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## The Boulding-Richardson model revisited

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# THE BOULDING-RICHARDSON MODEL REVISITED

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# **The Boulding-Richardson Model Revisited**

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## **Zusammenfassung / Abstract**

We review, and extend, one of the classic dynamic models of conflict in economics by Richardson (1919) and Boulding (1962). It turns out that the stability properties of the model change if one takes a more realistic “incrementalist” view, and that chance / friction can easily be incorporated into the standard model by defining a probability of (de-)escalation. This analysis is not just a study in the history of economic thought, but also relevant for the development of simulation models for the analysis of conflict dynamics.

**JEL-Klassifikation / JEL-Classification:** B25; D74

**Schlagworte / Keywords:** Conflict dynamics, psychology of aggression, escalation, stability, patterns of conflict.

# 1 Introduction

The formal, quantitative analysis of dynamic conflict in economics began in 1916, when Frederick Lanchester (1956) developed his linear and square laws of attrition. Such models consist of systems of ordinary or partial differential equations (ODEs/PDEs), whose dynamic behaviour – the **conflict dynamics** – and stability properties are analysed by solving the systems analytically or by phase diagramme techniques. Other examples for such models include the Lotka-Volterra-Goodwin equations of predator-prey conflict (Goodwin, 1967) and the Boulding-Richardson model (Boulding, 1962). The latter is in the focus of the present analysis.

One serious lack of this first group of models is that they are effectively “macro” models (even when used to illustrate the behaviour of individual parties to a conflict) with no microeconomic foundations, i.e. they do not contain explicit optimisation. It comes as no surprise then that the rise of game theory that began in the 1940s led to a shift of interest away from the first generation of models and towards dynamic games and, in particular, differential games (Isaacs, 1954). As far as static patterns of conflict are concerned, game theory provided a convincing taxonomy (Rapoport and Guyer, 1966),<sup>1</sup> and dynamic game theory yielded deep insight into such features of conflict dynamics as the initiative, signals, and reputation.

However, a general taxonomy of conflict dynamics proved elusive, and differential game theory hit a conceptual wall when dealing with the non-linearities that are pervasive in conflict theory (Beckmann and Reimer, 2014). These problems as well as improvements in the raw computing power available to scholars led to increased reliance on simulation methods (Fontana, 2006). And for simulation purposes, both the aforementioned theoretical limits and the insights of behavioural economics recommend some version of boundedly rational optimising. It is in this context that first-generation models may return as more than just a subject for the historian of economic thought.

In the present paper, we propose to re-visit the Richardson<sup>2</sup> equations (Richardson, 1919) as used in Boulding (1962), referring to this in its entirety as the “BR model”. The ultimate aim of the analysis is to assess whether the BR model can be used as a building block for modern conflict simulation, and for a taxonomy of dynamic conflict. *En passant*, we may also contribute to the history of economic thought.

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<sup>1</sup>See also the recent book by Robinson and Goforth (2005).

<sup>2</sup>Lewis Fry Richardson is little known in economics, and without the work of his fellow Quaker Boulding his work may not have resounded in our field at all. He is, however, well remembered for his contributions to other disciplines. On this, see Hunt (1995).

We begin by re-stating the Richardson equations and illustrating the dynamic properties of the original model (section 2). We find that the standard formulation is deficient in two respects, one having to do with the **psychological** predominance of escalation over the level of aggression, the other dealing with the probabilistic nature of escalation. These extensions and modifications of the BR-model are discussed in turn in section 3, noting that the resulting dynamics are both more stable and more plausible. Section 4 concludes.

## 2 The basic B-R model

The standard B-R setup describes the joint dynamics of the *aggressiveness* (or escalation level) of two parties to a conflict. Denote by  $a$  ( $b$ ) a measure of party A's (B's) aggressiveness and assume that without interaction, this reverts over time to a base level  $\hat{a}$  ( $\hat{b}$ ). This base level is, however, not the long-term equilibrium because of the interaction effect: each party's aggressiveness increases exponentially as a function of the competitor's escalation measure. This gives the Richardson equations

$$\dot{a} = k_a(\hat{a} - a) + r_a b \quad (1)$$

$$\dot{b} = k_b(\hat{b} - b) + r_b a \quad (2)$$

where the strictly positive parameters  $k_i$  and  $r_i$  represent the parties' speed of adjustment to the base level and sensitivity to aggression, respectively.

We can explicitly solve this system of linear ODEs for the time paths  $a(t), b(t)$  of aggressiveness.<sup>3</sup> For example, in the symmetric case where  $r_a = r_b = r$  and  $k_a = k_b = k$  (assuming  $r \neq k$ ), we find

$$a(t) = \frac{k(\hat{a}k + \hat{b}r)}{k^2 - r^2} + e^{-kt}(c_1 \cosh(tr) + c_2 \sinh(tr)) \quad (3)$$

where  $c_1$  and  $c_2$  are constants. If we additionally assume that  $a(0) = b(0) = 0$ , we have

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<sup>3</sup>An appendix detailing the solutions to the various systems of differential equations is available from the authors' web site at <http://beckmann.hsu-hh.de/>.

$$a(t) = \frac{e^{2rt} - e^{(k+r)t}}{\frac{r-k}{k}e^{(k+r)t}}\hat{a} \quad (4)$$

and likewise for  $b$ .

However, the general properties of this model are better studied using phase diagramme techniques. Letting  $\dot{a} = 0$  and  $\dot{b} = 0$ , we obtain the isoclines (written as functions of  $a$  for easier plotting)

$$b = \frac{k_a}{r_a}(a - \hat{a}) \quad (5)$$

$$b = \hat{b} + \frac{r_b}{k_b}a \quad (6)$$

Note that both isoclines are upward sloping in  $(a, b)$ -space and that the isocline for  $a$  has a negative intercept on the  $b$  axis, while the other isocline cuts the ordinate at  $\hat{b} > 0$ . This implies that there are just two possible configurations (see figure 1). If  $\frac{k_a}{r_a} > \frac{r_b}{k_b}$ , the two isoclines intersect in the positive orthant (left-hand panel in figure 1) and there exists a stable stationary equilibrium at

$$a^* = \frac{k_b(\hat{a}k_a + \hat{b}r_b)}{k_ak_b - r_ar_b} \quad (7)$$

and

$$b^* = \frac{k_a(\hat{b}k_b + \hat{a}r_b)}{k_ak_b - r_ar_b} \quad (8)$$

Otherwise, there is no such intersection and aggressiveness explodes in the positive orthant (right-hand panel in figure 1).

Note that a *symmetry* assumption, i.e.  $r_a = r_b$  and  $k_a = k_b$ , would generate a borderline case where the isoclines are parallel. The consequences are much the same as in case 2 below, with an explosion of aggression in the first orthant.

The endless escalation of conflict in case 2 may appear implausible because infinite aggression levels are an unwieldy concept. However, in interpreting the B-R model, one can assume that there exists a threshold level of escalation beyond which the conflict in question changes its nature (i.e., an open outbreak of military hostilities). One can also add an additional constraint to the model – for

instance,  $a$  and  $b$  could represent the *share* of two competing new media devoted to a particular conflict, or a particular scandal. The latter modification would give rise to a stable corner solution in case 2.

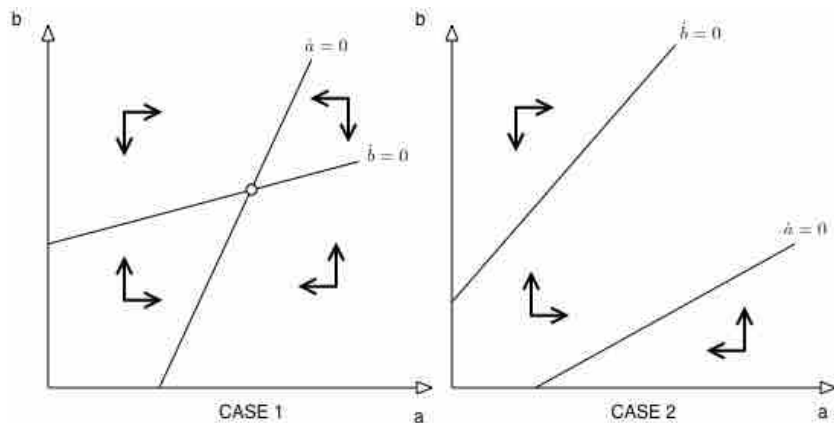


Figure 1: The two possible scenarios in the B-R model

### 3 Extensions and variations

We now propose two variants of the B-R model, which we explore in turn:

1. a version which incorporates the idea that it may be escalation rather than the stock of aggressiveness which determines the interaction effect,
2. a model which replaces the deterministic interaction effect with a probabilistic version, taking account of Clausewitzian friction and other sources of uncertainty.

#### 3.1 An incrementalist B-R model

In our first variation on the B-R theme, we recognise that it can be the *change* in enemy aggression levels, i.e. the *escalation* of conflict, which drives conflict dynamics. We retain the assumption that aggression levels will return to base values  $\hat{a}, \hat{b}$  over time, but replace the stock levels of aggression with their time derivatives  $\dot{a}, \dot{b}$ . This leads to the following model:

$$\dot{a} = k_a(\hat{a} - a) + r_a \dot{b} \quad (9)$$



$$\dot{b} = k_b(\hat{b} - b) + r_b \dot{a} \quad (10)$$

As was the case for the baseline model, we can solve this system of differential equations explicitly, obtaining complete time paths for the two variables of interest, given the parameters and starting values  $a(0), b(0)$ . Using the symmetric example from section 2, we find

$$a(t) = \left(1 - e^{\frac{k(1+r)t}{r^2-1}}\right) a(0) \quad (11)$$

with an analogous solution for  $b$ . Again, however, we find it more instructive to take a conventional approach using phase diagrams to illustrate system behaviour over time for more general parameter values.

Substituting  $\dot{b}$  into the first equation of the model and rearranging, we can express the change in  $a$  and in  $b$  as a function of the state variables

$$\dot{a} = \frac{k_a(\hat{a} - a) + r_a k_b(\hat{b} - b)}{1 - r_a r_b} \quad (12)$$

$$\dot{b} = \frac{k_b(\hat{b} - b) + r_b k_a(\hat{a} - a)}{1 - r_a r_b} \quad (13)$$

Setting these equations to zero yields the following equations in  $(a, b)$ -space for the isoclines:

$$b = \frac{\hat{a} k_a + \hat{b} k_b r_a - k_a a}{k_b r_a} \quad (14)$$

$$b = \frac{\hat{b} k_b + \hat{a} k_a r_b - k_a r_b a}{k_b r_a} \quad (15)$$

for  $\dot{a} = 0$  and  $\dot{b} = 0$ , respectively. Solving the simple system formed by the isoclines yields the stationary point at  $a^* = \hat{a} \wedge b^* = \hat{b}$ . This implies that contrary to the standard BR model, the isoclines always intersect in the positive orthant. Note, however, that the slope of the isocline for  $\dot{a} = 0$  is steeper than the other isocline iff  $r_b < 1$ . Also note that the denominator in both equations of motion becomes *negative* for  $r_a r_b > 1$ .

All in all, this leaves us with four possible dynamic configurations shown in figure 2 below. Case 1 exhibits a stable stationary state, whereas case 2 is characterised by instability. However, case 2 differs from the unstable case in the original model in that a corner solution at the origin is also a possibility. Cases 3 and 4 – where  $r_b > 1$  – have saddlepoint stable equilibria.

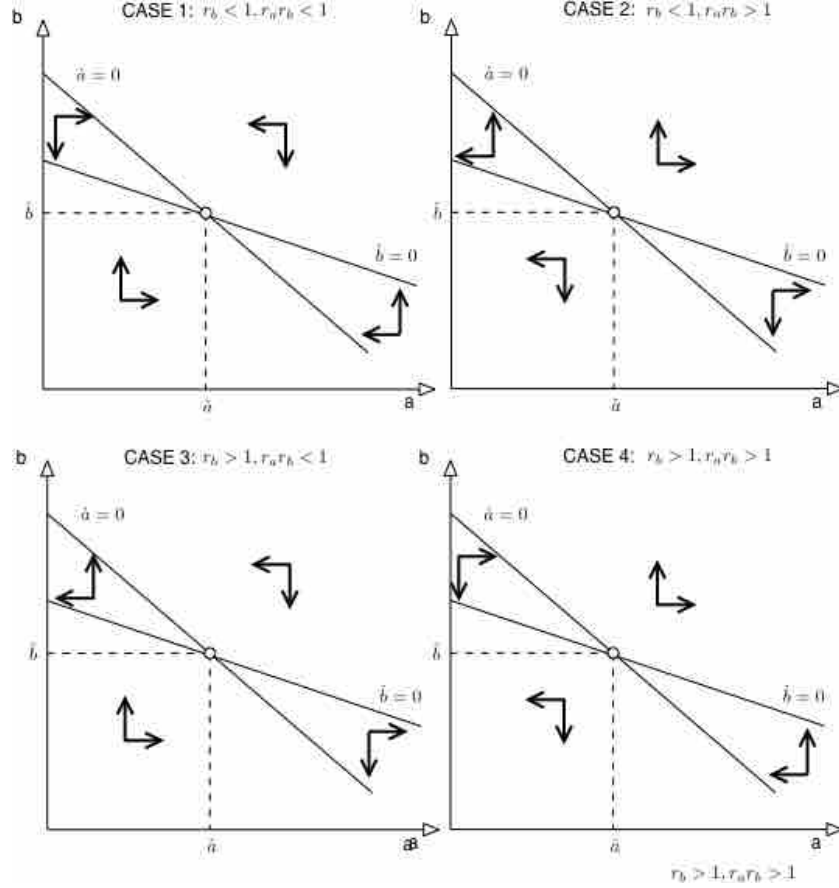


Figure 2: The four scenarios in the incrementalist BR model

In the original BR model, it was the *relative* size of adaptation  $k$  and reaction coefficients  $r$  that determined the dynamic pattern of conflict. Now, it is the *absolute* value of the reaction coefficients alone that proves crucial. It is sufficient for convergence to a stable equilibrium at the “normal” aggro level  $\hat{a}, \hat{b}$  that both parties do not respond “in kind” to an enemy escalation, but with an  $r < 1$ . This case is illustrated in the flow plot in figure 3.<sup>4</sup>

<sup>4</sup>We assume a symmetric solution with the following parameter values:  $k_a = k_b = \frac{1}{5}$ ,  $r_a = r_b = \frac{1}{2}$ ,  $\hat{a} = \hat{b} = 10$ . The plot was produced using *Mathematica*.

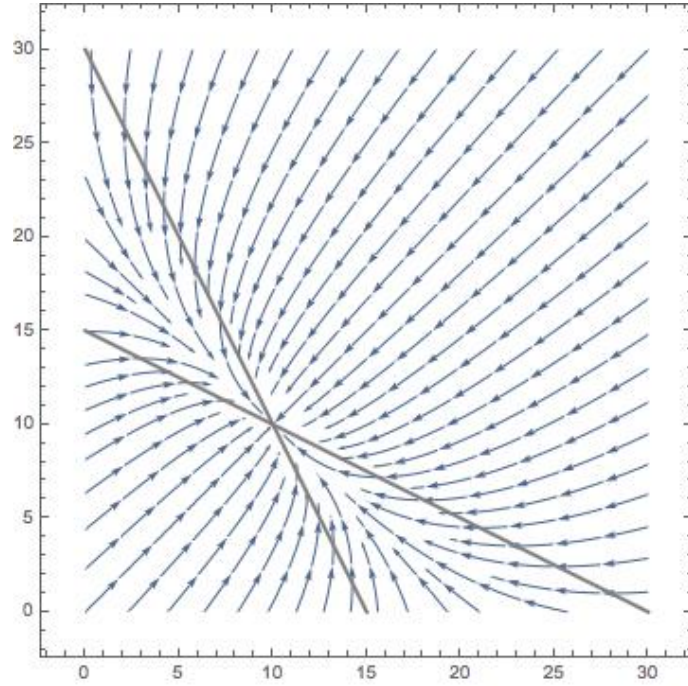


Figure 3: Stable equilibrium in the incrementalist BR model

This feature of the model appears more plausible than the results we obtained for the original formulation. In addition, the incrementalist model allows for a “pacifist” party (with low  $r$ ) to compensate for the existence of an aggressive opponent in a very plausible manner.

### 3.2 Probabilistic interaction

At the end of our journey, let us briefly consider how to incorporate *randomness* – and Clausewitzian “friction” – into the simple framework. As a large conflict unfolds, there will be several small interactions during which either side can either escalate, de-escalate, or ignore the other side’s aggression. Let  $a$ ’s probability  $p$  of escalation depend on  $b$ ’s aggro level according to a probability function  $p(b)$  with  $p' > 0$  and vice versa. For a large number of such interactions per unit of time, the BR equations of motion can then be amended by just plugging in the probability functions for  $r_a b$  and  $r_b a$ , respectively. We then obtain the following system of equations

$$\dot{a} = k_a(\hat{a} - a) + s_a p(b) - s_a(1 - p(b)) \quad (16)$$

$$\dot{b} = k_b(\hat{b} - b) + s_b p(a) - s_b(1 - p(a)) \quad (17)$$

where the  $s_i$  represent party  $i$ 's “step size” of (de-)escalation, assumed to be a constant for simplicity.

We require a specific probability function for plotting or explicit solutions, although basic phase diagrammes such as the ones in figures 1 and 2 could be derived with just some assumptions regarding the curvature of  $p$ . Borrowing from the literature on conflict success functions,<sup>5</sup> we employ a *logistic function*

$$p(a) = \frac{1}{1 + e^{\kappa(\hat{a}-a)}} \quad (18)$$

where  $\hat{a}$  denotes the reference level of aggression by A (i.e., the level where escalation and de-escalation are just as likely), and the parameter  $\kappa$  determines the steepness of the probability function.

One important difference from the variants discussed previously is that the isoclines are now non-linear. Also, the fact that the limits of the logistic function are zero for  $a, b \rightarrow -\infty$  and one for  $a, b \rightarrow +\infty$  together with the structure of the system imply that there exists a stable intersection in the positive orthant. Figure 4 below illustrates this for the symmetric case.<sup>6</sup>

## 4 Conclusion

Richardson (1919) approach was a **psychological** one, modelling how one party's aggression level depends on the *perceived* aggression by the other party, and vice versa. We believe that this model can plausibly be applied to various scenarios – the treatment of scandals by the media may serve as an example.

However, we find that the dynamic properties of the model can be improved upon and the model be made more realistic by letting a party's level of aggression depend on the **change** in the other party's aggro level (“escalation”) rather than on the stock variable.

This change follows the methodology used in research using machine-generated coded event data (Goldstein and Pevehouse, 1997, 1999) where the reaction on

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<sup>5</sup>The classic treatment is the book by Hirshleifer (2001).

<sup>6</sup>We use the same parameters as in the last example:  $k_a = k_b = \frac{1}{5}$ ,  $r_a = r_b = \frac{1}{2}$ ,  $\hat{a} = \hat{b} = 10$ .

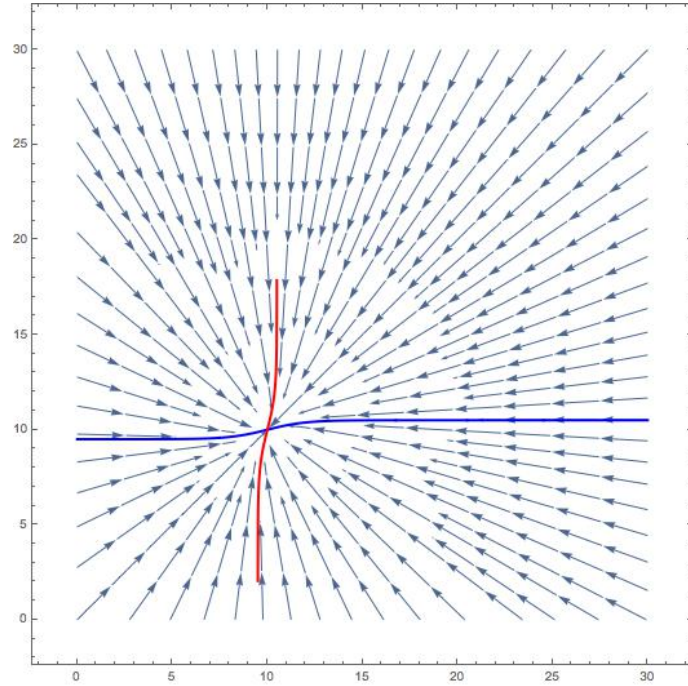


Figure 4: A stable stationary point in the probabilistic model

events is measured by using a numerical conflict scale for events data. The Goldstein Weight (Goldstein 1992) assigns each conflictary (and cooperative) event a flow-variable indicating the impact on relations between the involved parties. This scale is used in big event data sets like GDELT<sup>7</sup> and could be used for further analysis of the incrementalist model using time-series analysis.

We also note that the probabilistic nature of conflict, which has figured prominently in the military literature since Clausewitz' (1873) made friction a core element of his theory, and which is also reflected by the modern vision of "hybrid" warfare, can be integrated into the BR model. Such integration can be shown to improve the stability properties of the BR model.

The BR model thus appears to be a robust framework that is apt to be used in future analysis, in particular as a basis for simulation studies.

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<sup>7</sup>The GDELT Project - Global Database of Events, Language and Tone - <http://gdeltproject.org>

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