of the endogenous variables that may arise. Thus, monetary policy is anchoring expectations on its own. There is no need for help from fiscal policy and no requirement of fiscal insolvency along some off-equilibrium paths. To my knowledge, this paper is the first to present a monetary policy regime that produces global determinacy when fiscal policy is passive/Ricardian and the ZLB binds. This does not imply that fiscal rules or the fiscal theory of the price level are unimportant in theory or not relevant in practice. I do not analyze active/non-Ricardian rules, so my results are silent regarding the relative merits of eliminating indeterminacy by means of an active monetary policy versus an active fiscal policy. The need to study the combined fiscal-monetary regime remains as important as ever².

² See Leeper (1991), Sims (1994), Benhabib, Schmitt-Grohé, and Uribe (2001), Woodford (2001), Benhabib,

Schmitt-Grohé and Uribe (2014) propose an interest rate based strategy to escape liquidity traps that entails temporarily deviating from a Taylor rule by increasing nominal interest rates in a deterministic and non-state contingent way until it reaches a pre-specified target. This Neo-Fisherian strategy succeeds in setting a floor for inflation without the need for a “non-Ricardian” fiscal stance, although it does not lead to globally determinate outcomes. Cochrane (2015) shows that, under a broad variety of assumptions, NK models in a liquidity trap have equilibria with the Neo-Fisherian feature that when the central bank raises interest rates, inflation rises both in the short and the long run. He shows how to construct Taylor rules that implement any such equilibria in a locally determinate way. In this paper, I present an example in which global determinacy is attained by Neo-Fisherian means. The central bank commits to raising interest rates at particular dates in the future with little concern for what level of inflation will prevail at the time. The strong commitment to increasing rates in the future dispels any deflationary expectations by the same logic as in Schmitt-Grohé and Uribe (2014) and Cochrane (2015), allowing the economy to exit the expectational trap. For these Neo-Fisherian rules, it is still necessary to have a stage-contingent liftoff for global determinacy, an ingredient absent in Schmitt-Grohé and Uribe (2014) and Cochrane (2015).

At any rate, my main results focus on rules that are not Neo-Fisherian to keep monetary policy as close to current practice as possible, attempting to minimize the distance to real-world implementation. Even if Neo-Fisherianism ends up being empirically vindicated or becomes a consensus theory, it is currently far from central bank orthodoxy and does not seem to be under serious consideration by any major central banks. There are many other proposals on how monetary policy should be conducted in a liquidity trap. Some prominent examples include: Svensson (2004), who advocates a currency depreciation combined with a calibrated crawling peg; McCallum (2011), Sumner (2014) and Romer (2011), who recommend nominal GDP targeting; and Blanchard, DellAriccia, and Mauro (2010) and Ball (2014), who promote increasing the inflation target by contending that the trade-off of higher steady-state inflation and less frequent visits to the ZLB is worth undertaking. For the class of rules I consider in this paper, there are no suboptimal trade-offs and no need to change central bank communication to accommodate new monetary or price aggregates, price-level or inflation targets, “shadow” rates, exchange rates, the central bank’s balance sheet, or the quantity or price of other assets.

Schmitt-Grohé, and Uribe (2002a), Cochrane (2011) among many other.

2 The Canonical New Keynesian Model with a ZLB

I use the framework of [Werning \(2011\)](#), a standard deterministic New Keynesian model in continuous time, log-linearized around a zero-inflation steady state.³ The economy is described by:

$$\dot{x}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t)), \quad (1)$$

$$\dot{\pi}(t) = \rho\pi(t) - \kappa x(t), \quad (2)$$

$$i(t) \geq 0. \quad (3)$$

Markets are complete. The variables $x(t)$ and $\pi(t)$ are the output gap and the inflation rate, respectively. The output gap is the log-deviation of actual output from the hypothetical output that would prevail in the flexible price, efficient allocation. Henceforth, for brevity, I refer to the output gap simply as output. The central bank's policy instrument is the path for the nominal short-term (instantaneous) interest rate $i(t)$, which must remain non-negative at all times. The variable $r(t)$ is the exogenous natural rate of interest, defined as the real interest rate that would prevail in the flexible price, efficient economy with $x(t) = 0$ for all t . The process for $r(t)$ is

$$r(t) = \begin{cases} r_l < 0 & , \quad 0 \leq t < T \\ r_h > 0 & , \quad t \geq T \end{cases}.$$

The constants $T > 0$, $r_l < 0$ and $r_h > 0$ are given. The economy starts in a liquidity trap and exits it with certainty at time T . None of the results in this paper change if the path for $r(t)$ is different as long as $r(t) < 0$ for $t < T$ and $r(t) > 0$ for $t \geq T$.

Equation (1) is the IS curve, the log-linearized Euler equation of the representative consumer. The constant $\sigma^{-1} > 0$ is the elasticity of intertemporal substitution. Equation (2) is the New Keynesian Phillips Curve (NKPC), the log-linear version of firms' first-order conditions when they maximize profits by picking the price of consumption goods subject to consumers' demand and Calvo pricing. The constant $\rho > 0$ is the representative consumer's discount rate and $\kappa > 0$ is related to the amount of price stickiness in the economy. As

³ See [Woodford \(2003\)](#) or [Galí \(2009\)](#) for details.

Although the majority of analysis of determinacy in New Keynesian models is done in log-linearized models, [Braun, Körber, and Waki \(2012\)](#) contend that conclusions would differ in the non-linear model. On the other hand, [Christiano and Eichenbaum \(2012\)](#) show that the additional equilibria that arise from non-linearities in [Braun et al. \(2012\)](#) are not E-learnable. In addition, [Christiano and Eichenbaum \(2012\)](#) show that the linear approximation is accurate except on extreme cases, such as when output deviates by more than 20 percent from steady state.

While important, I do not seek to address these issues here and simply use the log-linear model (plus the ZLB), the standard practice in the literature.

$\kappa \rightarrow \infty$, the economy converges to a fully flexible price economy while prices are completely rigid when $\kappa \rightarrow 0$.

Definition. An *equilibrium* consists of bounded paths for output, inflation and the nominal interest rate $\{x(t), \pi(t), i(t)\}_{t \geq 0}$ that, given a path $\{r(t)\}_{t \geq 0}$ for the natural rate, satisfy equations (1)-(3).

There are three elements in the definition that are worth discussing in the present context. First, the requirement that output and inflation remain bounded at all times is equivalent to the asymptotic conditions

$$\lim_{t \rightarrow \infty} |x(t)| < \infty, \quad (4)$$

$$\lim_{t \rightarrow \infty} |\pi(t)| < \infty. \quad (5)$$

The justification and role that equation (5) plays for determinacy of equilibria is controversial in the literature.⁴ In the specific setup of this paper, inflation explodes if and only if the output gap explodes, making it impossible to differentiate nominal from real explosions. Thus, equation (5) can be omitted from the definition of equilibrium without any consequences. In the non-linear version of the simple model I consider, unbounded inflation leads to real consumption exploding or becoming zero, connecting nominal and real variables asymptotically. Equations (4) and (5) then capture transversality conditions. If consumption explodes, the behavior of the representative agent is suboptimal, as he can reallocate consumption intertemporally and consume more in every period. Similarly, zero asymptotic consumption implies suboptimality since transferring consumption from today, when marginal utility is finite, to the distant future, when marginal utility is unbounded, increases utility. In a more general model, however, it may be possible to separate nominal and real explosions and the conditions in equations (4) and (5) may not correspond directly to economically meaningful transversality conditions in the non-linear model.

Second, paths for $x(t)$ and $\pi(t)$ that satisfy equations (1) and (2) must be continuous.⁵ If there were any jumps and because markets are complete, the representative consumer's

⁴ Cochrane (2011) argues that there is no good economic reason to prevent nominal explosions. McCallum (2009) and Atkeson, Chari, and Kehoe (2009) agree and, among others in an active area of research, propose different criteria to eliminate or select equilibria. Woodford (2003), Wren-Lewis (2013) and others defend the approach of using equation (5).

⁵ Under the usual "classical" definition, differentiability of $x(t)$ and $\pi(t)$ would also be required for all t . I instead use derivatives in the distribution sense or in the sense of "Filippov solutions" (Filippov (2013)), a weaker solution concept that allows for non-differentiability in a set of measure zero and is a natural choice for the problem at hand. However, any other of the usual weaker notions of derivative (such as viscosity solutions) would yield the same results as long as derivatives are finite everywhere.

Without this purely technical modification, the assumed discontinuous process for the natural rate $r(t)$ would imply that (1) and (2) have no solution. More importantly, even if $r(t)$ were smooth, the central bank's control problem of Section 3 would have no solution since the solution requires a jump in the control $i(t)$.

Euler equation (in its IS incarnation) would be violated due to the existence of arbitrage opportunities. On the other hand, there are no smoothness requirements for $i(t)$ to be part of an equilibrium, as it is a choice variable for the central bank.

Third, neither the definition of equilibrium nor the dynamics of the economy in equations (1) and (2) make any explicit reference to fiscal policy although, as stressed by [Woodford \(1995\)](#), [Sims \(1994\)](#), [Benhabib et al. \(2001\)](#), [Cochrane \(2011\)](#) and others, determinacy or the lack thereof is a result of the joint monetary-fiscal regime. The implicit fiscal behavior I assume is that of a “Ricardian” (in the terminology of [Woodford \(2001\)](#)) or “passive” (in the terminology of [Leeper \(1991\)](#)) regime: The fiscal authority always adjusts taxes or spending ex-post to validate any path of the endogenous variables that may arise. I consider a passive fiscal policy to highlight that the determinacy results I present are a direct consequence of central bank policy and do not need any specific rules for government spending, debt or taxes to eliminate undesirable equilibria.

3 The Optimal Equilibrium

The social welfare loss function for the economy is

$$V = \frac{1}{2} \int_0^{\infty} e^{-\rho t} (x(t)^2 + \lambda \pi(t)^2) dt. \quad (6)$$

The constant $\lambda > 0$ is a preference parameter that dictates the relative importance of deviations of output and inflation from their desired value of zero. This quadratic loss objective function can be obtained as a second order approximation around zero inflation to the economy’s true social welfare function when the flexible price equilibrium is efficient ([Woodford, 2003](#)). An optimal equilibrium is an equilibrium that minimizes (6).

[Werning \(2011\)](#) solves for the optimal equilibrium $\{x^*(t), \pi^*(t), i^*(t)\}_{t \geq 0}$. He finds that it is unique and that the optimal path for the nominal interest rate is

$$i^*(t) = \begin{cases} 0 & , \quad 0 \leq t < t^* \\ (1 - \kappa\sigma\lambda) \pi^*(t) + r_h & , \quad t \geq t^* \end{cases}, \quad (7)$$

where the optimal liftoff date, t^* , is a constant that can be found as a function of the parameters of the model. Importantly, $t^* > T$. The optimal policy is to commit to zero nominal interest rates for longer than the natural rate $r(t)$ is negative — one of the main aspects of forward guidance. However, equation (7) is not a policy rule. Indeed, the optimal path $(1 - \kappa\sigma\lambda) \pi^*(t) + r(t)$ is a single fixed path, a function of time only. It describes one particular equilibrium. It is neither contingent on the actions of the central bank nor on

whether realized inflation, output or private sector expectations happen to take one value or another. As such, it addresses neither the on nor the off-equilibrium behavior of the central bank and hence says nothing about implementability or indeterminacy.

Plugging (7) into (1) and (2) gives the optimal paths for inflation and output for all $t > 0$. Given $x^*(0)$ and $\pi^*(0)$, equations (1), (2) and (7) then determine the entire optimal equilibrium. One way to find $x^*(0)$, $\pi^*(0)$ and t^* is to use the maximum principle, as in Werning (2011).

Figure 1 shows representative optimal paths for three parameter configurations in a π - x phase diagram. It is most easily understood in three stages, starting from the last one and working backwards in time. The beginning and end times for each stage are determined by the discontinuities of $r(t)$ and $i^*(t)$.

In the third and last stage, when $t \geq t^*$, the economy has positive natural and nominal rates. The dynamics of $(\pi^*(t), x^*(t))$ are saddle-path stable for $t \geq t^*$, with saddle given by

$$x = \phi\pi, \tag{8}$$

where

$$\phi = \frac{1}{2\kappa} \left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2} \right) > 0. \tag{9}$$

To ensure that inflation and output remain bounded so as to satisfy the asymptotic conditions (4) and (5), the economy must move towards the unique steady-state $(\pi_{ss}, x_{ss}) = (0, 0)$ along the saddle path. If and only if $\kappa\sigma\lambda = 1$, the economy reaches its steady state exactly at t^* and therefore does not need to travel along the saddle path to reach the steady-state. The blue path in figure 1 shows such a case. The other two cases keep all parameters unchanged except for λ , the relative weight given to deviations of output and inflation from zero. As can be seen in the figure, the optimal paths with larger λ tolerate smaller deviations of inflation from zero. Because the ZLB does not bind after t^* , the solution in this third stage is always independent of the process for the natural rate.

The second stage is $t \in [T, t^*)$, when the natural rate is positive but the nominal rate is zero. Starting at a given $(\pi(T), x(T))$, inflation and output move so as to minimize the time it takes to reach the saddle path $x = \phi\pi$. This is accomplished by pegging nominal rates to zero. When $x(t)$ and $\pi(t)$ hit the saddle path, the third stage begins.

In the first stage, given by $t \in [0, T)$, the natural rate is negative and the nominal rate is at the ZLB. The zero nominal rates in the first two periods imply a low real interest rate that reduces the incentive to save, producing high consumption. As a result, inflation and output eventually become positive before T . The initial point $(\pi^*(0), x^*(0))$ is determined by optimality conditions that trade off deviations of inflation and output from zero at each of the three stages.

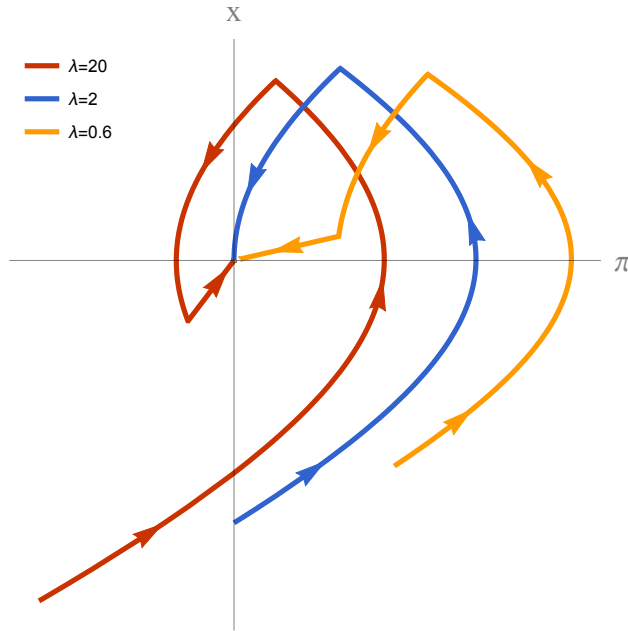


Figure 1: Optimal paths for inflation (in the horizontal axis) and the output gap (in the vertical axis) for three values of λ , the weight that the central bank places on inflation relative to output. The rest of the parameters used are taken from [Werning \(2011\)](#) ($T = 2$, $\sigma = 1$, $\kappa = 0.5$, $\rho = 0.01$, $r_h = 0.04$ and $r_l = -0.04$).

4 Calendar-based Forward Guidance Leads to Indeterminacy

In this section, I study a central bank that attempts —yet fails— to implement the optimal equilibrium as the unique equilibrium of the economy by means of an interest rate rule in which the time of liftoff is a constant. From now on and for the rest of the paper, the central bank has perfect commitment and credibility.⁶

An initial natural candidate rule is based on equation (7):

$$i(t) = \begin{cases} 0 & , \quad 0 \leq t < t^* \\ (1 - \kappa\sigma\lambda)\pi^*(t) + r_h & , \quad t \geq t^* \end{cases} . \quad (10)$$

Although equations (7) and (10) look very similar, they are conceptually different. While equation (7) describes the single optimal path $i^*(t)$, equation (10) is a rule —a policy response function— by which the central bank commits to set interest rates in all possible states of the world. It therefore provides both the on and off-equilibrium behavior of the central bank. For this particular rule, the behavior of the central bank is the same for all states of the world. The central bank announces that that interest rates will follow

⁶ Without them, there is no hope of implementing the optimal equilibrium.

the optimal path of nominal interest rates described in the last section, which is the same in all states of the world, come what may. If the central bank follows rule (10), then it can clearly implement the optimal equilibrium whenever $(\pi(0), x(0)) = (\pi^*(0), x^*(0))$. However, [Cochrane \(2013\)](#) shows that many other equilibria are also consistent with this rule, leading to an indeterminate outcome. In fact, he goes further and shows that if the central bank commits to *any* given non-explosive path of nominal rates, irrespective of its optimality or the central bank’s commitment ability, the economy suffers from indeterminacy. More formally, rules of the form $i(t) = f(t)$ that are explicitly only dependent on t (as opposed to dependent on $x(t)$ or $\pi(t)$) produce indeterminacy for any bounded choice of f . This means that forward guidance that makes no explicit reference to the state of the economy can never anchor expectations.

The intuition for this result is as follows. By virtue of equations (1) and (2), the choice of $i(t)$ directly affects —we may even say controls— inflation and output for all $t > 0$, but not for $t = 0$. The path of nominal interest rates only affects *changes* in x and π starting at $t = 0$, but cannot directly influence $x(0)$ or $\pi(0)$. Initial inflation and output, instead of being control variables like in the last section, are now non-predetermined or “jump” variables. The inability to affect current inflation and output with current interest rates, economically speaking, stems from the forward-looking nature of the IS equation. Can the central bank nevertheless induce $x(0)$ and $\pi(0)$ to be optimal in some way? For the rules we examine in this section, which are functions of time only, the answer is no. In the following sections, I characterize a class of rules that can indeed uniquely implement the desired equilibrium by guaranteeing that $(\pi(0), x(0)) = (\pi^*(0), x^*(0))$.

To understand the inability of the central bank to influence $x(0)$ or $\pi(0)$ with a rule like (10), one can interpret $i(t) = (1 - \kappa\sigma\lambda)\pi^*(t) + r_h$ or, more generally, $i(t) = f(t)$ as a Taylor rule with a time-varying intercept equal to $f(t)$ and coefficients of zero on inflation and output. Such a rule does not satisfy the Taylor principle. If $f(t)$ is such that the ZLB does not bind after t^* , as is the case for the rule in equation (10), the classical analysis of determinacy in log-linear New Keynesian model without a ZLB applies. The dynamics of $(\pi(t), x(t))$ are saddle-path stable for $t \geq t^*$. The existence of a saddle path breeds indeterminacy. Indeed, there is one equilibrium for each point in the saddle path. Pick a point $(\tilde{\pi}, \tilde{x})$ on the saddle path and consider a candidate equilibrium with $(\pi(t^*), x(t^*)) = (\tilde{\pi}, \tilde{x})$. For $t \geq t^*$, the economy follows the dynamics of the IS equation and the NKPC with $i(t) = (1 - \kappa\sigma\lambda)\pi^*(t) + r(t)$ by moving along the saddle path towards the steady state (or, if $(\tilde{\pi}, \tilde{x}) = (\pi_{ss}, x_{ss})$, it hits the steady-state directly at t^*). Now trace the dynamics of the IS and the NKPC backwards in time with $i(t) = 0$, from $t = t^*$ to $t = T$, starting at $(\pi(t^*), x(t^*)) = (\tilde{\pi}, \tilde{x})$ and ending at some $(\pi(T), x(T))$. Again, setting nominal rates to zero, follow the dynamics of the IS and the NKPC backwards in time from $t = T$ to $t = 0$,

now with initial condition $(\pi(T), x(T))$ inherited from the previous step. The resulting path is bounded, continuous, obeys the IS equation, the NKPC and the ZLB: It is an equilibrium.

Because of the linearity of the system, the saddle path and the set of initial conditions $x(0)$ and $\pi(0)$ that put the system on the saddle path at t^* are both lines in the π - x plane. The equilibria are thus indexed by points in a line, which we can take to be the saddle path or the $\pi(0)$ - $x(0)$ line that gets the economy to the saddle path at t^* . Figure 2 shows these two lines together with paths for inflation and output from equilibria that start at different $x(0)$ and $\pi(0)$. All of these equilibria are obtained using identical parameters and the same interest rate rule, given by (10). They are different equilibria of the same economy, unlike those in figure 1, which show the optimal equilibria for different parameter configurations. The green line is the optimal equilibrium that starts at $(\pi^*(0), x^*(0))$. The parameters in figure 2 are the same as those used to get the green line in figure 1, and so the green paths in both figures are the same. The red line is an equilibrium that reaches steady state $(\pi_{ss}, x_{ss}) = (0, 0)$ at t^* , just when the ZLB period ends. The yellow line is one of the “local-to-frictionless” equilibria from Cochrane (2013) in which inflation and output do not explode backwards in time. The remaining paths have two arbitrary values for $\pi(0)$ and illustrate the kinds of behavior that the various equilibria can exhibit.

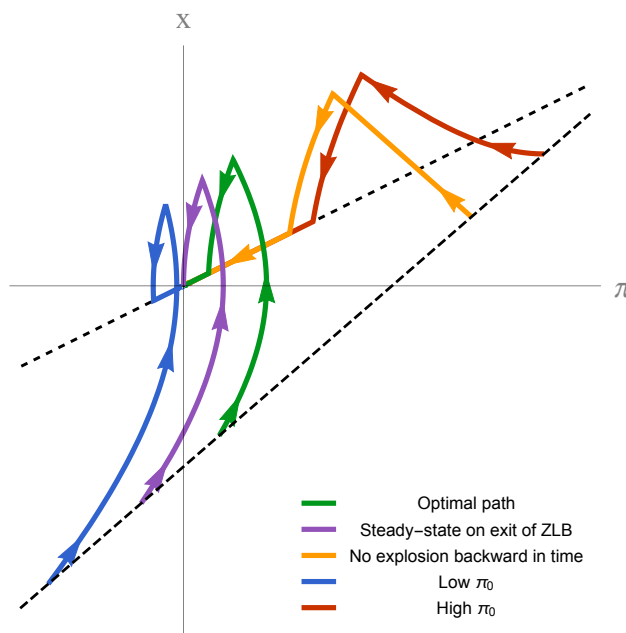


Figure 2: Multiple equilibria when the central bank chooses a rule for interest rates that sets them equal to the optimal path for all realizations of inflation, output and their expectations. Interest rate rules that are not contingent on the state of the economy never anchor expectations.

I now extend the results from Cochrane (2013) to Taylor rules. Consider a central bank that commits to keeping interest rates at zero until some time \underline{t} and follows a Taylor rule

that respects the ZLB forever after. I call \underline{t} the liftoff date even though the Taylor rule may prescribe zero interest rates for some time after \underline{t} . Therefore, liftoff may or may not coincide with interest rates becoming positive for the first time after the liquidity trap is over. Liftoff in this context is defined as the end of a commitment to zero interest rates, not as the actual end of zero interest rates.

The next proposition establishes that following any Taylor rule after a constant liftoff date does not solve the problem of indeterminacy. One direct implication is that following the Taylor principle outside the ZLB does not guarantee a unique equilibrium, as is the case in models without the ZLB. I allow the Taylor rule coefficients to be arbitrary functions of inflation and output realized at time zero, $R_0 = (\pi(0), x(0))$, to highlight that the key obstacle for determinacy is a constant liftoff date and not the choice of coefficients. When the liftoff date is constant and the economy is deterministic, conditioning the Taylor rule coefficients on R_0 is the same as conditioning on the entire path $\{\pi(t), x(t)\}_{t \in [0, \infty)}$. Although not exactly the same as allowing for a general non-linear rule, being able to have state-dependent Taylor-rule coefficients offers the central bank significant flexibility. For example, it gives the central bank the freedom to pick coefficients as if it knew which of the many potential equilibria will be realized. Despite these faculties, indeterminacy obtains. Calendar-based forward guidance, in which liftoff is announced for a particular date without making reference to the state of the economy, can therefore never anchor expectations, irrespective of what Taylor rule is followed afterwards.

Proposition 1 (Calendar-based forward guidance leads to indeterminacy). *Let $\underline{t} \geq T$ be a constant⁷. Let $\xi_\pi(R_0)$ and $\xi_x(R_0)$ be arbitrary functions. If $\kappa\sigma\lambda \neq 1$, the rule*

$$i(t) = \begin{cases} 0 & , \quad 0 \leq t < \underline{t} \\ \max\{0, \xi_\pi(R_0)\pi(t) + \xi_x(R_0)x(t) + r_h\} & , \quad \underline{t} \leq t < \infty \end{cases} \quad (11)$$

cannot implement the optimal equilibrium as the unique equilibrium of the economy.

Proof. Assume that rule (11) implements the optimal equilibrium. I show that it also implements a different second equilibrium. Because the rule (11) implements the optimal equilibrium, we have $\underline{t} = t^*$. Consider the path that starts at $(\pi(0), x(0))$ and reaches $(\pi_{ss}, x_{ss}) = (0, 0)$ at $t = t^*$ when following (1), (2) and (11). This path always exists, since we can find it by starting at $(0, 0)$ at $t = t^*$ and running time backwards until $t = 0$ using $i(t) = 0$ throughout. Since the point $(0, 0)$ is a steady-state after t^* for any choice of $\xi_\pi(R_0)$

⁷ Throughout the paper, I show results that assume the liftoff date does not occur before T , as even without commitment a central bank trying to minimize the social loss function in equation (6) would set $i(t) = 0$ for $t < T$, when the natural rate is negative. However, none of my results depend on this assumption; extending results to allow for any liftoff date is straightforward. The key for the relevant proofs is that the liftoff day is bounded below (which it always is, by zero).

and $\xi_x(R_0)$, $(\pi(t), x(t))$ remains bounded. If $\kappa\sigma\lambda \neq 1$, $(x(\underline{t}), \pi(\underline{t})) = (\pi(t^*), x(t^*)) = (0, 0) \neq (\pi^*(t^*), x^*(t^*))$ (see [Werning \(2011\)](#) for a proof). Hence, when $\kappa\sigma\lambda \neq 1$, the path that starts at the $(\pi(0), x(0))$ that reaches $(0, 0)$ at $t = t^*$ constitutes an equilibrium different from $(x^*(t), \pi^*(t))$ for any choice of functions $\xi_\pi(R_0)$ and $\xi_x(R_0)$. When $\kappa\sigma\lambda = 1$, the optimal equilibrium happens to have $(x^*(t), \pi^*(t)) = (0, 0)$ for all $t \geq t^*$ and the optimal equilibrium is indeed implementable as the unique equilibrium ([Appendix A.2](#) shows how). \square

5 Rules with State-Dependent Liftoff Date

I now introduce interest rate rules that allow for liftoff to be contingent on the state of the economy. As in the last section, liftoff refers to the time when the central bank switches from committing to zero interest rates to following a Taylor rule and not necessarily to the first time interest rates turn positive after the liquidity trap. Unlike the case of a constant liftoff time, the initial state of the economy, $(\pi(0), x(0))$, is no longer a full description of the state of the economy. Expectations of the future can be important determinants of when liftoff occurs, even after $x(0)$ and $\pi(0)$ are realized. In order to study equilibria in which expectations matter, I start by defining a rational expectations equilibrium.

5.1 Rational Expectations Equilibrium

Let t_1 be the actual time of liftoff that is realized in the economy and $R_s = (\pi(0), x(0), \pi(s), x(s))$ the vector containing realized inflation and output at times $t = 0$ and $t = s$. Conditional on time t -information, the private sector has subjective expectations given by a conditional expectations operator $E_t[\cdot]$. Private sector expectations are public knowledge. As of time $t = 0$, the private sector expects liftoff from the ZLB at time $E_0[t_1]$. The time-0 expectation for the values that inflation and output will take at $t = 0$ and $t = t_1$ are $E_0[R_{t_1}] = (\pi(0), x(0), E_0[\pi(t_1)], E_0[x(t_1)])$.

Before $t = 0$, a central bank with perfect commitment announces a liftoff rule. The rule is a function $f : \mathbb{R}^4 \rightarrow \mathbb{R}$ that specifies a time of liftoff for each possible combination of current private sector expectations of time-0 and time- t_1 inflation and output. The rule for liftoff works as follows. At $t = 0$, inflation and output are realized to some specific values, say π_0 and x_0 . Also at $t = 0$, the private sector forms expectations of R_{t_1} , say $E_0[R_{t_1}] = (\pi_0, x_0, \pi_1, x_1)$, which are observed by the central bank. The central bank then computes the number $f(\pi_0, x_0, \pi_1, x_1)$. If $f(\pi_0, x_0, \pi_1, x_1) = 0$, the central bank lifts off now (at $t = 0$) and, by definition, $t_1 = 0$. If $f(\pi_0, x_0, \pi_1, x_1) \neq 0$, the central bank does not lift off and repeats the same procedure in the future. Specifically, for any time s such that liftoff has not yet

occurred, having observed private sector expectations $E_s [R_{t_1}]$ given by, say, $(\tilde{\pi}_0, \tilde{x}_0, \tilde{\pi}_1, \tilde{x}_1)$, the central bank computes the number $f(\tilde{\pi}_0, \tilde{x}_0, \tilde{\pi}_1, \tilde{x}_1)$. If $f(\tilde{\pi}_0, \tilde{x}_0, \tilde{\pi}_1, \tilde{x}_1) = s$, then the central bank lifts off at time s and consequently $t_1 = s$. If $f(E_t [R_{t_1}]) \neq t$ for all t , then the central bank never lifts off and $i(t) = 0$ for all t . As before, I assume $f \geq T$.

Definition. A *rational expectations equilibrium* is an equilibrium in which private sector expectations are validated, i.e. for all $t, s \geq 0$

$$E_s [x(t)] = x(t), \quad (12)$$

$$E_s [\pi(t)] = \pi(t), \quad (13)$$

$$E_s [f(R_{t_1})] = f(R_{t_1}), \quad (14)$$

and central bank announcements are credible, i.e. for all $t \geq 0$ nominal interest rates follow the central bank's announced rule and

$$t_1 = f(R_{t_1}). \quad (15)$$

The condition that $t_1 = f(R_{t_1})$ in equation (15) is not tautologically satisfied. Indeed, the central bank can choose a rule f for which there is no t_1 such that $t_1 = f(R_{t_1})$. To drive home this idea, consider the function $f(R_{t_1}) = t_1 + 1$. The central bank commits to liftoff one year from the current date, for all future dates. Clearly, there is no solution to $t_1 = f(R_{t_1}) = t_1 + 1$. In this case, liftoff never occurs⁸. More generally, the liftoff date t_1 in a rational expectations equilibrium is a fixed point of

$$t_1 = f(\pi(0), x(0), \pi(t_1), x(t_1)). \quad (16)$$

Because, as described earlier, the central bank lifts off at the earliest time for which $t_1 = f(R_{t_1})$, t_1 is actually the smallest fixed point of equation (16).

Because I only analyze rational expectations equilibria, from now on I directly write $x(t)$ instead of $E_s [x(t)]$ and $\pi(t)$ instead of $E_s [\pi(t)]$. However it is useful to keep in mind for the interpretation of many of the results below that $x(t_1)$ and $\pi(t_1)$ are the actual realization of inflation and output at t_1 just as much as they are, as of some time $t < t_1$, expectations of future inflation and output at liftoff.

⁸ One technicality is worth pointing out. If $t_1, f(R_{t_1}) \in \mathbb{R}$, the definition of rational expectations equilibrium does not allow for an equilibrium in which liftoff never occurs, since condition (15) would not hold. Allowing for $t_1, f(E) = \infty$ restores this possibility. When liftoff never occurs, the economy converges to the non-optimal deflationary steady-state $(\pi_{zlb}, x_{zlb}) = (-r_h, -\frac{\rho}{\kappa} r_h) \neq (0, 0)$ and, in addition, there always is indeterminacy.

A central bank hoping to avoid non-optimal equilibria and/or indeterminacy will never pick a rule where this happens, so there is little loss in assuming liftoff must occur in finite time.

5.2 A Neo-Fisherian Rule

The smallest necessary deviation from the constant liftoff date rule (10) that produces a determinate optimal equilibrium is to change liftoff so that it is equal to t^* , $t^* + 1$ or $t^* + 2$ depending on the values of $x(0)$ and $\pi(0)$. More precisely, the interest rate rule is

$$i(t) = \begin{cases} 0 & , \quad 0 \leq t < f(R_{t_1}) \\ (1 - \kappa\sigma\lambda)\pi^*(t) + r_h & , \quad t \geq f(R_{t_1}) \end{cases} \quad (17)$$

where

$$f(R_{t_1}) = \begin{cases} t^* & , \quad \text{if } (\pi(0), x(0)) = (\pi^*(0), x^*(0)) \text{ or} \\ & \quad Ax(0) + B\pi(0) \neq C \\ t^* + 1 & , \quad \text{if } (\pi(0), x(0)) \neq (\pi^*(0), x^*(0)) \text{ and} \\ & \quad Ax(0) + B\pi(0) = C \text{ and } Dx(0) + E\pi(0) \neq F \\ t^* + 2 & , \quad \text{otherwise} \end{cases} \quad (18)$$

for some constants A, B, C, D, E, F given explicitly in Appendix A.3. This example shows that a state-dependent liftoff time is a powerful tool to fight indeterminacy at the ZLB. It also shows that when a binding ZLB is introduced, following the Taylor principle is no longer necessary for determinacy (in the next section, I show it is also not sufficient). The rule in equation (17) can be thought of as a Taylor rule with coefficients of zero on inflation and output and a time-varying intercept equal to $\hat{r}(t) = (1 - \kappa\sigma\lambda)\pi^*(t) + r(t) > 0$.

In this rule, the central bank commits to following the optimal path after liftoff irrespective of the state of the economy. The only state-dependent part of the rule is the time of liftoff. Since the optimal rule has positive interest rates after t^* , the economy under the rule given by equations (17)-(18) is always outside the ZLB for all $t \geq t_1$ and thus has the same behavior as the optimal equilibrium explained in section 3: The dynamics of the economy are saddle path stable, with a linear saddle that goes through the steady state $(\pi_{ss}, x_{ss}) = (0, 0)$.

When $\pi(0) = \pi^*(0)$ and $x(0) = x^*(0)$, the rule gives $f(R_{t_1}) = t^*$, $i(t) = i^*(t)$ and therefore the optimal equilibrium is a rational expectations equilibrium. In all other cases, the choice of f ensures no equilibrium is possible. To see why, consider a candidate equilibrium with initial conditions $\pi(0)$ and $x(0)$ different from $(\pi^*(0), x^*(0))$. There are three cases. First, if the economy is not on the saddle path at t^* , then the rule prescribes liftoff at t^* . The equation $Ax(0) + B\pi(0) \neq C$ in rule (18) describes the set of points $(\pi(0), x(0))$ for which this happens. Figure (2) is a useful guide. The condition $Ax(0) + B\pi(0) \neq C$ means that $(x(0), \pi(0))$ cannot be on the line where all the paths in the figure begin. But if paths do not start on the line $Ax(0) + B\pi(0) = C$, they do not end up on the saddle path, the line to which all the paths in the figure go to. Because the economy is not on its

saddle path at the time of liftoff, either inflation and output instantaneously jump a discrete amount to reach the saddle path, or else inflation and output become unbounded. In either case, an equilibrium cannot form. For the second case, focus on the points $(\pi(0), x(0))$ that take the economy to the saddle path at t^* but not at $t^* + 1$. In rule (18), this corresponds to the case $Ax(0) + B\pi(0) = C$ and $Dx(0) + E\pi(0) \neq C$. For these points, the rule assigns $f(R_{t_1}) = t^* + 1$ and equilibria are precluded by the same argument as in the first case. The points $(\pi(0), x(0))$ that reach the saddle path at $t^* + 1$ constitute the third case, for which the central bank picks $f(R_{t_1}) = t^* + 2$. There is at most one point in this category, since it is given by the intersection of two distinct lines: The line of initial conditions that reaches the saddle path at t^* and the one that reaches it at $t^* + 1$. This point, if it exists, cannot reach the saddle path at $t^* + 2$, the time of liftoff, and is therefore not an equilibrium. Since I have covered all possible $\pi(0)$ and $x(0)$ in the plane, I have shown that the rule does not implement any sub-optimal equilibrium.

The way equilibria are eliminated can alternatively be cast in terms of rational expectations. Suppose agents have expectations π_0^e and x_0^e for initial inflation and output (these can be ex-ante expectations, or concurrent with $t = 0$). If the central bank is credibly committed to rule (18), liftoff will be rationally expected to be at $f(\pi_0^e, x_0^e)$. On the other hand, an equilibrium also requires bounded paths for inflation and output. Therefore, agents have two rational ways to form expectations for $x(t_1)$ and $\pi(t_1)$. The first is to trace the evolution of $x(t)$ and $\pi(t)$ from $t = 0$ until $t = f(\pi_0^e, x_0^e)$ starting at (π_0^e, x_0^e) , giving an expected outcome of (π_1^e, x_1^e) . The second is to realize that non-explosive paths are expected to be on the saddle path at t_1 , giving a second set $(\tilde{\pi}_1^e, \tilde{x}_1^e)$ of expectations for $x(t_1)$ and $\pi(t_1)$. If (π_1^e, x_1^e) is not on the saddle path, we cannot have an equilibrium, since then $(\pi_1^e, x_1^e) \neq (\tilde{\pi}_1^e, \tilde{x}_1^e)$ and the two rational computations give contradictory expectations. By appropriately picking the value of $f(\pi_0^e, x_0^e)$, the central bank can always create the expectation that liftoff will occur at a time when the economy is not on the saddle path, thus eliminating any undesired equilibria.

An alternative perspective is to start by considering only bounded paths. Inflation and output expected to prevail at t_1 , (π_1^e, x_1^e) , must be on the saddle path. Assume $(\pi_1^e, x_1^e) \neq (\pi^*(t^*), x^*(t^*))$ and consider first expectations of liftoff occurring at t^* . Starting at the given (π_1^e, x_1^e) on the saddle path, trace the dynamics of the economy backwards using the IS, the NKPC and $i(t) = (1 - \kappa\sigma\lambda)\pi^*(t) + r_h$ for $t \geq t^*$ and $i(t) = 0$ for $t < t^*$. At time $t = 0$, the economy is expected to be at (π_0^e, x_0^e) . However, by equation (18), $f(\pi_0^e, x_0^e, \pi_1^e, x_1^e) \neq t^* = t_1$ and thus t^* cannot be the time of liftoff in a rational expectations equilibrium. Now consider liftoff at $t^* + 1$ or $t^* + 2$. The same logic indicates these cannot be rational expectations equilibria liftoff times. In brief, the proposed interest rate rule guarantees that, except for $(\pi^*(t^*), x^*(t^*))$, there are no initial conditions dire enough to

justify the degree of forward guidance stimulus promised by the central bank (or, if I had picked values lower than t^* for liftoff, there would be no initial conditions good enough to justify the lack of stimulus).

Although instructive and simple in many dimensions, a rule like (18) raises some important practical challenges. The core of this rule is Neo-Fisherian and is the same in spirit to that in Schmitt-Grohé and Uribe (2014). It relies on the idea that if the central bank raises interest rates, inflation will follow. The key for determinacy is that there are paths for which the central bank makes nominal interest rates positive even in the presence of a large deflation and negative output. While being currently studied and taken seriously by the literature, it remains controversial⁹. More importantly, Neo-Fisherianism, while it may turn out to be borne out by empirical evidence, is far from central bank orthodoxy. Judging by the lack of mention of Neo-Fisherism in official statements, it appears that most major central banks at the ZLB today would be reluctant to put this theory into practice or at least are not currently considering it¹⁰. One of the main motivations of this paper is to put forth a rule that can be readily applied and that requires as little deviation from current practices as possible. Irrespective of whether Neo-Fisherian ideas become the consensus among academics or policymakers in the future, the rule in equation (18) is certainly a leap from current practice, while the rule I present in the next section is not.

⁹ Schmitt-Grohé and Uribe (2014) and Schmitt-Grohé and Uribe (2012) have nominal interest rates that increase in a pre-determined fashion (irrespective of the behavior and output) and succeed in increasing inflation. Cochrane (2015) explores in depth many theoretical variations in which the Neo-Fisherian logic holds. Cochrane (2014) shows how fiscal-monetary cooperation in a model with interest on reserves also has higher interest rates causally raising then inflation.

In a model with a "liquidity effect", Andolfatto and Williamson (2015) obtain Neo-Fisherian results after a permanent increase in interest rates.

Kocherlakota (2016b) and Kocherlakota (2016a) question the the validity of the neo-Fisherian hypothesis by arguing that, in finite horizon models, its relevance depends on an empirical question: how do long-run inflation expectations depend on the central bank's peg?

Evans and McGough (2015b), Evans and McGough (2015a) and Evans, Honkapohja, and Mitra (2015) contend that introducing learning argues against the Neo-Fisherian view.

Garca-Schmidt and Woodford (2015) introduce a new boundedly rational cognitive process for agents and show that it eliminates the Neo-Fisherian predictions.

¹⁰ The president of the Federal Reserve Bank of St. Louis, James Bullard, considers Neo-Fisherianism in Bullard (2016). Although his remarks are personal and do not represent the views of the Federal Reserve, they can perhaps be viewed as Neo-Fisherian ideas percolating to some policymaking circles. Narayana Kocherlakota, the former president of the Federal Reserve Bank of Minneapolis, has expressed skepticism to Neo-Fisherianism in Kocherlakota (2016b) and Kocherlakota (2016a). To my knowledge, these are the only two public instances of a discussion of Neo-Fisherian issues by someone connected to a large central bank.

5.3 A Rule for Today's Central Banks

For the rest of the paper, I focus on a rule that is much closer to current central bank practice and is given by

$$i(t) = \begin{cases} 0 & , \quad 0 \leq t < f(R_{t_1}) \\ \max\{0, \xi_\pi(R_{t_1})\pi(t) + \xi_x(R_{t_1})x(t) + r_h\} & , \quad f(R_{t_1}) \leq t < \infty \end{cases}, \quad (19)$$

where f , ξ_π and ξ_x are functions of R_{t_1} chosen by the central bank. The rule has a forward guidance period followed by a standard Taylor rule period. The Taylor rule guarantees that interest rates do not become positive for “unconventional” states of the economy, precluding the Neo-Fisherian behavior of the rule in equations (17)-(18). In addition to a state-dependent liftoff rule, I allow for Taylor rule coefficients that depend on R_{t_1} . Similarly to the optimal equilibrium, this rule can be understood in three stages. The first stage is from $t = 0$ to T , when the natural rate is negative. The central bank sets $i(t) = 0$ for the entirety of this stage (recall $f(R_{t_1}) \geq T$ by assumption). The dynamics of the economy are exactly as in the “first stage” of the optimal equilibrium in section 3. In this stage, the central bank never changes its behavior. Each point $(\pi(0), x(0))$ maps to one and only one $(\pi(T), x(T))$ and the mapping is unaffected by expectations or outcomes. From T to t_1 , the dynamics follow those of the “second stage” of the optimal equilibrium, with the only difference that the dynamics are maintained until t_1 , which depends on R_{t_1} , instead of until the constant liftoff time t^* . In equilibrium $t_1 = f(R_{t_1})$ so the duration of this stage depends on the announced liftoff rule. In the last stage, from t_1 onward, the central bank follows a Taylor rule that respects the ZLB, allowing for endogenous entry and exit from the ZLB. Just as before, it is easiest to understand the last stage first and then work backwards, as I now do.

To study the economy after t_1 , I split the π - x plane into two regions depending on whether the ZLB is binding:

$$\Omega_{zlb}(R_{t_1}) = \{(x, \pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h \leq 0\}, \quad (20)$$

$$\Omega_{ss}(R_{t_1}) = \{(x, \pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h > 0\}, \quad (21)$$

where *zlb* stands for zero lower bound and *ss* stands for “intended steady-state”, as the region $\Omega_{ss}(R_{t_1})$ contains $(\pi_{ss}, x_{ss}) = (0, 0)$, the steady-state that the central bank would like the economy to converge to in the long run if the optimal equilibrium obtains. The boundary between the regions $\Omega_{zlb}(R_{t_1})$ and $\Omega_{ss}(R_{t_1})$ is a line

$$\partial\Omega(R_{t_1}) = \{(x, \pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h = 0\} \subset \Omega_{zlb}(R_{t_1}). \quad (22)$$

Henceforth, I suppress the dependence of ξ_x , ξ_π , Ω_{zlb} , Ω_{ss} and $\partial\Omega$ on R_{t_1} for ease of notation whenever it is not relevant.

Time changes in output and inflation, $(\dot{\pi}(t), \dot{x}(t))$, inherit the properties of $i(t)$ and are therefore not differentiable on $\partial\Omega$ as a function of time. However, they are always continuous, ensuring a continuous path for $(\pi(t), x(t))$ ¹¹. Clearly, after t_1 ,

$$i(t) = 0 \iff (\pi(t), x(t)) \in \Omega_{zlb} \quad (23)$$

$$i(t) = \xi_\pi \pi(t) + \xi_x x(t) + r_h \iff (\pi(t), x(t)) \in \Omega_{ss} \quad (24)$$

When using equations (23) and (24) in the IS and NKPC, the dynamics of the economy inside each of the two regions are given by a system of linear first-order ordinary differential equations in $x(t)$ and $\pi(t)$, each of which is easy to analyze inside its respective region with common methods. However, when analyzing the two regions together, the global dynamics are piecewise linear. Behavior of piecewise linear dynamical systems can, in general, exhibit a rich variety of non-linear phenomena such as limit cycles, bifurcations and chaos. The global properties can also behave quite differently from those of each individual region. For example, it is possible to construct bounded orbits for systems in which each region has explosive dynamics¹². To tackle the non-linearities of the New Keynesian economy at hand, I first analyze the properties of each of the two regions as if they held in the entire π - x plane and then combine them to understand the global behavior of the economy. Readers familiar with New Keynesian models without a ZLB should find the analysis of each of the separate regions familiar. The differences in this third stage, when $t \geq t_1$, arise when I combine the two regions and when I consider the effects of making Taylor rule coefficients depend on R_{t_1} .

Inside Ω_{ss} , there is always a single steady state, $(\pi_{ss}, x_{ss}) = (0, 0)$. The dynamic behavior of the economy depends on the choice of Taylor rule coefficients ξ_π and ξ_x . I focus on Taylor rule coefficient combinations that, in the absence of the ZLB, give either saddle-path or unstable dynamics, since the central bank would not pick stable dynamics that have no explosive paths as they always lead to indeterminacy¹³. The Taylor principle is the key concept needed to differentiate between saddle and unstable dynamics, and between *locally*

¹¹ In fact, $(\dot{x}(t), \dot{\pi}(t))$ is Lipschitz continuous in $(x(t), \pi(t))$ and continuous in t , guaranteeing global existence and uniqueness of the continuous solution.

¹² For one standard reference, see [Bernardo, Budd, Champneys, and Kowalczyk \(2008\)](#).

¹³ See [Lemma 9](#). I also exclude, for the same reason, the knife-edge case in which there is a line whose points are all steady-states, but the dynamics are otherwise explosive.

determinate and indeterminate equilibria. The *Taylor principle holds* if and only if

$$\begin{aligned} \kappa(\xi_\pi(R_{t_1}) - 1) + \rho\xi_x(R_{t_1}) &> 0 \\ \text{and} & \\ \xi_x(R_{t_1}) + \sigma\rho &> 0. \end{aligned} \tag{25}$$

Note that whether the Taylor principle holds depends on R_{t_1} so the same economy can have on and off-equilibrium paths that do not all satisfy the Taylor principle. In other words, the central bank can decide whether to follow the Taylor principle depending on the state of the economy (but must announce the specific form of this dependence before $t = 0$ and commit to it). When $\xi_x = 0$, the Taylor principle is equivalent to $\xi_\pi > 1$, one of its most used forms.

If the dynamics of the Ω_{ss} region were extended to the entire plane and nominal interest rates were allowed to be negative, or if I considered a small enough neighborhood of (π_{ss}, x_{ss}) that lies entirely in Ω_{ss} , the dynamics of the system would be unstable when the Taylor principle holds. All paths would be unbounded—or exit the small neighborhood—unless $(\pi(t), x(t)) = (\pi_{ss}, x_{ss}) = (0, 0)$ for all t . Figure 3 shows representative phase diagrams of two such economies. On the left-hand side diagram, the Taylor rule coefficients satisfy $(\xi_x - \sigma\rho)^2 - 4\kappa\sigma(\xi_\pi - 1) < 0$ and paths “slowly” spiral outward from the steady-state. When the reverse inequality holds, the steady-state is instead a source, shown in the right-hand side diagram. Without a ZLB, the Taylor principle is a necessary and sufficient condition for local determinacy. When the ZLB is re-introduced, since there always is a small enough neighborhood of (π_{ss}, x_{ss}) that it is contained entirely in Ω_{ss} , the Taylor principle is still a necessary and sufficient condition for local determinacy of an equilibrium with $(\pi(t), x(t)) = (\pi_{ss}, x_{ss})$ for all $t \geq t_1$. This is no longer the case for global determinacy, as was briefly discussed before and will be expanded on later.

The *Taylor principle does not hold* if and only if

$$\kappa(\xi_\pi(R_{t_1}) - 1) + \rho\xi_x(R_{t_1}) < 0. \tag{26}$$

If the dynamics of the Ω_{ss} region were extended to the entire plane and interest rates were allowed to be negative, or if I considered a small enough neighborhood of (π_{ss}, x_{ss}) that lies entirely in Ω_{ss} , the dynamics of the system would be saddle path stable. I denote the *ss saddle path* by Υ_{ss} . It is a line through the origin whose slope depends on ξ_x and ξ_π . Paths are bounded—or stay in the small neighborhood of (π_{ss}, x_{ss}) —if and only if they start on the saddle. Figure 4 displays a typical phase diagram when the Taylor principle does not hold.

Inside Ω_{zlb} , the system is always saddle path stable. Since $i(t) = 0$ inside Ω_{zlb} , the

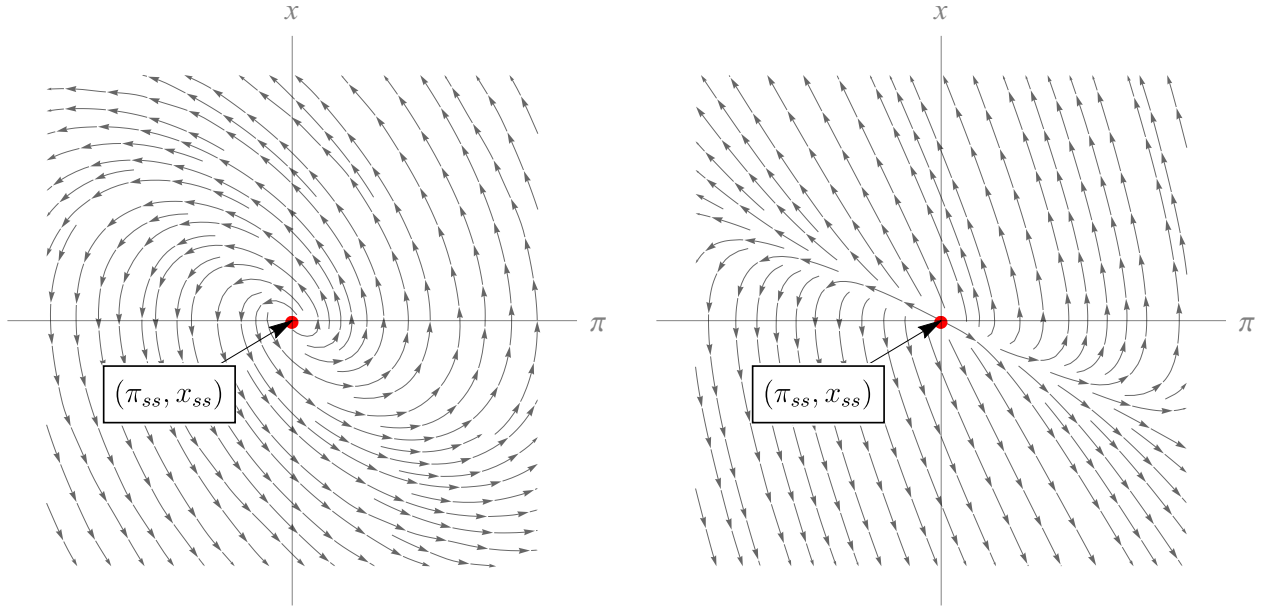


Figure 3: Dynamics of the economy when the Taylor principle holds and there is no ZLB. The dynamics of the left-hand side diagram have imaginary eigenvalues while the right-hand side has real ones. When there is no ZLB, the Taylor principle is necessary and sufficient for local determinacy. Unless the economy starts at $(0, 0)$, inflation and output become unbounded.

dynamics are akin to those of following a Taylor rule with $\xi_\pi = \xi_x = 0$ for which the Taylor principle does not hold, but with a different steady state. A large part of the challenge of ridding the economy of indeterminacy stems from the unresponsiveness of interest rates to inflation and output inside Ω_{zlb} . I denote the stable *zlb saddle path* by

$$\Upsilon_{zlb} = \left\{ (\pi, x) : x = \frac{\phi_1}{\kappa}\pi - \frac{\phi_2}{\kappa}r_h \right\}, \quad (27)$$

where

$$\phi_1 = \frac{1}{2}\rho + \frac{1}{2}\sqrt{\rho^2 + 4\frac{\kappa}{\sigma}} > 0, \quad (28)$$

$$\phi_2 = \frac{1}{2}\rho - \frac{1}{2}\sqrt{\rho^2 + 4\frac{\kappa}{\sigma}} < 0, \quad (29)$$

are the two eigenvalues of the system dynamics inside Ω_{zlb} . The *unstable zlb saddle path*,

$$\Psi_{zlb} = \left\{ (\pi, x) : x = \frac{\phi_2}{\kappa}\pi - \frac{\phi_1}{\kappa}r_h \right\}, \quad (30)$$

is the saddle path that would be stable if the system evolved backward in time. The *zlb saddle path*, a line with positive slope, intersects the *unstable zlb saddle path*, a line with

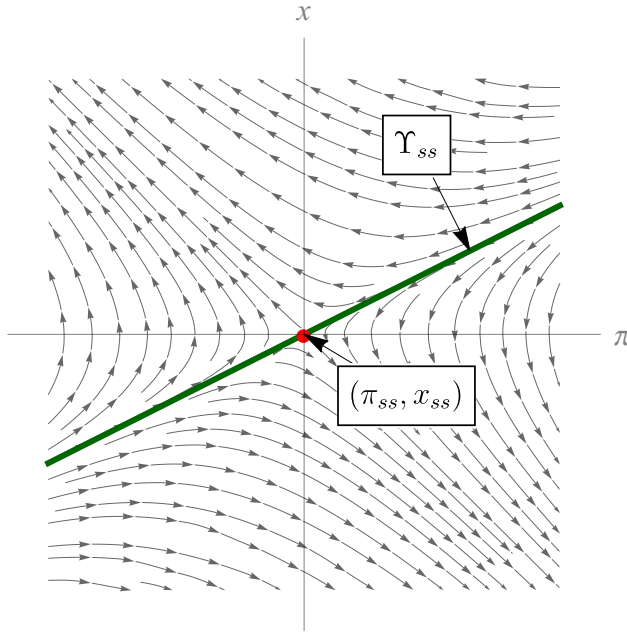


Figure 4: Dynamics of the economy when the Taylor principle does not hold and there is no ZLB. Unless the economy starts on its saddle path Υ_{ss} , shown in green, inflation and output become unbounded.

negative slope, at the *zlb steady state*

$$(\pi_{zlb}, x_{zlb}) = \left(-r_h, -\frac{\rho r_h}{\kappa} \right). \quad (31)$$

The point (π_{zlb}, x_{zlb}) is the only possible steady-state under the *zlb* dynamics and always lies in the third quadrant of the π - x plane. Neither the location of (π_{zlb}, x_{zlb}) nor the slopes of Υ_{zlb} or Ψ_{zlb} depend on Taylor rule coefficients or other policy choices of the central bank; they are fully specified by the parameters κ , ρ , σ and r_h . Figure 5 shows the phase diagram of $x(t)$ and $\pi(t)$ when the *zlb* dynamics hold in the entire π - x plane.

After having analyzed the Ω_{ss} and Ω_{zlb} regions separately, I combine the two and describe some of the global properties of the economy. The left-hand side panel of figure 6 shows an example in which the Taylor principle holds, while the right-hand side panel shows an example in which the Taylor principle does not hold. When the Taylor principle holds, the dynamics inside Ω_{ss} look like those in figure 3. When the Taylor principle does not hold, they look like those in figure 4. Of course, whether the Taylor principle holds or not, the dynamics in Ω_{zlb} always look like those in figure 5, as they are not affected by ξ_x or ξ_π . However, the coefficients ξ_x and ξ_π do have a crucial effect on the behavior of the economy at the ZLB, as they determine the location of the boundary $\partial\Omega$ and, consequently, whether an undesirable deflationary steady-state can exist in this economy. If the Taylor principle holds, the *zlb* steady-state (π_{zlb}, x_{zlb}) is in Ω_{zlb} , while when the Taylor principle does not

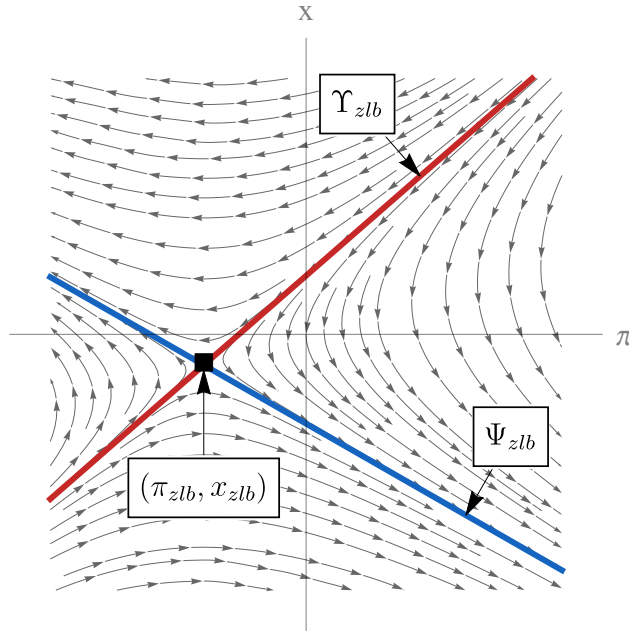


Figure 5: Dynamics of the economy when $i(t) = 0$ for all $\pi(t)$ and $x(t)$ and $r(t) = r_h > 0$. The red line, labeled Υ_{zlb} , is the saddle path. If the economy starts on Υ_{zlb} , it converges to the deflationary steady-state (π_{zlb}, x_{zlb}) . The blue line, labeled Ψ_{zlb} , is the “unstable saddle path”. If the economy starts on Ψ_{zlb} , it stays on Ψ_{zlb} and moves away from the deflationary steady-state (π_{zlb}, x_{zlb}) . If the economy is not on Υ_{zlb} , then inflation and output become unbounded.

hold, it is not. To see this, compute

$$\xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h = \xi_\pi (-r_h) + \xi_x \left(-\frac{\rho r_h}{\kappa} \right) + r_h \quad (32)$$

$$= -\frac{r_h}{\kappa} (\kappa (\xi_\pi - 1) + \rho \xi_x) \quad (33)$$

The sign of this expression determines whether the zlb steady state (π_{zlb}, x_{zlb}) is inside or outside Ω_{zlb} . In turn, the sign of $\kappa (\xi_\pi - 1) + \rho \xi_x$ is determined by whether the Taylor principle holds. When the Taylor principle holds, (π_{zlb}, x_{zlb}) is a steady state of the global dynamics. Because of its saddle dynamics, equilibria are locally indeterminate around (π_{zlb}, x_{zlb}) . Together with the intended steady-state (π_{ss}, x_{ss}) , they are the two global steady-states of the economy. On the other hand, if the Taylor principle does not hold, $(\pi_{zlb}, x_{zlb}) \in \Omega_{ss}$. Under the Ω_{ss} dynamics, the point (π_{zlb}, x_{zlb}) is not a steady-state. In this case, the only steady state for the global dynamics is the desired one, (π_{ss}, x_{ss}) .

The result that following the Taylor principle outside the ZLB induces a deflationary steady-state inside Ω_{zlb} is similar to [Benhabib et al. \(2001\)](#). They show that an infinite number of equilibria arise that can start arbitrarily close to the intended steady-state (π_{ss}, x_{ss})

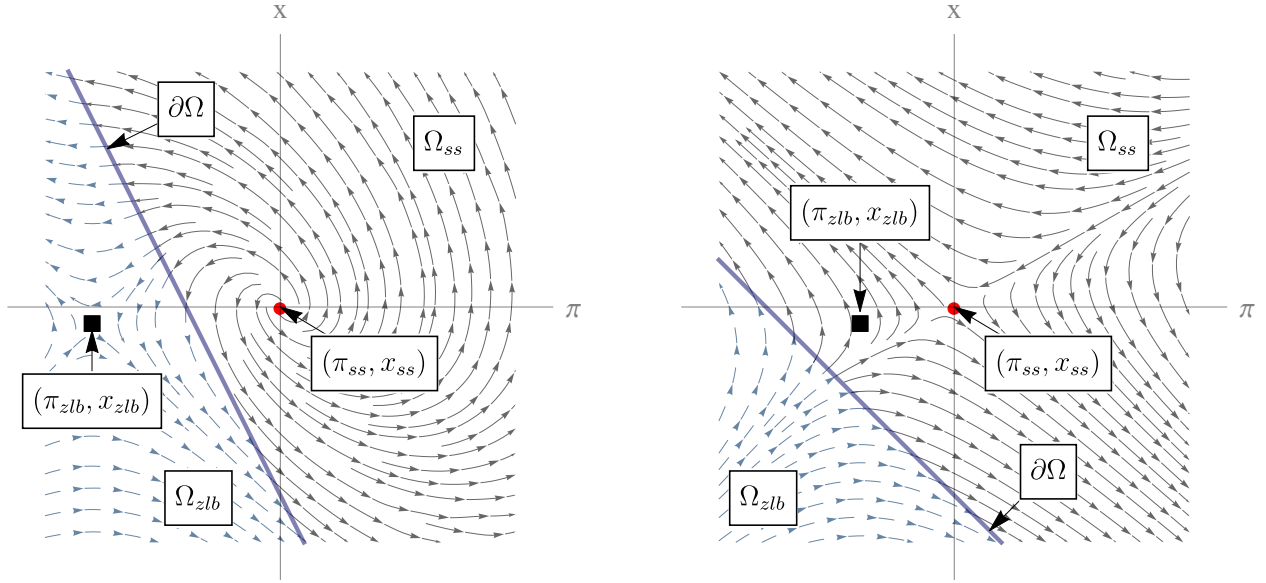


Figure 6: Non-linear dynamics of the economy after liftoff. The central bank follows the Taylor rule $i(t) = \max \{0, \xi_\pi \pi(t) + \xi_x x(t) + r_h\}$. When $\xi_\pi \pi(t) + \xi_x x(t) + r_h > 0$, the economy is in the region Ω_{ss} and follows the solid black flow lines. When $\xi_\pi \pi(t) + \xi_x x(t) + r_h \leq 0$, it is in the region Ω_{zlb} and follows the dashed blue flow lines. The boundary between the two regions is the line $\partial\Omega$. The point (π_{ss}, x_{ss}) , shown in red, is always a steady-state of the economy. The point (π_{zlb}, x_{zlb}) , shown as a black square, is a steady-state of the economy if and only if the Taylor principle holds, as in the left-hand side panel. When the Taylor principle does not hold, as in the right-hand side panel, (π_{zlb}, x_{zlb}) is not in Ω_{zlb} and is therefore not a steady-state.

yet still fall into an expectational trap and converge to (π_{zlb}, x_{zlb}) . The same possibility is present here. To construct these equilibria in the NK model I analyze, first pick two numbers r and q such that $q, r > T$ and $r - q > T$ and let¹⁴ $(\pi_b, x_b) = \partial\Omega \cap \Upsilon_{zlb}$. Assume the Taylor principle holds. For the next steps, refer to figure 7. Using $(\pi(r), x(r)) = (\pi_b, x_b)$ as the starting point, trace the dynamics of $(\pi(t), x(t))$ backwards in time using the interest rate specified by (24) for a length of time q . As in Benhabib et al. (2001), these equilibria can get arbitrarily close to the intended steady-state. Because the dynamics of $(\pi(t), x(t))$ are unstable when going forward in time, they are stable backward in time and $(\pi(t), x(t))$ converges to (π_{ss}, x_{ss}) as $q \rightarrow \infty$ ¹⁵. At time $r - q$, trace the dynamics of $(\pi(t), x(t))$ backward

¹⁴ If $\kappa\xi_\pi + \phi_1\xi_x \neq 0$, then the zlb boundary and the zlb saddle path do not intersect, i.e. $\partial\Omega \cap \Upsilon_{zlb} = \emptyset$. Albeit not a general strategy to eliminate all non-optimal equilibria, picking ξ_x, ξ_π such that $\kappa\xi_\pi + \phi_1\xi_x = 0$ does preclude this particular class of equilibria from forming for any choice of f . This possibility was not present in Benhabib et al. (2001) as their model did not have both inflation and output as states of the economy.

¹⁵ The result is not immediate, since it may have been possible that $(x(t), \pi(t))$ exits Ω_{ss} before getting close to (x_{ss}, π_{ss}) and then follow the Ω_{zlb} dynamics for which (x_{ss}, π_{ss}) is no longer a sink (flowing backwards in time). However, I show in Appendix A.5, Lemma 8, item 4 that this never happens. For

in time using $(\pi(r), x(r))$ as the starting point and $i(t) = 0$ throughout, until $t = 0$, when the path reaches $(\pi(0), x(0))$. Of course, the natural rate is negative before T and positive after T , so the dynamics change. Set $t_1 = r - q > T$. By construction, the path starting at $(\pi(0), x(0))$ reaches (π_b, x_b) at time r when following the interest rate rule in equation (19). Now going forward in time, for $t \geq r$, $(\pi(t), x(t)) \in \Upsilon_{ss} \subset \Omega_{zlb}$, which means the economy travels on the *zlb* saddle path towards the unintended steady state (π_{zlb}, x_{zlb}) . The path constructed is continuous, bounded and has consistent expectations: It is a rational expectations equilibrium. All equilibria in this class can be obtained by picking different q and r .

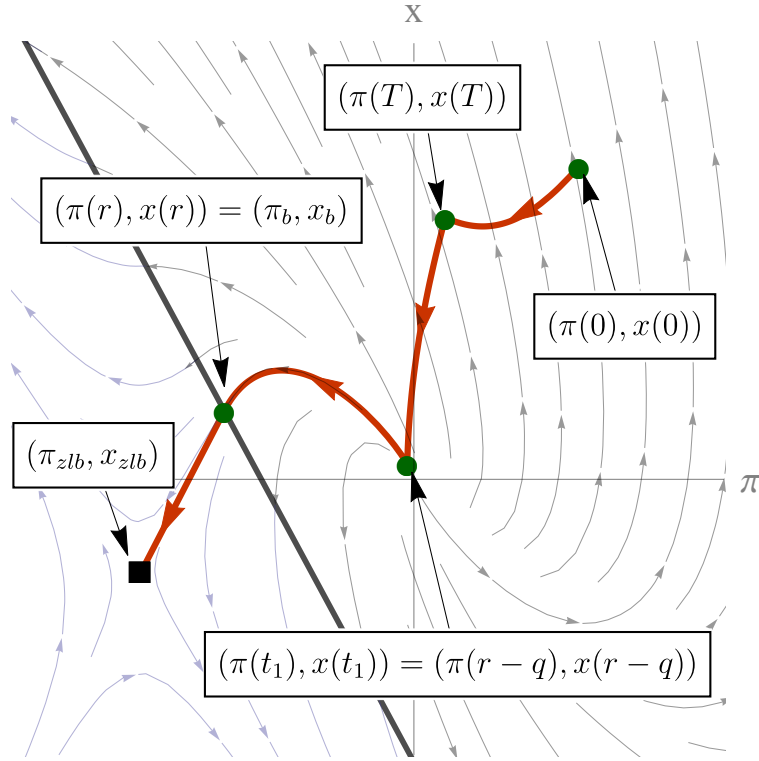


Figure 7: An equilibrium analogous to the one studied by Benhabib et al. (2001). The flow lines correspond to dynamics after liftoff, which occurs at t_1 . Because the Taylor principle holds, there is a deflationary steady-state (π_{zlb}, x_{zlb}) , shown as a black square. At time t_1 , even though the economy is outside the ZLB and can get arbitrarily close to the “desired” steady-state $(\pi_{ss}, x_{ss}) = (0, 0)$, it still converges to the “unintended” steady-state (π_{zlb}, x_{zlb}) . At time r , the economy hits the ZLB and $i(t) = 0$ forever after.

When $\xi_x = 0$ and $\xi_\pi > 1$, the interest rate rule in equation (19) obeys the Taylor principle, is non-negative and non-decreasing with respect to $\pi(t)$. In Benhabib et al. (2001), these conditions were sufficient to produce indeterminacy. Attempting to eliminate the undesired deflationary equilibria by not following the Taylor principle does succeed in Benhabib et al.

any q , the path of $(x(t), \pi(t))$ remains entirely in Ω_{ss} .

(2001)'s framework, but at the cost of introducing another type of multiple equilibria arising from the local indeterminacy around the intended steady state. In contrast, in section 6.3, I show that all Benhabib et al. (2001) equilibria can be ruled out while following the Taylor principle by picking an appropriate liftoff rule and that not following the Taylor principle does not necessarily lead to Benhabib et al. (2001)-like equilibria.

The dynamics of the economy between T and t_1 , the second stage, are the same as those in Ω_{zlb} during $t \geq t_1$ but now apply to the entire π - x plane. The central bank is committed to $i(t) = 0$ between T and t_1 for any realizations of inflation and output. Two key objects to interpret expectations and the interest rate rule between T and t_1 are:

$$d_{exit}(t) = x(t) - \frac{\phi_1}{\kappa} \pi(t) + \frac{\phi_2 r_h}{\kappa} \quad (34)$$

$$= [x(t) - x_{zlb}] - \frac{\phi_1}{\kappa} [\pi(t) - \pi_{zlb}] \quad (35)$$

$$d_{trap}(t) = x(t) - \frac{\phi_2}{\kappa} \pi(t) + \frac{\phi_1 r_h}{\kappa} \quad (36)$$

$$= [x(t) - x_{zlb}] - \frac{\phi_2}{\kappa} [\pi(t) - \pi_{zlb}] \quad (37)$$

The value of d_{exit} is a measure of the (signed) distance to the stable zlb saddle path Υ_{zlb} defined in equation (27). Similarly, d_{trap} gives a measure of distance from $(\pi(t), x(t))$ to the unstable saddle path Ψ_{zlb} , defined in equation (30). A d_{exit} closer to zero means the economy is closer to Υ_{zlb} and hence will behave more similarly to an economy that is on Υ_{zlb} , at least for some time. A d_{exit} closer to zero also implies that the economy at some point gets closer to the unintended steady-state (π_{zlb}, x_{zlb}) . In fact, $d_{exit}(t) = 0$ indicates the economy is exactly on Υ_{zlb} at time t and converging towards (π_{zlb}, x_{zlb}) . In contrast, a d_{trap} closer to zero implies the dynamics of the economy look more like those of the unstable saddle path Ψ_{zlb} , pushing the economy further away from (π_{zlb}, x_{zlb}) . These informal observations can be made precise and given clearer economic meaning by understanding the eigenvectors of the system. For $t \in [T, t_1)$, the eigenvectors of the system are

$$v_{exit} = \begin{bmatrix} -\frac{\phi_2}{\phi_1 - \phi_2} \\ -\frac{\kappa}{\phi_1 - \phi_2} \end{bmatrix} \text{ and } v_{trap} = \begin{bmatrix} \frac{\phi_1}{\phi_1 - \phi_2} \\ \frac{\kappa}{\phi_1 - \phi_2} \end{bmatrix}. \quad (38)$$

The eigenvector v_{exit} is associated with the ‘‘explosive’’ eigenvalue $\phi_1 > 0$ and lies on the unstable saddle path Ψ_{zlb} . The eigenvector v_{trap} is associated with the ‘‘stabilizing’’ eigenvalue $\phi_2 < 0$ and lies on the stable saddle path Υ_{zlb} . The dynamics of the economy can be described as the combination of two forces: v_{trap} is the ‘‘trap factor’’ driving the economy into the expectational trap and (π_{zlb}, x_{zlb}) , while v_{exit} is the ‘‘exit factor’’ pulling the economy away from it. After a change of coordinates that makes (π_{zlb}, x_{zlb}) the origin and

the eigenvectors of the system the basis vectors, the vector $(\pi(t), x(t))$ has coordinates $(d_{exit}(t), d_{trap}(t))$

$$\begin{bmatrix} -\frac{\phi_2}{\phi_1-\phi_2} & \frac{\phi_1}{\phi_1-\phi_2} \\ -\frac{\kappa}{\phi_1-\phi_2} & \frac{\kappa}{\phi_1-\phi_2} \end{bmatrix}^{-1} \begin{bmatrix} x(t) - x_{zlb} \\ \pi(t) - \pi_{zlb} \end{bmatrix} = \begin{bmatrix} d_{exit}(t) \\ d_{trap}(t) \end{bmatrix} \quad (39)$$

and

$$\begin{bmatrix} x(t) - x_{zlb} \\ \pi(t) - \pi_{zlb} \end{bmatrix} = d_{exit}(t) v_{exit} + d_{trap}(t) v_{trap}. \quad (40)$$

In other words, projecting $(\pi(t), x(t))$ onto the eigenvalues gives loadings of $(d_{exit}(t), d_{trap}(t))$. This linear two-factor representation is exact in the sense that there is no residual left once $x(t)$ and $\pi(t)$ are expressed as a linear combination of the factors plus a constant. Because eigenvectors are defined only up to a multiplicative constant, it is the the ratio of loadings at different points in time and not their specific magnitude at some particular time that gives the relevant information about the strength of expectations to either go towards or away from the unintended deflationary state. The necessary and sufficient conditions for determinacy in Section 6.3 are most simply expressed as functions of these ratios, highlighting not just the mathematical convenience but also the economic importance of d_{exit} and d_{trap} . The level of current inflation and output are, on their own, not very informative about whether the economy is in a liquidity trap, at risk of converging to (π_{zlb}, x_{zlb}) or on a desirable policy path, as has been already pointed out through various other arguments by [Benhabib et al. \(2001\)](#), [Werning \(2011\)](#) and others. The above decomposition suggests that, in the context of the basic New Keynesian model, the right empirical objects to pay attention to are the linear combinations d_{exit} and d_{trap} in equations (34) and (36).

The case in which $d_{exit} = 0$ is of special importance to questions of determinacy, so it is worth developing some economic intuition about it. Falling into the expectational trap requires being on Υ_{zlb} at a future time, which is equivalent to a zero future loading on the exit factor v_{exit} , or a future distance of $d_{exit} = 0$ to Υ_{zlb} . In this case, the expectations of a future $d_{exit} = 0$ are always strong enough to feed back into today's expectations and sustain a rational expectations equilibrium that ends up in (π_{zlb}, x_{zlb}) . We can see this directly in the IS and NKPC with $i(t) = 0$ by evaluating them at $t = s - dt$ and re-arranging, to get

$$\pi(s - dt) = (1 - \rho dt) \pi(s) + \kappa x(s) dt \quad (41)$$

$$x(s - dt) = x(s) + \sigma^{-1} (r_h + \pi(s)) dt \quad (42)$$

These two equations show the expectations of inflation and output at time $s - dt$ that can sustain some given expectations of inflation and output at time s . With expectations of $\pi(s)$

and $x(s)$ such that

$$d_{exit}(s) = 0, \quad (43)$$

the IS and NKPC become

$$\pi(s - dt) = \pi(s) - \frac{\phi_2 \kappa}{\phi_1 - \phi_2} d_{trap}(s) dt \quad (44)$$

$$x(s - dt) = x(s) - \frac{\phi_2 \phi_1}{\phi_1 - \phi_2} d_{trap}(s) dt \quad (45)$$

Looking backward from time s to $s - dt$, the last two equations reveal that $d_{trap}(s)$ determines what expectations of inflation and output at $s - dt$ can sustain the prospective time- s inflation and output. These time $s - dt$ expectations satisfy

$$d_{exit}(s - dt) = x(s - dt) - \frac{\phi_1}{\kappa} \pi(s - dt) + \frac{\phi_2 r_h}{\kappa} = 0 \quad (46)$$

Equation (46) means that expectations of not being able to exit the trap at some time s are possible to sustain in equilibrium solely with earlier beliefs of not being able to exit the trap. Repeating the same procedure shows that, $d_{exit}(s - \Delta) = 0$ for all $\Delta > 0$ (such that $s - \Delta$ is in the second stage I am analyzing, between T and t_1). This is the exact sense in which expectations of a future $d_{exit} = 0$ have a “strong enough” feedback to sustain $d_{exit} = 0$ today. Standing at time $s - dt$, equations (44) and (45) say that inflation and output over the next instant dt are expected to change only to the extent that future expectations of falling into the trap, given by $d_{trap}(s)$, are different from zero. Interpreted this way, the only reason inflation and output are moving is because of the expectation of falling into the trap. The coefficients in front of $d_{trap}(s)$ in equations(44) and (45) are both positive. A $d_{trap}(s)$ further from zero implies $x(s - dt)$ and $\pi(s - dt)$ are further from $x(s)$ and $\pi(s)$. It takes stronger expectations of falling into the trap when the economy is further away from it for expectations to be supportive of the economy actually falling into the trap. As the economy approaches (π_{zlb}, x_{zlb}) , d_{trap} endogenously gets closer to zero. When there are no expectations of falling into the trap, $d_{trap}(s) = 0$ implies $x(s - dt) = x(s)$. The economy is in steady state and must already be at (π_{zlb}, x_{zlb}) . As $d_{exit} = d_{trap} = 0$, there are no expectations of either falling into or escaping the liquidity trap so the only option is to be at (π_{zlb}, x_{zlb}) . From the perspective of a time before $s - dt$, $d_{trap}(s)$ conveys the revision in expectations from $s - dt$ to s needed to support the equilibrium with $d_{exit}(s) = 0$. Of course,

in a purely mathematical sense, all I have done in this paragraph is to trace the dynamics of the system backward or forward, deducing that an economy already on Υ_{zlb} at some time s must have been there before and will also be there after.

The magnitude of the coefficients in front of d_{trap} in equations (44) and (45) are also

important to assess how fast expectations must be changing over time and with respect to the current state of the economy in order to be rational. Since ϕ_2 is the eigenvalue associated with v_{trap} ,

$$\dot{d}_{trap}(s) = \phi_2 d_{trap}(s). \quad (47)$$

The eigenvalue $\phi_2 < 0$ is the modulator for how strongly expectations of falling into the trap must be if they are rational. The eigenvalue ϕ_2 is decreasing in κ and σ^{-1} but increasing in ρ . Given an initial level of expectations, more price flexibility and a larger elasticity of intertemporal substitution imply larger changes in expectations for a given length of time and a faster convergence to (π_{zlb}, x_{zlb}) . A more patient representative consumer, on the other hand, revises expectations more slowly.

Finally, in the first stage, when $t < T$, the economy behaves exactly as in the optimal equilibrium, since $i(t) = 0$ by assumption and there are no other decisions for the central bank to make. Figure (8) shows the phase portrait. It is similar to the phase portrait for the second stage, but because the natural rate is negative, the steady-state, labeled (π_l, x_l) in the figure, is in the first quadrant.

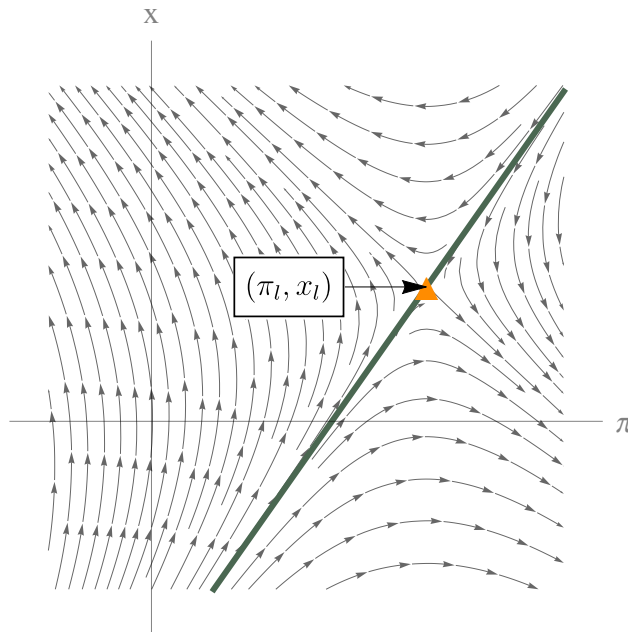


Figure 8: Dynamics of the economy when $i(t) = 0$ for all $\pi(t)$ and $x(t)$ and $r(t) = r_l < 0$. The green line is the saddle path and the orange triangle, labeled (π_l, x_l) is the steady-state.

6 How to Implement the Optimal Equilibrium without Indeterminacy

In this section, I prove the main result of the paper: Allowing the liftoff rule f to depend on R_{t_1} , the state of the economy, makes it possible to implement the optimal equilibrium without indeterminacy by means of the rule in equation (19).

Having assumed that expectations are rational from the beginning, there are, by definition, two other ways in which paths that follow the IS equation, the NKPC and the interest rate rule in equation (19) can fail to be an equilibrium. They can be unbounded or discontinuous. Understanding when paths are continuous and bounded is therefore a necessary step towards finding a rule that can always guarantee that the optimal equilibrium is the only equilibrium of the economy.

6.1 Continuous Pasting

In each of the intervals $[0, T)$, $[T, t_1)$ and $[t_1, \infty)$, paths for $(\pi(t), x(t))$ are automatically continuous because they are the solution to a standard system of linear ODEs in which $(\dot{\pi}(t), \dot{x}(t))$, given by equations (1), (2) and (19), are appropriately smooth functions of $\pi(t)$, $x(t)$ and t . At $t = T$, when $r(t)$ jumps, and perhaps at $t = t_1$, when $i(t)$ may jump, $(\pi(t), x(t))$ is not necessarily continuous. To have a continuous path for $(\pi(t), x(t))$, the solutions to the classical ODEs on $t \in [0, T)$, $t \in [T, t_1)$ and $t \in [t_1, \infty)$ have to agree at T and t_1 . The continuous pasting conditions for this to happen are

$$\lim_{t \uparrow T} (\pi(t), x(t)) = (\pi(T), x(T)), \quad (48)$$

$$\lim_{t \uparrow t_1} (\pi(t), x(t)) = (\pi(t_1), x(t_1)). \quad (49)$$

In economic terms, as explained in section 2, the continuous pasting conditions are no arbitrage conditions that must hold because of the representative consumer's Euler equation (or its log-linear version, the IS equation). Equivalently, continuous pasting means that expectations do not change abruptly

$$\lim_{t \uparrow T} (d_{exit}(t), d_{trap}(t)) = (d_{exit}(T), d_{trap}(T)), \quad (50)$$

$$\lim_{t \uparrow t_1} (d_{exit}(t), d_{trap}(t)) = (d_{exit}(t_1), d_{trap}(t_1)). \quad (51)$$

Let \bar{t}_1 be a number greater than or equal to T and let $\bar{R} = (\pi_0, x_0, \pi_1, x_1)$ be a vector of four numbers. I characterize the constraints that \bar{R} and \bar{t}_1 must satisfy for a path with $R_{t_1} = \bar{R}$ and $f(R_{t_1}) = \bar{t}_1$ to be continuous. To do so, I first solve the system of ODEs given

by equations (1), (2) and (19) for $t \in [0, T)$ using (π_0, x_0) as initial conditions. I then take the limit of this solution as $t \rightarrow T$ to get (π_T, x_T) and use this point as the initial condition when solving the ODEs for $t \in [T, \bar{t}_1)$. Continuous pasting at \bar{t}_1 requires that the limit of this last solution as $t \rightarrow \bar{t}_1$ is (π_1, x_1) . Graphically, continuous pasting requires stitching together the dynamics of figures (8) and (5) at T and of figures (5) and (6) at t_1 .

The goal of this section is to use the continuous pasting conditions to derive

$$\mathcal{P}(\bar{R}) = 0, \quad (52)$$

$$\bar{t}_1 \in \mathcal{T}(\bar{R}), \quad (53)$$

where \mathcal{P} is a function and \mathcal{T} is a set, both of which I explicitly find below. Assuming continuous pasting already happens at T , equations (52) and (53) are equivalent to continuous pasting at \bar{t}_1 . Equation (52) depends only on \bar{R} and not on \bar{t}_1 . It gives a family of points x_0, π_0, x_1, π_1 that can be part of a continuous path for some $\bar{t}_1 \geq T$. Equation (53) then describes the specific points in that family that are consistent with liftoff at a specific \bar{t}_1 . Separating \bar{R} and \bar{t}_1 in this way will make it easier and more intuitive to prove determinacy. On a first pass, the reader can skip to the last paragraph in this section without any loss of continuity.

To derive equations (52) and (53), and to solve for \mathcal{P} and \mathcal{T} , I consider four cases separately. The first case corresponds to the economy reaching (π_{zlb}, x_{zlb}) at \bar{t}_1 ; the second and third, to the economy reaching, respectively, Υ_{zlb} and Ψ_{zlb} at \bar{t}_1 ; the fourth case considers all remaining \bar{R} .

The first case, shown in green in figure 9, is defined by (π_0, x_0) such that¹⁶

$$d_{exit}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_1}{\kappa} [\pi_0 - \pi_{zlb}] = \frac{\phi_2}{\kappa} (r_h - r_l) (1 - e^{-T\phi_1}), \quad (54)$$

$$d_{trap}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_2}{\kappa} [\pi_0 - \pi_{zlb}] = \frac{\phi_1}{\kappa} (r_h - r_l) (1 - e^{-T\phi_2}). \quad (55)$$

In the figure, the line defined by equation (54) is the black dashed line while the line in equation (55) is the dashed gray line. This first case corresponds to (π_0, x_0) at the intersection of the two lines. The economy reaches the zlb steady-state (π_{zlb}, x_{zlb}) at $t = T$. Since between T and \bar{t}_1 the point (π_{zlb}, x_{zlb}) is a steady-state, the economy just sits there for all $t \in [T, \bar{t}_1)$.

¹⁶ With slight abuse of notation, in this section I write $d_{exit}(\pi_0, x_0)$ instead of $d_{exit}(t)$ to emphasize that $d_{exit}(\pi_0, x_0)$ is not a function of time since (π_0, x_0) is a vector of two numbers, as opposed to $(\pi(0), x(0))$, which is a function of time evaluated at $t = 0$. This is important because it clarifies that the continuous pasting conditions are not time-varying and that \mathcal{P} and \mathcal{T} do not depend on the actual path R_{t_1} , which is needed in some of the proofs. The same notation applies to d_{trap} and π_T, x_T, π_1, x_1 .

The continuous pasting conditions are

$$d_{exit}(\pi_T, x_T) = d_{trap}(\pi_T, x_T) = 0, \quad (56)$$

$$d_{exit}(\pi_1, x_1) = d_{trap}(\pi_1, x_1) = 0, \quad (57)$$

i.e. $(\pi_T, x_T) = (\pi_1, x_1) = (\pi_{zlb}, x_{zlb})$. In figure 9, the lines described in equations (56) and (57) are shown in the solid black and gray lines, and correspond to Υ_{zlb} and Ψ_{zlb} . Expectations to fall into or exit the expectational trap are both zero. Equations (56) and (57) define the function \mathcal{P} for this case

$$\mathcal{P}(\bar{R}) = 1 \{d_{trap}(\pi_1, x_1) \neq 0 \text{ or } d_{trap}(\pi_1, x_1) \neq 0\},$$

where $1\{E\}$ is the indicator function for event E , which is equal to 1 if E is true and zero otherwise. $\mathcal{P}(\bar{R}) = 0$ if and only if $d_{trap}(\pi_1, x_1)$ and $d_{trap}(\pi_1, x_1)$ are both zero and if and only if equations (56) and (57) hold. Graphically, the set of \bar{R} such that $\mathcal{P}(\bar{R}) = 0$ are the two points in figure 9 where the lines intersect and where the green path begins and ends. Changing \bar{t}_1 can change neither the time of arrival to (π_{zlb}, x_{zlb}) , which is equal to T , nor the actual set of points in the π - x plane visited by the economy from $t = 0$ to \bar{t}_1 . A larger \bar{t}_1 only guarantees that the economy is at (π_{zlb}, x_{zlb}) longer. Once (56) and (57) hold, any $\bar{t}_1 \geq T$ is consistent with continuous pasting and thus

$$\mathcal{T}(\bar{R}) = [T, \infty).$$

Because $\bar{t}_1 \in \mathcal{T}(\bar{R})$ does not introduce any additional restrictions on \bar{t}_1 , this case will present an important obstacle for a central bank trying to avoid the deflationary expectational trap by means of a liftoff rule. In the three remaining cases, $\mathcal{T}(\bar{R})$ is single-valued and depends on \bar{R} .

In the second case, the economy reaches some point in the zlb saddle path Υ_{zlb} at \bar{t}_1 , except for (π_{zlb}, x_{zlb}) , which was just analyzed. This case, shown in red in figure 9, occurs when

$$d_{exit}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_1}{\kappa} [\pi_0 - \pi_{zlb}] = \frac{\phi_2}{\kappa} (r_h - r_l) (1 - e^{-T\phi_1}) \quad (58)$$

$$d_{trap}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_2}{\kappa} [\pi_0 - \pi_{zlb}] \neq \frac{\phi_1}{\kappa} (r_h - r_l) (1 - e^{-T\phi_2}) \quad (59)$$

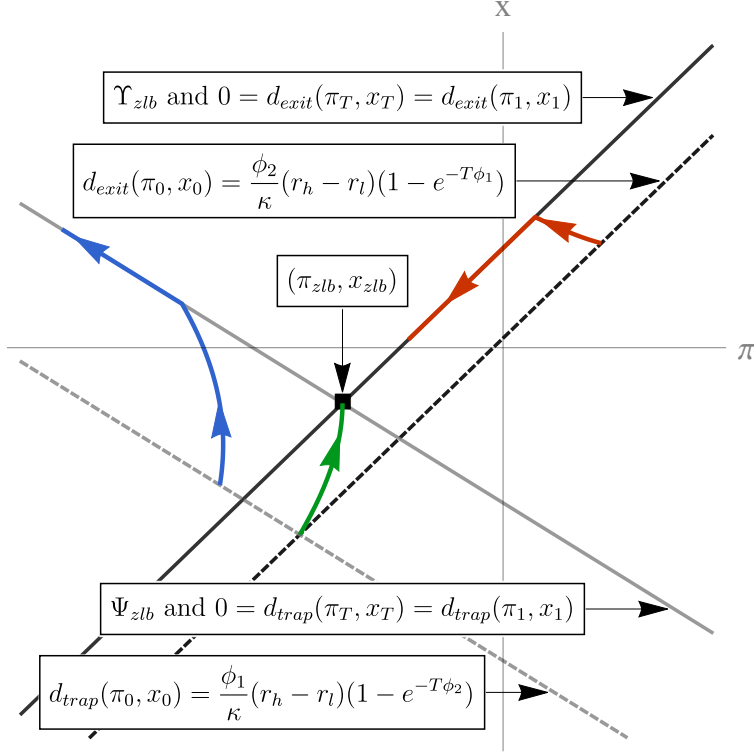


Figure 9: Continuous pasting conditions when (π_1, x_1) is in Υ_{zlb} or Ψ_{zlb} . Continuous pasting requires that paths that are on the zlb saddle path (the black solid line) at times T and t_1 , such as the red one, be on the $d_{exit}(\pi_0, x_0) = \frac{\phi_2}{\kappa}(r_h - r_l)(1 - e^{-T\phi_1})$ line (the black dashed line) at time $t = 0$. For these paths, expectations of falling into the deflationary equilibrium (π_{zlb}, x_{zlb}) are self-fulfilling. To be continuous, paths like the blue that start on the $d_{trap}(\pi_0, x_0) = \frac{\phi_1}{\kappa}(r_h - r_l)(1 - e^{-T\phi_2})$ line (the gray dashed line) at $t = 0$ must be on Ψ_{zlb} at times T and t_1 . These paths escape the deflationary trap. The green path originates at the intersection of the two dashed lines. Continuous pasting requires that it reaches the deflationary steady-state at T .

Continuous pasting at T requires

$$d_{exit}(\pi_T, x_T) = [x_T - x_{zlb}] - \frac{\phi_1}{\kappa} [\pi_T - \pi_{zlb}] = 0, \quad (60)$$

$$T = \frac{1}{\phi_2} \log \left(\frac{d_{trap}(\pi_T, x_T)}{d_{trap}(\pi_0, x_0) + \frac{\phi_1}{\kappa}(r_h - r_l)(e^{-T\phi_2} - 1)} \right). \quad (61)$$

Continuous pasting at \bar{t}_1 requires

$$d_{exit}(\pi_1, x_1) = [x_1 - x_{zlb}] - \frac{\phi_1}{\kappa} [\pi_1 - \pi_{zlb}] = 0 \quad (62)$$

$$\bar{t}_1 = T + \frac{1}{\phi_2} \log \left(\frac{d_{trap}(\pi_1, x_1)}{d_{trap}(\pi_T, x_T)} \right) \quad (63)$$

$$= T + \frac{1}{\phi_2} \log \left(\frac{\pi_1 - \pi_{zlb}}{\pi_T - \pi_{zlb}} \right) \quad (64)$$

where the last line uses equations (60) and (62) to write the expression in the previous line as a function of π_T and π_1 only. Equations (58), (60) and (62) describe the continuous pasting constraints on (π_0, x_0) without any reference to \bar{t}_1 . Combinations (π_0, x_0) that satisfy these equations can be part of a continuous path for some \bar{t}_1 . Equations (61) and (63) then show which specific point is reachable with a specific \bar{t}_1 . Any continuous path in this case must start in the black dashed line of figure 9 and be on the solid black line at times T and \bar{t}_1 . For a specific \bar{t}_1 , or for a specific point in one of the two lines, only one path is continuous.

As discussed in the previous section, when expectations of exiting the trap are zero at some point between T and \bar{t}_1 , they must be zero for the entire interval $[T, \bar{t}_1]$. Continuity at \bar{t}_1 requires that the same is true at \bar{t}_1 , at the endpoint of the interval. Since $t_1 \geq T$ and $\phi_2 < 0$,

$$\frac{d_{trap}(\pi_1, x_1)}{d_{trap}(\pi_T, x_T)} = \frac{\pi_1 - \pi_{zlb}}{\pi_T - \pi_{zlb}} \leq 1. \quad (65)$$

There are no continuous paths that reach Υ_{zlb} at T and go through π_T and π_1 when the above inequality does not hold. When equation (62) holds but the inequality in equation (65) does not hold, (π_1, x_1) is on Υ_{zlb} but not on the segment of Υ_{zlb} between (π_T, x_T) and (π_{zlb}, x_{zlb}) . Since $(\pi(t), x(t))$ always moves from (π_T, x_T) towards (π_{zlb}, x_{zlb}) , there is no way to reach (π_1, x_1) after T . In figure 9, this would occur if (π_1, x_1) were on the black line, but to the northwest of the red path. Equation (65) reaffirms that for expectations to be self-fulfilling, being further away from (π_{zlb}, x_{zlb}) requires stronger expectations of falling into the trap in order to support future expectations of falling into the trap. Additionally

$$\frac{d_{trap}(\pi_1, x_1)}{d_{trap}(\pi_T, x_T)} = \frac{\pi_1 - \pi_{zlb}}{\pi_T - \pi_{zlb}} > 0 \quad (66)$$

is required to make the logarithm in equations (63) and (64) well-defined. If the sign of $d_{trap}(\pi_1, x_1)$ and $d_{trap}(\pi_T, x_T)$ are different, expectations at T and \bar{t}_1 are contradictory in the direction from which the economy is expected to approach (π_{zlb}, x_{zlb}) when traveling on Υ_{zlb} . For values that satisfy equations (65) and (66), the continuous pasting condition in equation (63) describes the exact change in expectations required to move from π_T to π_1 in an amount of time $\bar{t}_1 - T$. When π_T is further from π_1 , it is intuitive that the time required to

go from one to the other increases. The size of the change in expectations required to support the path from T to \bar{t}_1 with rational beliefs is controlled by ϕ_2 , the eigenvalue associated with v_{trap} . In fact, (64) is equivalent to

$$d_{trap}(\pi_T, x_T) = e^{\phi_2(\bar{t}_1 - T)} d_{trap}(\pi_1, x_1)$$

which is just a different incarnation of equation (47) with continuous pasting at \bar{t}_1 . Lastly, combining equations (58)-(63) gives

$$\begin{aligned} \mathcal{P}(\bar{R}) &= d_{exit}(\pi_1, x_1) + 1 \{d_{trap}(\pi_1, x_1) = 0\}, \\ \mathcal{T}(\bar{R}) &= \frac{1}{\phi_2} \log \left(\frac{d_{trap}(\pi_1, x_1)}{d_{trap}(\pi_0, x_0) + \frac{\phi_1}{\kappa} (r_h - r_l) (e^{-T\phi_2} - 1)} \right). \end{aligned}$$

The indicator $1 \{d_{trap}(\pi_1, x_1) = 0\}$ in the equation for \mathcal{P} is there to guarantee that equation (58) holds. Graphically, the points \bar{R} such that $\mathcal{P}(\bar{R}) = 0$ are given by the dashed and solid black lines, with the exception of the points where the black lines intersect the gray lines.

The third case is similar to the second and is represented by the blue line in figure 9. Instead of reaching Υ_{zlb} at \bar{t}_1 , the economy reaches the unstable zlb saddle path Ψ_{zlb} at \bar{t}_1 , with the exception of (π_{zlb}, x_{zlb}) , which was already covered. This case is defined by

$$d_{exit}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_1}{\kappa} [\pi_0 - \pi_{zlb}] \neq \frac{\phi_2}{\kappa} (r_h - r_l) (1 - e^{-T\phi_1}) \quad (67)$$

$$d_{trap}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_2}{\kappa} [\pi_0 - \pi_{zlb}] = \frac{\phi_1}{\kappa} (r_h - r_l) (1 - e^{-T\phi_2}) \quad (68)$$

Continuous pasting at T occurs if and only if

$$d_{trap}(\pi_T, x_T) = [x_T - x_{zlb}] - \frac{\phi_2}{\kappa} [\pi_T - \pi_{zlb}] = 0, \quad (69)$$

$$T = \frac{1}{\phi_1} \log \left(\frac{d_{exit}(\pi_T, x_T)}{d_{exit}(\pi_0, x_0) + \frac{\phi_2}{\kappa} (r_h - r_l) (e^{-T\phi_2} - 1)} \right). \quad (70)$$

while continuous pasting at t_1 occurs if and only if

$$d_{trap}(\pi_1, x_1) = [x_1 - x_{zlb}] - \frac{\phi_2}{\kappa} [\pi_1 - \pi_{zlb}] = 0 \quad (71)$$

$$\bar{t}_1 = T + \frac{1}{\phi_1} \log \left(\frac{d_{exit}(\pi_1, x_1)}{d_{exit}(\pi_T, x_T)} \right) \quad (72)$$

$$= T + \frac{1}{\phi_1} \log \left(\frac{\pi_1 - \pi_{zlb}}{\pi_T - \pi_{zlb}} \right) \quad (73)$$

It follows that

$$\begin{aligned}\mathcal{P}(\bar{R}) &= d_{trap}(\pi_1, x_1) + 1 \{d_{exit}(\pi_1, x_1) = 0\}, \\ \mathcal{T}(\bar{R}) &= \frac{1}{\phi_1} \log \left(\frac{d_{exit}(\pi_1, x_1)}{d_{exit}(\pi_0, x_0) + \frac{\phi_2}{\kappa} (r_h - r_l) (e^{-T\phi_2} - 1)} \right).\end{aligned}$$

Compared to the last case, because the unstable saddle has dynamics that move $(\pi(t), x(t))$ away from (π_{zlb}, x_{zlb}) , the inequalities and comparative statics are reversed.

The fourth and last case corresponds to all remaining choices for \bar{R} that can be part of a continuous path. The continuous pasting conditions are

$$\begin{aligned}\left(\frac{d_{trap}(\pi_0, x_0) + \frac{\phi_1}{\kappa} (r_h - r_l) (e^{-T\phi_2} - 1)}{d_{trap}(\pi_T, x_T)} \right)^{\phi_1} \\ = \left(\frac{d_{exit}(\pi_0, x_0) + \frac{\phi_2}{\kappa} (r_h - r_l) (e^{-T\phi_1} - 1)}{d_{exit}(\pi_T, x_T)} \right)^{\phi_2},\end{aligned}\quad (74)$$

$$T = \frac{1}{\phi_2} \log \frac{d_{trap}(\pi_T, x_T)}{d_{trap}(\pi_0, x_0) + \frac{\phi_1}{\kappa} (r_h - r_l) (e^{-T\phi_2} - 1)},\quad (75)$$

for T and

$$\left(\frac{d_{trap}(\pi_1, x_1)}{d_{trap}(\pi_T, x_T)} \right)^{\phi_1} = \left(\frac{d_{exit}(\pi_1, x_1)}{d_{exit}(\pi_T, x_T)} \right)^{\phi_2}\quad (76)$$

$$t_1 = T + \frac{1}{\phi_2} \log \left(\frac{d_{trap}(\pi_1, x_1)}{d_{trap}(\pi_T, x_T)} \right)\quad (77)$$

for \bar{t}_1 . As in the last two cases, the continuous pasting conditions can be separated into those that does not involve \bar{t}_1 , equations (74), (75) and (76), and one that explicitly solves for \bar{t}_1 , equation (77). Unlike the previous cases, none of the equations without \bar{t}_1 are linear in $x_0, \pi_0, x_T, \pi_T, x_1$ and π_1 , since the economy is not on either saddle path, which are the only linear orbits of the economy between T and \bar{t}_1 . Figure 10 is helpful in gaining some intuition. The black and gray lines are the same as in figure 9. The solid lines intersect at (π_{zlb}, x_{zlb}) and split the π - x plane into four sections. The dashed lines also split the plane into four sections. Points starting below both of the dashed lines are in the “south” region and have $d_{trap}(\pi_0, x_0) + \frac{\phi_1}{\kappa} (r_h - r_l) (e^{-T\phi_2} - 1) < 0$ and $d_{exit}(\pi_0, x_0) + \frac{\phi_2}{\kappa} (r_h - r_l) (e^{-T\phi_1} - 1) < 0$. They always paste continuously to points (π_T, x_T) and (π_1, x_1) in the corresponding south region that is below the solid lines, given by $d_{exit}(\pi_T, x_T) < 0$ and $d_{trap}(\pi_T, x_T) < 0$. The same is true for each of the remaining three regions. For example, the blue path has (π_0, x_0) in the “east” region of the dashed lines, so $(\pi(T), x(T))$ and $(\pi(\bar{t}_1), x(\bar{t}_1))$ must be in the

corresponding east region of the solid lines.

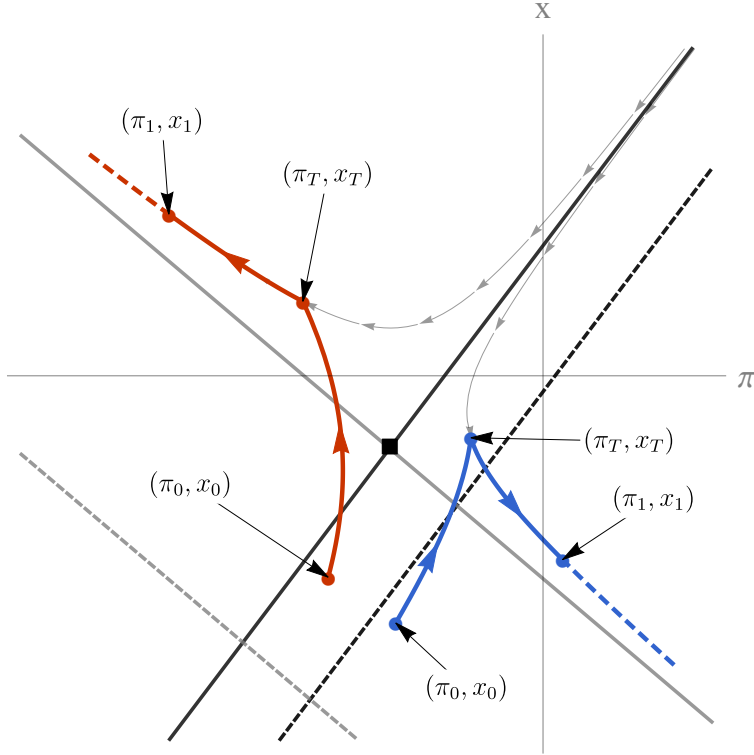


Figure 10: Continuous pasting conditions when (π_1, x_1) is neither in Υ_{zlb} nor in Ψ_{zlb} . The black and gray lines are as in figure 9. The black square at the intersection of the two solid lines is (π_{zlb}, x_{zlb}) , the deflationary steady-state. For a given (π_0, x_0) , the set of points (π_1, x_1) that can be part of a continuous path are those reachable by following the flow line determined by (π_T, x_T) . The time of liftoff then determines which point in this family occurs in the economy.

Assuming continuous pasting at T , equations (74)-(76) reveal the set of points (π_0, x_0, π_1, x_1) that can be reached through continuous paths for some \bar{t}_1 , which then give

$$\mathcal{P}(\bar{R}) = \left(\frac{d_{exit}(\pi_1, x_1)}{d_{exit}(\pi_0, x_0) + \frac{\phi_2}{\kappa}(r_h - r_l)(e^{-T\phi_1} - 1)} \right)^{\phi_2} - \left(\frac{d_{trap}(\pi_1, x_1)}{d_{trap}(\pi_0, x_0) + \frac{\phi_1}{\kappa}(r_h - r_l)(e^{-T\phi_2} - 1)} \right)^{\phi_1} + 1 \{d_{exit}(\pi_1, x_1) = 0 \text{ or } d_{trap}(\pi_1, x_1) = 0\}$$

As before, the indicator function in the last term is there only to exclude previously considered cases. To understand what $\mathcal{P}(\bar{R}) = 0$ implies, refer to the blue path in figure 10. When the economy starts at the blue dot labeled $(\pi(0), x(0))$, continuous paths must go through the blue dot labeled $(\pi(T), x(T))$. Between T and \bar{t}_1 , the economy follows its dynamics with

$r(t) = r_h > 0$ and $i(t) = 0$, represented by the flow line that goes through $(\pi(T), x(T))$. Continuous paths have $(\pi(\bar{t}_1), x(\bar{t}_1))$ on the same flow line as $(\pi(T), x(T))$. Which exact point in the flow line is realized in the economy is determined by equation (77), which gives

$$\mathcal{T}(\bar{R}) = \frac{1}{\phi_1} \log \frac{d_{exit}(\pi_1, x_1)}{d_{exit}(\pi_0, x_0) + \frac{\phi_2}{\kappa} (r_h - r_l) (e^{-T\phi_1} - 1)}.$$

As \bar{t}_1 grows larger, the economy spends more time traveling on the flow line. Figure 10 shows one specific choice for \bar{t}_1 . A larger \bar{t}_1 would put $(\pi(\bar{t}_1), x(\bar{t}_1))$ somewhere in the dashed blue line.

As $\bar{t}_1 \rightarrow \infty$, the economy approaches the unstable saddle and $d_{trap}(\pi_1, x_1) \rightarrow 0$ while $d_{exit}(\pi_1, x_1) \rightarrow \infty$. This aspect is crucial for the analysis of determinacy. Delaying the time of liftoff steers expectations away from the trap factor and into the exit factor. Because $\bar{t}_1 \geq T$, the side of flow line to the north of (π_T, x_T) is not reachable when starting at the given (π_0, x_0) . The red path shows another continuous path that starts in the “north” region of the dashed lines and converges to Ψ_{zlb} in the opposite direction of the blue path.

To sum up, I have derived a function \mathcal{P} and a set \mathcal{T} that characterize continuity of paths. Assuming paths are already continuous at T , $\mathcal{P}(\pi_0, x_0, \pi_1, x_1) = 0$ gives the collection of points (π_0, x_0, π_1, x_1) that can be part of some continuous path that starts at (π_0, x_0) and reaches (π_1, x_1) at some time larger than or equal to T . For (π_0, x_0, π_1, x_1) such that $\mathcal{P}(\pi_0, x_0, \pi_1, x_1) = 0$, $\mathcal{T}(\pi_0, x_0, \pi_1, x_1)$ gives the time it takes to go from (π_0, x_0) to (π_1, x_1) . Note that neither \mathcal{P} nor \mathcal{T} depend on $f(R_{t_1})$ or the choice of Taylor rule coefficients $\xi_\pi(R_{t_1})$, $\xi_x(R_{t_1})$. They are fully determined by (π_0, x_0, π_1, x_1) and parameters of the model. Of course, in a rational expectations equilibrium, paths are continuous and thus

$$0 = \mathcal{P}(\pi(0), x(0), \pi(t_1), x(t_1)), \quad (78)$$

$$f(\pi(0), x(0), \pi(t_1), x(t_1)) = t_1 = \mathcal{T}(\pi(0), x(0), \pi(t_1), x(t_1)). \quad (79)$$

It is these equilibrium conditions that connect the policy actions that the central bank takes after time T to the continuous pasting conditions.

6.2 Boundedness of Continuous Paths

Because of the non-linear dynamics after t_1 , characterizing boundedness of continuous paths is the most difficult step towards obtaining a rule that gives global determinacy. When the Taylor principle holds, the results in this section rely on the Poincaré-Bendixson theorem, a powerful tool that classifies all possible dynamics of two dimensional systems. On the other hand, when the Taylor principle does not hold, the results can be obtained by straightforward

computations.

Before stating the propositions, recall that whether the Taylor principle holds depends on R_{t_1} . In addition, when paths are continuous, R_{t_1} defines the entire path. The first two elements of R_{t_1} determine $(\pi(0), x(0))$ and the last two give $(\pi(t_1), x(t_1))$; equation (79) then gives t_1 . Let $\bar{\Omega}_{ss} = \Omega_{ss} \cup \partial\Omega$ be the closure of Ω_{ss} .

Proposition 2 (Continuous bounded paths when the Taylor principle does not hold). *For a continuous path defined by R_{t_1} , if the Taylor principle does not hold, the path is bounded if and only if there exists $r \geq t_1$ such that $(\pi(r), x(r)) \in \Upsilon_{ss} \cap \bar{\Omega}_{ss}$.*

Proof. Appendix A.4. □

When the Taylor principle does not hold, continuous paths are bounded if and only if they reach the ss saddle path outside (or on the boundary of) the ZLB region at t_1 or later. There are two ways in which this can happen, depicted in figure 11. The flow lines in the background show the dynamics after t_1 . The green path shows the first case. The economy finds itself outside the ZLB at t_1 . If it is on the saddle path Υ_{ss} , then it stays on it and travels towards (π_{ss}, x_{ss}) , remaining bounded as in the figure. If the economy is not on the saddle path Υ_{ss} , paths for $(\pi(t), x(t))$ can either become unbounded or eventually enter the ZLB region Ω_{zlb} . Because the Taylor principle does not hold, the zlb steady-state is not inside Ω_{zlb} . Thus, when the economy enters Ω_{zlb} , paths can either become unbounded (when they stay inside Ω_{zlb}) or eventually exit the ZLB and return to Ω_{ss} . Proposition 2 states that the process of moving from one region to the next has to stop at some point — there are no limit cycles. The proof in the appendix reveals that, in fact, there can be at most one transition; once the economy leaves Ω_{ss} and enters Ω_{zlb} , it remains there. The red line in figure 11 shows the second way in which the economy can reach Υ_{ss} . At t_1 , the economy is inside the ZLB region Ω_{zlb} . As it follows the dynamics specified by $i(t) = 0$, it eventually hits the boundary $\partial\Omega$ at some time $r > t_1$ exactly where it intersects Υ_{ss} . From there on, the analysis is identical to that of the first case. Proposition 2 shows the benefits and challenges of not following the Taylor principle. On the one hand, for continuous bounded paths that can be part of an equilibrium, the economy always exits the ZLB. On the other, the saddle dynamics outside the ZLB are conducive to (both local and global) indeterminacy, since being on any point of the saddle at t_1 , or on any point of the path inside the ZLB that eventually reaches the saddle, gives continuous bounded paths and hence potential equilibria.

In contrast, when the Taylor principle holds, the benefit is to have no saddle path equilibria outside the ZLB, but at the cost of introducing them inside the ZLB, as I show next.

Proposition 3 (Continuous bounded paths when the Taylor principle holds). *For a continuous path defined by R_{t_1} , if the Taylor principle holds, the path is bounded if and only if either $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$ or there exists $r \geq t_1$ such that $(\pi(r), x(r)) \in \Upsilon_{zlb} \cap \Omega_{zlb}$.*

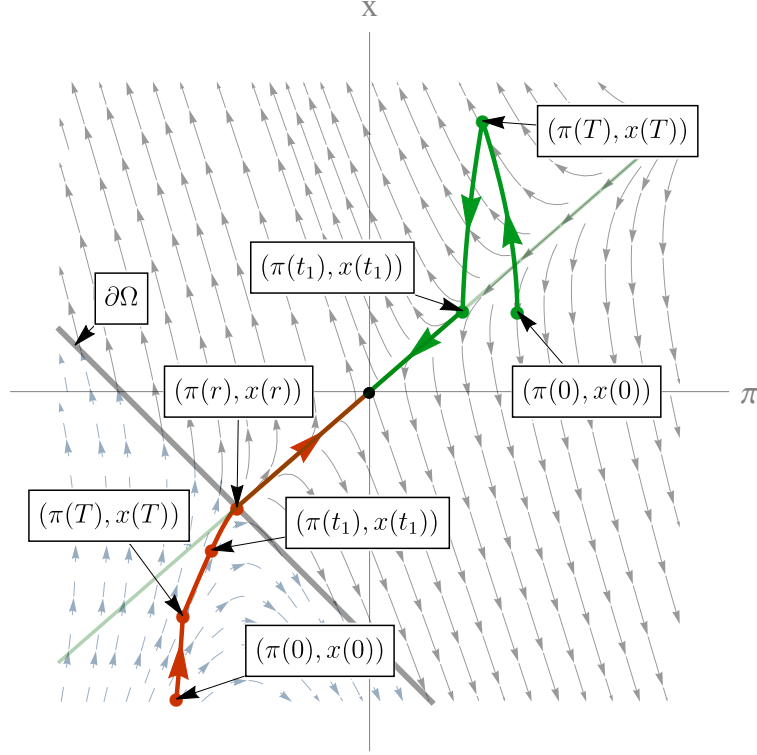


Figure 11: When the Taylor principle does not hold, there are only two kinds of continuous bounded paths, shown in the figure. The flow lines in the background are for the dynamics after liftoff, which occurs at t_1 .

Proof. Appendix A.5. □

When the Taylor principle does not hold, Proposition 3 states that continuous paths are bounded if and only if they either reach the intended steady state (π_{ss}, x_{ss}) exactly at t_1 , or if they eventually reach the zlb saddle path Υ_{zlb} while inside the ZLB region Ω_{zlb} (the boundary $\partial\Omega$ is part of Ω_{zlb}). If $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$, then paths are clearly bounded since (π_{ss}, x_{ss}) is a steady-state. If $(\pi(t), x(t))$ is in Ω_{ss} but is not equal to (π_{ss}, x_{ss}) , the only way for continuous paths to remain bounded is to eventually enter Ω_{zlb} , since the dynamics in Ω_{ss} are otherwise explosive. Once in Ω_{zlb} , boundedness requires that the economy follows the saddle path towards the zlb steady-state (π_{zlb}, x_{zlb}) . The red line in figure 12 illustrates this case. Because the Taylor principle holds, (π_{zlb}, x_{zlb}) is inside Ω_{zlb} and is therefore a steady-state of the system, as discussed first in section 5.3. The economy converges to (π_{zlb}, x_{zlb}) as $t \rightarrow \infty$. This is the Benhabib et al. (2001)-type equilibrium shown in figure 7. The closer $(\pi(t), x(t))$ gets to (π_{ss}, x_{ss}) , the longer it will take the economy to reach Ω_{zlb} . Although listed separately and at first sight different, the case for which $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$ can be seen as the limit of this type of equilibria when the time to reach Ω_{zlb} goes to infinity. Proposition 3 reveals another type of potential equilibrium that converges to the unintended

steady-state (π_{zlb}, x_{zlb}) and shown in the green line of figure 12. The economy is always at the ZLB. At T , the economy is already on Υ_{zlb} and stays on it for all $t \geq T$, traveling towards (π_{zlb}, x_{zlb}) . At t_1 , even though liftoff occurs, the Taylor rule prescribes $i(t_1) = 0$, since the economy is in Ω_{zlb} . Proposition 3 also proves that there are no continuous bounded paths other than the ones I have described. In particular, there are no closed loops, no limit cycles and no chaos, unlike the setup in Benhabib et al. (2002b), in which the Taylor principle gives rise to chaotic trajectories. One direct implication is that whether following the Taylor principle leads to chaotic interest rate rules depends on the specific setup. For this economy, the Taylor principle is actually the exact condition needed to make the divergence of (π_T, x_T) positive everywhere. By Green's theorem, a positive divergence automatically eliminates bounded orbits, since the line integral around a closed loop is positive¹⁷. Without a ZLB, a positive divergence is equivalent to the Taylor principle and to explosive dynamics. Proposition 3 shows that having a ZLB does not break this link. Thus, even though the system includes a binding ZLB, the Taylor principle is still the right concept to assess the tendency of the system to move away from the intended steady-state. The major change from the case without the ZLB is that the Taylor principle now induces (π_{zlb}, x_{zlb}) to be a steady-state of the global dynamics.

6.3 Implementing the Optimal Equilibrium

Even though the optimal equilibrium is unique, I now show there are many choices for the interest rate rule I study to implement it. Let $R^* = (\pi^*(0), x^*(0), \pi^*(t^*), x^*(t^*))$ be the vector R_{t_1} evaluated at the optimal equilibrium.

Proposition 4 (Implementation of the optimal equilibrium). *If $\kappa\sigma\lambda \neq 1$, the rule in equation (19) implements the optimal equilibrium if and only if*

$$f(R^*) = t^* \tag{80}$$

$$\xi_\pi(R^*) + \phi\xi_x(R^*) = (1 - \kappa\sigma\lambda) \tag{81}$$

If $\kappa\sigma\lambda = 1$, the rule implements the optimal equilibrium if and only if equation (80) holds (i.e. equation (81) is no longer needed and any $\xi_\pi(R^)$, $\xi_x(R^*)$ implement the optimal equilibrium).*

Proof. Appendix A.6. □

The proposition states that to implement the optimal equilibrium, liftoff has to occur at t^* when $R_{t_1} = R^*$. Since the optimal equilibrium features $i^*(t) > 0$ for $t \geq t^*$, liftoff after

¹⁷ The proof of item 6 in Appendix A.5 has more details.

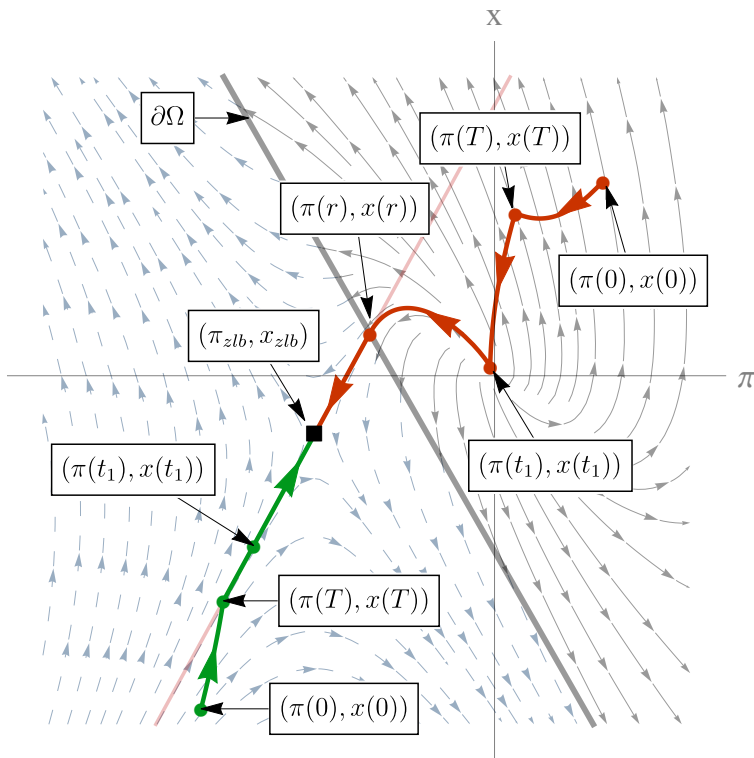


Figure 12: When the Taylor principle holds, there are three kinds of continuous bounded paths. Two of them are shown in the figure. The flow lines in the background are for the dynamics after liftoff, which occurs at t_1 . In the third kind, the economy reaches the steady-state $(\pi_{ss}, x_{ss}) = (0, 0)$ at t_1 and stays there forever after.

t^* would imply $i(t) = 0$ when $i^*(t) > 0$. Conversely, liftoff before t^* would imply $i(s) > 0$ for all $s < t^*$ such that $\xi_\pi \pi(s) + \xi_x x(s) + r_h > 0$, while $i^*(s) = 0$. Such an s always exists because the optimal equilibrium features forward guidance. [Werning \(2011\)](#) shows that the optimal equilibrium features a promise to zero interest rates when $(1 - \kappa\sigma\lambda) \pi^*(t) + r(t) > 0$ for at least some $t \in (T, t^*)$.

When $\kappa\sigma\lambda \neq 1$, the Taylor rule coefficients have to be picked according to equation (81), where recall that ϕ is the slope of the optimal saddle path defined in equation (8) and $1 - \kappa\sigma\lambda$ is the coefficient in front of $\pi^*(t)$ in the optimal interest rate path in equation (7). Picking $\xi_\pi(R^*)$ and $\xi_x(R^*)$ that generate the correct slope for the optimal saddle path is all that is required to implement the optimal equilibrium. Because there are two coefficients and a single slope to match, there is one degree of freedom in the choice of rule at R^* . When $\kappa\sigma\lambda = 1$, the optimal equilibrium never travels along the saddle path, as the economy reaches the steady state exactly at t^* . In this case, it does not matter what the slope of the saddle path is; all that matters is that the economy reaches (π_{ss}, x_{ss}) at t^* , which is guaranteed by equation (80). For this case only, the optimal equilibrium is therefore implementable while following the Taylor principle.

Even though there is one degree of freedom in picking $\xi_\pi(R^*)$ and $\xi_x(R^*)$, condition (81) does imply that, when $\kappa\sigma\lambda \neq 1$, the Taylor principle does not hold. Thus, any implementation of the optimal equilibrium with $\kappa\sigma\lambda \neq 1$ necessarily requires something akin to local indeterminacy around (π_{ss}, x_{ss}) . However, local indeterminacy in this context says nothing about uniqueness of equilibria. In fact, the Taylor principle not holding is neither necessary nor sufficient for uniqueness of equilibria, as the next section shows. Suppose the economy at time $t \geq t_1$ finds itself close enough to (π_{ss}, x_{ss}) and traveling on the optimal saddle path. Locally, the economy looks like an economy without a binding ZLB in which the Taylor principle does not hold. Standard analysis may then seem to suggest that there are multiple equilibria in the economy. However, unlike in the standard analysis, $x(t)$ and $\pi(t)$ are not jump (or non-predetermined) variables. Actually, $x(t)$ and $\pi(t)$ are predetermined variables for all $t > 0$. The only jump variables are $x(0)$ and $\pi(0)$. After the initial instant, inflation and output are both predetermined by the history of the economy. Of course, if at some time $t > 0$ the central bank reneged on its commitment and announced a new rule that does not satisfy the Taylor principle, or if anything else suddenly changed in the economy, then it is possible that new expectations can form in which local indeterminacy actually translates to multiplicity of equilibria. But this is strictly outside the model; if anything unexpected can happen after $(\pi(0), x(0))$ or if equilibria are required to be robust to perturbations after t_1 , then the analysis must be performed in a different setting.

6.4 Eliminating Non-Optimal Equilibria

What should the central bank do for $R_{t_1} \neq R^*$? What should its off-equilibrium threats be if it wants to eliminate indeterminacy? The next proposition gives the necessary and sufficient conditions to rule out non-optimal equilibria.

Proposition 5 (Eliminating non-optimal equilibria). *The rule in equation (19) implements no equilibria with $R_{t_1} \neq R^*$ if and only if:*

1. *The Taylor principle does not hold when $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$.*

2. *The liftoff rule satisfies*

$$f(R_{t_1}) \neq \mathcal{T}(R_{t_1}) \tag{82}$$

for any $R_{t_1} \neq R^$ fulfilling one of the three conditions below:*

(a) *The Taylor principle holds and there exists $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{zlb}$;*

(b) *The Taylor principle does not hold and $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss} \cap \Upsilon_{ss}$;*

(c) The Taylor principle holds and $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$.

3. The liftoff rule satisfies

$$f(R_{t_1}) > \frac{1}{\phi_1} \log \left(\frac{d_{exit}(r)}{d_{exit}(t_1)} \right) \quad (83)$$

for any $R_{t_1} \neq R^*$ such that the Taylor principle does not hold and there exists $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$.

Proof. I present a sketch of the proof, the full one is in Appendix 5. For R_{t_1} such that $\mathcal{P}(R_{t_1}) \neq 0$, paths are not continuous and hence not an equilibrium. Consider then all $R_{t_1} \neq R^*$ such that $\mathcal{P}(R_{t_1}) = 0$. Items 1, 2a-2c and 3 classify these R_{t_1} into five disjoint sets. Using Propositions 2 and 3, items 1-3 guarantee that continuous paths are unbounded and that bounded paths are discontinuous.

Item 1: If the Taylor principle holds and $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$, paths are always continuous and bounded. This case corresponds to the green path in figure 12. Earlier or later liftoff while inside Ω_{zlb} does not change paths in any way since $\max\{0, \xi_\pi \pi(t) + \xi_x x(t) + r_h\} = 0$. When the Taylor principle holds, $(\pi(t), x(t))$ remains in $\Omega_{zlb} \cap \Upsilon_{zlb}$ for all $t \geq t_1$. Thus, picking a different f cannot eliminate these equilibria. The only option is to not have the Taylor principle hold so that $(\pi_{zlb}, x_{zlb}) \notin \Omega_{zlb}$ and $(\pi(t), x(t))$ eventually exits Ω_{zlb} and becomes unbounded.

Item 2: By Propositions 2 and 3, paths are continuous and bounded if and only if equation (82) does not hold. The case in item 2a corresponds to the red path in figure 12 while the case in item 2b corresponds to the green path in figure 11.

Item 3: This case corresponds to the red path in figure 11. Liftoff occurs inside Ω_{zlb} if and only if inequality (83) does not hold (note $r > t_1$). In this case, paths are continuous without further conditions on f since $\max\{0, \xi_\pi \pi(t) + \xi_x x(t) + r_h\} = 0$ and Proposition 2 shows such paths are bounded. \square

The proposition offers a taxonomy of potential equilibria and how to prevent them from forming by choosing Taylor rule coefficients and an appropriate liftoff rule. It can be seen as a menu for central banks that prescribes different communication strategies for different private sector expectations.

The first class of equilibria, described in item 1, is defined by $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$. In this class, the economy is against the ZLB at all times. Expectations of falling into (π_{zlb}, x_{zlb}) are the strongest they can be. In the introduction, I referred to this case as the one with “strong” expectations of falling into the deflationary trap. Indeed, the loading d_{exit} on the exit factor v_{exit} is zero for all $t \geq T$ and the economy is solely driven by the trap factor v_{trap} . For all other cases in Proposition 5, the economy is in Ω_{ss} —and the

loading on v_{exit} is non-zero— for at least some of the time. When expectations are such that $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$, the only way to avoid the expectational trap is to have the Taylor principle not hold. A central bank that reacts weakly enough to deviations of inflation from zero outside the ZLB is necessary and sufficient to eliminate this undesirable class of deflationary equilibria. Neither a time nor a state-dependent commitment to delay liftoff can rescue the economy from reaching (π_{zlb}, x_{zlb}) . The promise of a longer ZLB period only confirms the already strong deflationary expectations.

Item 2 defines what in the introduction I called “weak” expectations of falling into the deflationary trap. Unlike the case in item 1, in which not following the Taylor principle is the only option, there are two different ways to eliminate the class of equilibria in items 2 and 3. The first way is to choose Taylor rule coefficients that make the Taylor principle hold or not hold depending on the case at hand. For example, for all continuous paths for which there exist $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{zlb}$, item 2a shows that equilibria can be precluded by *not* following the Taylor principle. The second way to eliminate equilibria is to choose an appropriate rule for liftoff by satisfying equations (82) and (83).

Whenever the Taylor principle does not hold, equilibria involving the expectational trap are eliminated, as discussed before. Following the Taylor principle is an equally powerful tool to rid the economy of suboptimal equilibria that converge to the intended steady-state (π_{ss}, x_{ss}) . By Proposition 1, not all suboptimal equilibria can be eliminated simultaneously by an appropriate choice of Taylor rule coefficients. Thus, despite the broad effectiveness of using appropriate Taylor rule coefficients to eliminate equilibria, some of the potential equilibria must be ruled out by a state-contingent liftoff rule. This conclusion can also be reached directly in Proposition 5. For example, when $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$, if item 2b does not hold, then item 2c holds, and vice-versa. Following or not following the Taylor principle merely changes which case is the relevant one. For paths with $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$, the only way to stop equilibria is to abide by equation (82).

Item 2a corresponds to the Benhabib et al. (2001) equilibria discussed in Section 5.3. Although the economy has succeeded in escaping Ω_{zlb} by time t_1 and $(\pi(t_1), x(t_1))$ can get arbitrarily close to (π_{ss}, x_{ss}) , self-fulfilling expectations of reaching (π_{zlb}, x_{zlb}) nevertheless take hold. At some time $r > t_1$, the economy enters Ω_{zlb} through Υ_{zlb} and converges to the unintended steady-state. For $t \geq r$, the loading on the exit factor is zero and trap expectations are maximal. The expectation as of time t_1 of a future (at time r) loading of zero on the exit factor requires a small enough loading on the exit factor at t_1 in order to be validated. On the other hand, the exit factor is large enough between T and t_1 to allow the economy to be outside the ZLB between t_1 and r . In this class of equilibria, expectations of entering and exiting the trap are both present, and the former end up dominating. Because there is at least some tension between the trap and exit factors, this class of equilibria is

not as severe as the one in item 1 in terms of avoiding (π_{zlb}, x_{zlb}) . As such, it is possible to preclude it without having to abandon the Taylor principle if f is chosen so as to satisfy equation (82). This result is different from Benhabib et al. (2001), where the Taylor principle necessarily implied the existence of these equilibria. The difference arises because I consider a broader set of rules that the central bank can follow by allowing state-dependent liftoff. Despite this difference, item 1 does support the conclusion in Benhabib et al. (2001) that it is not possible to have determinacy while always following the Taylor principle outside the ZLB.

Why does equation (82) preclude equilibria for item 2a? For each R_{t_1} , by Proposition 3 and the continuous pasting condition in equation A.27, there is a single value of $f(R_{t_1})$ that leads to continuous bounded paths. Liftoff at any other time therefore impedes these type of equilibria from happening; a well-timed liftoff by the central bank avoids validating expectations of a suboptimal path. When $f(R_{t_1}) < \mathcal{T}(R_{t_1})$, the economy undershoots $\partial\Omega \cap \Upsilon_{zlb}$ at r , while when $f(R_{t_1}) > \mathcal{T}(R_{t_1})$, the economy overshoots it. Both under and overshooting $\partial\Omega \cap \Upsilon_{zlb}$ lead to a non-zero loading on the exit factor at time r , which is enough to stop the economy from falling into (π_{zlb}, x_{zlb}) . Alternatively, the central bank can preclude the equilibria in item 2a by not following the Taylor principle, a result also consistent with Benhabib et al. (2001). Unlike Benhabib et al. (2001), however, not following the Taylor principle does not necessarily introduce other sub-optimal equilibria.

Item 2b describes an economy that converges to the desired steady-state (π_{ss}, x_{ss}) after having successfully escaped Ω_{zlb} by time t_1 . Expectations of escaping (π_{zlb}, x_{zlb}) , captured by the value of the loading d_{exit} on the exit factor, are always strong enough to take the economy far enough from Υ_{zlb} during $t \in [T, t_1)$, allowing it to escape the expectational trap. Even though the economy converges to the intended steady-state, it does so by following a suboptimal path. These equilibria can be precluded by following the Taylor principle or by choosing $f(R_{t_1}) \neq \mathcal{T}(R_{t_1})$. Just as for item 2a, $f < \mathcal{T}$ and $f > \mathcal{T}$ correspond to under and overshooting the trajectories that result in continuous bounded paths.

Item 2c considers equilibria in which the Taylor principle holds and the economy is on (π_{ss}, x_{ss}) exactly at liftoff. The intuition for why equilibria are eliminated by using equation (82) are analogous to the other cases in item 2.

The case described in item 3 is somewhat different from the previous ones. It corresponds to the case of “medium” strength expectations of falling into the deflationary trap. For each R_{t_1} that satisfies the assumptions in item 3, any low enough value of t_1 gives a continuous bounded path. When the economy is inside Ω_{zlb} , pegging interest rates at zero or following $i(t) = \max\{0, \xi_\pi \pi(t) + \xi_x x(t) + r_h\}$ are equivalent. Liftoff while $\xi_\pi \pi(t) + \xi_x x(t) + r_h < 0$ produces the same path as lifting off exactly at time r , when $\xi_\pi \pi(r) + \xi_x x(r) + r_h = 0$. At time r , the economy is on $\partial\Omega \cap \Upsilon_{ss}$. For all liftoff times $t_1 \leq r$, the economy follows the stable

saddle-path towards (π_{ss}, x_{ss}) , staying bounded and continuous. To stop this equilibrium from forming, the central bank first computes how long it takes to reach $\partial\Omega \cap \Upsilon_{ss}$ starting at the $(\pi(0), x(0))$ given by the first two components of R_{t_1} . It then picks a liftoff that exceeds that time. Equation (83) expresses exactly this strategy. The lower bound for $f(R_{t_1})$ in equation (83) is a function of how strong the trap factor expectations are at the time of exiting the ZLB compared to those at liftoff. For a given $(\pi(t_1), x(t_1))$ or, equivalently, for a given $d_{exit}(t_1)$, larger trap expectations at time r , given by $d_{exit}(r)$, imply a longer time inside the ZLB. On the other hand, for a given $d_{exit}(r)$, a larger loading $d_{exit}(t_1)$ on the exit factor implies a shorter time at the ZLB.

Even though Proposition 5 gives exact ways in which to eliminate each type of potential equilibria, it also offers broader, more conceptual lessons. It gives general properties any interest rate rule must have and can have to produce a determinate optimal outcome. When combined with Proposition 4, it gives the following result.

Proposition 6 (General properties for global determinacy of the optimal equilibrium). *If $\kappa\sigma\lambda \neq 1$, when the optimal equilibrium is the unique equilibrium:*

1. *The liftoff rule cannot be purely forward-looking, i.e. $f(R_{t_1})$ cannot be constant in $(\pi(0), x(0))$, the first two elements of R_{t_1} .*
2. *The interest rate rule can be purely backward-looking, i.e. $\xi_\pi(R_{t_1})$, $\xi_x(R_{t_1})$ and $f(R_{t_1})$ can be constant in $(\pi(t_1), x(t_1))$, the last two elements of R_{t_1} .*
3. *The interest rate rule can be memoryless after liftoff, i.e. Taylor rule coefficients $\xi_\pi(R_{t_1})$ and $\xi_x(R_{t_1})$ be constant (and hence state and path independent).*
4. *If the interest rate rule is continuous with respect to the state of the economy, i.e. if $\xi_\pi(R_{t_1})$, $\xi_x(R_{t_1})$ and $f(R_{t_1})$ are continuous in R_{t_1} :*
 - (a) *The Taylor principle never holds.*
 - (b) *The interest rate rule must be forward and backward-looking, i.e. $\xi_\pi(R_{t_1})$, $\xi_x(R_{t_1})$ and $f(R_{t_1})$ can be constant neither in $(\pi(0), x(0))$ nor in $(\pi(t_1), x(t_1))$.*
 - (c) *Proposition 5 holds by replacing equation (82) with*

$$f(R_{t_1}) > \mathcal{T}(R_{t_1}). \quad (84)$$

Proof. Appendix A.8. □

The first two items in Proposition 6 state that an interest rate rule that brings about global determinacy of the optimal equilibrium must be path dependent but need not be forward looking. More precisely, there exist functions $f(R_{t_1})$, $\xi_x(R_{t_1})$ and $\xi_\pi(R_{t_1})$ that do not depend on (are constant in) $x(t_1)$ and $\pi(t_1)$ that produce a unique optimal equilibrium, but the same is not true for functions that do not depend on (are constant in) $\pi(0)$ and $x(0)$. When the central bank explains why it will promise to keep interest rates at zero for some time, determinacy requires that it refers to the level of inflation and output that prevail at the beginning of the liquidity trap. Even more, it requires that the length of time promised for zero interest rates changes as a function of $\pi(0)$ and $x(0)$. Note that if t_1 is known, either $(\pi(0), x(0))$ or $(\pi(t_1), x(t_1))$ fully determine the path of the economy. If $f(R_{t_1})$ is purely backward looking, knowing $(\pi(0), x(0))$ immediately pins down t_1 and thus the entire path. If $f(R_{t_1})$ is purely forward looking, the same reasoning applies. Given $(\pi(t_1), x(t_1))$, liftoff is then determined by $f(R_{t_1})$, which is solely a function of $(\pi(t_1), x(t_1))$. The relevant difference for determinacy between purely backward-looking and purely forward-looking rules is not whether there is a unique path for a given $(\pi(0), x(0))$ or $(\pi(t_1), x(t_1))$. The key difference is that, for purely forward-looking rules, no matter what form $f(R_{t_1})$ takes, there always are some $(\pi(t_1), x(t_1))$ that support non-optimal equilibria. Perhaps the easiest example is when $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$. By the definition of rational expectations equilibrium, $f(R_{t_1}) < \infty$ for all R_{t_1} . Consider the value $\hat{t}_1 = f(a, b, \pi_{ss}, x_{ss})$, which will be the same for all a, b since the rule is purely forward-looking. Using the continuous pasting conditions, it is always possible to find $(\hat{\pi}_0, \hat{x}_0)$ such that when $(\pi(0), x(0)) = (\hat{\pi}_0, \hat{x}_0)$, the economy follows a continuous path that reaches (π_{ss}, x_{ss}) at $t = \hat{t}_1$. Indeed, all it takes is to trace the IS and NKPC backward in time starting at (π_{ss}, x_{ss}) for a period of time \hat{t}_1 while setting $i(t) = 0$ throughout. Because $\hat{t}_1 < \infty$, it is always possible to find such $(\hat{\pi}_0, \hat{x}_0)$. The key for this argument to work is that \hat{t}_1 is the same for all a and b . The path of the economy when tracing the dynamics backwards in time is fully determined by the value of $(\pi(t_1), x(t_1))$. If the rule has a backward-looking component, finding the path by traveling backward in time from liftoff until $t = 0$ is a fixed-point problem, since the amount of time elapsed before reaching $(\pi(0), x(0))$ depends on $(\pi(0), x(0))$ itself. If the central bank picks an appropriate rule, the fixed-point problem can be guaranteed to have no solution. In economic terms, no solution to the fixed-point problem means that there are no rational expectations for $(\pi(0), x(0))$ that get the economy to (π_{ss}, x_{ss}) at time $f(\pi(0), x(0), \pi_{ss}, x_{ss})$. This type of logic implies that if $f(R_{t_1})$, instead of being a function of R_{t_1} , were a function of $(\pi(s), x(s), \pi(t_1), x(t_1))$ for some $s \in (0, t_1)$, Propositions 5 and 6 would still apply. It is thus not necessary for the central bank to make its policy contingent on exactly $(\pi(0), x(0))$. It can make it contingent on inflation and output for some other time s , as long as $s < t_1$. If the central bank is willing to accept an indeterminate path between $t = 0$ and s , there is

actually no need to announce f until s .

Item 3 of Proposition 6 shows that it is possible to make monetary policy after liftoff be independent of what happens before liftoff and still provide determinacy. Forward guidance need not affect monetary policy forever, as would occur if, for example, the central bank increased the inflation target. What is crucial is that there is path dependence for some time after the natural rate becomes positive, but not necessarily after the *nominal* rate becomes positive. The proof in Appendix A.8 makes clear that determinacy can be obtained with any combination of constant coefficients that do not satisfy the Taylor principle. Of course, if the central bank wishes to implement the optimal equilibrium, then it must pick coefficients in accordance with Proposition 4. Lastly, constant Taylor rule coefficients may be desirable from a communications standpoint, as the complexity of the rule is reduced.

Another way to conceivably simplify communication is to have a rule that is continuous with respect to the state of the economy. Empirically, at least during non-crisis times, small changes in economic conditions generally lead to changes in monetary policy of commensurate size. To the best of my knowledge, all interest rate and targeting rules in the literature also share this reasonable feature. On the other hand, Proposition 5 allows for discontinuity of the rule even though the fundamentals of the economy, $x(t)$ and $\pi(t)$, are continuous as a function of time and $i(t)$ and $r(t)$ each jump at most once. The rule in equation (18), in which the liftoff times were arbitrary as long as they avoided a specific value, reveals why it is possible to have a discontinuous f and still get determinacy. Taylor rule coefficients are also not required to be continuous by Proposition 4. As long as the central bank follows or does not follow the Taylor principle when required, determinacy can be achieved.

When $f(R_{t_1})$, $\xi_x(R_{t_1})$ and $\xi_\pi(R_{t_1})$ are assumed to be continuous, 6 shows that the rules become more dovish. Item 4a states that the Taylor principle cannot hold for any R_{t_1} . If the Taylor principle holds for one R_{t_1} , it must hold for all R_{t_1} and the same is true when the Taylor principle does not hold. Because implementing the optimal equilibrium requires that the Taylor principle does not hold for R^* (except when $\kappa\sigma\lambda = 1$, which is excluded from Proposition 6), the Taylor principle never holds. If the Taylor principle holds for some points but not for others, we have that $\kappa(\xi_\pi - 1) + \rho\xi_x > 0$ for some points and $\kappa(\xi_\pi - 1) + \rho\xi_x < 0$ for some others. By continuity of ξ_π and ξ_x , there must then be a point for which $\kappa(\xi_\pi - 1) + \rho\xi_x = 0$. For that point, there always is an equilibrium of the kind described in item 1 of Proposition 5 with the only difference that when $\kappa(\xi_\pi - 1) + \rho\xi_x = 0$, the unintended steady state (π_{zlb}, x_{zlb}) is in $\Omega_{zlb} \cap \Upsilon_{zlb}$, so any continuous path with $(\pi(t_1), x(t_1)) \in \Upsilon_{zlb}$ is an equilibrium.

According to item 4b, continuity also implies that the rule must now necessarily be both forward and backward-looking. By item 1, we already know the rule cannot be purely forward-looking. The reason it can no longer be purely backward-looking is that suboptimal

equilibria would emerge in a neighborhood of R^* , the optimal equilibrium. Because the optimal equilibrium (with $\kappa\sigma\lambda \neq 1$) must be implemented without following the Taylor principle, it always induces saddle-path dynamics. First imagine that Taylor rule coefficients are constant, so that the slope of the saddle path is the same for all R_{t_1} . With a continuous f , points $(\pi(0), x(0))$ around $(\pi^*(0), x^*(0))$ always land close to $(\pi^*(t^*), x^*(t^*))$. In a neighborhood of $(\pi^*(t^*), x^*(t^*))$, the dynamics are saddle-path stable. For a small enough $\varepsilon > 0$, there always is a point $(\pi^*(t_1), x^*(t_1)) \in \Upsilon_{ss}$ that is close enough to $(\pi^*(t^*), x^*(t^*))$ and can be reached at time $t^* \pm \varepsilon$ when starting at an $(\pi(0), x(0))$ close to $(\pi^*(0), x^*(0))$. This situation is similar to that in Figure 2, only that instead of a constant liftoff time, liftoff times change an arbitrarily small amount as the economy approaches R^* . The small change in liftoff time is not enough to overcome the indeterminacy. The presence of a saddle path breeds indeterminacy; liftoff slightly earlier or later simply puts the economy on different nearby points on the saddle. This is one of the reasons why the discontinuous Neo-Fisherian rule in equation (18) has discrete jumps (t^* , $t^* + 1$ and $t^* + 2$) for the liftoff times in the three different cases. Paths with R_{t_1} very close to R^* look so much like the optimal equilibrium that they themselves are equilibria. When the Taylor rule coefficients are not constant but instead a continuous function of $(\pi(0), x(0))$, the only change is that now the saddle path starting at $(\pi^*(0), x^*(0))$ is slightly different from the saddle path starting at $(\pi(0), x(0))$. This is still not enough to eliminate the equilibria. Points near R^* can still be part of an equilibrium with liftoff arbitrarily close to t^* and a saddle path arbitrarily close to the optimal one.

Lastly, item 4c sharpens equation (82) and states that for the cases described by item 2 of Proposition 5, liftoff must be “late enough”. Because for item 3 of the same Proposition they also need to be “late enough”, continuous rules can produce determinacy by promising a long enough period of zero interest rates together with a commitment to not following the Taylor principle once the promise expires. The precise meaning of “long enough” is given by equations (83) and (84). Even though these two lower bounds for liftoff are relatively simple functions of expectations of falling and exiting the expectational trap, the freedom to overshoot the lower bound and still get determinacy significantly reduces the informational requirements of the central bank. It is not necessary to know the precise parameters of the model or the functional form of \mathcal{T} .

Why does continuity change equation (82) into equation (84)? Consider a point $R = (\pi_0, x_0, \pi_1, x_1)$ that fulfills one of the three conditions in item 2 of Proposition 5 but does not satisfy equation (84). Let \mathfrak{p} be the set of points $(\pi(t), x(t), \pi_1, x_1)$ with $0 < t < \mathcal{T}(R)$ and $(\pi(0), x(0)) = (\pi_0, x_0)$. In other words $(\pi(t), x(t))$ is in the path between (π_0, x_0) and (π_1, x_1) . If there is some $P = (\pi(p), x(p), \pi_1, x_1) \in \mathfrak{p}$ that satisfies equation (84), then by continuity there is some point in the path between (π_0, x_0) and $(\pi(p), x(p))$ with $f = \mathcal{T}$, a

suboptimal equilibrium. So equation (84) not holding for R means it does not hold for all points in \mathfrak{p} . But this cannot happen, since the point with $\mathcal{T} = T$ cannot have $f < \mathcal{T} = T$, since liftoff cannot happen before T ¹⁸. So R cannot violate inequality (84) and eliminate indeterminacy.

Less formally, for all paths that reach the same (π_1, x_1) at time \mathcal{T} , the central bank cannot have liftoff be above \mathcal{T} for some paths and below \mathcal{T} for some other. For if it does, continuity implies at least one point has liftoff at \mathcal{T} , which produces a suboptimal equilibrium. So all paths must have either early liftoff (before \mathcal{T}) or late liftoff (after \mathcal{T}). If there is early liftoff for all paths, then as we consider shorter and shorter paths (paths with smaller and smaller \mathcal{T}), then liftoff also must be happening earlier and earlier. But there is always a path with $\mathcal{T} = T$ for which the central bank, by assumption, cannot set $f < T$. Promising early liftoff for one state forced the central bank to promise an even earlier liftoff for similar but less dire states. As the central bank considers states that are less and less dire, the promise of liftoff gets so short that, for some states, it must happen before T . But it does not make sense to lift off when the natural rate is negative, so it does not make sense to have early liftoff for any states¹⁹.

7 Conclusion

I have presented necessary and sufficient conditions for global determinacy of the optimal equilibrium in a New Keynesian economy with a binding ZLB. The central bank sets interest rates according to a state-dependent forward guidance rule followed by a traditional Taylor rule. Forward guidance entails a promise to keep interest rates pegged at zero for some time; making that length of time depend on private sector expectations, inflation and the output gap make the forward-guidance state-dependent. When deflationary expectations of the private sector are the strongest they can be, the only way to eliminate self-fulfilling deflations is to commit to not following the Taylor principle once the forward guidance period is over. When deflationary expectations are present but not as strong, a state-dependent extension of the forward guidance period becomes the crucial element.

The model I have used is one of the simplest models in which New Keynesian forces and rational expectations are present. The relative simplicity makes it possible to write down the solution of the model in closed form and fully characterize global determinacy. On the other hand, there are a myriad of important ways examined in the literature in which this baseline model can be extended, refined and modified. Investment dynamics, imports

¹⁸ The same argument applies if liftoff is allowed to happen even before T . The key is that liftoff is bounded below, which it always is, by $t = 0$.

¹⁹ As mentioned in the introduction, the key assumption is not that $f \geq T$ but that f is bounded below. Since liftoff cannot occur before $t = 0$, f is always bounded below.

and exports, coordination with fiscal policy, limited commitment, imperfect credibility, heterogeneous agents, financial intermediation and financial stability concerns, informational frictions, learning and other departures from rational expectations are, among other, components of a more realistic version of the model than can affect determinacy and optimality of equilibria. Nevertheless, the “three equation” New Keynesian model seems like the natural place to start, a necessary step towards deeper analysis, as the mechanisms presented in this paper are likely at play in any model with a New Keynesian core. Assessing the robustness and welfare properties of the interest rate rules I have considered in various versions on the model—and different models as well—is a required next step towards real-world use.

8 Bibliography

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A Proofs of Propositions

A.1 Preliminaries

Transition times. Let $\tau(t)$ be the largest of t_1 and the last time $(\pi(t), x(t))$ entered the region Ω_{zlb} from Ω_{ss} before or at time t . More formally²⁰

$$\tau(t) = \max\{t_1, \tau_{zlb}(t)\}, \quad (\text{A.1})$$

$$\tau_{zlb}(t) = \sup\{s \leq t \mid (x(s), \pi(s)) \in \partial\Omega \text{ and } (x(s^-), \pi(s^-)) \in \Omega_{ss}\}, \quad (\text{A.2})$$

Let $\eta(t)$ be the largest of t_1 and the last time $(\pi(t), x(t))$ exits the region Ω_{zlb} (or enters the region $\overline{\Omega_{ss}}$) before or at time t , i.e.

$$\eta(t) = \max\{t_1, \eta_{ss}(t)\}, \quad (\text{A.3})$$

$$\eta_{ss}(t) = \sup\{s \leq t \mid (x(s), \pi(s)) \in \partial\Omega \text{ and } (x(s^+), \pi(s^+)) \in \Omega_{ss}\}. \quad (\text{A.4})$$

Note that $\tau(t)$ and $\eta(t)$ are piece-wise constant and thus for all $t \neq \tau(t)$ we have $\dot{\tau}(t) = 0$ and for all $t \neq \eta(t)$ we have $\dot{\eta}(t) = 0$.

Solution to IS and NKPC. Let

$$A_{zlb} = \begin{bmatrix} 0 & -\frac{1}{\sigma} \\ -\kappa & \rho \end{bmatrix},$$

$$A_{ss} = \begin{bmatrix} \frac{1}{\sigma}\xi_x & \frac{1}{\sigma}(\xi_\pi - 1) \\ -\kappa & \rho \end{bmatrix}.$$

The matrix A_{zlb} has eigenvalues ϕ_1 and ϕ_2 defined in equations (28)-(29). The eigenvalues of A_{ss} are

$$\alpha_1 = \frac{1}{2\sigma} \left(\xi_x + \sigma\rho + \sqrt{(\xi_x - \sigma\rho)^2 - 4\kappa\sigma(\xi_\pi - 1)} \right), \quad (\text{A.5})$$

$$\alpha_2 = \frac{1}{2\sigma} \left(\xi_x + \sigma\rho - \sqrt{(\xi_x - \sigma\rho)^2 - 4\kappa\sigma(\xi_\pi - 1)} \right). \quad (\text{A.6})$$

Because stable dynamics always produce indeterminacy, I restrict all analysis to cases in which either $\det A_{ss} > 0$ and $\text{trace } A_{ss} > 0$, or $\det A_{ss} < 0$.

²⁰ Recall that the supremum of the empty set is $-\infty$. If there is no s such that $(x(s), \pi(s)) \in \partial\Omega$ and $(x(s^-), \pi(s^-)) \in \Omega_{ss}$, then $\tau_{zlb}(t) = -\infty$ and $\tau(t) = t_1$.

For $t \in [0, T)$ the solution to (1)-(2) is

$$\begin{aligned} x(t) &= -\frac{\phi_2}{(\phi_1 - \phi_2)} \left(d_{exit}(0) - \frac{\phi_2}{\kappa} (r_h - r_l) \right) e^{\phi_1 t} \\ &\quad + \frac{\phi_1}{(\phi_1 - \phi_2)} \left(d_{trap}(0) - \frac{\phi_1}{\kappa} (r_h - r_l) \right) e^{\phi_2 t} - \frac{\rho}{\kappa} r_l, \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} \pi(t) &= -\frac{\kappa}{(\phi_1 - \phi_2)} \left(d_{exit}(0) - \frac{\phi_2}{\kappa} (r_h - r_l) \right) e^{\phi_1 t} \\ &\quad + \frac{\kappa}{(\phi_1 - \phi_2)} \left(d_{trap}(0) - \frac{\phi_1}{\kappa} (r_h - r_l) \right) e^{\phi_2 t} - r_l. \end{aligned} \quad (\text{A.8})$$

For $t \in [T, t_1)$ the solution is

$$x(t) = -\frac{\phi_2}{(\phi_1 - \phi_2)} d_{exit}(T) e^{\phi_1(t-T)} + \frac{\phi_1}{(\phi_1 - \phi_2)} d_{trap}(T) e^{\phi_2(t-T)} - \frac{\rho}{\kappa} r_h \quad (\text{A.9})$$

$$\pi(t) = -\frac{\kappa}{(\phi_1 - \phi_2)} d_{exit}(T) e^{\phi_1(t-T)} + \frac{\kappa}{(\phi_1 - \phi_2)} d_{trap}(T) e^{\phi_2(t-T)} - r_h \quad (\text{A.10})$$

For $t \in [t_1, \infty)$, when $(\pi(t), x(t)) \in \Omega_{zlb}$, the solution is

$$x(t) = -\frac{\phi_2 d_{exit}(\tau(t))}{(\phi_1 - \phi_2)} e^{\phi_1(t-\tau(t))} + \frac{\phi_1 d_{trap}(\tau(t))}{(\phi_1 - \phi_2)} e^{\phi_2(t-\tau(t))} - \frac{\rho}{\kappa} r_h, \quad (\text{A.11})$$

$$\pi(t) = -\frac{\kappa d_{exit}(\tau(t))}{(\phi_1 - \phi_2)} e^{\phi_1(t-\tau(t))} + \frac{\kappa d_{trap}(\tau(t))}{(\phi_1 - \phi_2)} e^{\phi_2(t-\tau(t))} - r_h. \quad (\text{A.12})$$

For $t \in [t_1, \infty)$, when $(\pi(t), x(t)) \in \Omega_{ss}$, I distinguish three cases:

Case I: $\xi_\pi \neq 1$ and $4\kappa\sigma(\xi_\pi - 1) \neq (\xi_x - \sigma\rho)^2$.

Case II: $\xi_\pi = 1$ and $4\kappa\sigma(\xi_\pi - 1) \neq (\xi_x - \sigma\rho)^2$.

Case III: $4\kappa\sigma(\xi_\pi - 1) = (\xi_x - \sigma\rho)^2$.

For Case I, the solution is

$$\begin{aligned} x(t) &= -\frac{(1 - \xi_\pi) \pi_{\eta(t)} + (\sigma\alpha_2 - \xi_x) x_{\eta(t)}}{\sigma(\alpha_1 - \alpha_2)} e^{\alpha_1(t-\eta(t))} \\ &\quad + \frac{(1 - \xi_\pi) \pi_{\eta(t)} + (\sigma\alpha_1 - \xi_x) x_{\eta(t)}}{\sigma(\alpha_1 - \alpha_2)} e^{\alpha_2(t-\eta(t))}, \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned} \pi(t) &= \frac{(1 - \xi_\pi) \pi_{\eta(t)} + (\sigma\alpha_2 - \xi_x) x_{\eta(t)}}{\sigma(\xi_\pi - 1)(\alpha_1 - \alpha_2)} (\xi_x - \sigma\alpha_1) e^{\alpha_1(t-\eta(t))} \\ &\quad - \frac{(1 - \xi_\pi) \pi_{\eta(t)} + (\sigma\alpha_1 - \xi_x) x_{\eta(t)}}{\sigma(\xi_\pi - 1)(\alpha_1 - \alpha_2)} (\xi_x - \sigma\alpha_2) e^{\alpha_2(t-\eta(t))}. \end{aligned} \quad (\text{A.14})$$

For Case II, the solution is

$$x(t) = x_{\eta(t)} e^{\frac{1}{\sigma} \xi_x (t - \eta(t))}, \quad (\text{A.15})$$

$$\pi(t) = \frac{\pi_{\eta(t)} (\xi_x - \sigma\rho) + \kappa \sigma x_{\eta(t)}}{\xi_x - \sigma\rho} e^{\rho(t - \eta(t))} - \frac{\kappa \sigma x_{\eta(t)}}{\xi_x - \sigma\rho} e^{\frac{1}{\sigma} \xi_x (t - \eta(t))}. \quad (\text{A.16})$$

For Case III, the solution is

$$x(t) = \left(1 + \frac{1}{2\sigma} (\xi_x - \sigma\rho) (t - t_1)\right) x_{\eta(t)} e^{\frac{1}{2}(\rho + \frac{1}{\sigma} \xi_x)(t - \eta(t)} \\ + \frac{1}{\kappa} \left(\frac{1}{2\sigma} (\sigma\rho - \xi_x)\right)^2 (t - t_1) \pi_{\eta(t)} e^{\frac{1}{2}(\rho + \frac{1}{\sigma} \xi_x)(t - \eta(t))}, \quad (\text{A.17})$$

$$\pi(t) = -\kappa (t - t_1) x_{\eta(t)} e^{\frac{1}{2}(\rho + \frac{1}{\sigma} \xi_x)(t - \eta(t)} \\ + \left(1 - \frac{1}{2\sigma} (\xi_x - \sigma\rho) (t - t_1)\right) \pi_{\eta(t)} e^{\frac{1}{2}(\rho + \frac{1}{\sigma} \xi_x)(t - \eta(t))}. \quad (\text{A.18})$$

Continuous pasting. Because solutions to (1)-(2) must be continuous, any equilibrium path $(\pi(t), x(t))$ must paste continuously at $t = T$ and at $t = t_1$:

$$\lim_{t \uparrow T} (\pi(t), x(t)) = (\pi(t), x(t)), \quad (\text{A.19})$$

$$\lim_{t \uparrow t_1} (\pi(t), x(t)) = (\pi(t_1), x(t_1)). \quad (\text{A.20})$$

After t_1 , continuous paths require that whenever $(\pi(t), x(t))$ transitions between Ω_{ss} and Ω_{zlb} , the paths paste continuously at the boundary $\partial\Omega$:

$$\lim_{t - \tau_t \downarrow 0} (\pi(t), x(t)) = (x(\tau_t), \pi(\tau_t)), \quad (\text{A.21})$$

$$\lim_{t - \eta_t \downarrow 0} (\pi(t), x(t)) = (x(\eta_t), \pi(\eta_t)). \quad (\text{A.22})$$

In equilibrium, the continuous pasting conditions at $t = T$ and at $t = t_1$ imply

$$x(0) = \frac{\phi_1 e^{-\phi_2 t_1} - \phi_2 e^{-\phi_1 t_1}}{\phi_1 - \phi_2} x(t_1) - \frac{1}{\sigma} \frac{e^{-\phi_1 t_1} - e^{-\phi_2 t_1}}{\phi_1 - \phi_2} \pi(t_1) \quad (\text{A.23}) \\ + \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 t_1} - \phi_2^2 e^{-\phi_1 t_1}}{\phi_1 - \phi_2} + \left(\frac{r_h - r_l}{\kappa}\right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa},$$

$$\pi(0) = -\kappa \frac{e^{-\phi_1 t_1} - e^{-\phi_2 t_1}}{\phi_1 - \phi_2} x(t_1) + \frac{\phi_1 e^{-\phi_1 t_1} - \phi_2 e^{-\phi_2 t_1}}{\phi_1 - \phi_2} \pi(t_1) \quad (\text{A.24}) \\ + r_h \frac{\phi_1 e^{-\phi_2 t_1} - \phi_2 e^{-\phi_1 t_1}}{\phi_1 - \phi_2} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l.$$

Solving for $(\pi(t_1), x(t_1))$ gives, in matrix notation,

$$\begin{bmatrix} x(t_1) \\ \pi(t_1) \end{bmatrix} = \frac{e^{(\phi_1+\phi_2)t_1}}{\phi_1 - \phi_2} \begin{bmatrix} \phi_1 e^{-\phi_1 t_1} - \phi_2 e^{-\phi_2 t_1} & \frac{1}{\sigma} (e^{-\phi_1 t_1} - e^{-\phi_2 t_1}) \\ \kappa (e^{-\phi_1 t_1} - e^{-\phi_2 t_1}) & \phi_1 e^{-\phi_2 t_1} - \phi_2 e^{-\phi_1 t_1} \end{bmatrix} \begin{bmatrix} x(0) - h(t_1) \\ \pi(0) - m(t_1) \end{bmatrix}, \quad (\text{A.25})$$

where

$$\begin{aligned} h(t_1) &= r_h \frac{\phi_1^2 e^{-\phi_2 t_1} - \phi_2^2 e^{-\phi_1 t_1}}{\kappa(\phi_1 - \phi_2)} + (r_h - r_l) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\kappa(\phi_1 - \phi_2)} - \frac{r_l \rho}{\kappa}, \\ m(t_1) &= r_h \frac{\phi_1 e^{-\phi_2 t_1} - \phi_2 e^{-\phi_1 t_1}}{(\phi_1 - \phi_2)} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{(\phi_1 - \phi_2)} - r_l. \end{aligned}$$

Solving for $e^{-\phi_1 t_1}$ and $e^{-\phi_2 t_1}$, and then eliminating t_1 from one of the equations, gives the result of section 6.1 that equilibrium paths that are already continuous at T are also continuous at t_1 if and only if

$$0 = \mathcal{P}(R_{t_1}), \quad (\text{A.26})$$

$$t_1 = \mathcal{T}(R_{t_1}), \quad (\text{A.27})$$

where

$$\mathcal{P}(R_{t_1}) = \begin{cases} 1 \{d_{trap}(t_1) \neq 0 \text{ or } d_{trap}(t_1) \neq 0\} & , \text{ if } d_{exit}(0) = \frac{\phi_2}{\kappa} (r_h - r_l) (1 - e^{-T\phi_1}) \\ & \text{and } d_{trap}(0) = \frac{\phi_1}{\kappa} (r_h - r_l) (1 - e^{-T\phi_2}) \\ \\ d_{exit}(t_1) + 1 \{d_{trap}(t_1) = 0\} & , \text{ if } d_{exit}(0) = \frac{\phi_2}{\kappa} (r_h - r_l) (1 - e^{-T\phi_1}) \\ & \text{and } d_{trap}(0) \neq \frac{\phi_1}{\kappa} (r_h - r_l) (1 - e^{-T\phi_2}) \\ \\ d_{trap}(t_1) + 1 \{d_{exit}(t_1) = 0\} & , \text{ if } d_{exit}(0) \neq \frac{\phi_2}{\kappa} (r_h - r_l) (1 - e^{-T\phi_1}) \\ & \text{and } d_{trap}(0) = \frac{\phi_1}{\kappa} (r_h - r_l) (1 - e^{-T\phi_2}) \\ \\ \left(\frac{d_{exit}(t_1)}{d_{exit}(0) + \frac{\phi_2}{\kappa} (r_h - r_l) (e^{-T\phi_1} - 1)} \right)^{\phi_2} & , \text{ if } d_{exit}(0) \neq \frac{\phi_2}{\kappa} (r_h - r_l) (1 - e^{-T\phi_1}) \\ & \text{and } d_{trap}(0) \neq \frac{\phi_1}{\kappa} (r_h - r_l) (1 - e^{-T\phi_2}) \\ \\ - \left(\frac{d_{trap}(t_1)}{d_{trap}(0) + \frac{\phi_1}{\kappa} (r_h - r_l) (e^{-T\phi_2} - 1)} \right)^{\phi_1} & \end{cases},$$

and

$$\mathcal{T}(R_{t_1}) = \begin{cases} [T, \infty) & , \quad \text{if } d_{exit}(0) = \frac{\phi_2}{\kappa}(r_h - r_l)(1 - e^{-T\phi_1}) \\ & \text{and } d_{trap}(0) = \frac{\phi_1}{\kappa}(r_h - r_l)(1 - e^{-T\phi_2}) \\ \frac{1}{\phi_2} \log \frac{d_{trap}(t_1)}{d_{trap}(0) + \frac{\phi_1}{\kappa}(r_h - r_l)(e^{-T\phi_2} - 1)} & , \quad \text{if } d_{exit}(0) = \frac{\phi_2}{\kappa}(r_h - r_l)(1 - e^{-T\phi_1}) \\ & \text{and } d_{trap}(0) \neq \frac{\phi_1}{\kappa}(r_h - r_l)(1 - e^{-T\phi_2}) \\ \frac{1}{\phi_1} \log \frac{d_{exit}(t_1)}{d_{exit}(0) + \frac{\phi_2}{\kappa}(r_h - r_l)(e^{-T\phi_1} - 1)} & , \quad \text{otherwise} \end{cases} .$$

Saddle path in Ω_{ss} . The saddle path Υ_{ss} is the set of points (π, x) such that

$$\pi = \begin{cases} \frac{(\xi_x - \sigma\alpha_2)}{(1 - \xi_\pi)} x & , \quad \text{if } \det A_{ss} < 0 \text{ and } \xi_\pi \neq 1 \\ \frac{\kappa\sigma}{(\sigma\rho - \xi_x)} x & , \quad \text{if } \det A_{ss} < 0 \text{ and } \xi_\pi = 1 \\ \frac{\kappa}{\rho} x & , \quad \text{if } \det A_{ss} = 0 \text{ and } \text{trace } A_{ss} \geq 0 \\ \emptyset & , \quad \text{otherwise} \end{cases} . \quad (\text{A.28})$$

A.2 Case $\kappa\sigma\lambda = 1$ in Proposition 1

The next example shows that when $\kappa\sigma\lambda = 1$, it is indeed possible to implement the optimal equilibrium uniquely with a constant liftoff time. Let $\underline{t} = t^*$ and $\kappa\sigma\lambda = 1$. Pick

$$(\xi_x(R_0), \xi_\pi(R_0)) = \begin{cases} (0, 0) & , \quad \text{if } (x(\underline{t}), \pi(\underline{t})) = (0, 0) \text{ or } \rho\pi(\underline{t}) \neq \kappa x(\underline{t}) \\ (1, -\frac{\rho}{\kappa}) & , \quad \text{otherwise} \end{cases} . \quad (\text{A.29})$$

Then the rule

$$i(t) = \begin{cases} 0 & , \quad 0 \leq t < \underline{t} \\ \max\{0, \xi_\pi(R_0)\pi(t) + \xi_x(R_0)x(t) + r(t)\} & , \quad \underline{t} \leq t < \infty \end{cases} \quad (\text{A.30})$$

implements the optimal equilibrium as the unique equilibrium of the economy.

When $(\pi(\underline{t}), x(\underline{t})) = 0$, the rule implements the optimal path. First, since $\underline{t} = t^*$ and $(\pi(\underline{t}), x(\underline{t})) = (\pi^*(t^*), x^*(t^*)) = (0, 0)$, $(\pi(t), x(t)) = (\pi^*(t), x^*(t))$ for $t < t^*$. By equation (A.29), $\xi_x(R^*) = \xi_\pi(R^*) = 0$ and thus $i(t) = i^*(t) = r_h > 0$ for $t \geq t_1$. As $(0, 0)$ is a steady state, $(\pi(t), x(t)) = (0, 0)$ for all $t \geq t_1$, which shows that $(\pi(t), x(t)) = (\pi^*(t), x^*(t))$.

No other equilibrium exists since, for all $R_0 \neq R^*$, continuous paths are unbounded. If $\rho\pi(\underline{t}) \neq \kappa x(\underline{t})$, equation (A.29) gives $(\xi_x, \xi_\pi) = (0, 0)$ and by equation (A.28) the saddle path is $\rho\pi = \kappa x$. It follows that $(\pi(\underline{t}), x(\underline{t})) \notin \Upsilon_{ss}$. In addition, $i(t) = r_h > 0$ for $t \geq \underline{t}$ and thus $(\pi(t), x(t)) \in \Omega_{ss}$ for all $t \geq \underline{t}$ since Ω_{zlb} is empty. The global saddle path dynamics and $(\pi(\underline{t}), x(\underline{t})) \notin \Upsilon_{ss}$ imply that $(\pi(t), x(t))$ explodes as $t \rightarrow \infty$. If $(\pi(\underline{t}), x(\underline{t})) \neq 0$

and $\rho\pi(\underline{t}) = \kappa x(\underline{t})$, $(\xi_x, \xi_\pi) = (1, -\frac{\rho}{\kappa})$ implies that the Taylor principle does not hold, since $\kappa(\xi_\pi - 1) + \rho\xi_x = -\kappa < 0$. In addition, $(\pi(\underline{t}), x(\underline{t})) \in \Omega_{ss}$ since $\xi_\pi\pi(\underline{t}) + \xi_x x(\underline{t}) + r(\underline{t}) = r_h > 0$ and $(\pi(\underline{t}), x(\underline{t})) \notin \Upsilon_{ss}$ by equation (A.28). Because the dynamics are saddle path stable and $(\pi(\underline{t}), x(\underline{t}))$ is not on the saddle path, $(\pi(t), x(t))$ either explodes or enters Ω_{zlb} in finite time. But by item 3 in Proposition 7, if $(\pi(t), x(t))$ enters Ω_{zlb} it also explodes.

A.3 Constants in the Neo-Fisherian Rule of Section 5.2

To be on the saddle at time t_1 , $\pi(t_1) = \phi x(t_1)$. Using the continuous pasting conditions in equation (A.25) to express $\pi(t_1) = \phi x(t_1)$ in terms of $x(0)$ and $\pi(0)$ gives

$$p(t_1)x(0) + q(t_1)\pi(0) = v(t_1),$$

where

$$\begin{aligned} p(t_1) &= \kappa\sigma\left((\phi_1 - \kappa\phi)e^{-\phi_1 t_1} - (\phi_2 - \kappa\phi)e^{-\phi_2 t_1}\right), \\ q(t_1) &= \kappa\left((\sigma\phi\phi_2 + 1)e^{-\phi_1 t_1} - (\sigma\phi\phi_1 + 1)e^{-\phi_2 t_1}\right), \\ v(t_1) &= -\left((\kappa + \sigma\phi_1(\rho - \kappa\phi))r_l + \frac{(r_h - r_l)\phi_1}{\phi_1 - \phi_2}(\kappa + \sigma\phi_1^2 - \kappa\sigma\phi(\phi_1 - \phi_2))e^{-T\phi_2}\right)e^{-\phi_1 t_1} \\ &\quad + \left((\kappa + \sigma\phi_2(\rho - \kappa\phi))r_l - \frac{(r_h - r_l)\phi_2}{\phi_1 - \phi_2}(\kappa + \sigma\phi_2^2 + \kappa\sigma\phi(\phi_1 - \phi_2))e^{-T\phi_1}\right)e^{-\phi_2 t_1} \\ &\quad + \frac{(\sigma\rho^2 + 4\kappa)(\rho - \kappa\phi)r_h}{(\phi_1 - \phi_2)}e^{-\phi_1 t_1}e^{-\phi_2 t_1}. \end{aligned}$$

Evaluating these functions at $t_1 = t^*$ and $t_1 = t^* + 1$ determines the constants A, B, C, D, E, F .

A.4 Proof of Proposition 2

I first prove a lemma and then proceed to the proof of Proposition 2.

Lemma 7. *When the Taylor principle does not hold, the following are true:*

1. $(\pi_{zlb}, x_{zlb}) \notin \Omega_{zlb}$.
2. If $(\pi(q), x(q)) \in \Omega_{zlb}$ for some $q \geq t_1$, $(\pi(t), x(t))$ either explodes as $t \rightarrow \infty$ or exits Ω_{zlb} in finite time.
3. If $(\pi(s), x(s)) \in \Omega_{ss}$ for some $s \geq t_1$ and $(\pi(q), x(q)) \in \Omega_{zlb}$ for some $q > s$, then $(\pi(t), x(t))$ explodes as $t \rightarrow \infty$.

4. If $(\pi(q), x(q)) \in \Omega_{zlb}$ for some $q \geq t_1$ and there exist no $r \geq q$ such that $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$, then $(\pi(t), x(t))$ explodes as $t \rightarrow \infty$.
5. If $(\pi(q), x(q)) \in \Omega_{zlb}$ for some $q \geq t_1$ and there exist $r > q$ such that $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$, then there exist no $t \in [q, r)$ such that $(\pi(t), x(t)) \in \Upsilon_{ss}$.

Proof of Lemma 7. 1. Plugging the steady-state from equation (31) into the Taylor rule gives

$$\begin{aligned}
\xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h &= \xi_\pi (-r_h) + \xi_x \left(-\frac{\rho}{\kappa} r_h \right) + r_h, \\
&= -\frac{r_h}{\kappa} (\kappa (\xi_\pi - 1) + \rho \xi_x), \\
&= -\frac{r_h \sigma}{\kappa} \det A_{ss}, \\
&> 0.
\end{aligned}$$

where $\det A_{ss} < 0$ because the Taylor principle does not hold.

2. The saddle path stable dynamics in Ω_{zlb} and $(\pi_{zlb}, x_{zlb}) \notin \Omega_{zlb}$ immediately imply that paths starting in Ω_{zlb} either explode or exit Ω_{zlb} in finite time.
3. Let \hat{n} be a unit vector normal to $\partial\Omega$ pointing towards Ω_{ss} . Because $(\pi(t), x(t))$ transitions from Ω_{ss} to Ω_{zlb} and its path is continuous, there exist $\omega \in (s, q]$ such that

$$\xi_\pi \pi_\omega + \xi_x x_\omega + r_h = 0, \quad (\text{A.31})$$

$$\hat{n} \cdot (\dot{\pi}_\omega, \dot{x}_\omega) \leq 0. \quad (\text{A.32})$$

Equation (A.31) says that $(x_\omega, \pi_\omega) \in \partial\Omega$. The non-positive dot product in equation (A.32) says that (x_ω, π_ω) is not moving towards Ω_{ss} (and moving towards Ω_{zlb} when the dot product is negative). Writing out the dot product gives

$$\begin{bmatrix} \frac{\xi_x}{\sqrt{\xi_\pi^2 + \xi_x^2}} \\ \frac{\xi_\pi}{\sqrt{\xi_\pi^2 + \xi_x^2}} \end{bmatrix}^T \begin{bmatrix} -\frac{1}{\sigma} (\pi_\omega + r_h) \\ \rho \pi_\omega - \kappa x_\omega \end{bmatrix} = -\frac{\pi_\omega (\xi_x - \sigma \rho \xi_\pi) + \xi_x r_h + \kappa \sigma \xi_\pi x_\omega}{\sigma \sqrt{\xi_\pi^2 + \xi_x^2}} \leq 0,$$

or, simplifying,

$$\pi_\omega (\xi_x - \sigma \rho \xi_\pi) + \xi_x r_h + \kappa \sigma \xi_\pi x_\omega \geq 0. \quad (\text{A.33})$$

The Taylor principle (TP) not holding, equation (A.31), equation (A.33) and $\phi_1, r_h > 0$

imply that

$$\begin{aligned}
& \underbrace{-r_h(\kappa(\xi_\pi - 1) + \rho\xi_x)}_{<0 \text{ as TP does not hold}} - \underbrace{\kappa(r_h + \xi_\pi\pi_\omega + \xi_x x_\omega)}_{=0 \text{ by eq. (A.31)}} \\
& \quad + \phi_1 \underbrace{(\pi_\omega(\xi_x - \sigma\rho\xi_\pi) + \xi_x r_h + \kappa\sigma\xi_\pi x_\omega)}_{\geq 0 \text{ by eq. (A.33)}} > 0. \quad (\text{A.34})
\end{aligned}$$

Equation (A.34) is a sufficient condition for (π_T, x_T) to be in Ω_{zlb} for all $t \geq \omega$. To see this, use the dynamics in equations (A.11) and (A.12) to write

$$\xi_\pi \pi_t + \xi_x x_t + r_h = W(t - \omega),$$

where

$$\begin{aligned}
W(t - \omega) &= Ae^{\phi_1(t-\omega)} + Be^{\phi_2(t-\omega)} + C, \\
A &= -(\phi_1\pi_\omega - \kappa x_\omega - \phi_2 r_h) \frac{\xi_x - \sigma\phi_1\xi_\pi}{\sigma\phi_1(\phi_1 - \phi_2)}, \\
B &= (\phi_2\pi_\omega - \kappa x_\omega - \phi_1 r_h) \frac{\xi_x - \sigma\phi_2\xi_\pi}{\sigma\phi_2(\phi_1 - \phi_2)}, \\
C &= \frac{r_h}{\kappa} (\kappa(1 - \xi_\pi) - \rho\xi_x).
\end{aligned}$$

By definition, $(\pi(t), x(t)) \in \Omega_{zlb}$ iff $W(t - \omega) \leq 0$. Therefore, if $W(t - \omega)$ has no zeros for $t > \omega$, $(\pi(t), x(t))$ remains in Ω_{zlb} forever. Since $\phi_2 < 0 < \phi_1$, $W(0) = 0$ by (A.31) and $W'(0) \leq 0$ by (A.32), a sufficient condition for $W(u)$ to have no zeros for $u > 0$ is that $A < 0$. After some manipulations, it can be seen that (A.34) is equivalent to $A < 0$. By 3, since (π_T, x_T) never transitions to Ω_{ss} after ω , it follows that $(\pi(t), x(t))$ explodes as $t \rightarrow \infty$.

4. By 3, if $(\pi(t), x(t))$ does not exit Ω_{zlb} , it explodes. If $(\pi(t), x(t))$ exits Ω_{zlb} at some time η and (π_η, x_η) is not on the ss saddle path, due to the saddle path dynamics inside Ω_{ss} , $(\pi(t), x(t))$ either explodes or returns to Ω_{zlb} in finite time. If $(\pi(t), x(t))$ returns to Ω_{zlb} , by item 3, it explodes.
5. By equations (A.28), Υ_{ss} is a line through the origin, which can be written as $A\pi - x = 0$ with $A \neq 0$. If $\xi_\pi \neq 1$ then $A = (1 - \xi_\pi) / (\xi_x - \sigma\alpha_2)$ and if $\xi_\pi = 1$ then $A = (\sigma\rho - \xi_x) / \kappa\sigma$. Let

$$F(t) = A\pi_t - x_t. \quad (\text{A.35})$$

The path of $(\pi(t), x(t))$ intersects Υ_{ss} at some time \bar{t} iff $F(\bar{t}) = 0$.

Using equation (22) in the IS and NKPC after t_1 , it can be seen that (π_T, x_T) is

continuous on $\partial\Omega$ for all $t \geq t_1$. It follows that the right and left derivatives of $F(t)$ are equal at $t = r$. By equations (A.13)-(A.18) and (A.28), (π_T, x_T) remains on Υ_{ss} after intersecting it at $t = r$. Hence, the right derivative of $F(t)$ at $t = r$ is zero. To find the left derivative, first note that between times q and r , the dynamics of $(\pi(t), x(t))$ are given by equations (A.11) and (A.12). Using these dynamics in equation (A.35) gives

$$F(t) = Pe^{\phi_1(t-\tau(t))} + Qe^{\phi_2(t-\tau(t))} + R, \quad (\text{A.36})$$

where

$$\begin{aligned} P &= \frac{\phi_2 - A\kappa}{\kappa(\phi_1 - \phi_2)} (\phi_2 r_h - \phi_1 \pi_{\tau(t)} + \kappa x_{\tau(t)}), \\ Q &= -\frac{\phi_1 - A\kappa}{\kappa(\phi_1 - \phi_2)} (\phi_1 r_h - \phi_2 \pi_{\tau(t)} + \kappa x_{\tau(t)}), \\ R &= \frac{r_h(\rho - A\kappa)}{\kappa}. \end{aligned}$$

Note that P and Q cannot both be zero. Indeed, $P = Q = 0$ implies $R = 0$ since $F(r) = 0$. But then $(x_{\tau(t)}, \pi_{\tau(t)}) = (\pi_{zlb}, x_{zlb}) = (x_q, \pi_q) \in \Omega_{zlb}$ contradicts item 1. Using equation (A.36) to compute the left derivative of $F(t)$ at $t = r$ and setting it equal to zero (the value of the right derivative) gives

$$F'(r) = 0 = \phi_1 P e^{\phi_1(r-\tau(r))} + \phi_2 Q e^{\phi_2(r-\tau(r))}. \quad (\text{A.37})$$

In other words, the path for $(\pi(t), x(t))$ must be tangent to the ss saddle path at $t = r$. Since $\phi_2 < 0 < \phi_1$, equation (A.37) implies that P and Q have the same sign (and the sign is not zero since P and Q cannot both be zero). In turn, P and Q having the same sign implies that

$$F''(t) = \phi_1^2 P e^{\phi_1(t-\tau(t))} + \phi_2^2 Q e^{\phi_2(t-\tau(t))} \quad (\text{A.38})$$

has the same (non-zero) sign for all $t \in [q, r]$, so $F'(t)$ is strictly monotonic. A continuous and strictly monotonic $F'(t)$ in $t \in [q, r]$, together with $F(r) = F'(r) = 0$, imply that the only solution to $F(t) = 0$ for $t \in [q, r]$ is r . \square

Proof of Proposition 2. Consider the following condition:

$$\begin{aligned} &(\pi(t_1), x(t_1)) \in \Omega_{ss} \cap \Upsilon_{ss}, \\ &\text{or} \\ &(\pi(t_1), x(t_1)) \in \Omega_{zlb} \text{ and } (\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss} \text{ for some } r \in [t_1, \infty). \end{aligned} \quad (\text{A.39})$$

I first show that condition (A.39) implies paths are not explosive. If $(\pi(t_1), x(t_1)) \in \Omega_{ss}$ and $(\pi(t_1), x(t_1)) \in \Upsilon_{ss}$, then $(\pi_T, x_T) \in \Upsilon_{ss}$ for all $t \geq t_1$. If $(x_{t_1}, \pi_{t_1}) \in \Omega_{zlb}$ and $(x_r, \pi_r) \in \partial\Omega \cap \Upsilon_{ss}$ for some $r \in [t_1, \infty)$, $(\pi(t), x(t)) \in \Upsilon_{ss}$ for all $t \geq r$. In either case, the path converges to $(0, 0)$ and therefore does not explode.

To prove the converse, I prove the contrapositive. There are three cases to consider.

Case 1: If $(\pi(t_1), x(t_1)) \notin \Omega_{ss}$ and there exist no $r \in [t_1, \infty)$ such that $(x_r, \pi_r) \in \partial\Omega \cap \Upsilon_{ss}$, then (π_T, x_T) explodes by item 5 of Lemma 7.

Case 2: If $(\pi(t_1), x(t_1)) \notin \Upsilon_{ss}$ and $(\pi(t_1), x(t_1)) \notin \Omega_{zlb}$, (π_T, x_T) either explodes or enters Ω_{zlb} . If it enters Ω_{zlb} , it also explodes by item 3 of Lemma 7.

Case 3: If $(\pi(t_1), x(t_1)) \notin \Upsilon_{ss}$ and there exist no $r \in [t_1, \infty)$ such that $(x_r, \pi_r) \in \partial\Omega \cap \Upsilon_{ss}$, consider two sub-cases. The first sub-case, given by $(\pi(t_1), x(t_1)) \notin \Omega_{ss}$, was already covered in case 1. For the second sub-case, given by $(\pi(t_1), x(t_1)) \notin \Omega_{zlb}$, case 2 implies $(\pi(t_1), x(t_1))$ explodes. \square

A.5 Proof of Proposition 3

I first prove a lemma and then proceed to the proof of Proposition 3.

Lemma 8. *When the Taylor principle holds, the following are true:*

1. $(\pi_{zlb}, x_{zlb}) \in \Omega_{zlb}$.
2. If $(x_m, \pi_m) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ with $m \geq T$, then $(\pi_T, x_T) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ for all $t \geq m$.
3. There exist (π_0, x_0) such that $(\pi(T), x(T)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$.
4. If $(\pi(s), x(s)) \in \partial\Omega$ for some $s \geq t_1$, there is no $p > 0$ such that $(\pi(t), x(t)) \in \Omega_{ss}$ for $t \in (s, s+p)$ and $(\pi(s+p), x(s+p)) \in \Upsilon_{zlb} \cap \partial\Omega$.
5. If $(\pi(q), x(q)) \in \Omega_{ss}$ for $q \geq t_1$ with $(\pi(q), x(q)) \neq (\pi_{ss}, x_{ss})$ and there is no $p > 0$ such that $[(\pi(t), x(t)) \in \Omega_{ss}$ for $t \in (q, q+p)$ and $(\pi(q+p), x(q+p)) \in \Upsilon_{zlb} \cap \partial\Omega]$, then $(\pi(t), x(t))$ explodes as $t \rightarrow \infty$.
6. There is no chaos (in the sense of R. Devaney²¹).

²¹ See Banks, Brooks, Cairns, Davis, and Stacey (1992) for a definition.

Proof of Lemma 8. 1. Plugging the steady-state (31) into the Taylor rule gives

$$\begin{aligned}
\xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h &= \xi_\pi (-r_h) + \xi_x \left(-\frac{\rho}{\kappa} r_h \right) + r_h, \\
&= -\frac{1}{\kappa} r_h (\kappa (\xi_\pi - 1) + \rho \xi_x), \\
&= -\frac{r_h \sigma}{\kappa} \det A_{ss}, \\
&< 0.
\end{aligned}$$

where $\det A_{ss} > 0$ because the Taylor principle holds.

2. Let (x, π) be a point in the line segment with endpoints (x_m, π_m) and (π_{zlb}, x_{zlb}) , i.e. (x, π) is in the portion of the *zlb* saddle path between (x_m, π_m) and (π_{zlb}, x_{zlb}) . Then

$$(x, \pi) = a(x_m, \pi_m) + (1 - a)(\pi_{zlb}, x_{zlb}),$$

for some $a \in [0, 1]$. It follows that

$$\begin{aligned}
\xi_\pi \pi + \xi_x x + r_h &= \xi_\pi (a\pi_m + (1 - a)\pi_{zlb}) + \xi_x (ax_m + (1 - a)x_{zlb}) + r_h, \\
&= a(\xi_\pi \pi_m + \xi_x x_m + r_h) + (1 - a)(\xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h), \\
&< 0,
\end{aligned} \tag{A.40}$$

where the last line uses that (x_m, π_m) and (π_{zlb}, x_{zlb}) are both in Ω_{zlb} . The line segment with endpoints (x_m, π_m) and (π_{zlb}, x_{zlb}) is thus entirely in Ω_{zlb} . For $t \in [T, t_1)$, the dynamics of $(\pi(t), x(t))$ are given by (A.9)-(A.10) and thus (π_T, x_T) travels along the *zlb* saddle path. For $t \geq t_1$, equation (A.40) implies that $\max\{0, \xi_\pi \pi_t + \xi_x x_t + r_h\} = 0$ so that $(\pi(t), x(t))$ follows the same dynamics given by (A.11)-(A.12), which means $(\pi(t), x(t))$ stays on the *zlb* saddle path and travels on it towards (π_{zlb}, x_{zlb}) .

3. Because $(\pi(T), x(T))$ is in Ω_{zlb} and in Υ_{zlb} , it satisfies

$$\begin{aligned}
0 &\geq \xi_\pi \pi_T + \xi_x x_T + r_h, \\
x_T &= \frac{\phi_1}{\kappa} \pi_T - \frac{\phi_2}{\kappa} r_h.
\end{aligned}$$

Solving for $(\pi(T), x(T))$ gives

$$x_T = \frac{\phi_1}{\kappa} \pi_T - \frac{\phi_2}{\kappa} r_h, \tag{A.41}$$

$$(\kappa \xi_\pi + \xi_x \phi_1) \pi_T \leq r_h (\phi_2 \xi_x - \kappa). \tag{A.42}$$

If $\kappa \xi_\pi + \xi_x \phi_1 \neq 0$, it is easy to find (π_T, x_T) that satisfies (A.41) and (A.42). If $\kappa \xi_\pi +$

$\xi_x \phi_1 = 0$, equation (A.42) holds because the Taylor principle holds. Any pair (π_T, x_T) that satisfies equation (A.41) will be in Ω_{zlb} and in Υ_{zlb} . To find the corresponding $(\pi(0), x(0))$, use the dynamics of (π_T, x_T) for $t \in [0, T]$ given by (A.7)-(A.7).

4. By direct computation, the set of points $(x, \pi) \in \Upsilon_{zlb} \cap \partial\Omega$ are

$$(x, \pi) = \begin{cases} \left(-r_h \frac{\phi_1 + \phi_2 \xi_\pi}{\kappa \xi_\pi + \phi_1 \xi_x}, -r_h \frac{\kappa - \phi_2 \xi_x}{\kappa \xi_\pi + \phi_1 \xi_x} \right) & \text{if } \kappa \xi_\pi + \phi_1 \xi_x \neq 0 \\ \emptyset & \text{if } \kappa \xi_\pi + \phi_1 \xi_x = 0 \end{cases}. \quad (\text{A.43})$$

If $\kappa \xi_\pi + \phi_1 \xi_x = 0$, there is clearly no $p > 0$ such that $(\pi(s+p), x(s+p)) \in \Upsilon_{zlb} \cap \partial\Omega$. If $\kappa \xi_\pi + \phi_1 \xi_x \neq 0$, I analyze three cases according to the three different dynamics that (π_T, x_T) can follow in Ω_{ss} given in Section A.1).

Case I. Let $t = s + p$. Then $\eta(t) = s$ and $(x_{s+p}, \pi_{s+p}) \in \Upsilon_{zlb} \cap \partial\Omega$ gives

$$x_{s+p} = -r_h \frac{\phi_1 + \phi_2 \xi_\pi}{\kappa \xi_\pi + \phi_1 \xi_x} = -\frac{(1 - \xi_\pi) \pi_s + (\sigma \alpha_2 - \xi_x) x_s}{\sigma (\alpha_1 - \alpha_2)} e^{\alpha_1 p} + \frac{(1 - \xi_\pi) \pi_s + (\sigma \alpha_1 - \xi_x) x_s}{\sigma (\alpha_1 - \alpha_2)} e^{\alpha_2 p},$$

$$\pi_{s+p} = -r_h \frac{\kappa - \phi_2 \xi_x}{\kappa \xi_\pi + \phi_1 \xi_x} = \frac{(1 - \xi_\pi) \pi_s + (\sigma \alpha_2 - \xi_x) x_s}{\sigma (\xi_\pi - 1) (\alpha_1 - \alpha_2)} (\xi_x - \sigma \alpha_1) e^{\alpha_1 p} - \frac{(1 - \xi_\pi) \pi_s + (\sigma \alpha_1 - \xi_x) x_s}{\sigma (\xi_\pi - 1) (\alpha_1 - \alpha_2)} (\xi_x - \sigma \alpha_2) e^{\alpha_2 p}.$$

Solving for $(\pi(s), x(s))$ as a function of p gives

$$x_s(p) = -\frac{r_h (-\kappa + \kappa \xi_\pi + \phi_1 \xi_x + \phi_2 \xi_x - \sigma \alpha_2 \phi_1 - \sigma \alpha_2 \phi_2 \xi_\pi)}{\sigma (\kappa \xi_\pi + \phi_1 \xi_x) (\alpha_1 - \alpha_2)} e^{-p \alpha_1} - \frac{r_h (\kappa - \kappa \xi_\pi - \phi_1 \xi_x - \phi_2 \xi_x + \sigma \alpha_1 \phi_1 + \sigma \alpha_1 \phi_2 \xi_\pi)}{\sigma (\kappa \xi_\pi + \phi_1 \xi_x) (\alpha_1 - \alpha_2)} e^{-p \alpha_2},$$

$$\pi_s(p) = \frac{r_h (\xi_x - \sigma \alpha_1) (-\kappa + \kappa \xi_\pi + \phi_1 \xi_x + \phi_2 \xi_x - \sigma \alpha_2 \phi_1 - \sigma \alpha_2 \phi_2 \xi_\pi)}{\sigma (\xi_\pi - 1) (\kappa \xi_\pi + \phi_1 \xi_x) (\alpha_1 - \alpha_2)} e^{-p \alpha_1} + \frac{r_h (-\xi_x + \sigma \alpha_2) (-\kappa + \kappa \xi_\pi + \phi_1 \xi_x + \phi_2 \xi_x - \sigma \alpha_1 \phi_1 - \sigma \alpha_1 \phi_2 \xi_\pi)}{\sigma (\xi_\pi - 1) (\kappa \xi_\pi + \phi_1 \xi_x) (\alpha_1 - \alpha_2)} e^{-p \alpha_2}.$$

Let

$$F(p) = -\frac{r_h(-\xi_x + \sigma\alpha_1\xi_\pi)(-\kappa + \kappa\xi_\pi + \phi_1\xi_x + \phi_2\xi_x - \sigma\alpha_2\phi_1 - \sigma\alpha_2\phi_2\xi_\pi)}{\sigma(\xi_\pi - 1)(\kappa\xi_\pi + \phi_1\xi_x)(\alpha_1 - \alpha_2)}e^{-p\alpha_1} \\ - \frac{r_h(\xi_x - \sigma\alpha_2\xi_\pi)(-\kappa + \kappa\xi_\pi + \phi_1\xi_x + \phi_2\xi_x - \sigma\alpha_1\phi_1 - \sigma\alpha_1\phi_2\xi_\pi)}{\sigma(\xi_\pi - 1)(\kappa\xi_\pi + \phi_1\xi_x)(\alpha_1 - \alpha_2)}e^{-p\alpha_2} \\ + \frac{r_h\sigma(\xi_\pi - 1)(\alpha_1 - \alpha_2)(\kappa\xi_\pi + \phi_1\xi_x)}{\sigma(\xi_\pi - 1)(\kappa\xi_\pi + \phi_1\xi_x)(\alpha_1 - \alpha_2)}.$$

Then,

$$F(p) = \xi_\pi\pi_s(p) + \xi_x x_s(p) + r_h,$$

and since $(\pi(s), x(s)) \in \partial\Omega$, it follows that $\xi_\pi\pi_s(0) + \xi_x x_s(0) + r_h = 0 = F(0)$. I show there is no $p > 0$ that satisfies $F(p) = 0$.

First, note that x_s and π_s are always real, even when α_1 and α_2 are complex. By direct computation, I find that

$$F(0) = 0. \quad (\text{A.44})$$

$$F'(p) = 0 \text{ has at most one solution for } p > 0, \quad (\text{A.45})$$

$$F'(0) = \frac{r_h}{\sigma\phi_1}(\kappa(\xi_\pi - 1) + \rho\xi_x) > 0, \quad (\text{A.46})$$

$$\lim_{p \rightarrow \infty} F(p) = r_h > 0. \quad (\text{A.47})$$

$F'(0) > 0$ because the Taylor principle holds. Together, equations (A.44)-(A.47) and continuity of $F(p)$ show that there is no solution to $F(p) = 0$ for $p > 0$.

Case II. Let $t = s + p$. Then $\eta(t) = s$ and

$$-r_h \frac{\rho}{\kappa + \phi_1\xi_x} = x_s e^{\frac{1}{\sigma}\xi_x p}, \\ -r_h \frac{\kappa - \phi_2\xi_x}{\kappa + \phi_1\xi_x} = \frac{\pi_s(\xi_x - \sigma\rho) + \kappa\sigma x_s}{\xi_x - \sigma\rho} e^{\rho p} - \frac{\kappa\sigma x_{\eta(t)}}{\xi_x - \sigma\rho} e^{\frac{1}{\sigma}\xi_x p}.$$

Using equation (A.43) and that $\xi_\pi\pi_s + \xi_x x_s + r_h = 0$, I get

$$-r_h \frac{\rho}{\kappa + \phi_1\xi_x} = x_s e^{\frac{1}{\sigma}\xi_x p}, \\ -r_h \frac{\kappa - \phi_2\xi_x}{\kappa + \phi_1\xi_x} = \left(\frac{(\xi_x - \sigma\rho)\xi_x - \kappa\sigma}{(\xi_x - \sigma\rho)(\kappa + \phi_1\xi_x)} r_h \rho e^{-\frac{1}{\sigma}\xi_x p} - r_h \right) e^{\rho p} \\ + \frac{r_h \kappa \sigma \rho}{(\xi_x - \sigma\rho)(\kappa + \phi_1\xi_x)}.$$

Solving for x_s in the first equation and plugging it into the second equation gives

$$x_s = \xi_x (\kappa - \phi_2 \xi_x + \sigma \rho \phi_2), \quad (\text{A.48})$$

$$\begin{aligned} \xi_x (\kappa - \phi_2 \xi_x + \sigma \rho \phi_2) &= e^{p\rho} (\xi_x - \sigma \rho) (\kappa + \phi_1 \xi_x) \\ &\quad + e^{\frac{p}{\sigma}(\sigma\rho - \xi_x)} \rho (-\xi_x^2 + \kappa\sigma + \sigma\rho\xi_x). \end{aligned} \quad (\text{A.49})$$

I now show that there is no $p > 0$ such that equations (A.48)-(A.49) hold. If $\xi_x = \sigma\phi_1$, then $-\xi_x^2 + \kappa\sigma + \sigma\rho\xi_x = 0$ and the equations become

$$\begin{aligned} x_s &= -\rho r_h \frac{e^{-p\phi_1}}{\sigma\phi_1^2 + \kappa}, \\ 1 &= e^{p\rho}. \end{aligned}$$

The last equation has no solution for $p > 0$. If $\xi_x \neq \sigma\phi_1$, and recalling that $\xi_x \neq \sigma\phi_2$ so that $\Upsilon_{zlb} \cap \partial\Omega$ is non-empty, then $-\xi_x^2 + \kappa\sigma + \sigma\rho\xi_x \neq 0$ and equations (A.48)-(A.49) become

$$\begin{aligned} x_s &= -\frac{r_h \rho}{\kappa + \phi_1 \xi_x} e^{-\frac{1}{\sigma} \xi_x p}, \\ 0 &= \frac{\phi_2 \xi_x}{\xi_x - \sigma\phi_1} (e^{p\rho} - 1) + \rho e^{p\rho} \left(e^{-\frac{\xi_x}{\sigma} p} - 1 \right). \end{aligned}$$

Let

$$F(p) = \frac{\phi_2 \xi_x}{\xi_x - \sigma\phi_1} (e^{p\rho} - 1) + \rho e^{p\rho} \left(e^{-\frac{\xi_x}{\sigma} p} - 1 \right).$$

Compute

$$\begin{aligned} F'(p) &= \frac{\rho \xi_x}{\sigma} \left(e^{-\frac{p}{\sigma} \xi_x} - \frac{\sigma \phi_2}{(\xi_x - \sigma\phi_1)} e^{-p\rho} \right), \\ F''(p) &= -\frac{\rho \xi_x}{\sigma^2} \left(\xi_x e^{-\frac{p}{\sigma} \xi_x} - \frac{\sigma^2 \rho \phi_2}{(\xi_x - \sigma\phi_1)} e^{-p\rho} \right), \end{aligned}$$

and

$$F(0) = 0, \quad (\text{A.50})$$

$$F'(0) = \frac{\rho \xi_x (\xi_x - \sigma \rho)}{\sigma (\xi_x - \sigma \phi_1)}, \quad (\text{A.51})$$

$$F'(p) = 0 \Rightarrow e^{(\rho - \frac{\xi_x}{\sigma})p} = \frac{\sigma \phi_2}{(\xi_x - \sigma \phi_1)}, \quad (\text{A.52})$$

$$\lim_{p \rightarrow \infty} F(p) = \phi_1 \frac{\xi_x - \sigma \rho}{\xi_x - \sigma \phi_1}, \quad (\text{A.53})$$

$$\lim_{p \rightarrow \infty} F'(p) = 0. \quad (\text{A.54})$$

If $\xi_x - \sigma \phi_1 > 0$, $F(p)$ is monotonic, which combined with $F(0) = 0$ gives no solutions to $F(p) = 0$ for $p > 0$. If $\xi_x - \sigma \phi_1 < 0$ and $\xi_x - \sigma \rho < 0$, then the unique local maximum occurs for some $p > 0$ and F is positive at that maximum. Using (A.53) and (A.54) then shows that there is no solution to $F(p) = 0$ for $p > 0$. If $\xi_x - \sigma \phi_1 < 0$ and $\xi_x - \sigma \rho > 0$, an analogous argument applies but instead of a unique maximum, there is a unique minimum.

Case III. Let $t = s + p$. Then $\eta(t) = s$ and

$$\begin{aligned} x_p &= \left(\left(1 + \frac{1}{2\sigma} (\xi_x - \sigma \rho) p \right) x_s + \frac{1}{\kappa} \left(\frac{1}{2\sigma} (\sigma \rho - \xi_x) \right)^2 p \pi_s \right) e^{\frac{1}{2}(\rho + \frac{1}{\sigma} \xi_x)p}, \\ \pi_p &= \left(-\kappa p x_s + \left(1 - \frac{1}{2\sigma} (\xi_x - \sigma \rho) p \right) \pi_s \right) e^{\frac{1}{2}(\rho + \frac{1}{\sigma} \xi_x)p}. \end{aligned}$$

Using equation (A.43) and that $\xi_\pi \pi_s + \xi_x x_s + r_h = 0$,

$$\begin{aligned} -r_h \frac{\phi_1 + \phi_2 \xi_\pi}{\kappa \xi_\pi + \phi_1 \xi_x} &= \left(1 + \frac{1}{2\sigma} (\xi_x - \sigma \rho) p \right) x_s e^{\frac{1}{2}(\rho + \frac{1}{\sigma} \xi_x)p} \\ &\quad + \frac{1}{\kappa} \left(\frac{1}{2\sigma} (\sigma \rho - \xi_x) \right)^2 p \left(-\frac{(\xi_x x_s + r_h)}{\xi_\pi} \right) e^{\frac{1}{2}(\rho + \frac{1}{\sigma} \xi_x)p}, \quad (\text{A.55}) \end{aligned}$$

$$\begin{aligned} -r_h \frac{\kappa - \phi_2 \xi_x}{\kappa \xi_\pi + \phi_1 \xi_x} &= -\kappa p x_s e^{\frac{1}{2}(\rho + \frac{1}{\sigma} \xi_x)p} \\ &\quad + \left(1 - \frac{1}{2\sigma} (\xi_x - \sigma \rho) p \right) \left(-\frac{(\xi_x x_s + r_h)}{\xi_\pi} \right) e^{\frac{1}{2}(\rho + \frac{1}{\sigma} \xi_x)p}. \quad (\text{A.56}) \end{aligned}$$

Combining the two equations, I solve for x_s as a function of p

$$x_s = \frac{A_0 + A_1 p}{B_0 + B_1 p}, \quad (\text{A.57})$$

where

$$\begin{aligned}
A_0 &= -4\sigma r_h (\phi_2 \xi_x^2 + \sigma^2 \rho^2 \phi_2 + 4\kappa \sigma \rho - 2\sigma \rho \phi_2 \xi_x), \\
A_1 &= 2r_h (\xi_x^2 - \sigma^2 \rho^2) (2\kappa - \phi_2 \xi_x + \sigma \rho \phi_2), \\
B_0 &= 4\kappa \sigma (\xi_x^2 + \sigma^2 \rho^2 + 4\kappa \sigma - 2\sigma \rho \xi_x + 4\sigma \phi_1 \xi_x), \\
B_1 &= (\xi_x + \sigma \rho) (2\kappa - \phi_2 \xi_x + \sigma \rho \phi_2) (\sigma^2 \rho^2 + 4\kappa \sigma - \xi_x^2).
\end{aligned}$$

Plugging equation (A.57) into equation (A.55), I get

$$F(p) = 0,$$

where

$$F(p) = e^{\frac{1}{2}(\rho + \frac{1}{\sigma}\xi_x)p} - 1 + \frac{(\xi_x + \sigma \rho) (2\kappa - \phi_2 \xi_x + \sigma \rho \phi_2)}{4\kappa \sigma} \left(\frac{\xi_x - \sigma (\phi_1 - \phi_2)}{\xi_x + \sigma (\phi_1 - \phi_2)} \right) p.$$

Since

$$\begin{aligned}
F(0) &= 0, \\
F'(0) &= -\frac{\phi_2}{4\kappa \sigma} (\xi_x + \sigma \rho)^2 > 0, \\
F''(p) &= \frac{1}{4} \left(\rho + \frac{1}{\sigma} \xi_x \right)^2 e^{\frac{1}{2}(\rho + \frac{1}{\sigma}\xi_x)p} > 0,
\end{aligned}$$

the equation $F(p) = 0$ has no solution for $p > 0$.

5. The assumptions required for the Poincaré-Bendixson Theorem, Theorem 12 in Section B, hold. Indeed, because

$$\dot{x}_t = \sigma^{-1} (\max \{0, \xi_x x_t + \xi_\pi \pi_t + r(t)\} - r(t) - \pi_t), \quad (\text{A.58})$$

$$\dot{\pi}_t = \rho \pi_t - \kappa x_t, \quad (\text{A.59})$$

are, as functions of x_t and π_t , continuous and differentiable almost everywhere, they are Lipschitz. The rest of the conditions are easy to check.

I show that the ω -limit set²² of $(\pi(q), x(q))$ contains no steady-states and is not a periodic orbit. By Theorem 12, (π_T, x_T) then explodes.

Because (π_{ss}, x_{ss}) is a locally unstable steady-state (by the Taylor principle) and $(\pi(q), x(q)) \neq (\pi_{ss}, x_{ss})$, the ω -limit set of (x_q, π_q) does not contain (π_{ss}, x_{ss}) , as

²² See Section B for definitions of ω -limit sets and other concepts needed to state the Poincaré-Bendixson Theorem.

(π_T, x_T) is bounded away from (π_{ss}, x_{ss}) for all $t \geq q$. Because (π_{zlb}, x_{zlb}) is locally a saddle-path steady-state, the only paths converging to (π_{zlb}, x_{zlb}) as $t \rightarrow \infty$ must eventually be in $\Upsilon_{zlb} \cap \Omega_{zlb}$. By hypothesis, $(x_{\tau(t)}, \pi_{\tau(t)}) \notin \Upsilon_{zlb} \cap \partial\Omega$, where recall $\tau(t)$ is the time of first entry into Ω_{zlb} after t . By item 5, if $(\pi(t), x(t))$ enters Ω_{zlb} a second time after $\tau(t)$ (of course, by first visiting Ω_{ss}) it is not through $\Upsilon_{zlb} \cap \partial\Omega$. It follows that the ω -limit set of $(\pi(q), x(q))$ does not contain (π_{zlb}, x_{zlb}) , as the orbit of (x_q, π_q) never intersects $\Upsilon_{zlb} \cap \Omega_{zlb}$.

I now show that there are no closed orbits. The divergence of (π_T, x_T) computed in the distribution sense is

$$\operatorname{div}(\dot{x}_t, \dot{\pi}_t) = \frac{\partial \dot{x}_t}{\partial x_t} + \frac{\partial \dot{\pi}_t}{\partial \pi_t} = \begin{cases} \rho & , \text{ if } (\pi(t), x(t)) \in \Omega_{zlb} \setminus \partial\Omega \\ \frac{\xi_x}{2\sigma} + \rho & , \text{ if } (\pi(t), x(t)) \in \partial\Omega \\ \frac{1}{\sigma}\xi_x + \rho & , \text{ if } (\pi(t), x(t)) \in \Omega_{ss} \end{cases} ,$$

where $\Omega_{zlb} \setminus \partial\Omega$ denotes the interior of Ω_{zlb} .

The Taylor principle and $\rho > 0$ imply that $\operatorname{div}(\pi_T, x_T) > 0$ for all $(\pi(t), x(t))$. By Theorem 11, there are no closed orbits²³.

6. The result that there is no chaos is a direct consequence of Theorem 12, which tightly restricts the behavior of bounded solutions to two cases, none of which is chaotic. For continuous systems, strange attractors and other chaotic behavior can only emerge when the dimension of the phase space is three or more. Note that the concept of chaos I consider here is different from chaos in the sense of Li and Yorke (1975) used in Benhabib et al. (2002b), which is more appropriate for a discrete time setting.

□

²³ The version of the Poincaré-Bendixson theorem I have used is stronger than needed since our vector field is continuous (but non-differentiable) in $\partial\Omega$ while the theorem allows for discontinuities across the boundary between regions.

In addition, I have used one particular generalized derivative, the "derivative in the distribution sense". However, since the vector field under consideration is continuous, any generalized derivative (such as viscosity solutions) would still give a finite value for $\operatorname{div}(\dot{x}_t, \dot{\pi}_t)$. When the value of $\operatorname{div}(\dot{x}_t, \dot{\pi}_t)$ is finite along $\partial\Omega$, because $\partial\Omega$ has measure zero, its value does not contribute to the line integral along a closed loop. By Green's theorem, it then does not matter which concept of generalized derivative I use for this particular purpose.

Proof of Proposition 3. Consider the following condition:

$$\begin{aligned}
(\pi(t_1), x(t_1)) &= (\pi_{ss}, x_{ss}), \\
&\text{or} \\
(\pi(t_1), x(t_1)) &\in \Omega_{zlb} \cap \Upsilon_{zlb}, \\
&\text{or} \\
(\pi(t_1), x(t_1)) &\in \overline{\Omega}_{ss} \text{ and } (\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{zlb} \text{ for some } r \in (t_1, \infty).
\end{aligned} \tag{A.60}$$

I first prove that if condition (A.60) holds, then (π_T, x_T) is bounded. I consider three cases.

Case 1: If $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$, then $(\pi(t), x(t))$ is bounded because (π_{ss}, x_{ss}) is a steady state.

Case 2: If $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$, then item 3 of Proposition 8 shows, by picking $m = t_1$, that $(\pi_T, x_T) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ for all $t \geq t_1$. The dynamics in equations (A.11)-(A.12) then show $(\pi(t), x(t)) \rightarrow (\pi_{zlb}, x_{zlb})$.

Case 3: If $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss}$ and $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{zlb}$ for some $r \in (t_1, \infty)$, item 3 of Proposition 8 shows, by picking $m = r$, that $(\pi_T, x_T) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ for all $t \geq r$. The dynamics in equations (A.11)-(A.12) then show $(\pi(t), x(t)) \rightarrow (\pi_{zlb}, x_{zlb})$.

To prove the converse, I prove the contrapositive. Assume $(x_{t_1}, \pi_{t_1}) \neq (\pi_{ss}, x_{ss})$ and $(\pi(t_1), x(t_1)) \notin \Omega_{zlb} \cap \Upsilon_{zlb}$. I consider two cases.

Case 1: $(\pi(t_1), x(t_1)) \notin \overline{\Omega}_{ss}$. Because of the saddle path dynamics in Ω_{zlb} , if $(x_{t_1}, \pi_{t_1}) \notin \Upsilon_{zlb}$, then (π_T, x_T) either explodes or enters Ω_{ss} in finite time. If it enters Ω_{ss} by intersecting $\partial\Omega$ at some time r , then item 6 of Proposition 8 shows $(\pi(t), x(t))$ explodes. We can apply this item of Proposition 8 because $(x_r, \pi_r) \neq (\pi_{ss}, x_{ss})$ (since (π_{ss}, x_{ss}) is bounded away from $\partial\Omega$) and because, by item 5 of Proposition 8, there is no $p > 0$ such that $(\pi_T, x_T) \in \Omega_{ss}$ for $t \in (r, r+p)$ and $(x_{r+p}, \pi_{r+p}) \in \Upsilon_{zlb} \cap \partial\Omega$.

Case 2: There is no $r \in (t_1, \infty)$ such that $(x_r, \pi_r) \in \partial\Omega \cap \Upsilon_{zlb}$. If $(\pi(t_1), x(t_1)) \notin \overline{\Omega}_{ss}$, case 1 shows $(\pi(t), x(t))$ explodes. If $(x_{t_1}, \pi_{t_1}) \in \overline{\Omega}_{ss}$, given that $(\pi(t_1), x(t_1)) \neq (\pi_{ss}, x_{ss})$, $(\pi(t), x(t))$ either explodes or enters Ω_{zlb} . By assumption, if it enters Ω_{zlb} , it does not intersect Υ_{zlb} . This means $(\pi(t), x(t))$ is eventually in the interior of Ω_{zlb} but not in Υ_{zlb} . The same logic applied in case 1 shows that $(\pi(t), x(t))$ explodes. \square

A.6 Proof of Proposition 4

Assume the rule implements the optimal equilibrium, i.e. $\{x_t, \pi_t, i_t\} = \{x^*(t), \pi^*(t), i^*(t)\}$ when the central bank follows the rule in equation (19). Werning (2011) shows that $i^*(t) =$

$(1 - \kappa\sigma\lambda) \pi^*(t) + r(t) > 0$ for $t \geq t^*$. It follows that $t_1(R^*) \leq t^*$. In addition, $t_1(R^*) \geq s$ for all $s \leq t^*$ such that $(1 - \kappa\sigma\lambda) \pi^*(t) + r(t) > 0$, since otherwise the rule (19) would prescribe $i_t > 0$ while $i^*(t) = 0$. Pick $s = t^*$ to get $t_1(R^*) \geq t^*$ since $(1 - \kappa\sigma\lambda) \pi_{t^*}^* + r_{t^*} > 0$. Because $t_1(R^*) \leq t^*$ and $t_1(R^*) \geq t^*$, it follows that $t_1(R^*) = t^*$, and equation (80) holds.

To prove (81), I use $t_1(R^*) = t^*$ to get that for all $t \geq t^*$

$$\begin{aligned} \max \{0, \xi_\pi(R^*) \pi^*(t) + \xi_x(R^*) x^*(t) + r(t)\} &= \xi_\pi(R^*) \pi^*(t) + \xi_x(R^*) x^*(t) + r_h, \\ &= (1 - \kappa\sigma\lambda) \pi^*(t) + r_h, \end{aligned} \quad (\text{A.61})$$

since otherwise $i_t = i^*(t)$ would not hold. If $\kappa\sigma\lambda \neq 1$, use $x^*(t) = \phi\pi^*(t)$ in equation (A.61) and then equation (81) follows immediately, as $\pi^*(t) \neq 0$ for $t \in [t^*, \infty)$. If $\kappa\sigma\lambda = 1$, any $\xi_\pi(R^*), \xi_x(R^*)$ implement the optimal equilibrium as $(0, 0)$ is a steady state for all $\xi_\pi(R^*), \xi_x(R^*)$.

Now assume that equations (80)-(81) hold. I show rule (19) implements the optimal equilibrium. When $(\pi_0, x_0) = (x_0^*, \pi_0^*)$, clearly $i_t = i^*(t) = 0$ and $(\pi(t), x(t)) = (x^*(t), \pi^*(t))$ for $t < t^*$. Because (π_T, x_T) and $(x^*(t), \pi^*(t))$ are continuous as a function of time and their paths coincide in $[t^* - \varepsilon, t^*)$ for any $\varepsilon > 0$, $(x_{t^*}, \pi(t^*)) = (x^*(t^*), \pi_{t^*}^*)$. As $x^*(t^*) = \phi\pi^*(t^*)$ for $t = t^*$,

$$x(t^*) = \phi\pi(t^*). \quad (\text{A.62})$$

If $\kappa\sigma\lambda = 1$, $(x(t^*), \pi(t^*)) = (x^*(t^*), \pi^*(t^*)) = (0, 0)$, because $(0, 0)$ is a steady state, $(\pi_T, x_T) = (x^*(t), \pi^*(t)) = (0, 0)$ for all $t \geq t^*$ and any $\xi_\pi(R^*), \xi_x(R^*)$. In addition, if $(\pi_T, x_T) = (0, 0)$,

$$i_t = \max \{0, \xi_\pi(R^*) \pi_t + \xi_x(R^*) x_t + r(t)\} = r_h = i^*(t)$$

for all $t \geq t^*$.

When $\kappa\sigma\lambda \neq 1$, using equations (A.13)-(A.18), it can be checked by direct computation that $(\pi_T, x_T) = (x^*(t), \pi^*(t))$ for all $t \geq t^*$ where $(x^*(t), \pi^*(t))$ is given by

$$x^*(t) = x_1^* \exp\left(-\frac{\kappa\lambda}{\phi}(t - t_1)\right), \quad (\text{A.63})$$

$$\pi^*(t) = \pi_1^* \exp\left(-\frac{\kappa\lambda}{\phi}(t - t_1)\right). \quad (\text{A.64})$$

The following relations may be helpful for the computations: If $\xi_\pi(R^*) < \kappa\sigma\lambda + \sigma\phi\rho + 1$,

then

$$\alpha_1 = \rho + \frac{1 - \xi_\pi(R^*)}{\sigma\phi}, \quad (\text{A.65})$$

$$\alpha_2 = -\frac{\kappa\lambda}{\phi}. \quad (\text{A.66})$$

If $\xi_\pi(R^*) > \kappa\sigma\lambda + \sigma\phi\rho + 1$, then

$$\alpha_1 = -\frac{\kappa\lambda}{\phi}, \quad (\text{A.67})$$

$$\alpha_2 = \rho + \frac{1 - \xi_\pi(R^*)}{\sigma\phi}. \quad (\text{A.68})$$

If $\xi_\pi(R^*) = \kappa\sigma\lambda + \sigma\phi\rho + 1$

$$\alpha_1 = \alpha_2 = -\frac{\kappa\lambda}{\phi}. \quad (\text{A.69})$$

A.7 Proof of Proposition 5

I first assume the rule implements no equilibrium with $R \neq R^*$ and prove items 1-3 hold.

Item 1: By Proposition 3, if the Taylor principle holds, there exist continuous bounded paths with $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$. Since for these paths $(\pi(t), x(t)) \rightarrow (\pi_{zlb}, x_{zlb})$, they constitute non-optimal equilibria. By item 2 of Proposition 8, $(\pi(t), x(t)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ for all $t \geq t_1$, irrespective of the choice of t_1 . It follows that the only way to preclude these type of equilibria is to have the Taylor principle not hold for $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$.

Items 2a, 2b and 2c: If there are continuous paths that satisfy the hypotheses of items 2a, 2b or 2c, they are bounded by Propositions 2 and 3 and constitute non-optimal equilibria. Thus, all paths that satisfy the hypothesis in items 2a, 2b and 2c must be discontinuous, which implies equation (82) holds.

Item 3: If $\partial\Omega \cap \Upsilon_{ss} = \emptyset$, then the item is vacuously true. If $\partial\Omega \cap \Upsilon_{ss}$ is non-empty then

$$(\pi(r), x(r)) = \begin{cases} \left(r h \frac{(\xi_\pi - 1)}{\xi_x - \sigma\alpha_2\xi_\pi}, -r h \frac{(\xi_x - \sigma\alpha_2)}{\xi_x - \sigma\alpha_2\xi_\pi} \right) & , \text{ if } \det A_{ss} < 0 \text{ and } \xi_\pi = 1 \\ \left(-r h \frac{(\xi_x - \sigma\rho)}{\xi_x^2 - \kappa\sigma - \sigma\rho\xi_x}, \kappa\sigma \frac{r h}{\xi_x^2 - \sigma\rho\xi_x - \kappa\sigma} \right) & , \text{ if } \det A_{ss} < 0 \text{ and } \xi_\pi \neq 1 \end{cases} \quad (\text{A.70})$$

Assume that the Taylor principle does not hold for $(x_{t_1}, \pi_{t_1}) \in \Omega_{zlb}$ and that there exist some $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega_{zlb} \cap \Upsilon_{ss}$. I show that if equation (82) does not hold, then there exist a non-optimal equilibrium. By assumption, $t_1 \in [T, r)$. Let P be the set of points in the continuous path between $(\pi(T), x(T))$ and $(\pi(r), x(r))$, which can be obtained by running the system dynamics backward in time while respecting continuous pasting. By the dynamics in equations (A.11)-(A.12) and equation (A.70), the time q at

which $(\pi(t_1), x(t_1)) \in \Omega_{zlb}$ reaches $(\pi(r), x(r))$ while following a continuous path is

$$q = \frac{1}{\phi_1} \log \left(\frac{d_{exit}(r)}{d_{exit}(t_1)} \right) \quad (\text{A.71})$$

Note that q is not necessarily equal to r since the hypotheses of item 3 do not require that paths are continuous. Because equation (83) does not hold, $T \leq t_1 \leq q$ and thus $(\pi(t_1), x(t_1)) \in P$. By Theorem 2, the continuous path going through $(\pi(t_1), x(t_1))$ and $(\pi(r), x(r))$ is bounded for $t \geq t_1$. Using the continuous pasting conditions in Section 6.1, the path can be continuously extended from $(\pi(t_1), x(t_1))$ to $(\pi(0), x(0))$ to get a continuous bounded path for all $t \geq 0$. This equilibrium is non-optimal since no optimal path has $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \setminus \partial\Omega$.

Conversely, I now assume items 1-3 hold and prove that the rule implements no equilibria with $R_{t_1} \neq R^*$. By Proposition 3 and item 1, there are no equilibria with $(x_{t_1}, \pi_{t_1}) \in \Omega_{zlb} \cap \Upsilon_{zlb}$. By Propositions 2 and 3, items 2a-2c and the continuous pasting conditions in Section 6.1, there are no equilibria when: The Taylor principle holds for $(\pi(t_1), x(t_1)) \in \Omega_{ss}$ and there exist some $r \in (t_1, \infty)$ such that $(x_r, \pi_r) \in \partial\Omega \cap \Upsilon_{zlb}$, the Taylor principle holds for $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$, or the Taylor principle does not hold for $(x_{t_1}, \pi_{t_1}) \in \bar{\Omega}_{ss} \cap \Upsilon_{ss}$. Item 5 of Proposition 2 and item 3 imply that there is no continuous path from $(\pi(t_1), x(t_1)) \in \Omega_{zlb}$ to $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$ that remains bounded after r , and thus there are no equilibria when the Taylor principle does not hold for $(\pi(t_1), x(t_1)) \in \Omega_{zlb}$ and there exist some $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega_{zlb} \cap \Upsilon_{ss}$. By Propositions 2 and 3, all other cases lead to paths that are discontinuous or unbounded.

A.8 Proof of Proposition 6

Item 1. Assume $f(R_{t_1})$ is constant in $(\pi(0), x(0))$. I show there always exist a non-optimal equilibrium. Denote the value of $f(\cdot, \cdot, 0, 0)$ by \hat{t} (because f is constant in its first two arguments, $f(a, b, 0, 0) = \hat{t}$ for all a, b). Define (π_0, x_0) by

$$\hat{x}_0 = \frac{r_h \phi_1^2 e^{-\phi_2 \hat{t}} - \phi_2^2 e^{-\phi_1 \hat{t}}}{\kappa (\phi_1 - \phi_2)} + \left(\frac{r_h - r_l}{\kappa} \right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa}, \quad (\text{A.72})$$

$$\hat{\pi}_0 = r_h \frac{\phi_1 e^{-\phi_2 \hat{t}} - \phi_2 e^{-\phi_1 \hat{t}}}{\phi_1 - \phi_2} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l. \quad (\text{A.73})$$

The continuous path starting at (π_0, x_0) reaches $(0, 0)$ at time \hat{t} by equations (A.23) and (A.24). Since $(0, 0)$ is a steady-state, $(\pi_t, x_t) = (0, 0)$ for all $t \geq \hat{t}$. The path for $(\pi(t), x(t))$ is continuous, bounded and follows the IS, the NKPC and the interest rate rule: It is an equilibrium. If $\kappa\sigma\lambda \neq 1$, the equilibrium is not optimal.

Item 2. Consider a rule with

$$\begin{aligned}\xi_\pi(R_{t_1}) &= 1 - \kappa\sigma\lambda \\ \xi_x(R_{t_1}) &= 0 \\ t_1(R^*) &= t^* \\ t_1(R) &= \tau(\pi(0), x(0))\end{aligned}$$

for some function $\tau : \mathbb{R}^2 \rightarrow \mathbb{R}$. The choice of ξ_x, ξ_π implies

$$\alpha_1 = \frac{1}{2} \left(\rho + \sqrt{\rho^2 + 4\kappa^2\lambda} \right) = \kappa\phi > 0 \quad (\text{A.74})$$

$$\alpha_2 = \frac{1}{2} \left(\rho - \sqrt{\rho^2 + 4\kappa^2\lambda} \right) < 0 \quad (\text{A.75})$$

$$\alpha_1\alpha_2 = -\kappa^2\lambda < 0 \quad (\text{A.76})$$

$$\alpha_1 + \alpha_2 = \rho \quad (\text{A.77})$$

The Taylor principle never holds, as $\kappa(\xi_\pi - 1) + \rho\xi_x = -\kappa^2\sigma\lambda < 0$.

Item 1 of Proposition 5 is true because the Taylor principle does not hold. Items 2a and 2c do not apply, since the Taylor principle does not hold. I now analyze items 2b and 3 and show they can be satisfied with an appropriate choice of τ .

Item 2b of Proposition 5. Because $(\pi(t), x(t)) \in \Upsilon_{ss}$,

$$\pi_1 = \frac{1}{\phi}x_1 \quad (\text{A.78})$$

and because $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss}$

$$(1 - \kappa\sigma\lambda)\pi_1 + r_h \geq 0.$$

Consider the four cases in the continuous pasting condition in equation (A.26).

Case 1: If $d_{exit}(t_1) = 0$ and $d_{trap}(t_1) = 0$, $x_1 = x_{zlb} = -\frac{1}{\kappa}r_h\rho$ and $\pi_1 = \pi_{zlb} = -r_h$, which contradicts equation (A.78) and hence there is no equilibrium.

Case 2: If $d_{exit}(t_1) = 0$ and $d_{trap}(t_1) \neq 0$,

$$x_1 = \frac{\phi_1}{\kappa}\pi_1 - \frac{r_h\phi_2}{\kappa}. \quad (\text{A.79})$$

If $\sigma\lambda\kappa = 1$, there is no (π_1, x_1) that satisfies equations (A.79) and (A.78) simultaneously. If

$\sigma\lambda\kappa \neq 1$, equations (A.79) and (A.78) imply

$$\begin{aligned} x_1 &= r_h \frac{\phi - \sigma\lambda\phi_2}{(\kappa\sigma\lambda - 1)}, \\ \pi_1 &= \frac{r_h}{\phi} \frac{\phi - \sigma\lambda\phi_2}{(\kappa\sigma\lambda - 1)} \end{aligned}$$

But then, since $\phi_2 < 0$,

$$\xi_x x_1 + \xi_\pi \pi_1 + r_h = (1 - \kappa\sigma\lambda) \pi_1 + r_h = \frac{\lambda\sigma\phi_2 r_h}{\phi} < 0$$

contradicts that $(\pi(t_1), x(t_1)) \in \bar{\Omega}_{ss}$ and thus there is no equilibrium.

Case 3: If $d_{exit}(t_1) \neq 0$ and $d_{trap}(t_1) = 0$,

$$x_1 = \frac{\phi_2}{\kappa} \pi_1 - \frac{r_h \phi_1}{\kappa}. \quad (\text{A.80})$$

If $\sigma\lambda\kappa = 1$, there is no (π_1, x_1) that satisfies equations (A.79) and (A.78) simultaneously. If $\sigma\lambda\kappa \neq 1$, equations (A.79) and (A.78) imply

$$x_1 = r_h \frac{\phi - \sigma\lambda\phi_1}{(\kappa\sigma\lambda - 1)}, \quad (\text{A.81})$$

$$\pi_1 = \frac{r_h}{\kappa\sigma\lambda - 1} \frac{(\phi - \sigma\lambda\phi_1)}{\phi}. \quad (\text{A.82})$$

The pasting condition in equation (A.27) and equation (A.82) give

$$\mathcal{T}(R_{t_1}) = -\frac{1}{\phi_1} \log \frac{\pi_0 + \left(r_l + (r_h - r_l) \frac{\phi_1 e^{-T\phi_2} - \phi_2 e^{-T\phi_1}}{\phi_1 - \phi_2} \right)}{(\pi_1 + r_h)} \quad (\text{A.83})$$

$$= -\frac{1}{\phi_1} \log \frac{\pi_0 + \left(r_l + (r_h - r_l) \frac{\phi_1 e^{-T\phi_2} - \phi_2 e^{-T\phi_1}}{\phi_1 - \phi_2} \right)}{\left(\frac{r_h}{\kappa\sigma\lambda - 1} \frac{(\phi - \sigma\lambda\phi_1)}{\phi} + r_h \right)} \quad (\text{A.84})$$

$$= \mathcal{T}(\pi_0, x_0) \quad (\text{A.85})$$

Setting

$$\tau(\pi(0), x(0)) \neq \mathcal{T}(\pi_0, x_0)$$

precludes any equilibrium.

Case 4: If $d_{exit}(t_1) \neq 0$ and $d_{trap}(t_1) \neq 0$, using equation (A.78), the continuous pasting

condition $\mathcal{P}(R_{t_1}) = 0$ is

$$\begin{aligned} -\frac{1}{\phi_1} \log \frac{x_0 - \frac{\phi_1}{\kappa} \pi_0 + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa} (r_h - r_l) (e^{-T\phi_1} - 1)}{\left(\phi - \frac{\phi_1}{\kappa}\right) \pi_1 + \frac{\phi_2 r_h}{\kappa}} \\ = -\frac{1}{\phi_2} \log \frac{x_0 - \frac{\phi_2}{\kappa} \pi_0 + \frac{\phi_1 r_h}{\kappa} + \frac{\phi_1}{\kappa} (r_h - r_l) (e^{-T\phi_2} - 1)}{\left(\phi - \frac{\phi_2}{\kappa}\right) + \frac{\phi_1 r_h}{\kappa}} \end{aligned} \quad (\text{A.86})$$

Let

$$\begin{aligned} f(\pi_1, \pi_0, x_0) = -\frac{1}{\phi_1} \log \frac{x_0 - \frac{\phi_1}{\kappa} \pi_0 + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa} (r_h - r_l) (e^{-T\phi_1} - 1)}{\left(\phi - \frac{\phi_1}{\kappa}\right) \pi_1 + \frac{\phi_2 r_h}{\kappa}} \\ + \frac{1}{\phi_2} \log \frac{x_0 - \frac{\phi_2}{\kappa} \pi_0 + \frac{\phi_1 r_h}{\kappa} + \frac{\phi_1}{\kappa} (r_h - r_l) (e^{-T\phi_2} - 1)}{\left(\phi - \frac{\phi_2}{\kappa}\right) \pi_1 + \frac{\phi_1 r_h}{\kappa}} \end{aligned}$$

Then $f(\pi_1, \pi_0, x_0) = 0$ iff equation (A.86) holds. Since

$$\frac{\partial f(\pi_1, \pi_0, x_0)}{\partial \pi_1} = 0 \iff r_h = (\kappa\sigma\lambda - 1) \pi_1$$

the implicit function theorem implies that we can write $f(\pi_1, \pi_0, x_0) = 0$ as

$$\pi_1 = g(\pi_0, x_0)$$

for some function $g(\pi_0, x_0)$, except when

$$\kappa\sigma\lambda - 1 \neq 0 \text{ and } \pi_1 = \frac{r_h}{(\kappa\sigma\lambda - 1)} \quad (\text{A.87})$$

If equation (A.87) does not hold, then the continuous pasting conditions are given by

$$\begin{aligned} \pi_1 &= g(\pi_0, x_0) \\ \mathcal{T}(R_{t_1}) &= -\frac{1}{\phi_1} \log \frac{x_0 - \frac{\phi_1}{\kappa} \pi_0 + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa} (r_h - r_l) (e^{-T\phi_1} - 1)}{\left(\phi - \frac{\phi_1}{\kappa}\right) \pi_1 + \frac{\phi_2 r_h}{\kappa}} \\ &= -\frac{1}{\phi_1} \log \frac{x_0 - \frac{\phi_1}{\kappa} \pi_0 + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa} (r_h - r_l) (e^{-T\phi_1} - 1)}{\left(\phi - \frac{\phi_1}{\kappa}\right) g(\pi_0, x_0) + \frac{\phi_2 r_h}{\kappa}} \\ &= \mathcal{T}(\pi_0, x_0) \end{aligned}$$

If equation (A.87) holds, $f(\pi_1, \pi_0, x_0) = f\left(\frac{r_h}{\kappa\sigma\lambda - 1}, \pi_0, x_0\right)$. If there is no $(\pi(0), x(0))$ so that

$f\left(\frac{r_h}{(\kappa\sigma\lambda-1)}, \pi_0, x_0\right) = 0$, then there are no equilibria since no path is continuous. If there exist $(\pi(0), x(0))$ such that $f\left(\frac{r_h}{(\kappa\sigma\lambda-1)}, \pi_0, x_0\right) = 0$, continuous pasting gives

$$\begin{aligned}\mathcal{T}(R_{t_1}) &= -\frac{1}{\phi_1} \log \frac{x_0 - \frac{\phi_1}{\kappa}\pi_0 + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa}(r_h - r_l)(e^{-T\phi_1} - 1)}{\left(\phi - \frac{\phi_1}{\kappa}\right) \frac{r_h}{(\kappa\sigma\lambda-1)} + \frac{\phi_2 r_h}{\kappa}} \\ &= \mathcal{T}(\pi_0, x_0)\end{aligned}$$

In either case (when equation (A.87) holds and when it does not hold), setting

$$\tau(\pi(0), x(0)) \neq \mathcal{T}(\pi_0, x_0)$$

precludes any equilibrium.

Item 3 of Proposition 5. If $\kappa\sigma\lambda = 1$, $\Upsilon_{ss} \cap \partial\Omega_{zlb} = \emptyset$ and thus there are no equilibria. If $\kappa\sigma\lambda \neq 1$, $(\pi(r), x(r)) \in \Upsilon_{ss} \cap \partial\Omega_{zlb}$ implies

$$x(r) = \frac{1}{2\kappa} \frac{r_h}{\kappa\sigma\lambda - 1} \left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right) \quad (\text{A.88})$$

$$\pi(r) = \frac{r_h}{\kappa\sigma\lambda - 1} \quad (\text{A.89})$$

Using (A.88)-(A.89) and that $(\pi(t), x(t)) \in \Omega_{zlb}$ for $t \in [t_1, r]$, the continuous pasting equations (A.23) and (A.24) imply that

$$\begin{aligned}x_0 &= \frac{\phi_1 e^{-\phi_2 r} - \phi_2 e^{-\phi_1 r}}{\phi_1 - \phi_2} \left(\frac{1}{2\kappa} \frac{r_h}{\kappa\sigma\lambda - 1} \left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2} \right) \right) \\ &\quad - \frac{1}{\sigma} \frac{e^{-\phi_1 r} - e^{-\phi_2 r}}{\phi_1 - \phi_2} \frac{r_h}{\kappa\sigma\lambda - 1} + \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 r} - \phi_2^2 e^{-\phi_1 r}}{\phi_1 - \phi_2} \\ &\quad + \left(\frac{r_h - r_l}{\kappa} \right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa},\end{aligned}$$

$$\begin{aligned}\pi_0 &= -\kappa \frac{e^{-\phi_1 r} - e^{-\phi_2 r}}{\phi_1 - \phi_2} \left(\frac{1}{2\kappa} \frac{r_h}{\kappa\sigma\lambda - 1} \left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2} \right) \right) \\ &\quad + \frac{\phi_1 e^{-\phi_1 r} - \phi_2 e^{-\phi_2 r}}{\phi_1 - \phi_2} \frac{r_h}{\kappa\sigma\lambda - 1} + r_h \frac{\phi_1 e^{-\phi_2 r} - \phi_2 e^{-\phi_1 r}}{\phi_1 - \phi_2} \\ &\quad + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l.\end{aligned}$$

Solving for r gives

$$r = \nu(\pi_0, x_0)$$

where

$$A = \frac{1}{\kappa} \rho r_l + \frac{1}{\kappa} (r_h - r_l) \frac{\phi_1^2 e^{-T\phi_2} - \phi_2^2 e^{-T\phi_1}}{\phi_1 - \phi_2}$$

$$B = r_l + (r_h - r_l) \frac{\phi_1 e^{-T\phi_2} - \phi_2 e^{-T\phi_1}}{\phi_1 - \phi_2}$$

are two constants and

$$\nu(\pi(0), x(0)) = -\frac{1}{\phi_2} \log \left(\kappa \frac{\sigma \phi \phi_2 + 1}{r_h (\rho - \kappa \phi)} (x_0 + A) - \frac{\phi_2 + \kappa \phi + \sigma \rho \phi \phi_2}{r_h (\rho - \kappa \phi)} (\pi_0 + B) \right)$$

is a function of x_0 and π_0 only (not of x_1 , π_1 or t_1). Setting

$$\tau(\pi(0), x(0)) > \nu(\pi_0, x_0)$$

precludes all equilibria.

Item 3. The rule in the last item has constant Taylor rule coefficients.

Item 4a. I start with a Lemma.

Lemma 9. *If $\det A_{ss}(R) = 0$ for some $R = (\pi_0, x_0)$, then there exist a non-optimal equilibrium.*

Proof of Lemma 9. When $\det A_{ss}(R) = 0$, $(\pi_{zlb}, x_{zlb}) \in \partial\Omega$. A continuous path with $(\pi(T), x(T)) \in \Upsilon_{zlb} \cap \Omega_{zlb}$ is bounded for any choice of f , since $(\pi(t), x(t)) \in \Omega_{zlb}$ for all t and it converges to (π_{zlb}, x_{zlb}) , which is a steady-state of the economy. \square

Since $\xi_\pi(R_{t_1})$ and $\xi_x(R_{t_1})$ are continuous, then either the Taylor principle holds for all R_{t_1} , or the Taylor principle does not hold for all R_{t_1} . To see this, assume for the sake of contradiction that there exists R_{TP} satisfies the Taylor principle and R_{no-TP} that does not. Then

$$\det A_{ss}(R_{TP}) = \kappa (\xi_\pi(R_{TP}) - 1) + \rho \xi_x(R_{TP}) > 0,$$

$$\det A_{ss}(R_{no-TP}) = \kappa (\xi_\pi(R_{no-TP}) - 1) + \rho \xi_x(R_{no-TP}) < 0.$$

By the Intermediate Value Theorem, there exist an R_0 such that $\det A_{ss}(R_0) = \kappa (\xi_\pi(R_0) - 1) + \rho \xi_x(R_0) = 0$. By Lemma 9, there exist a non-optimal equilibrium.

By Proposition 4, when $\kappa \sigma \lambda \neq 1$, the Taylor principle does not hold for R^* . Because the Taylor principle does not hold for one R , then it does not hold for all R .

Item 4b. By item 1, the rule cannot be purely forward-looking.

I show that if the rule is purely backward-looking, that is, if f , ξ_x and ξ_π are constant in their last two arguments, $x(t_1)$ and $\pi(t_1)$, then there exists an equilibrium with $R \neq R^*$.

By item 4a, the Taylor principle never holds. I look for an equilibrium with

$$\pi(t_1) = c(R_{t_1})x(t_1) \quad (\text{A.90})$$

with the function $c(R_{t_1}) > 0$ defined by equation A.28. Because ξ_x and ξ_π are continuous in R_{t_1} and constant in $x(t_1)$, $\pi(t_1)$, so is c . To see that c is continuous when $\xi_\pi = 1$, compute

$$\begin{aligned} \lim_{\xi_\pi \rightarrow 1} c(R_{t_1}) &= \lim_{\xi_\pi \rightarrow 1} \frac{(\xi_x - \sigma\alpha_2)}{(1 - \xi_\pi)} \\ &= \lim_{\xi_\pi \rightarrow 1} -\frac{1}{2(\xi_\pi - 1)} \left(\xi_x - \sigma\rho + \sqrt{\xi_x^2 + \sigma^2\rho^2 + 4\kappa\sigma - 4\kappa\sigma\xi_\pi - 2\sigma\rho\xi_x} \right) \\ &= \lim_{\xi_\pi \rightarrow 1} \kappa\sigma (\xi_x^2 + \sigma^2\rho^2 + 4\kappa\sigma - 4\kappa\sigma\xi_\pi - 2\sigma\rho\xi_x)^{-\frac{1}{2}} \\ &= \frac{\kappa\sigma}{|\xi_x - \sigma\rho|} \\ &= \frac{\kappa\sigma}{\sigma\rho - \xi_x} \end{aligned}$$

The third line follows by L'Hospital's Rule; in a small enough neighborhood of $\xi_\pi = 1$, the Taylor principle not holding implies $\xi_x < 0$ and thus both numerator and denominator in the second line go to zero as $\xi_\pi \rightarrow 1$. The last line follows because $\xi_x < 0$ when $\xi_\pi = 1$, again because the Taylor principle does not hold. When $\xi_\pi \neq 1$, c is continuous by equation A.28.

Let

$$\begin{aligned} M(\pi(0), x(0)) &= \frac{\phi_1 e^{-\phi_2 f} - \phi_2 e^{-\phi_1 f}}{\phi_1 - \phi_2} c - \frac{1}{\sigma} \frac{e^{-\phi_1 f} - e^{-\phi_2 f}}{\phi_1 - \phi_2} \\ N(\pi(0), x(0)) &= -\kappa \frac{e^{-\phi_1 f} - e^{-\phi_2 f}}{\phi_1 - \phi_2} c + \frac{\phi_1 e^{-\phi_1 f} - \phi_2 e^{-\phi_2 f}}{\phi_1 - \phi_2} \\ P(\pi(0), x(0)) &= \frac{r_h \phi_1^2 e^{-\phi_2 f} - \phi_2^2 e^{-\phi_1 f}}{\kappa (\phi_1 - \phi_2)} \\ Q(\pi(0), x(0)) &= r_h \frac{\phi_1 e^{-\phi_2 f} - \phi_2 e^{-\phi_1 f}}{\phi_1 - \phi_2} \\ A &= \left(\frac{r_h - r_l}{\kappa} \right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa} \\ B &= (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l \end{aligned}$$

The functions M , N , P and Q are continuous and depend only on $x(0), \pi(0)$ (and not on $x(t_1), \pi(t_1)$) because f and c are continuous and constant in $\pi(t_1), x(t_1)$. The continuous

pasting conditions (A.23)-(A.24) give

$$x(0) = M\pi_{t_1} + P + A \quad (\text{A.91})$$

$$\pi(0) = N\pi_{t_1} + Q + B \quad (\text{A.92})$$

If $M \neq 0$ and $N \neq 0$, the last two equations give

$$\pi(t_1) = \frac{x(0) - P - A}{M} \quad (\text{A.93})$$

$$\pi(0) = \frac{N}{M}(x(0) - P - A) + Q + B \quad (\text{A.94})$$

Fix $x(0)$ to $\hat{x}_0 = x^*(0) + \varepsilon$ with $\varepsilon > 0$. The right hand-side of equation (A.94) is a function of $\pi(0)$ only. It is bounded above and below, as $f \in [T, \infty)$ and

$$\begin{aligned} & \lim_{f \rightarrow \infty} \frac{N}{M} (\hat{x}_0 - P - A) + Q + B, \\ = & \frac{\kappa\sigma(\rho - 2\phi_1)(A - \hat{x}_0) + B(2\kappa + \sigma\rho\phi_1)}{\phi_1 - \phi_2} \lim_{f \rightarrow \infty} \frac{c}{(c\sigma\phi_1 + 1)} \\ & - \frac{(2\kappa + \sigma\rho\phi_2)(A - \hat{x}_0) + B(\rho - 2\phi_1)}{(\phi_1 - \phi_2)} \lim_{f \rightarrow \infty} \frac{1}{(c\sigma\phi_1 + 1)}, \end{aligned}$$

is finite since $c > 0$. The left-hand side of equation (A.94), on the other hand, tends to $\pm\infty$ as $\pi(0) \rightarrow \pm\infty$. This means, since N , M , Q and B are continuous in $\pi(0)$, that there is at least one $\pi(0)$, say $\hat{\pi}_0$, that satisfies equation (A.94). Plugging $(\hat{\pi}_0, \hat{x}_0)$ into equations (A.90) and (A.93) give values for $(\pi(t), x(t))$, say $(\hat{\pi}_1, \hat{x}_1)$. By construction, the path defined by $(\hat{\pi}_0, \hat{x}_0)$ is continuous. Picking ε small enough guarantees that $(\hat{\pi}_1, \hat{x}_1) \in \Omega_{ss}$, since $(\pi^*(t^*), x^*(t^*)) \in \Omega_{ss}$ is bounded away from $\partial\Omega$. Equation (A.90) implies $(\hat{\pi}_1, \hat{x}_1) \in \Upsilon_{ss}$. Proposition [[xx]] then shows the path defined by $(\hat{\pi}_0, \hat{x}_0)$ is bounded and hence an equilibrium. Because $\varepsilon \neq 0$, the equilibrium is not the optimal equilibrium.

If $M = 0$ and $N \neq 0$, the continuous pasting conditions (A.23)-(A.24) give

$$x_0 = P + A \quad (\text{A.95})$$

$$\pi_{t_1} = \frac{\pi_0 - (Q + B)}{N} \quad (\text{A.96})$$

$$\frac{r_h \phi_1^2 e^{-\phi_2 f} - \phi_2^2 e^{-\phi_1 f}}{\kappa (\phi_1 - \phi_2)}$$

But $M = 0$ implies

$$e^{-f\phi_2} = \frac{c\sigma\phi_2 + 1}{c\sigma\phi_1 + 1} e^{-f\phi_1}$$

and thus

$$\begin{aligned}
\lim_{f \rightarrow \infty} P + A &= \lim_{f \rightarrow \infty} \frac{r_h \phi_1^2 e^{-\phi_2 f} - \phi_2^2 e^{-\phi_1 f}}{\kappa (\phi_1 - \phi_2)} + A \\
&= \lim_{f \rightarrow \infty} \frac{r_h (\rho - c\kappa)}{\kappa + c\kappa\sigma\phi_1} e^{-f\phi_1} + A \\
&= A
\end{aligned}$$

is finite. An argument analogous to the one used for $M \neq 0$ and $N \neq 0$ shows the existence of a non-optimal equilibrium. The case $M \neq 0$ and $N = 0$ can be treated the same way and $M = N = 0$ cannot happen.

Item 4c. By Proposition 5, if items 1-3 hold but with equation (82) replaced with (84), then there is no equilibrium with $R \neq R^*$, since (84) implies (82).

Conversely, assume there is no equilibrium with $R \neq R^*$. I show equation (84) holds. To do so, I first show that the Intermediate Value Theorem is applicable and then use it to prove equation (84). Let

$$\Theta = \{R \in \mathbb{R}^4 : \mathcal{P}(R) = 0 \text{ and } R \neq R^*\}$$

Because f , ξ_x and ξ_π are continuous, their restriction to Θ are also continuous. In addition, Θ is path-connected because the solution to the ODE (1)-(2) is continuous with respect to time, the mapping from $(\pi(0), x(0))$ to $(\pi(t_1), x(t_1))$ is a continuous bijection for a fixed t_1 , $f(R_{t_1}) = t_1$ is continuous in R_{t_1} and the exclusion of R^* from Θ does not destroy path-connectedness because it is a zero-dimensional set while the dimension of Θ is 3. Because f , ξ_x and ξ_π are continuous in Θ and Θ is path-connected, we can apply the Intermediate Value Theorem. Assume, for the sake of contradiction, that there exists $R_{low} \in \Theta$ with $f(R_{low}) < \mathcal{T}(R_{low})$. The inequality $f(R_{low}) < \mathcal{T}(R_{low})$ implies $f(R) < \mathcal{T}(R)$ for all $R \in \Theta$ since otherwise, by the Intermediate Value Theorem, there would be some $R_0 \in \Theta$ with $f(R_0) = \mathcal{T}(R_0)$, contradicting that there is no equilibrium with $R \neq R^*$. Consider the point $R_T = (\pi_0, x_0)$ defined by

$$\begin{aligned}
\hat{x}_0 &= \frac{r_h \phi_1^2 e^{-\phi_2 T} - \phi_2^2 e^{-\phi_1 T}}{\kappa (\phi_1 - \phi_2)} + \left(\frac{r_h - r_l}{\kappa} \right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa}, \\
\hat{\pi}_0 &= r_h \frac{\phi_1 e^{-\phi_2 T} - \phi_2 e^{-\phi_1 T}}{\phi_1 - \phi_2} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l, \\
\hat{x}_1 &= 0, \\
\hat{\pi}_1 &= 0.
\end{aligned}$$

By the continuous pasting conditions in equations (A.23)-(A.24), $R_T \in \Theta$ and $f(R_T) = T =$

$\mathcal{T}(R_T)$, contradicting that $f(R) < \mathcal{T}(R)$ for all $R \in \Theta$.

B Non-Linear dynamics – Poincaré-Bendixson Theorem

Assume $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Consider the two-dimensional system

$$\dot{x}_t = f(x_t). \quad (\text{B.1})$$

Let $\phi_t(p)$ be a solution to (B.1) for $t \geq 0$ with initial condition $x_0 = p$. We assume that for each p , there is a unique solution $\phi(t, p)$. This is the case, for example, if f is Lipschitz.

The *positive semi-orbit* of f through p is defined as

$$\gamma^+(p) = \{x \in \mathbb{R}^2 : x = \phi_t(p) \text{ for some } t \in [0, \infty)\}.$$

Similarly, the *negative semi-orbit* through p is

$$\gamma^-(p) = \{x \in \mathbb{R}^2 : x = \phi_t(p) \text{ for some } t \in (-\infty, 0]\}.$$

The *orbit* of f through p is the union

$$\gamma(p) = \gamma^+(p) \cup \gamma^-(p).$$

A *periodic solution* is one for which $\phi_{t+T}(p) = \phi_t(p)$ for some $T > 0$ and all $t \in R$. A *periodic orbit* is the orbit $\gamma(p)$ of periodic solution $\phi_t(p)$.

The *ω -limit set* of p , denoted by $\omega(p)$, is the set

$$\omega(p) = \{x \in \mathbb{R}^2 : \exists \{t_k\}_{k=0}^{\infty}, t_k \in R \text{ with } t_k \rightarrow \infty \text{ such that } \phi_{t_k}(p) \rightarrow x \text{ as } k \rightarrow \infty\}.$$

Consider the following four assumptions:

1. Ω is an open domain in \mathbb{R}^2 , divided into a finite number of open sub-domains Ω_i such that $\bigcup \bar{\Omega}_i = \bar{\Omega}$.
2. If $\bar{\Omega}_i$ and $\bar{\Omega}_j$ are not disjoint and $i \neq j$, then $\bar{\Omega}_i \cap \bar{\Omega}_j = \Gamma_{ij}$, where Γ_{ij} (joint boundaries) are piecewise smooth.
3. f is Lipschitz in all sub-domains Ω_i and possibly discontinuous along Γ_{ij} .

4. The vector field f defines a direction at each point in Ω . In particular, at every point of Γ_{ij} the vector field $f(x)$ specifies into which Ω_i the flow is directed.

Theorem 10 (Extension of the Poincaré-Bendixson theorem). *Consider the planar autonomous system (B.1). Let the conditions 1-4 be satisfied and let f be bounded in Ω . Suppose that K is a compact region in Ω , containing no fixed points of (B.1). If the solution of (B.1) is in K for all $t \geq t_0$, then (B.1) has a closed orbit in K .*

Theorem 11 (Extension of the Bendixson criterion). *Consider the planar autonomous system (B.1). Let the conditions 1-4 be satisfied and let f be bounded in the simply connected region Ω and C^1 in each Ω_i . If $\operatorname{div} f$ (the divergence of f calculated in the distribution sense) is of the same sign and is not identically zero in Ω , then (B.1) has no closed orbit in Ω .*

Remark The requirement that f is bounded is too strong; it suffices that

$$\iint_D \operatorname{div} f \text{ and } \int_C f \cdot n \, ds$$

are well-defined (in the distribution sense) for all smooth closed curves C , where D is the region enclosed by C and n is a unit vector normal to C .

A proof of both theorems can be found in Melin (2005). Compared to the classical Poincaré-Bendixson theorem, Melin (2005) allows for some discontinuities in f .

We have cited the theorems exactly as they appear in Melin (2005). However, in this context, it is perhaps more familiar for economists to refer to points for which $f = 0$ as steady-states instead of fixed points and to periodic orbits instead of closed orbits.

We now prove an immediate consequence of this "extended" Poincaré-Bendixson theorem.

Theorem 12. *Assume Theorem 10 holds. If a solution φ_t is bounded for all $t \geq 0$, then either*

1. $\omega(\varphi)$ contains a steady state

or

2. $\omega(\varphi)$ is a periodic orbit

Proof. First, note that because φ is bounded, $\omega(\varphi)$ is non-empty. Indeed, consider a sequence $x_i = \varphi_{t_i}(x)$ for some x . The sequence $\{x_i\}$ is bounded and infinite, so there exist a convergent subsequence. If such convergent subsequence converges to p , then $p \in \omega(\varphi)$ and thus $\omega(\varphi)$ is non-empty.

If $\omega(\varphi)$ contains a steady state, item 1 obtains. If $\omega(\varphi)$ contains no steady-states (no fixed points), then Theorem 10 implies that $\omega(\varphi)$ is a periodic orbit, corresponding to item 2 (note that because φ is bounded we can always find a compact set K that contains it). \square