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Das House-Kapital: A Theory of Wealth-to-Income Ratios

Preliminary version

Volker Grossmann* and Thomas Steger†

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Abstract

This paper presents a novel dynamic general equilibrium model to examine the evolution of two major wealth-to-income ratios – housing wealth and non-residential wealth – in advanced countries since WWII. Our theory rests on three premises: (1) the overall land endowment is fixed; (2) the production of new houses requires land as an essential input; (3) land employed for real estate development must be permanently withdrawn from alternative uses. The model distinguishes, for the first time, between the extensive and the intensive margin of housing production. The calibrated model replicates the post WWII increase in the two major wealth-to-income ratios. It also suggests a moderate further increase in wealth-to-income ratios that is associated with a considerable future surge in land prices and house prices. Higher population density and technological progress do, however, not affect long run wealth-to-income ratios. The model also accounts for the close connection of house prices to land prices in the data.

Key words: Wealth-to-Income Ratios; Housing Wealth; Non-Residential Wealth; House Prices; Land Prices.

JEL classification: E10, E20, O40.

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1 Introduction

The aggregate wealth-to-GDP ratio has increased in major industrialized countries since World War II (Piketty and Zucman, 2014a, 2015). Housing wealth appears of fundamental importance for understanding this development. Figure 1 shows the post World War II evolution of the two major components of private wealth (relative to GDP), housing wealth and non-residential wealth, for Germany, France, United Kingdom, and the US. Two characteristics are striking and widely-discussed. First, the rise was more pronounced in European economies as compared to the U.S. (e.g., Piketty and Zucman, 2014a). Second, housing wealth increased considerably more than non-residential wealth (e.g., Bonnet et al., 2014). This last observation is consistent with rising real house prices since World War II, as documented empirically by Knoll, Schularick and Steger (2014). They show that the surge in house prices goes hand in hand with rising land prices and argue that the price channel of rising land valuation is likely to be important for the observed surge in wealth-to-income ratios.

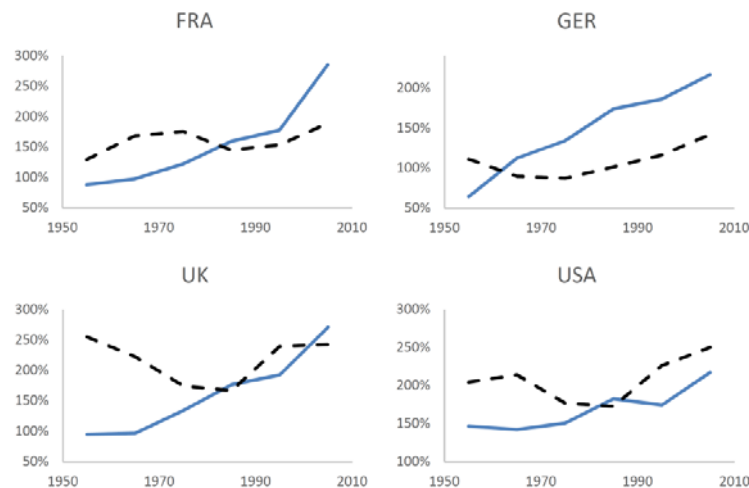


Figure 1. Evolution of the ratios of housing wealth to GDP (solid, blue lines) and non-residential wealth to GDP (dashed, black lines) in France, Germany, UK, and US.

Notes: Time series smoothed over decades. Non-residential wealth is defined as private wealth minus housing wealth minus net foreign assets. Housing wealth is defined as housing assets (incl. land). It includes mortgages and other financial liabilities.

Data source: Piketty and Zucman (2014b).

The evolution of the wealth-to-GDP ratio and its decomposition is important for at least two reasons. First, Piketty (2014) stresses that a rising wealth-to-GDP ratio, assuming that the interest rate remains largely constant, changes the functional income distribution to the advantage of capital income recipients. Moreover, given that wealth is not uniformly distributed across the population, a rising wealth-to-GDP ratio could trigger off a more unequal distribution of personal income. Ronglie (2015) demonstrates that the increase in the economy-wide capital income share is exclusively driven by the housing sector. Stiglitz (2015) points to the important role of rising land prices in the process of rising wealth-to-GDP ratios and rising inequality of wealth and income. Once more, both land prices and the housing sector appear pivotal. Second, a rising wealth-to-GDP ratio may also have important implications for macroeconomic volatility. Recent research has stressed wealth effects that may act as important amplification mechanisms. Given that a subset of households are credit constrained and use housing wealth as collateral, a reduction in, say, house prices forces those households to cut consumption expenditures (e.g., Kiyotaki and Moore, 1997; Favilukis, Ludvigson and van Nieuwerburgh, 2015).¹

We address the following research questions: How large is the private wealth-to-GDP ratio in the long run and how will it evolve?² How does the composition of the wealth-to-GDP ratio (housing and non-residential wealth) look like? How do population dynamics and technological progress contribute to the observed evolution of the two major wealth-to-GDP ratios? To discuss these questions, we construct a new dynamic macroeconomic model that is designed to analyze the evolution of different wealth components. Our theory puts much emphasis on modelling the housing sector and the factor land. This allows us to capture important mechanisms that impact on the evolution of wealth-to-income ratios. The suggested model, labelled House-Kapital Model, rests on three premises:

Premise 1 (Fixed Land Endowment) The overall land endowment is given by na-

¹Similarly, Liu, Wang and Zha (2013) reason that firms may be credit constrained and finance business investment by using land as collateral, implying that a positive shock to land prices leads to persistent increases in business investment, private consumption, and aggregate labor hours.

²Throughout this paper, we do not consider foreign capital. Foreign capital played a substantial role in Britain and France in the late 19th and early 20th century, but not in modern times. Moreover, we exclusively focus on private rather than public wealth.

ture. The total amount of land that can be used economically, therefore, is fixed in the long run.

Premise 2 (Land in Housing Production) (i) A house is a bundle of the underlying land plot and residential structures. Setting up new housing projects requires land as an essential input. (ii) Replacement investments do not, however, require land as an input.

Premise 3 (Land Rivalry) Land that is used as an input in the production of new houses is permanently withdrawn from alternative economic uses, unless the respective house is being demolished.

Premise 1 represents a general law of nature. This has been acknowledged by previous researchers who model land and long run economic growth, starting with Nichols (1970). A sceptic may argue that land-augmenting technical change, land reclamation, and land development due to infrastructure investment can enlarge the available amount of land.³ This is indeed plausible in the short to medium run. In the long run, however, the total amount of economically usable land is fixed by nature. Increasing land scarcity, already in contemporaneous times, is also compatible with rising prices of farmland and urban residential land, as documented by Knoll et al. (2014) for 14 advanced countries since WWII. *Premise 2 (i)* appears largely undisputed and taken into account by existing theories (e.g., Davis and Heathcote, 2005; Favilukis et al., 2015). *Premise 2 (ii)* is usually not taken into account. We capture the fact that replacement investment do not require land by distinguishing between the extensive and the intensive margin of residential investment. It will be argued that only the enlargement of the housing stock along the extensive margin (setting up new housing projects) requires land. This distinction enables us to avoid the long run inconsistency that arises from Premises 1 & 3 together with the (widely-employed) assumption that investment into the housing stock, including replacement investment, requires land. *Premise 3* describes simply the fact that land represents a rivalrous input: A land plot that is put underneath a house cannot, at the

³Zoning regulations and other restrictions on land use, on the other hand, have inhibited the utilization of additional land in recent decades (Glaeser, Gyourko, Saks 2005; Glaeser and Gyourko, 2003).

same time, host a manufacturing plant. As a result, land plots that are employed for setting up new housing projects have the characteristics of an exhaustible resource.

The interaction between scarce land and housing production in a growing economy, resulting from Premises 1-3, will show up in a modern version of Ricardo's (1817) famous principle of scarcity. Ricardo was mainly concerned with agricultural land and the production of corn to feed a growing population.⁴ In modern times, societies are more confronted with residential investments to meet the increasing demand for housing services under the constraint of scarce land. We argue that the modelling strategy, i.e. whether or not Premises 1-3 laid out above are captured by the model, has important implications for the evolution of the land price that directly determines the value of non-residential. In addition, the land price impacts on the house price and thereby on the evolution of housing wealth. For instance, if one assumes that a given amount of new pieces of land (building permits) becomes available for housing production each period, then the increasing scarcity of land cannot be captured in an appropriate manner.

Our model can be sketched as follows. We distinguish three different types of wealth: physical capital, non-residential land, and residential ("house") capital.⁵ House capital includes residential land that is required to set up a housing project and structures. The housing stock can be expanded along the extensive margin (setting up new housing projects) and along the intensive margin (enlarging existing housing projects by building larger residential buildings). As it becomes increasingly difficult to build houses upwards (i.e. constructing larger residential buildings on a given amount of land), there are decreasing returns to scale along the intensive margin. This creates profits (residual incomes) for housing services producers that provide incentives for real estate development in the first place, despite perfect competition. Land can either be allocated to the housing sector or used for the production of a non-residential consumption good. The overall land endowment is treated as a fixed factor. Moreover, we capture that land that is devoted to the production of new houses is permanently withdrawn from alternative

⁴Ricardo (1817) argued that, over the long run, economic growth profits landlords disproportionately, as the owners of the fixed factor. Since land is unequally distributed across the population, Ricardo reasoned that market economies would produce rising inequality.

⁵The model therefore enables us to quantify the major components of the overall wealth-to-GDP ratio, distinguishing between non-residential wealth and housing wealth, as displayed in Figure 1.

economic uses. In this sense, land devoted to housing production shares some characteristics of an exhaustible resource. The land price is fully endogenous and responds to economic development and rising population density that trigger increasing land scarcity.

By calibrating the model and solving it numerically, we are able to describe the evolution of the different wealth-to-GDP ratios, the land price, and the house price over time. The numerical analysis employs the relaxation algorithm to solve the model for transitional dynamics (Trimborn, Koch and Steger, 2008). This technique allows us to calculate the transitional dynamics implied by the non-linear differential equation system.⁶ The calibrated model replicates the post WWII increase in the two major wealth-to-GDP ratios surprisingly well.

The calibrated model also suggests a moderate further increase in wealth-to-GDP ratios that is associated with a considerable future surge in land prices and house prices. Assuming an intermediate capital income tax rate, such as in the U.S., and depending on the subjective discount rate, the implied long run housing-wealth-to-income ratio ranges from 275 to 360 percent. The implied total private wealth-to-GDP ratio ranges from 490 to 625 percent in the long run, when accounting for considerably lower land prices in rural areas.⁷ These values are somewhat below the values suggested by Piketty (2014). Assuming a net savings rate, s , of about 10 percent and real GDP per capita growth rate, g , of about 1.5 percent, Piketty argues that the (private) wealth-to-GDP ratio, s/g , would rise to 600-700 percent. Provided that the real interest rate, r , does not adjust, this would result in a rising capital income share (rs/g) and, given that capital is unequally distributed, in rising income inequality.⁸ His projections are based on a simple Solow-type framework that does not distinguish between housing wealth and non-residential wealth.

⁶This procedure is extended to analyze sizeable transitions due to large shocks in state variables and substantial exogenous movements in population and total factor productivity, as is appropriate with respect to the period after WWII.

⁷Average housing-wealth-to-GDP ratio (across FRA, GER, and UK) is about 255 percent, while it amounts to a somewhat lower 217 percent for the U.S., according to Figure 1. Similarly, average non-residential-wealth-to-GDP ratio is about 190 percent, while it amounts to a substantially higher 250 percent for the U.S. Average total private wealth-to-GDP ratio is about 450 percent, while it amounts to a somewhat higher 465 percent for the U.S.

⁸Krusell and Smith (2015) have debated these propositions. Our model is not subject to their criticisms.

There is an extensive and growing theoretical literature on housing and macroeconomics (Piazzesi and Schneider, 2016). Instead of providing an exhaustive overview, we highlight some of the more recent contributions. Hornstein (2008, 2009) employs a general equilibrium model to explain the surge in house prices in the US between 1975 and 2005. Davis and Heathcote (2005) build a multi-sector stochastic growth model with a housing sector to explain the business cycle dynamics of residential investment. Favilukis et al. (2015) construct a stochastic two-sector general equilibrium model of housing and non-housing production to explain the surge and the subsequent decline of the price-to-rent ratio in the US housing market between 2000 and 2010. Li and Zeng (2010) employ a two-sector neoclassical growth model with housing to explain a rising real house price driven by a comparably low technical progress in the construction sector. Borri and Reichlin (2016) use a two-sector, life-cycle economy with bequests to explain the increasing wealth-to-income ratio and wealth inequality driven by rising construction costs that are caused by a comparably slow technological progress in the construction sector. We depart from previous contributions in two important respects. First, we distinguish between the extensive and the intensive margin of housing production. This distinction is particularly important if there is depreciation of house capital and if land is treated as a fixed factor.⁹ Second, none of the previous contributions rests on the assumption that Premises 1-3 do hold simultaneously. We demonstrate why this is critical for understanding the dynamics of housing wealth in the long run.

The paper is organized as follows. Section 2 presents the new macroeconomic model with a housing sector. Section 3 derives important asset price implications, defines the general equilibrium and the variables of interest. It also characterizes the long run equilibrium analytically. Section 4 calibrates the model, derives numerical long-run implications, and provides the transition paths for the variables of interest. Section 5 compares our new long-run macroeconomic model to the canonical macroeconomic model with a housing sector. The last section concludes.

⁹Sachs and Boone (1988) circumvent the inherent difficulty associated with land as an input in the housing sector by assuming that the stock of house capital does not depreciate. Matsuyama (1990) does not explicitly model land as a production factor.

2 The Model

Consider a perfectly competitive, closed economy. Time is continuous and indexed by $t \geq 0$. The innovation of our macroeconomic model is the housing sector. This sector encompasses different types of firms that interact in the production of housing services. The technical and legal prerequisite to produce housing services is provided by real estate development firms. Their activity comprises the purchase of a piece of land, securitized in the form of a land ownership deed, and infrastructure investment to develop land for residential purposes. Land is treated as a fixed and exhaustible resource. Setting up new housing projects diminishes the amount of land that can be employed elsewhere in the economy. Housing services are produced by combining a developed real estate and residential buildings ("structures"). Structures are produced by employing materials (like cement) and labor. The non-residential sector produces a consumption (numeraire) good different to housing services by combining capital, labor, and land. Like in standard (one-sector) models, the numeraire good can be used for capital investment as well.

2.1 Firms

2.1.1 The Numeraire Good Sector

The *numeraire good* (Y -) *sector* produces final output, that can be used for consumption and investment purposes, according a standard Cobb-Douglas production function:

$$Y_t = B_t^Y (K_t^Y)^\alpha (L_t^Y)^\beta (Z_t^Y)^{1-\alpha-\beta}, \quad (1)$$

where K_t^Y , L_t^Y and Z_t^Y denote the amounts of physical capital, labor and land employed as input in the Y -sector at time t , respectively. Total factor productivity $B_t^Y > 0$ may increase over time and $\alpha > 0$, $\beta > 0$ denote constant technology parameters that satisfy $\alpha + \beta < 1$. Physical capital is only employed in the Y -sector. The capital resource constraint is given by $K_t^Y \leq K_t$, where K_t denotes the total supply of physical capital in terms of the numeraire good. In equilibrium $K_t^Y = K_t$ will turn out to hold. Given that capital depreciates at rate $\delta^K \geq 0$, gross physical capital investment reads as

$$I_t^K \equiv \dot{K}_t + \delta^K K_t.^{10}$$

2.1.2 Housing Sector and the Characteristics of Land

Producing housing services requires to combine activities along the extensive margin, i.e., real estate development in the form of an *ex ante* investment in the stock of houses, and along the intensive margin, i.e., producing new residential buildings that may depreciate over time. There is free entry into the housing sector. Enlarging the stock of houses along the extensive margin requires land that is modelled as a fixed and exhaustible resource. Real estate development inevitably implies that land is withdrawn from the alternative use in the Y -sector.

Real Estate Development (Extensive Margin) The "number" of houses (housing projects) at time t is denoted by N_t , a real number. This variable captures the extensive margin of the housing stock. Increasing N_t requires to purchase land (associated with a building permit) and to incur adjustment costs in terms of the numeraire good (e.g., private infrastructure investment). The number of land units that must be put underneath each house is given by $\psi > 0$.¹¹ Total land usage in the housing sector is given by $Z_t^N \equiv \psi N_t$. The resource constraint reads as $Z_t^N + Z_t^Y \leq Z$, i.e., the alternative use of land in the Y -sector is limited by $Z_t^Y \leq Z_t - \psi N_t$. Land input in the Y -sector, Z_t^Y , can increase at some point in time t if the stock of houses decreases ($\dot{N}_t < 0$).

Adjustment costs are convex in the change of the stock of houses, \dot{N}_t . Let P_t^Z denote the price per unit of land. The costs $\mathcal{C}(\tilde{I}_t^N, P_t^Z)$ of increasing the number of houses at time t by

$$\dot{N}_t = \tilde{I}_t^N \tag{2}$$

amounts to

$$\mathcal{C}(\tilde{I}_t^N, P_t^Z) = \psi P_t^Z \tilde{I}_t^N + \frac{\xi}{2} \left(\tilde{I}_t^N \right)^2, \tag{3}$$

$\xi > 0$, where $\frac{\xi}{2} (\tilde{I}_t^N)^2 \equiv I_t^N$ is the infrastructure investment at time t in terms of the

¹⁰A dot above a variable denotes the partial derivative with respect to time.

¹¹We abstract from heterogeneity of real estates. This simplification greatly simplifies the analysis as we do not have to keep track of the history of houses. It is furthermore clearly indicated as we are not interested in the size distribution of firms in the housing sector.

numeraire good. The convex adjustment cost component makes N_t a state variable. If we assumed that $\xi = 0$, then N_t would turn into a jump variable which appears economically less plausible. Investment costs \mathcal{C} for real estate development are financed by issuing equity and are eventually covered by the profit stream of new housing projects. The stock market valuation per house at time t is denoted by q_t^N .

As will become apparent, the land requirement per house, as measured by parameter ψ , does not affect the long run wealth-to-GDP ratios.¹² The distinction between the enlargement of the housing stock along the extensive margin (that requires a fixed amount of land) and the intensive margin (that does not require land) allows us to treat the total available amount of land as a fixed (*Premise 1 - land endowment*) despite continuous depreciation of the housing stock (along the intensive margin). This feature provides a key advantage vis-à-vis the canonical approach that is important when it comes to the analysis of the long run evolution, as explained in detail in Section 5.

Land that has not been used in the process of real estate development is devoted to the alternative land use (land area Z^Y), such as office space and land associated with production sites, including agriculture, manufacturing except construction, and services except housing.

Construction and Housing Services (Intensive Margin) Producing housing services requires to combine a developed real estate with residential buildings (structures). The developed real estate represents the fixed factor, whereas structures represent the variable input in the production of housing services. As it becomes increasingly difficult to build houses upwards, it appears natural to assume that the production of housing services, at the level of single housing project, is characterized by decreasing returns to scale. Let x_t denote the amount of structures per housing project at time t . An amount x_t of structures produces housing services h_t per house according to

$$h_t = B_t^h(x_t)^\gamma, \quad (4)$$

¹²The appendix demonstrates that ψ also does not affect the labor share in total income, the economy's savings rate and the factor allocation in the long run.

$0 < \gamma < 1$, where $B_t^h > 0$ is a (possibly time-variant) productivity parameter. Total supply of housing services is $N_t h_t$. There are two admissible institutional settings describing the relationship between housing services producers and real estate developers. First, both activities are organized in the same firm. Second, a housing services producer buys a real estate at time t from real estate developers at the price q_t^N that equals the present value of future profits accruing from a housing project.

There is a representative construction firm producing structures that are rented out to the housing services producers. It combines materials M_t and labor L_t^X according to a constant-returns-to-scale technology. We assume that the production of one unit of construction materials (e.g., cement) requires p_t^M units of final output at time t . That is, the extraction of construction material is implicitly assumed to require capital, labor, and land with a similar technology as in the numeraire good sector. Technological progress in the use of materials means a decrease in p_t^M or an increase in $B_t^M \equiv 1/p_t^M$. Let $\delta^X > 0$ denote the depreciation rate of structures (residential buildings) and

$$I_t^X = B_t^X (M_t)^\eta (L_t^X)^{1-\eta} \quad (5)$$

gross investment in structures, $0 < \eta < 1$, where $B_t^X > 0$ is a (possibly time-variant) productivity parameter. The total stock of residential structures, denoted by X_t , therefore evolves according to

$$\dot{X}_t = I_t^X - \delta^X X_t = B_t^X (M_t)^\eta (L_t^X)^{1-\eta} - \delta^X X_t. \quad (6)$$

The associated resource constraint is $N_t x_t \leq X_t$.

2.1.3 Households

Preferences and Non-accumulable Endowments There is an infinitely living, representative household with intertemporal utility

$$U = \int_0^\infty (\log C_t + \theta \log S_t) e^{-\rho t} dt, \quad (7)$$

where C_t and S_t denote total consumption of the numeraire good and housing services at time t , respectively, $\rho > 0$ is the subjective discount rate, and $\theta > 0$ indicates the relative preference for housing services.¹³

The household inelastically supplies the remaining land that has not (yet) been purchased by the housing sector to the numeraire good sector. Initially, with total land size Z , these are $Z_0^Y = Z - \psi N_0 > 0$ units of land. Households also supply inelastically L_t units of labor at time t to a perfect labor market. The labor resource constraint is $L_t^X + L_t^Y \leq L_t$. We allow L_t to increase temporarily and assume that it remains constant in the long run.

Assets, Taxes, and Intertemporal Budget Constraint Households own the entire stock of financial assets, consisting of shares issued by housing services firms ($q_t^N N_t$), ownership claims on construction firms ($q_t^X X_t$), and bonds that provide firms in the numeraire sector with physical capital (K_t). The total asset holding, A_t , of the representative individual is thus given by

$$A_t = K_t + q_t^N N_t + q_t^X X_t. \quad (8)$$

Although initial values of stocks K_0 , N_0 and X_0 are given, total initial asset holding, A_0 , is endogenous because asset prices q_0^N and q_0^X are endogenous.

For calibrating our model, it is important to account for capital income taxation. The reason is that a tax on capital income affects the rate at which the profit stream of firms and land returns are discounted, for a given subjective discount rate of households. We assume that both capital income and returns from land ownership are proportionally taxed at rate τ_r , whereas labor income is taxed at proportional tax rate τ_w . For simplicity, we do not model government consumption or public investment and assume that the tax revenue is redistributed lump sum to households.

¹³Since the instantaneous utility function is a (monotonic transformation of a) linearly homogenous function, preferences are homothetic. Thus, the distribution of land and asset holdings will not play a role for the evolution of aggregates in the economy. That is, there exists a (positive and normative) representative consumer. Although not the focus of this paper, our framework could therefore be employed to address the personal income distribution in addition to the functional income distribution (Caselli and Ventura, 2000).

Let w_t , r_t , R_t^Z , p_t , and T_t denote the wage rate, the interest rate, the rental rate of land, the (relative) price per unit of housing services, and the lump-sum transfer at time t , respectively. The intertemporal household budget constraint may then be expressed as

$$\dot{A}_t = (1 - \tau_r)(r_t A_t + R_t^Z Z_t^Y) + (1 - \tau_w)w_t L_t - C_t - p_t S_t + T_t, \quad (9)$$

where the value of initial assets is given by $A_0 = K_0 + q_0^N N_0 + q_0^X X_0$.

3 Equilibrium Analysis

This section highlights important equilibrium implications that result from the decisions of firms in the housing sector, defines the general equilibrium and characterizes the economy's steady state.

3.1 Decisions in the Housing Sector and Asset Prices

Let the rental rate of structures at time t be denoted by R_t^X . The instantaneous profit at time t resulting from a housing project on a developed real estate that accrues to *housing services producers*, noting production function (4), depends on the amount of employed structures and is given by $\pi_t \equiv p_t B_t^h(x_t)^\gamma - R_t^X x_t$. The necessary first order condition for profit maximization yields the inverse demand schedule for structures per house

$$R_t^X = p_t B_t^h(x_t)^{\gamma-1}, \quad (10)$$

implying positive profits that amount to

$$\pi_t = (1 - \gamma)p_t B_t^h(x_t)^\gamma. \quad (11)$$

At time t , the *representative real estate developer* maximizes the present discounted value (PDV) of housing services producers' profit stream, $N\pi$, minus real estate devel-

opment costs, \mathcal{C} , i.e. solves

$$\max_{\{\tilde{I}_s^N\}_{s=t}^{\infty}} \int_t^{\infty} \left(\pi_s N_s - \mathcal{C}(\tilde{I}_s^N; P_s^Z) \right) e^{\int_t^s -r_v dv} ds \quad \text{s.t.} \quad \dot{N}_s = \tilde{I}_s^N, \quad (3), \quad N_0 \geq 0 \text{ given.} \quad (12)$$

The shadow price per house associated with constraint (2) is equal to the stock market evaluation of a real estate, q_t^N .

Lemma 1 *The stock of houses and the stock market valuation of a real estate evolve according to*

$$\dot{N}_t = \frac{q_t^N - \psi P_t^Z}{\xi}, \quad (13)$$

$$\frac{\dot{q}_t^N}{q_t^N} + \frac{\pi_t}{q_t^N} = r_t. \quad (14)$$

All formal results are proven in Appendix. According to (13), if the value per house is sufficiently large relative to the land price (i.e. if $q_t^N/P_t^Z > \psi$), the number of houses is being enlarged, i.e., $\dot{N}_t > 0$. (14) is a no-arbitrage condition. It says that the sum of the growth rate in the share price (capital gains) and the dividend per share paid to the owners of real estate firms must equal the rate of return to bonds. In other words, a household has to be indifferent whether to invest in real estates or to hold bonds.¹⁴ Ruling out bubbles by imposing an appropriate endpoint condition, the stock market valuation of a real estate is given by the PDV of the profit stream:

$$q_t^N = \int_t^{\infty} \pi_s e^{\int_t^s -r_v dv} ds. \quad (15)$$

Construction firms rent the entire stock of structures to housing services producers by charging R_t^X per unit of structures. The *representative construction firm* maximizes the PDV of the cash flow, defined as the difference between rental income $R^X X$ and the costs of gross investment, $p^M M + wL^X$. That is, the representative construction firm solves

$$\max_{\{M_s, L_s^X\}_{s=t}^{\infty}} \int_t^{\infty} \left(R_s^X X_s - p_s^M M_s - w_s L_s^X \right) e^{\int_t^s -r_v dv} ds \quad \text{s.t.} \quad (6), \quad X_0 \geq 0 \text{ given.} \quad (16)$$

¹⁴Since all kinds of capital income are taxed at the same rate, τ_r does not enter (14).

Denote the shadow price per unit of X associated with constraint (6) by q_t^X .

Lemma 2 *The shadow price per unit of structures evolves according to*

$$\frac{\dot{q}_t^X}{q_t^X} + \frac{R_t^X}{q_t^X} - \delta^X = r_t. \quad (17)$$

Thus, ruling out bubbles by imposing an appropriate endpoint condition, the value of one unit of X equals the PDV of future rental returns, accounting for the depreciation rate as component of the discount rate, i.e.,

$$q_t^X = \int_t^\infty R_s^X e^{\int_t^s -(r_v + \delta^X) dv} ds. \quad (18)$$

Due to the constant returns to scale technology (5) and perfectly competitive markets, the value of total gross output in the construction sector must equal the total factor costs in construction, i.e.,

$$q_t^X I_t^X = p_t^M M_t + w_t L_t^X. \quad (19)$$

Finally, the price of land, P_t^Z , equals the PDV of income from renting one unit of land to the producers in the Y -sector. It reads as $P_t^Z = \int_t^\infty R_\tau^Z e^{\int_t^\tau -r_v dv} ds$, implying that the following no-arbitrage condition must hold:

$$\frac{\dot{P}_t^Z}{P_t^Z} + \frac{R_t^Z}{P_t^Z} = r_t. \quad (20)$$

3.2 Definition of Equilibrium

Definition 1. *A general equilibrium is a sequence of quantities, a sequence of prices, and a sequence of operating profits of housing services producers*

$$\begin{aligned} & \{Y_t, K_t^Y, X_t, N_t, x_t, h_t, M_t, L_t^Y, L_t^X, Z_t^Y, C_t, S_t, A_t\}_{t=0}^\infty, \\ & \{p_t, P_t^Z, q_t^N, q_t^X, w_t, r_t, R_t^Z, R_t^X\}_{t=0}^\infty, \{\pi_t\}_{t=0}^\infty \end{aligned}$$

for initial conditions $K_0 > 0$, $N_0 > 0$, $X_0 > 0$, such that

1. *the representative individual maximizes lifetime utility, i.e. solves*

$$\max_{\{C_t, S_t\}_{t=0}^{\infty}} \int_0^{\infty} (\log C_t + \theta \log S_t) e^{-\rho t} dt \quad \text{s.t. (9), } \lim_{t \rightarrow \infty} A_t \exp \left(- \int_0^t (1 - \tau_r) r_s ds \right) \geq 0; \quad (21)$$

2. *the representative firm in the numeraire goods (Y-) sector maximizes profits taking factor prices as given (i.e., factor prices equal marginal products);*
3. *the representative real estate developer solves profit maximization problem (12), taking the sequence of land prices $\{P_t^Z\}_{t=0}^{\infty}$ as given;*
4. *housing services producers maximize profits at each time t , taking the price of housing services, p_t , and the rental rate of structures, R_t^X , as given;*
5. *the representative firm in the construction (X-) sector solves profit maximization problem (16), taking the sequences of rental rate of structures $\{R_t^X\}_{t=0}^{\infty}$ and wage rates $\{w_t\}_{t=0}^{\infty}$ as given;*
6. *there are no arbitrage possibilities to use land, i.e. (20) holds;*
7. *the bond market, the land market, the market for structures clear and the land market clear at any t , i.e.,*

$$K_t^Y = K_t, \quad (22)$$

$$Z_t^Y = Z - \psi N_t, \quad (23)$$

$$N_t x_t = X_t, \quad (24)$$

$$L_t^X + L_t^Y = L_t; \quad (25)$$

8. *the financial asset market clears at any t , i.e. (8) holds;*
9. *the market for housing services clears at any t , i.e.,*

$$S_t = N_t h_t = N_t B^h(x_t)^\gamma; \quad (26)$$

10. the market for the numeraire good clears at any t , i.e.,¹⁵

$$Y_t = C_t + I_t^K + I_t^N + p_t^M M_t. \quad (27)$$

3.3 Aggregates

The gross domestic product (GDP) is given by the value of total consumption, $C_t + p_t S_t$, plus the value of total investment, $I_t^K + I_t^N + q_t^X I_t^X$, i.e.,¹⁶

$$GDP_t \equiv C_t + p_t S_t + I_t^K + I_t^N + q_t^X I_t^X. \quad (28)$$

Using (19), (26) and (27) gives us

$$GDP_t = Y_t + p_t N_t h_t + w_t L_t^X. \quad (29)$$

The house price is conceptualized as the sum of the value of a housing project (q_t^N) and the value of the employed structure associated with a real estate (valued at $q_t^X x_t$), i.e.,

$$P_t^H \equiv q_t^N + q_t^X x_t. \quad (30)$$

Total private wealth, W_t , is the sum of asset holdings, A_t , and the value of non-residential land $P_t^Z Z_t^Y$, i.e., $W_t \equiv A_t + P_t^Z Z_t^Y$. Noting (8) and (30) total private wealth may be expressed as

$$W_t = P_t^H N_t + K_t + P_t^Z Z_t^Y, \quad (31)$$

where $P_t^H N_t$ represents housing capital and $K_t + P_t^Z Z_t^Y$ captures non-residential wealth.

We are especially interested in the evolution of the housing wealth-to-GDP ratio, as displayed in Fig. 1. We use fracture scripture to denote wealth-to-GDP ratios. The housing wealth-to-GDP ratio is denoted by \mathfrak{H}_t (speak "fracture H ") and may be expressed

¹⁵Equilibrium condition 10 is redundant, according to Walras' law. To exclude conceptual or calculation errors, we prove, in Online-Appendix A, that the long run equilibrium derived from conditions 1-9 fulfill condition 10.

¹⁶In the long run, where $\dot{A} = 0$ and thus $C + pS = rA + wL + R^Z Z^Y$, according to (9), we have $rA + wL + R^Z Z^Y = GDP - \delta^K K - q^X \delta^X X$. Hence, total factor income is equal to the net domestic product (i.e. GDP minus depreciation).

as

$$\mathfrak{H}_t \equiv \frac{P_t^H N_t}{GDP_t} = \frac{q_t^N N_t + q_t^X X_t}{GDP_t} = \frac{A_t - K_t}{GDP_t}. \quad (32)$$

The non-residential wealth-to-GDP ratio is denoted by \mathfrak{N}_t ("fracture N ") and is the sum of the ratio of physical capital to income, $\mathfrak{K}_t \equiv K_t/GDP_t$, and the ratio of the value of non-residential land (farm land and other productive, non-residential land property) to income ("non-residential land wealth-to-GDP ratio"), $\mathfrak{Z}_t \equiv P_t^Z Z_t^Y / GDP_t$, i.e.,

$$\mathfrak{N}_t \equiv \frac{K_t + P_t^Z Z_t^Y}{GDP_t} = \mathfrak{K}_t + \mathfrak{Z}_t. \quad (33)$$

The total (private) wealth-to-GDP ratio is denoted \mathfrak{W}_t ("fracture W ") and reads as

$$\mathfrak{W}_t \equiv \frac{W_t}{GDP_t} = \mathfrak{H}_t + \mathfrak{N}_t. \quad (34)$$

3.4 The Long Run Equilibrium

We now derive analytical results for the long run equilibrium. Denote population density at time t as $D_t \equiv L_t/Z$, which is supposed to be time-invariant in the long run. Also suppose the vector of productivity parameters, $\mathbf{B}_t \equiv (B_t^Y, B_t^h, B_t^X, B_t^M)$, does not change in the long run. In this case, any long run equilibrium is characterized by zero growth rates of all stock variables. Superscript (*) denotes long run equilibrium values.

Proposition 1. (Existence of long run equilibrium) *Suppose that*

$$\varrho \equiv \frac{\beta}{1-\alpha} - \frac{\gamma(1-\eta)}{1-\gamma\eta} > 0 \quad (A1)$$

holds. Then there exists a unique, non-trivial long run equilibrium in which

$$\frac{L^{Y*}}{Z^{Y*}} > D. \quad (35)$$

Assumption (A1) holds for any reasonable calibration of the model. The allocation of labor and land is characterized by fractions $l^X \equiv L^X/L$, $l^Y \equiv L^Y/L$, $\mathfrak{z}^N \equiv Z^N/Z$, and $\mathfrak{z}^Y \equiv Z^Y/Z$. Equilibrium property (35) is equivalent to $\mathfrak{z}^{Y*} < l^{Y*}$ and $\mathfrak{z}^{N*} > l^{X*}$, which

means that in the housing sector the fraction of land is larger than the fraction of labor whereas the opposite holds in the rest of the economy. In this sense, the housing sector is endogenously land-intensive, whereas the rest of the economy is labor-intensive.

Proposition 2. (Long run prices). *Under (A1), in long run equilibrium,*

(i) *the price for housing services, p^* , is decreasing in B^X , B^h , B^M , increasing in both D and B^Y , and independent of ψ ;*

(ii) *the wage rate, w^* , is decreasing in D , independent of B^X , B^h , B^M , ψ , and increasing in B^Y ;*

(iii) *the return to land, R^{Z^*} , and the land price, P^{Z^*} are increasing in both D and B^Y , and independent of B^X , B^h , B^M , ψ ;*

(iv) *The long run interest rate reads as*

$$r^* = \frac{\rho}{1 - \tau_r}. \quad (36)$$

An increase in population density, D , means that labor becomes more abundant and land becomes scarcer, in turn lowering the wage rate and raising both the return per unit of land and its price. Consequently, since the housing sector is land-intensive, the price for housing services rises with D . An increase in TFP of the Y -sector, B^Y , raises output of the numeraire for a given factor allocation, thus raising the relative long run price for housing services, p^* . It also transmits into higher (long run) factor returns, w^* , R^{Z^*} , like in standard one-sector models. Higher productivity parameters in the housing sector (increase in B^X , B^h , or B^M) lowers the price of housing services. Consequently, rather than transmitting into higher long run factor prices, w^* and R^{Z^*} remain unchanged. Finally, the long run interest rate is determined by the Keynes-Ramsey rule that implies for the long run that the after-tax interest rate equals the subjective discount rate.

Proposition 3. (Long run wealth-to-GDP ratios). *Under (A1), in long run equilibrium, the housing wealth-to-GDP ratio is*

$$\mathfrak{H}^* = \frac{P^{H^*} N^*}{GDP^*} = \frac{1 + \frac{(1-\gamma)(1-\tau_r)\delta^X}{\rho}}{\frac{\rho}{1-\tau_r} + \left(\frac{\frac{\rho}{1-\tau_r} + \delta^X}{\theta} + \eta\gamma\delta^X \right) \frac{\frac{\rho}{1-\tau_r} + \delta^K}{\frac{\rho}{1-\tau_r} + (1-\alpha)\delta^K} + (1 + (1-\eta)\gamma)\delta^X}, \quad (37)$$

the physical-capital-to-GDP ratio is

$$\mathfrak{K}^* = \frac{K^*}{GDP^*} = \mathfrak{K}^* = \alpha \left(\frac{\rho}{1 - \tau_r} + \delta^K + \frac{\left(\frac{\rho}{1 - \tau_r} + \delta^X (1 + (1 - \eta)\gamma) \right) \left(\frac{\rho}{1 - \tau_r} + (1 - \alpha)\delta^K \right)}{\frac{\rho}{1 - \tau_r} + \delta^X + \eta\gamma\delta^X} \right)^{-1}, \quad (38)$$

the non-residential land wealth-to-GDP ratio is

$$\mathfrak{Z}^* = \frac{PZ^* ZY^*}{GDP^*} = \frac{(1 - \alpha - \beta)(1 - \tau_r)}{\rho} \left(1 + \frac{\frac{\rho}{1 - \tau_r} + (1 - \alpha)\delta^K}{\frac{\rho}{1 - \tau_r} + \delta^K} \frac{\frac{\rho}{1 - \tau_r} + \delta^X + (1 - \eta)\gamma\delta^X}{\frac{\rho}{1 - \tau_r} + \delta^X + \eta\gamma\delta^X} \right)^{-1}, \quad (39)$$

Thus, \mathfrak{H}^* , \mathfrak{K}^* , and \mathfrak{Z}^* are independent of D , \mathbf{B} , ψ .

Corollary 1. Under (A1), in long run equilibrium, the non-residential wealth-to-GDP ratio, $\mathfrak{N}^* = \mathfrak{K}^* + \mathfrak{Z}^*$, and the total wealth-to-GDP ratio, $\mathfrak{W}^* = \mathfrak{H}^* + \mathfrak{N}^*$, are independent of D , \mathbf{B} , ψ .

Proposition 3 and Corollary 1 reflect that the long run factor allocation is independent of population density, D , and technology parameters, \mathbf{B} and ψ . An increase in population density, D , changes the marginal product of labor and the return to land equally in the housing sector and the Y -sector, leaving the factor allocation unchanged. Moreover, reflecting the assumption of homothetic preferences, changes in technological parameters \mathbf{B} and ψ do not induce structural change and leave the factor allocation unaffected as well.¹⁷

By calibrating the model, we are able to quantify the long run wealth ratios, as discussed by Piketty (2014) and Piketty and Zucman (2014a, 2015), from a unifying theoretical model of different wealth components.

¹⁷That the long land allocation is unaffected by population density means that more structures are built per unit of land. That is, houses become "higher" rather than more numerous in more densely populated and in more advanced regions, reflecting the opportunity costs of land in its alternative use in the Y -sector. See Proposition A.1 in Online-Appendix A.

4 Numerical Analysis

We first calibrate our model to the US. The calibration strategy *does not* assume that the US currently *is* in long run equilibrium. Subsection 4.1 focuses on those parameters that shall be similar in other advanced countries and that, together with the capital income tax rate, fully determine housing wealth-to-GDP ratio, \mathfrak{H}_t , and the non-residential wealth-to-GDP ratio, \mathfrak{N}_t .¹⁸ Subsection 4.2 reports the quantitative long run results. Subsection 4.3 describes, according to the House-Kapital Model, the evolution of \mathfrak{H}_t and \mathfrak{N}_t as well as the evolution of land prices and house prices.

4.1 Calibration

Optimal household decisions imply that at any time t the marginal rate of substitution between the two consumption goods is equal the relative price, $pS/C = \theta$ (see also the proof of Proposition 1). We employ data on household consumption expenditure on housing provided by Knoll et al. (2014). The data suggests that the ratio of households' expenditure on housing to total consumption expenditures,

$$\frac{pS}{C + pS} = \frac{\theta}{1 + \theta}, \quad (40)$$

in the US is quite stable over time for the period 1960-2012 and equals, on average, about 18 percent. The value is very close to the average values for the UK, France and Germany. Setting the expression in (40) to 0.18 suggests $\theta = 0.22$.

We next calibrate the depreciation rates. For the housing sector (residential structures), Hornstein (2009, p. 13) suggests, by referring to data from the US Bureau of Economic Analysis (2004), that $\delta^X = 0.015$. The depreciation rate of physical capital, δ^K , can be inferred from the definition of gross investment in physical capital, $I^K = \dot{K} + \delta^K K$, and the associated investment rate, $\mathfrak{s}^K \equiv I^K/GDP$. Also using $\mathfrak{K} = K/GDP$, we have

$$\delta^K = \frac{I^K}{K} - \frac{\dot{K}}{K} = \frac{\mathfrak{s}^K}{\mathfrak{K}} - \frac{\dot{K}}{K}. \quad (41)$$

¹⁸As a side product, we also obtain the long run values of income shares, sectoral allocation shares for labor and land, and investment rates. These are reported in the Online-Appendix, Table A.1.

For the period 1969-2014, the average ratio of US non-residential investment to GDP, which we use as measure for \mathfrak{s}^K , was 12.6 percent, about the average value of the 1970s, 1980s, 1990s and 2000s (Bureau of Economic Analysis, 2015a, Tab. 1.1.10). To measure the physical-capital-to-GDP ratio, $\mathfrak{K} = \mathfrak{N} - \mathfrak{Z}$, we first calculate the non-residential wealth-to-GDP ratio, \mathfrak{N} , from the dataset provided by Piketty and Zucman (2014b), and find that \mathfrak{N} was 209 percent (on average) in the period 1950-1969. Unfortunately, we cannot readily observe the non-residential land wealth-to-GDP ratio, \mathfrak{Z} . Piketty and Zucman (2014b, Tab. US.6c, column [6]) report that the value of farm land as a fraction of GDP was, on average, 20 percent in the 1970s. We suppose that \mathfrak{Z} was 80 percent, which means that the value of farm land was about one quarter of the value of total productive land not used for residential purposes. This suggests that $\mathfrak{K} = \mathfrak{N} - \mathfrak{Z} = 1.29$ and thus $I^K/K = \mathfrak{s}^K/\mathfrak{K} = 12.6/129$ is about 10 percent. Assuming that the average annual growth rate of physical capital, \dot{K}/K , was about three percent (the sum of the long term GDP per capita growth rate of two percent and the population growth rate of one percent), we approximately arrive at $\delta^K = 0.07$, according to (41).¹⁹

From the first-order conditions of the Y -sector (see (69) and (70) in Appendix A), we have $\alpha = (r + \delta^K)K/Y$ and $\beta = wL^Y/Y$. Define $H \equiv pNh + wL^X$ as the value-added to GDP contributed by the housing sector (housing services and residential construction). According to (29), we can thus write $GDP = Y + H$, i.e. $Y/GDP = 1 - H/GDP$. This gives us

$$\alpha = \frac{(r + \delta^K)\mathfrak{K}}{1 - \frac{H}{GDP}}. \quad (42)$$

Similarly, using $L^Y/L = 1 - l^X$, $Y = GDP - H$, and denoting the labor share $\mathfrak{L} \equiv wL/GDP$, we get

$$\beta = \frac{(1 - l^X)\mathfrak{L}}{1 - \frac{H}{GDP}}. \quad (43)$$

According to Bureau of Economic Analysis (2015b), the average value-added of housing and (residential and non-residential) construction as percentage of GDP in the period 1998-2001 (prior to the housing boom) in the US was, on average, 9.1 percent and 4.4 percent, respectively. With respect to the second component, we approximate the value-

¹⁹The value also seems reasonable according to the evidence on depreciation rates for 36 manufacturing sectors in House and Shapiro (2008).

added of residential construction as fraction of GDP, in line with our model. The ratio of residential to total investment in structures in the period 1998-2001 was 61 percent (calculated from Bureau of Economic Analysis, 2015c), which gives us a value-added of residential construction to GDP, $q^X I^X / GDP$, of $0.61 \times 4.4 = 2.7$ percent. Thus, we set H/GDP to $9.1 + 2.7 \approx 12$ percent. Hypothesizing that the real interest rate was around seven percent in the 1970s, consistent with the long-term average real interest rate (including the equity premium) suggested by Mehra and Prescott (1985), recalling $\delta^K = 0.07$ and again using $\mathfrak{R} = 1.29$, this suggests $\alpha = 0.14 \times 1.29 / (1 - 0.12) = 0.21$, according to (42). According to Henderson (2015, Tab. 2.1), the US employment share in construction decreased from 4.8 percent in 2004 to 4.1 percent in 2014. Taking an intermediate value of 4.5 percent and multiplying it by the fraction of residential investment in total investment in structures (61 percent), we arrive at $l^X = 0.045 \times 0.61 = 0.027$. According to Karabarbounis and Neiman (2014, "CLS KN merged"), the corporate US labor share in total income was pretty stable in the period 1975-2008, only recently declining in a more pronounced way. The average value for the period 1975-2012 was 62 percent. Using $1 - H/GDP = 0.88$ and $l^X = 0.027$ we thus set $\beta = (1 - 0.027) \times 0.62 / 0.88 = 0.69$, according to (43). Therefore, $R^Z Z^Y / Y = 1 - \alpha - \beta = 0.1$.

According to (4), (11) and (26), the ratio of housing services producers' profits to their revenue is $\pi / (ph) = 1 - \gamma$. Unfortunately, we know little about the profit-to-revenue ratio in the housing services sector. We assume that it is equal to 10 percent, i.e. set $\gamma = 0.9$. We checked that our results are rather insensitive with respect to reasonable variations of γ .

We next address the output elasticity of materials in the construction sector, η . From the first-order condition with respect to labor in the construction sector (see (65) in Appendix A and use (5)), we have $wL^X = (1 - \eta)q^X I^X$. Using $wL^X = l^X \mathfrak{L} GDP$, we obtain

$$\eta = 1 - \frac{wL^X}{q^X I^X} = 1 - \frac{l^X \mathfrak{L}}{q^X I^X / GDP}. \quad (44)$$

Recalling $q^X I^X / GDP = 0.027$, $\mathfrak{L} = 0.62$, and $l^X = 0.027$, we obtain $\eta = 0.38$.²⁰

²⁰The amount to land per house, ψ , does not affect the long run factor allocation, factor prices, wealth-to-GDP ratios or income shares (see appendix). For transitional dynamics, we set the arbitrary value of $\psi = 1$. Similarly, the parameter that captures the importance of residential land development costs ξ

The long run interest rate, r^* , as given by (36), plays an important role for long run asset prices, including the land price. Clearly, asset prices approaches infinity if the subjective discount rate, ρ , goes to zero. As the return to equity contains a risk premium and land or firm ownership is at risk (albeit not modelled here explicitly) of expropriation by government action or devaluation because of natural disaster or environmental damage, we shall set the subjective discount rate at a value that is in the middle or at the upper end of the typical range used in calibration exercises. We compare results for $\rho = 0.02$ vis-à-vis $\rho = 0.03$.

Finally, we turn to policy instruments. The tax on wage income is purely redistributive and has no incentive effects. Thus, it does not enter the dynamical system (summarized in Online-Appendix A). To calibrate the capital income tax rate, τ_r , we take from the OECD (2015) tax database the net top statutory dividend tax rate "to be paid at the shareholder level, taking account of all types of reliefs and gross-up provisions at the shareholder level". For the year 2015, this gives us $\tau_r = 0.3$ for the US and the UK, $\tau_r = 0.26$ for Germany, and $\tau_r = 0.44$ for France.

4.2 Long Run Implications

Table 1 shows (in percent) the long run implications for wealth-to-GDP ratios as resulting from the calibrated House-Kapital Model, assuming alternative subjective discount rates, ρ , and capital income tax rates, τ_r .²¹

The annual average of the housing wealth-to-GDP ratio, \mathfrak{H} , in the 2000s was 217 percent in the US with a peak of 254 percent in 2006 before the financial crises. In the UK, \mathfrak{H} was 271 percent in the 2000s.²² The calibrated model under $\tau_r = 0.3$ implies the long run value, \mathfrak{H}^* , to be 276 percent for $\rho = 0.03$ and 362 percent for $\rho = 0.02$. Our analysis thus suggests that the US housing capital rises above the pre-crisis level and converges to the UK level if the capital income tax rates remain similar. The implied

does not have an impact on the long-run equilibrium either. It does, however, impact on the speed of N -dynamics along the transition. We set $\xi = 40$.

²¹In addition, Table A.1 in Online-Appendix A displays the long run values of income shares (\mathfrak{L}^* , $r^*\mathfrak{H}^*$, $r^*\mathfrak{N}^*$), sectoral allocation variables (l^{X^*} , \mathfrak{z}^{N^*}), as well as the total investment rate, $\mathfrak{s}^* \equiv (I^{K^*} + I^{N^*} + q^{X^*}I^{X^*})/GDP^*$ and its components $\mathfrak{s}^{K^*} = I^{K^*}/GDP^*$ and $\mathfrak{s}^{H^*} \equiv \mathfrak{s}^* - \mathfrak{s}^{K^*}$.

²²With respect to stylized facts on wealth-to-income ratios, we again refer to the data provided by Piketty and Zucman (2014b) in the remainder of the paper.

non-residential wealth-to-GDP ratio is $\mathfrak{N}^* = 345$ percent for $\rho = 0.03$ and to $\mathfrak{N}^* = 462$ percent for $\rho = 0.02$. About two fifth is due to physical capital and three fifth to non-residential land wealth. In the 2000s, \mathfrak{N} was 249 percent in the US and 241 percent in the UK. In total, if the current tax system remains in place, the implied wealth-to-GDP ratio, \mathfrak{W}^* , in the US and UK is in the range from 622 to 824 percent in the long run, depending on the subjective discount rate, ρ . The sensitivity of long run wealth-to-GDP ratios to ρ is rooted in the the fact that the PDV of asset values, particularly land that does not depreciate, is heavily dependent on the rate at which future returns are discounted.

ρ	τ_r	\mathfrak{H}^*	\mathfrak{K}^*	\mathfrak{Z}^*	$\bar{\mathfrak{Z}}^*$	\mathfrak{N}^*	\mathfrak{W}^*	$\bar{\mathfrak{W}}^*$
0.02	0.26	375	178	303	94	481	856	647
0.03	0.26	287	156	203	63	359	647	506
0.02	0.3	362	175	287	89	462	824	626
0.03	0.3	276	153	192	60	345	622	489
0.02	0.44	313	163	231	72	394	707	548
0.03	0.44	235	140	154	48	294	529	423

Table 1. Long run implications for wealth-to-GDP ratios (\mathfrak{H}^* , \mathfrak{K}^* , \mathfrak{N}^* , \mathfrak{W}^*).

Notes: All endogenous values in percent. In addition to the displayed values for ρ and τ_r , results are based on calibration $\alpha = 0.21$, $\beta = 0.69$, $\gamma = 0.9$, $\eta = 0.38$, $\theta = 0.22$, $\delta^X = 0.015$, $\delta^K = 0.07$. Recall $\mathfrak{H}^* = \frac{PH^*N^*}{GDP^*}$, $\mathfrak{K}^* = \frac{K^*}{GDP^*}$, $\mathfrak{Z}^* = \frac{PZ^*Z^*}{GDP^*}$, $\mathfrak{N}^* = \mathfrak{K}^* + \mathfrak{Z}^*$, $\mathfrak{W}^* = \frac{W^*}{GDP^*} = \mathfrak{H}^* + \mathfrak{N}^*$, $\bar{\mathfrak{Z}}^* = \kappa \mathfrak{Z}^*$ and $\bar{\mathfrak{W}}^* \equiv \mathfrak{H}^* + \mathfrak{K}^* + \kappa \mathfrak{Z}^*$ with land price correction factor $\kappa = 0.31$ as explained in Section 4.3.

We now consider the sensitivity of wealth-to-income ratios with respect to capital income taxation. We start with lowering τ_r to 26 percent, the current value in Germany. As the long run interest rate, r^* , is slightly reduced, the implied wealth-to-GDP ratios increase somewhat and the investment rates slightly rise. The net capital shares slightly decrease and the labor share remains unchanged. The fractions of labor and land in the housing sector decrease marginally. Whereas the German housing wealth-to-GDP ratio \mathfrak{H} was 217 percent in the 2000s, the calibrated model suggests a range of 287 to

375 percent in the long run. With respect to the non-residential wealth-to-GDP ratio, Germany seems in recent times far away from our long run predictions, with a value for \mathfrak{N} of just 139 percent in the 2000s. This discrepancy may reflect the considerably higher taxation of capital income that took place in Germany in the past.

Raising τ_r to 44 percent, the current value in France, the impact on income shares, the factor allocation and investment rates is marginal compared to the case where $\tau_r = 0.3$, whereas the wealth-to-GDP ratios decreases somewhat. In the 2000s, the French housing wealth-to-GDP ratio, \mathfrak{H} , was 285 percent which is in the range (between 235 and 313 percent) for the implied value of \mathfrak{H}^* . Again, like for the other countries, the implied non-residential wealth-to-GDP ratio is far from the current value. For France in the 2000s, \mathfrak{N} was just 190 percent, whereas the calibrated model implies for \mathfrak{N}^* a range from 294 to 394 percent. The reason is primarily the high implied level of the long run non-residential land wealth-to-GDP ratio, \mathfrak{Z}^* , reflecting the growing importance of land scarcity in the process of economic development and, as discussed in some detail in the next subsection, the fact that we do not distinguish rural and urban land.

4.3 Transitional Dynamics Post WWII

The calibrated model is employed to discuss the empirical time paths of the housing wealth-to-GDP ratio, $\mathfrak{H}_t = P_t^H N_t / GDP_t$, and the non-residential wealth-to-GDP ratio, $\mathfrak{N}_t = (K_t + P_t^Z Z_t^Y) / GDP_t$, as shown in Figure 1. The model generates transitory growth, that may extend over several centuries, through exogenous population growth, exogenous technical progress, and endogenous accumulation of various capital components. We specifically let the population size, L_t , increase smoothly over time (according to a logistic function) by a factor of three and let B_t^Y and B_t^X increase smoothly (according to a logistic function) by a factor of four.

4.3.1 The Evolution of Wealth-to-Income Ratios

With regard to state variables, we assume that the capital stock K_t , the number of housing projects N_t and the total amount of residential buildings X_t start below their respective

initial steady state values.²³ The set of parameters is as described in Section 4.1. For the subjective discount rate and the capital income tax rate, we assume intermediate values, $\rho = 0.025$ and $\tau_r = 0.3$.

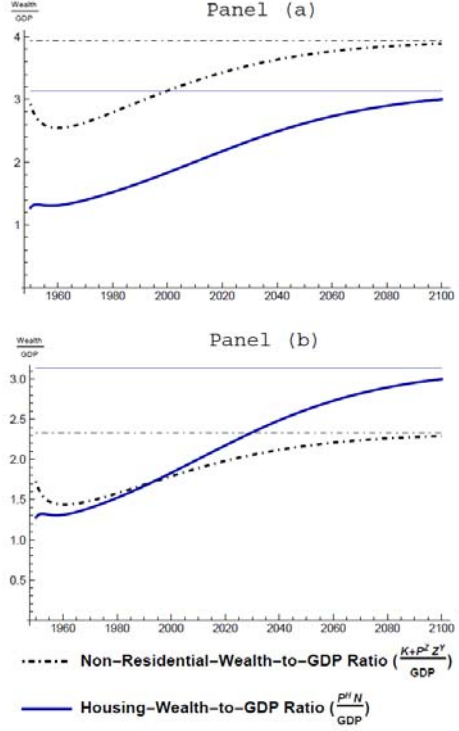


Figure 2. Evolution of housing wealth-to-GDP ratio (solid, blue lines) and non-residential wealth-to-GDP ratio (dashed, black lines) as resulting from the calibrated House-Kapital model.

Notes: Panel (a) and (b): K_t starts at 70 percent, while N_t and X_t start at 40 percent of their (initial) steady states (i.e. $K_0 = 0.7K_0^*$ etc.); L_t increases threefold and B_t^Y and B_t^X increase fourfold over time according to logistic functions; $B^h = B^M$, $\rho = 0.025$, $\tau_r = 0.3$, and other parameters as in Table 1 with $\psi = 1$ and $\xi = 40$. Panel (b): The land price P_t^Z is adjusted according to $\bar{P}_t^Z = \kappa P_t^Z$ as explained in the main text.

²³These initial state variables are specified as percentage of the initial long run equilibrium values, K_0^* , X_0^* , N_0^* , that would result for initial population size and productivity levels, L_0 , B_0^Y , B_0^X . This assumption appears especially plausible for European economies shortly after World War II.

Figure 2 displays the implied evolution of the housing wealth-to-GDP ratio, $\mathfrak{H}_t = P_t^H N_t / GDP_t$, and the non-residential wealth-to-GDP ratio, $\mathfrak{N}_t = (K_t + P_t^Z Z_t^Y) / GDP_t$, when defining the initial period as year 1950. The solid (blue) line demonstrates that housing wealth as fraction of GDP, \mathfrak{H}_t , starts at about 100 percent and nearly doubles between 1950 and 2010. The value of about 200 percent roughly matches the values in Germany and the US in 2010. The overall pattern of \mathfrak{H}_t therefore is broadly compatible with the empirical picture. Figure 2 (a) also suggests that \mathfrak{H}_t will continue to increase further until it approaches its long run value of about $\mathfrak{H}^* = 3.2$.

The dashed (black) line in Figure 2 (a) shows the evolution of non-residential wealth, relative to GDP, \mathfrak{N}_t . One striking pattern is the moderate U-shaped evolution that can also be recognized in the empirical post World War II series in Figure 1. The reason is that economic growth affects at first GDP (the denominator of the wealth-to-GDP ratios). The accumulation of physical capital as well as the rising valuation of land, however, lack behind the GDP development such that the numerator of the wealth-to-GDP ratio responds with some delay. The overall increase of \mathfrak{N}_t (about 60 percentage points between 1960 and 2010) and the U-shaped evolution capture important aspects of the empirical series.

However, the (initial) level of non-residential wealth, \mathfrak{N}_t , in Figure 2 (a) is obviously higher compared to the empirical series, shown in Figure 1. This implication is due the simplifying assumption that land is homogenous. A simple and conceptually straightforward possibility to discuss this issue is as follows. The House-Kapital Model considers one single region that hosts both production and housing. This region should be interpreted as urban, given that about 70 to 80 percent of the population in advanced countries lives in cities. The House-Kapital Model therefore suffers from a land valuation bias: It values the entire non-residential land, which comprises non-residential urban and non-residential rural land in reality, at the urban land price.²⁴ The same criticism applies, of course, to any other one-regional macro model with production and housing. As a result, non-residential wealth is overestimated due to an overvaluation of (non-residential) land, $P^Z Z^Y$. To be more precise, let us think in terms of a model economy that encompasses

²⁴The issue of land heterogeneity and associated land price differentials has already been brought up in the context of macroeconomics and housing by Sachs and Boone (1988).

two regions: one rural and one urban region. The value of land in this economy would be given by

$$\bar{P}^Z Z^Y = P_U^Z Z_U^Y + P_R^Z Z_R^Y, \quad (45)$$

where \bar{P}^Z denotes the average non-residential land price, P_U^Z the urban land price, P_R^Z the rural land price, Z_U^Y the quantity of urban land, and Z_R^Y the quantity of rural land. From the perspective of this two-region model, the House-Kapital Model makes the following mistake: It values rural (non-residential) land at the urban land price, i.e.,

$$P^Z Z^Y = P_U^Z Z_U^Y + \underbrace{P_U^Z}_{!} Z_R^Y. \quad (46)$$

When comparing model results to empirical data taken from wealth statistics, it seems appropriate to value overall non-residential land, Z^Y , at the average (across rural and urban regions) land price \bar{P}^Z instead of P^Z . Is there a systematic relationship between \bar{P}^Z and P^Z such that we can correct for this valuation bias?²⁵ Dividing (45) by (46) yields

$$\frac{\bar{P}^Z}{P^Z} = \frac{P_U^Z Z_U^Y}{P_U^Z Z_U^Y + P_U^Z Z_R^Y} + \frac{P_R^Z Z_R^Y}{P_U^Z Z_U^Y + P_U^Z Z_R^Y} \quad (47)$$

$$= \frac{1}{1 + \frac{Z_R^Y}{Z_U^Y}} + \frac{1}{\frac{P_U^Z Z_U^Y}{P_R^Z Z_R^Y} + \frac{P_U^Z}{P_R^Z}} \quad (48)$$

Hence, the average land price \bar{P}^Z , that does not suffer from the rural-urban bias, and the (biased) land price P^Z are related according to

$$\bar{P}^Z = \kappa P^Z \quad \text{with} \quad \kappa \equiv \frac{1}{1 + \frac{Z_R^Y}{Z_U^Y}} + \frac{1}{\frac{P_U^Z Z_U^Y}{P_R^Z Z_R^Y} + \frac{P_U^Z}{P_R^Z}}. \quad (49)$$

Employing $Z_R^Y/Z_U^Y = 3$, $P_U^Z Z_U^Y/P_R^Z Z_R^Y = 4$ (consistent with our calibration assuming that farmland is valued by 25 percent of the total non-residential land value) and $P_U^Z/P_R^Z = 12$ gives $\kappa = 0.31$. That is, under these assumptions, the land price resulting from the House-Kapital Model, P^Z , should be multiplied by 0.31 to obtain a measure for the average

²⁵The alternative procedure would be to extend the House-Kapital model to explicitly encompass a rural and an urban region. This is left for future research.

(non-residential) land price \bar{P}^Z that accounts for the urban-rural land price dispersion.

To see whether this correction of the (non-residential) land price produces a better match between the theoretical and the empirical non-residential wealth-to-GDP ratio, we conduct the following exercise. Starting from the previously calculated time paths of the endogenous variables (employing the House-Kapital Model), we redefine non-residential wealth to read $K + \bar{P}^Z Z^Y$, where \bar{P}^Z denotes the average land price according to (49). The dashed black line in Figure 2 (b) shows the non-residential wealth-to-GDP ratio, \mathfrak{N}_t , that now starts at about 170 percent, crosses the housing wealth-to-GDP ratio (solid blue line) in 1990 and amounts to roughly 190 percent in 2010. This pattern matches the empirical picture, displayed in Figure 1, remarkably well.

According to Table 1, a correction factor of $\kappa = 0.31$ for non-residential land wealth implies that the corrected long run non-residential wealth-to-GDP ratio, $\bar{\mathfrak{N}}^* \equiv \bar{\mathfrak{K}}^* + \kappa \bar{\mathfrak{Z}}^*$, is in the range of 213 and 264 percent, assuming a capital income tax of 30 percent. This means that, in total, the corrected long run wealth-to-GDP ratio, $\bar{\mathfrak{W}}^* \equiv \bar{\mathfrak{H}}^* + \bar{\mathfrak{N}}^*$, ranges from 489 to 626 percent.

4.3.2 Land Prices and House Prices

Land and the allocation of land play a prominent role in the underlying model. Given that the overall land endowment is fixed (Premise 1) and that land represents a rivalrous factor (Premise 3), land is becoming scarcer and more expensive as the economy grows. The model demonstrates that the price channel of increased land valuations plays a critical role in the process of pushing up wealth-to-GDP ratios. On the one hand, the ratio of non-residential wealth to GDP, \mathfrak{N}_t , rises in the process of economic growth simply because it contains a sizable land wealth component $P_t^Z Z_t^Y$. On the other hand, rising land prices push the house price up and this price channel triggers an increase in the ratio of housing wealth to GDP, \mathfrak{H}_t . Figure 3 (a) displays the evolution of the land price, P_t^Z , and the house price, P_t^H , as implied by the calibrated model (both series are normalized to unity for the initial period, the year 1950). The house price, P_t^H , (solid gray line) increases by a factor of five, whereas the land price, P_t^Z , (dashed green line) rises by a factor of about eleven between 1950 and 2010. This pattern is quite similar to the empirical

series. Figure 3 (b) displays the evolution of the mean (across 14 advanced countries) real house prices and mean real land prices between 1950 and 2010. The land price increases considerably more (about sevenfold) than the house price (about threefold).²⁶

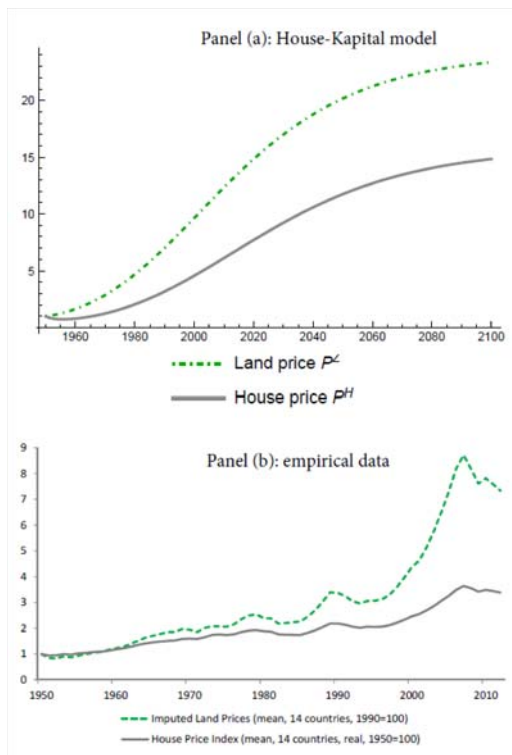


Figure 3. Evolution of land prices and house prices: calibrated model and data.

Notes: Panel (a): Evolution of the land price, P_t^L , and the house price, P_t^H , as resulting from the model (1950 values = 1). K_t starts at 70 percent, while N_t and X_t start at 40 percent of their (initial) steady state levels; L_t increases threefold and B_t^Y and B_t^X increase fourfold over time according to logistic functions; $B^h = B^M = 1$, $\rho = 0.025$, $\tau_r = 0.3$, and other parameters as in Table 1 with $\psi = 1$ and $\xi = 40$. Panel (b): Empirical land price (imputed) and real house price data (mean across 14 advanced countries), taken from Knoll et al. (2014).

²⁶Davis and Heathcote (2007) show that the price of residential land in the US has increased considerably more than house prices during 1975-2005 (almost fourfold), whereas the costs of structures have increased only slightly. Knoll et al. (2014) demonstrate, focusing on 14 countries between 1950 and 2012, that 20 percent of the increase in house prices can be attributed to rising construction costs and 80 percent to rising land prices.

5 The Canonical Macro-Housing Model

We now sketch a canonical version of the typical macroeconomic model with a housing sector, as it is used in the business cycle literature (Davis and Heathcote, 2005; Favilukis et al. 2015) and in Borri and Reichlin (2016). Subsequently, the major differences between the canonical model and our House-Kapital model are highlighted.

5.1 Setup

The economy is perfectly competitive and comprises two sectors. The *numeraire* (Y -) *sector* combines capital, K_t^Y , and labor, L_t^Y , but not land, to produce a final output good according to

$$Y_t = B_t^Y (K_t^Y)^\alpha (L_t^Y)^{1-\alpha}, \quad (50)$$

$B_t^Y > 0$, $0 < \alpha < 1$. The numeraire good can be either consumed or invested. The *housing sector* produces housing services that are sold to households. Housing services per period of time, S_t , are proportional to the stock of houses H_t , i.e., $S_t = bH_t$ with $b > 0$. Without loss of generality we set $b = 1$. The production of houses employs a fixed amount of (additional) land together with a variable amount of structures. The fixed amount of residential land becomes exogenously available each period. The stock of houses accumulates according to $\dot{H}_t = I_t^H - \delta^H H_t$, where I_t^H denotes gross investment and $\delta^H > 0$ the depreciation rate of the housing stock. Gross additions to the housing stock are described by a constant-returns to scale technology,

$$I_t^H = B_t^H X_t^\beta \bar{Z}^{1-\beta}, \quad (51)$$

$B_t^H > 0$, $0 < \beta < 1$, where X_t is the amount of structures that are combined with a fixed quantity of (additional) land, \bar{Z} , that is inelastically supplied each period. Residential structures (a flow) are produced according to

$$X_t = B_t^X (K_t^X)^\gamma (L_t^X)^{1-\gamma}, \quad (52)$$

$B^X > 0$, $0 < \gamma < 1$, by combining capital, K^X , and labor, L^X . Consequently, the evolution of the housing stock H is described by

$$\dot{H}_t = \underbrace{\tilde{B}_t^H (K_t^X)^{\beta\gamma} (L_t^X)^{\beta(1-\gamma)} \bar{Z}^{1-\beta}}_{=I_t^H} - \delta^H H, \quad (53)$$

where $\tilde{B}_t^H \equiv B_t^H (B_t^X)^\beta$. The exogenous and time-invariant supply of (additional) residential land, \bar{Z} , used as specific factor in housing production, is supposed to capture the idea that there is a time-invariant amount of (new) pieces of land available each period. According to Favilukis et al. (2015, p. 13), "a constant quantity [...] of new land/permits suitable for residential development is available each period". Davis and Heathcote (2005) point to the "declining relative returns to agricultural use" (p. 756) as a potential source of new land.²⁷

Households maximize utility and firms maximize profits. Intertemporal utility of the representative consumer is again given by (7). Let q_t^H denote the house price at time t . Total wealth, denoted by A_t , comprises capital and houses, i.e.,

$$A_t = K_t + q_t^H H_t, \quad (54)$$

where $K_0 > 0$ and $H_0 > 0$ are given. Again, let r_t denote the rate of return of financial assets, w_t the wage rate, R_t^Z the land rent, and p_t the price of housing services, respectively. Household wealth then accumulates according to

$$\dot{A}_t = r_t A_t + w_t L_t + R_t^Z \bar{Z} - p_t S_t - C_t, \quad (55)$$

where p_t denotes the price of one unit of housing services. The resource constraints

²⁷The canonical model could alternatively be interpreted as a model with two consumption goods, one flow good and one durable (capturing something else than house capital) that is produced by making use of an intermediate product and a specific factor that could as well be viewed as specific labor (like in the Ricardo-Viner model).

(holding with equality in equilibrium) are given by

$$K_t^X + K_t^Y \leq K_t, \quad (56)$$

$$L_t^X + L_t^Y \leq L_t, \quad (57)$$

and market clearing in the numeraire good sector requires

$$Y_t = C_t + I_t^K = C_t + \dot{K}_t + \delta^K K_t. \quad (58)$$

GDP_t for this economy is the sum of value-added of the numeraire good sector, Y_t , the housing services sector, $p_t H_t$, and the construction sector (building new houses with value $q_t^H I_t^H$).²⁸

$$GDP_t \equiv Y_t + p_t H_t + q_t^H I_t^H. \quad (59)$$

5.2 Comparison with the House-Kapital Model

The Canonical Model, due to its lean structure, represents an attractive and important analytical tool to study a large set of research questions. The House-Kapital Model, on the other hand, is a bit more complex and appears to have advantages especially when it comes to research questions that focus on the long run. We now highlight the major differences between the two models.

5.2.1 Land Availability and Land Allocation

Canonical Model The quantity of (additional) land that is employed every period in the housing sector, \bar{Z} , is exogenous and time invariant. Land is not used elsewhere in the economy, that is there is no endogenous land allocation. This automatically implies that the amount of land allocated to the housing sector does not change along the transition. What may change along the transition to the steady state is the quantity of complementary factors: K_t^X and L_t^X .

²⁸Alternatively, one can define GDP according to its use: $GDP_t = C_t + p_t S_t + I_t^K + q_t^H I_t^H$, which leads to (59), according to (58) and $S_t = H_t$.

House-Kapital Model The economy-wide amount of land is fixed. The land allocation is, however, endogenous. Land can be either employed in the Y -sector, Z_t^Y , or in the housing sector, amounting to $Z_t^N = \psi N_t$. The quantity of land allocated to the housing sector is time-varying and endogenous. This difference in the land allocation (exogenous and time-invariant vs. endogenous and time-varying) has important implications for the evolution of the land price, as explained below.

5.2.2 Cumulated Land in a Stationary Housing Sector

Canonical Model Let us assume a stationary steady state with a positive and constant housing stock and a positive depreciation rate $\delta^H > 0$. The cumulated amount of land that is incorporated in the housing sector converges to infinity as time goes to infinity. This is not compatible with Premise 1 (fixed overall land endowment) above and may be viewed as long run inconsistency.

House-Capital Model Let us assume a stationary steady state with a positive and constant housing stock and a positive depreciation rate of residential buildings $\delta^X > 0$. The cumulated quantity of land that is incorporated in the housing stock is $Z_t^N = \psi N_t$, a finite number, even for time approaching infinity. Hence, the House-Kapital Model is consistent with Premise 1 and the long run inconsistency is avoided.

This aspect may appear of second-order importance in the context of business cycle phenomena. When it comes to analyzing the long run evolution of residential and non-residential wealth, we think this point must be taken seriously.

5.2.3 Extensive and Intensive Margin of Housing Production

Canonical Model The housing stock is a one-dimensional object. It appears appropriate to interpret an increase in the housing stock as an increase along the extensive margin because this process requires land. The alternative interpretation would imply that there is a single house that is enlarged continuously upwards.²⁹

²⁹This interpretation is not crucial, however, for the description of the major differences between the two models.

House-Capital Model The housing stock can be enlarged along the extensive margin (increasing the number of houses) and along the intensive margin (increasing the size of the typical houses). Only the enlargement along the extensive margin requires land as an input (Premise 2).

The distinction between the extensive and intensive margin has two advantages: (i) It allows us to avoid the long run inconsistency, as explained above; (ii) It enables a distinction between the destruction of the housing stock either along the extensive margin (e.g., through a nuclear incident) or along the intensive margin (e.g., through a moderate earth-quake). This distinction may be employed in future research.

5.2.4 The Land Price

Canonical Model Given that each parcel of land used in housing production is permanently incorporated in the respective house, land plots can only be used once such that the land price, P_t^Z , equals the competitive land rent, R_t^Z , i.e., $P_t^Z = R_t^Z = p_t(\partial I_t^H / \partial \bar{Z})$ (Davis and Heathcote, 2005; Favilukis et al., 2015). How does the land price evolve in response to a destruction in the housing stock? Given that, by assumption, the initial housing stock is below the steady state level, $H_0 < H^*$, the economy allocates a large amount of capital, K_t^X , and labor, L_t^X , to the construction sector. This construction boom enables to build up the housing stock. As the economy converges to the steady state, the construction boom diminishes, implying that capital and labor are being reallocated to the alternative sector. Because land, along the transition to the steady state, is combined with less and less K_t^X and L_t^X the marginal productivity of land, $(\partial I_t^H / \partial \bar{Z})$, declines.³⁰ Moreover, the price of housing services, p_t , declines too as the supply in the housing market is being enlarged. Taken together, the land price $P_t^Z = p_t(\partial I_t^H / \partial \bar{Z})$ unambiguously decreases along the transition to the steady state, as illustrated in Figure 4 (a).

House-Capital Model Each unit of land can be either permanently incorporated in a house or can be employed for an infinite sequence of periods in the Y -sector. The

³⁰Recall that gross residential investments may be expressed as: $I_t^H = \tilde{B}_t^H (K_t^X)^{\beta\gamma} (L_t^X)^{\beta(1-\gamma)} \bar{Z}^{1-\beta}$.

equilibrium land price, excluding bubbles, equals the PDV of an infinite land rent earned in the Y -sector: $P_t^Z = \int_t^\infty R_\tau^Z e^{\int_t^\tau -r_v dv} ds$.

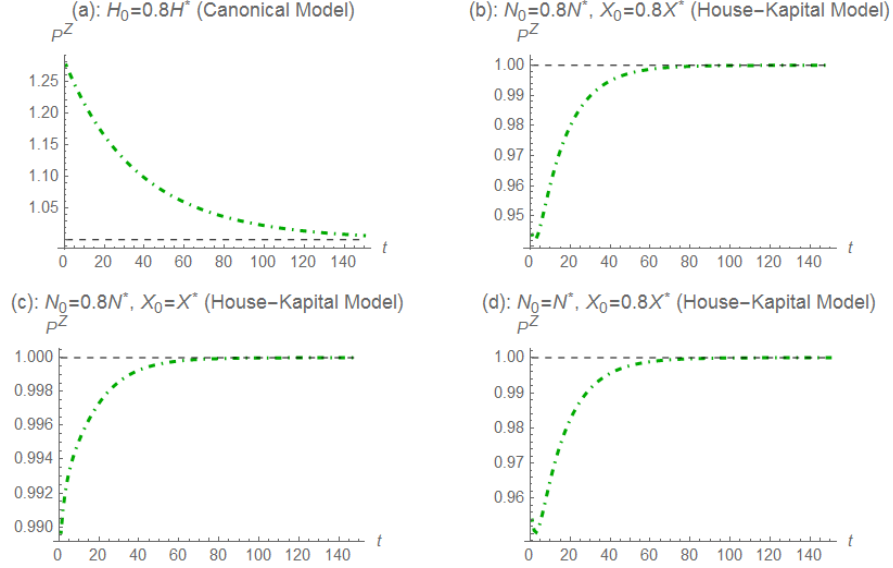


Figure 4. Evolution of the land prices in response to housing stock destruction: Canonical Model and House-Kapital Model.

Notes: Panel (a): Canonical Model, initial housing stock $H_0 = 0.8H^*$, set of parameters: $\rho = 0.025$, $\gamma = 0.7$, $\delta^H = 0.015$, $\delta^K = 0.07$, $\alpha = 0.4$, $\beta = 0.5$, $\theta = 0.22$, $L = Z = B^Y = B^X = 1$. Panel (b): House-Kapital Model: $N_0 = 0.8N^*$ and $X_0 = 0.8X^*$; Panel (c): House-Kapital Model: $N_0 = 0.8N^*$ and $X_0 = X^*$; Panel (d): House-Kapital Model: $N_0 = N^*$ and $X_0 = 0.8X^*$; Panel (b) - (c): set of parameters as in Figure 3 with $L = Z = B^Y = B^X = B^h = B^M = 1$.

Figures 4 (b) and 4 (c) show that the land price increases along the transition to the steady state in response to a destruction of the housing stock. The reasoning is basically the same as before, but now leads to a different conclusion. The construction boom, to rebuild the housing stock, leads to a temporary increase in the residential investment rate, $\mathfrak{s}_t^H \equiv (I_t^N + q_t^X I_t^X)/GDP_t$. This implies that (i) labor is reallocated to the construction sector and (ii) physical capital is temporarily decumulated (a standard implication of multi-sector models with multiple state variables). That is, non-residential land, Z^Y , is initially combined with a small amount of complementary factors, implying that R_t^Z is

low early on. As the construction boom diminishes, more and more labor returns to the Y -sector and K accumulates back to its initial steady state level. As both L^Y and K are complementary to Z^Y , the land rent and, with some delay, the land price increases. This pattern is largely independent of whether the shock applies to both margins, Figure 4 (b), to the extensive margin only, Figure 4 (c), or to the intensive margin only, Figure 4 (d). The implication of rising land prices in response to a destruction of the housing stock is economically plausible and clearly compatible with the empirical evidence of land price and house prices after WWII (Knoll et al., 2014).

5.2.5 Housing Wealth and Non-Residential Wealth

Canonical Model Housing wealth is given by $q_t^H H_t$. This wealth component comprises the cumulated quantity of land that is incorporated in the housing stock. The housing wealth-to-GDP ratio is $\mathfrak{H}_t = q_t^H H_t / GDP_t$. This is similar to the House-Kapital Model (both residential land and structures included). The non-residential wealth-to-GDP ratio is given by $\mathfrak{N}_t = K_t / GDP_t$. That is, land outside the housing sector, i.e., the value of non-residential land, is missing.³¹

House-Capital Model Residential land, $\psi N_t = Z_t - Z_t^Y$, enters housing wealth since land is incorporated in houses, recall $\mathfrak{H}_t = (q_t^N N_t + q_t^X X_t) / GDP_t$. Non-residential land, Z_t^Y , enters non-residential wealth, recall $\mathfrak{N}_t = (K_t + P_t^Z Z_t^Y) / GDP_t$. This matches the assignment of land in the different wealth components according to national accounting.

Models that do not capture land as an input in the non-residential sector (Y -sector) cannot adequately attribute rising non-residential wealth to rising land prices associated with land scarcity. Rising land prices are, however, important for both the evolution of housing wealth and the evolution of non-residential wealth.

³¹Sometimes K_t is interpreted to capture physical assets including land, like in standard macroeconomic models without explicit land considerations.

6 Conclusion

We have proposed a novel dynamic general equilibrium model to examine the evolution of two major wealth-to-income ratios (housing wealth and non-residential wealth), land prices, and house prices over time. Our theory rests on three premises: (1) fixed overall land endowment; (2) land as an essential input in housing production, except for replacement investment; (3) land rivalry between non-residential and residential production. The housing stock can be expanded along an extensive margin (setting up new housing projects) and along an intensive margin (enlarging existing housing projects). This distinction allows us to simultaneously capture Premises 1-3. The model is consistent with the close connection of land prices to house prices in the data. It points to the importance of land for understanding the dynamics of wealth and the dynamics of wealth distributions. This point has already been noted by Ricardo (1817), who reasoned that economic growth primarily benefits the owners of the fixed factor land.

The calibrated model replicates the post WWII experiences of wealth-to-GDP ratios, land prices and house prices in advanced countries remarkably well. It also suggests that wealth-to-income ratios, land prices, and house prices will considerably rise further in the future. Higher population density and technological progress do, however, not affect long run wealth-to-income ratios.

The analytical framework developed in this paper has many potential applications. The House-Kapital Model is potentially fruitful for investigating the interaction between the composition of wealth and the dynamics of the wealth distribution. The wealth composition may also be important for better understanding business cycles. For instance, specific capital goods, such as specialized machines, appear less suitable than land or houses as collateral for lending. Therefore, the wealth composition may impact on the propagation of business cycle shocks. The House-Kapital Model can also be employed to study public finance topics, like the effects of property taxation and regulation of house rental rates residential investment, house prices, and land prices. Finally, the model can be fruitfully applied to study the interaction between migration, house prices, and residential construction in a multi-region model.

Appendix: Proofs

Proof of Lemma 1. Using (3), the current-value Hamiltonian associated with real estate development problem (12) together with the necessary first-order conditions can then be expressed as

$$\mathcal{H}^N = \pi N - \psi P^Z \tilde{I}^N - \frac{\xi}{2} \left(\tilde{I}^N \right)^2 + q^N \tilde{I}^N, \quad (60)$$

$$\left[\frac{\partial \mathcal{H}^N}{\partial \tilde{I}^N} = \right] - \psi P^Z - \xi \left(\tilde{I}^N \right) + q^N = 0, \quad (61)$$

$$\left[-\frac{\partial \mathcal{H}^X}{\partial N} = \right] - \pi = \dot{q}^N - r q^N. \quad (62)$$

Combining (2) and (61) confirms (13). (14) follows from (62). This concludes the proof. ■

Proof of Lemma 2. The current-value Hamiltonian of the representative construction firm associated with (16) together with the necessary first-order conditions can then be expressed as

$$\mathcal{H}^X = R^X X - p^M M - w L^X + q^X \left[B^X M^\eta (L^X)^{1-\eta} - \delta^X X \right], \quad (63)$$

$$\left[\frac{\partial \mathcal{H}^X}{\partial M} = \right] - p^M + \eta q^X B^X \left(\frac{L^X}{M} \right)^{1-\eta} = 0, \quad (64)$$

$$\left[\frac{\partial \mathcal{H}^X}{\partial L^X} = \right] - w + (1-\eta) q^X B^X \left(\frac{M}{L^X} \right)^\eta = 0, \quad (65)$$

$$\left[-\frac{\partial \mathcal{H}^X}{\partial X} = \right] - R^X + \delta^X q^X = \dot{q}^X - r q^X. \quad (66)$$

(17) follows from (66). This concludes the proof. ■

Proof of Proposition 1. Define $D^Y \equiv L^Y / Z^Y$. We first establish³²

Lemma A.1. *Suppose that (A1) holds and the (relative) price of housing services, p , is given. Then there exists a unique, nontrivial solution for the long run labor-to-land*

³²We omit the time index t in Appendix A, which deals with the long run equilibrium.

ratio in the Y -sector, that is given by

$$D^Y = \left(\frac{(\eta B^M)^{\frac{\gamma\eta}{1-\gamma\eta}} \left(\frac{B^X\gamma}{r+\delta^K}\right)^{\frac{\gamma}{1-\gamma\eta}} (B^h p)^{\frac{1}{1-\gamma\eta}} \left(\frac{1-\eta}{\beta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}}}{\left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left(\frac{\psi(1-\alpha-\beta)}{1-\gamma}\right)^{\frac{1-\gamma}{1-\gamma\eta}}}\right)^{\frac{1}{\theta}} \equiv \tilde{D}^Y(p, \mathbf{B}). \quad (67)$$

Function \tilde{D}^Y is increasing in p , B^X , B^h , B^M , and decreasing in B^Y .

Proof. The typical final output firm maximizes profits given by

$$\Pi = B^Y (K^Y)^\alpha (L^Y)^\beta (Z^Y)^{1-\alpha-\beta} - (r + \delta^K) K^Y - wL^Y - R^Z Z^Y. \quad (68)$$

Using $K^Y = K$, the necessary first-order conditions are

$$\left[\alpha \frac{Y}{K^Y} = \right] \alpha B^Y K^{\alpha-1} (L^Y)^\beta (Z^Y)^{1-\alpha-\beta} = r + \delta^K, \quad (69)$$

$$\left[\beta \frac{Y}{L^Y} = \right] \beta B^Y K^\alpha (L^Y)^{\beta-1} (Z^Y)^{1-\alpha-\beta} = w, \quad (70)$$

$$\left[(1 - \alpha - \beta) \frac{Y}{Z^Y} = \right] (1 - \alpha - \beta) B^Y K^\alpha (L^Y)^\beta (Z^Y)^{-\alpha-\beta} = R^Z. \quad (71)$$

Combining (69) and (70), leads to

$$K = \frac{\alpha}{\beta} \frac{w}{r + \delta^K} L^Y. \quad (72)$$

Substituting (72) into (69) and (71), we obtain

$$w = \beta \left(\frac{\alpha}{r + \delta^K} \right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left(\frac{Z^Y}{L^Y} \right)^{\frac{1-\alpha-\beta}{1-\alpha}}, \quad (73)$$

$$R^Z = (1 - \alpha - \beta) B^Y \left(\frac{\alpha}{\beta} \frac{w}{r + \delta^K} \right)^\alpha \left(\frac{L^Y}{Z^Y} \right)^{\alpha+\beta}. \quad (74)$$

respectively.

The current-value Hamiltonian for the household optimization problem (equilibrium

condition 1 in Definition 1) reads as

$$\mathcal{H} = \log C + \theta \log S + \lambda [(1 - \tau_r)(rA + R^Z Z) + (1 - \tau_w)wL - C - pS], \quad (75)$$

where λ is the multiplier (co-state variable) associated with financial asset holding, A . Necessary optimality conditions are $\partial\mathcal{H}/\partial C = \partial\mathcal{H}/\partial S = 0$ (control variables), $\dot{\lambda} = \rho\lambda - \partial\mathcal{H}/\partial A$ (co-state variable), and the corresponding transversality condition. Thus,

$$\lambda = \frac{1}{S}, \quad (76)$$

$$\frac{\theta}{S} = \lambda p, \quad (77)$$

$$\frac{\dot{\lambda}}{\lambda} = \rho - (1 - \tau_r)r. \quad (78)$$

Combining (76) and (77), we have

$$C = \frac{pS}{\theta}, \quad (79)$$

whereas combining (76) and (78) yields

$$\frac{\dot{C}}{C} = (1 - \tau_r)r - \rho. \quad (80)$$

We seek for a steady state without long run growth. Setting $\dot{C} = 0$ in (80) gives us the long run interest rate

$$r^* = \frac{\rho}{1 - \tau_r}. \quad (81)$$

Setting $\dot{N} = 0$ in (13) implies, for the long run,

$$q^N = \psi P^Z. \quad (82)$$

Substituting (11) into (14) and using $\dot{q}^N = 0$, we have

$$q^N = \frac{(1 - \gamma)pB^h x^\gamma}{r}. \quad (83)$$

Setting $\dot{P}^Z = \dot{q}^X = 0$ in (20) and (17), respectively, we obtain, for the long run,

$$P^Z = \frac{R^Z}{r}, \quad (84)$$

$$q^X = \frac{R^X}{r + \delta^X}. \quad (85)$$

Setting $\dot{X} = 0$ in (6) and using $X = Nx$, we have

$$x = \frac{B^X M^\eta (L^X)^{1-\eta}}{\delta^X N}, \quad (86)$$

To prove Lemma A.1, where the relative consumer goods price p is taken as exogenously given, we ignore the consumer demand side. First, combine (64) with (85) and use $p^M = 1/B^M$ to obtain

$$M = L^X \left(\frac{\eta B^M R^X B^X}{r + \delta^X} \right)^{\frac{1}{1-\eta}}. \quad (87)$$

Substitute next (86) into (10) to find

$$R^X = p B^h \gamma \left(\frac{B^X M^\eta (L^X)^{1-\eta}}{\delta^X N} \right)^{\gamma-1}. \quad (88)$$

Substituting (88) into (87) yields a useful expression for M :

$$M = (L^X)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{\gamma \eta p B^M B^h (\delta^X N)^{1-\gamma} (B^X)^\gamma}{r + \delta^X} \right)^{\frac{1}{1-\eta}}. \quad (89)$$

Combining (65) with (85) and (87) we obtain

$$w = (1 - \eta) (\eta B^M)^{\frac{\eta}{1-\eta}} \left(\frac{R^X B^X}{r + \delta^X} \right)^{\frac{1}{1-\eta}}. \quad (90)$$

Substituting (88) into (90) leads to

$$w = (1 - \eta) (\eta B^M)^{\frac{\eta}{1-\eta}} \left(\frac{\gamma p B^h}{r + \delta^X} \right)^{\frac{1}{1-\eta}} (B^X)^{\frac{\gamma}{1-\eta}} (\delta^X N)^{\frac{1-\gamma}{1-\eta}} M^{\frac{\eta(\gamma-1)}{1-\eta}} (L^X)^{\gamma-1}. \quad (91)$$

Substituting (89) into (91) gives us a useful expression for w :

$$w = (1 - \eta) (\eta B^M)^{\frac{\gamma\eta}{1-\gamma\eta}} \left(\frac{\gamma B^h p}{r + \delta^X} \right)^{\frac{1}{1-\gamma\eta}} (B^X)^{\frac{\gamma}{1-\gamma\eta}} \left(\frac{N \delta^X}{L^X} \right)^{\frac{1-\gamma}{1-\gamma\eta}}. \quad (92)$$

Combining (92) with (73) and leads to

$$\begin{aligned} \frac{L^X}{N} &= \frac{(1 - \eta)^{\frac{1-\gamma\eta}{1-\gamma}} (\eta B^M)^{\frac{\gamma\eta}{1-\gamma}} \left(\frac{\gamma B^h p}{r + \delta^X} \right)^{\frac{1}{1-\gamma}} (B^X)^{\frac{\gamma}{1-\gamma}} \delta^X \left(\frac{L^Y}{Z^Y} \right)^{\frac{1-\alpha-\beta}{1-\alpha} \frac{1-\gamma\eta}{1-\gamma}}}{\left[\beta \left(\frac{\alpha}{r + \delta^K} \right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \right]^{\frac{1-\gamma\eta}{1-\gamma}}} \left(\frac{L^Y}{Z^Y} \right)^{\frac{1-\alpha-\beta}{1-\alpha} \frac{1-\gamma\eta}{1-\gamma}} \\ &\equiv \Phi \left(\frac{L^Y}{Z^Y}, p, \mathbf{B} \right), \end{aligned} \quad (93)$$

where $\mathbf{B} \equiv (B^Y, B^X, B^h, B^M)$ is the vector of productivity parameters. Note that Φ is increasing as a function of L^Y/Z^Y . Moreover, Φ is strictly concave as a function of L^Y/Z^Y if and only if (A1) holds. Substituting (83) and (84) into (82), we obtain

$$[\pi =] (1 - \gamma) p B^h x^\gamma = \psi R^Z. \quad (94)$$

Inserting (86) and (74) into (94), we get

$$\frac{(1 - \gamma) p B^h}{w} \left(\frac{B^X M^\eta (L^X)^{1-\eta}}{\delta^X N} \right)^\gamma = \frac{\psi (1 - \alpha - \beta) B^Y}{w^{1-\alpha}} \left(\frac{\alpha}{\beta} \frac{1}{r + \delta^K} \right)^\alpha \left(\frac{L^Y}{Z^Y} \right)^{\alpha+\beta}. \quad (95)$$

Substituting (73) and (91) into the right-hand side and left-hand side of (95), respectively, yields

$$\frac{(1 - \gamma) M^{\frac{\eta(1-\gamma\eta)}{1-\eta}} (L^X)^{1-\eta\gamma}}{(1 - \eta) (\eta p B^M B^h)^{\frac{\eta}{1-\eta}} \left(\frac{\gamma}{r + \delta^X} \right)^{\frac{1}{1-\eta}} (B^X)^{\frac{\gamma\eta}{1-\eta}} (\delta^X N)^{\frac{1-\gamma\eta}{1-\eta}}} = \frac{\psi (1 - \alpha - \beta) L^Y}{\beta Z^Y}. \quad (96)$$

Now substitute (89) into (96) to find

$$\frac{L^X}{N} = \psi v \frac{L^Y}{Z^Y}, \quad \text{with} \quad (97)$$

$$v \equiv \frac{(1 - \alpha - \beta) (1 - \eta) \gamma \delta^X}{\beta (1 - \gamma) (r + \delta^X)}. \quad (98)$$

According to (A1), $v < 1$.³³ Finally, combine the right-hand sides of (93) and (97) to confirm (67). ■

According to (97) and $L^Y/Z^Y = \tilde{D}^Y(p, \mathbf{B})$, we have

$$\frac{L^X}{N} = \psi v \tilde{D}^Y(p, \mathbf{B}), \quad (99)$$

Using $L^Y = D^Y Z^Y$ and $L^X = N\psi v D^Y$, $L^X + L^Y = L$ and $Z^Y = Z - \psi N$ we obtain

$$\frac{N}{Z} = \frac{1}{\psi(1-v)} \left(1 - \frac{D}{\tilde{D}^Y(p, \mathbf{B})} \right), \quad (100)$$

$$\mathfrak{z}^Y = \frac{Z^Y}{Z} = \frac{1}{1-v} \left(\frac{D}{\tilde{D}^Y(p, \mathbf{B})} - v \right), \quad (101)$$

$$\frac{N}{Z^Y} = \frac{1}{\psi} \frac{\frac{\tilde{D}^Y(p, \mathbf{B})}{D} - 1}{1 - v \frac{\tilde{D}^Y(p, \mathbf{B})}{D}}, \quad (102)$$

From $L^X = N\psi v D^Y$, $D = L/Z$ and (100), we also have

$$\frac{L^X}{L} = \frac{v}{1-v} \left(\frac{\tilde{D}^Y(p, \mathbf{B})}{D} - 1 \right). \quad (103)$$

Thus, as $L^Y/L = 1 - L^X/L$,

$$\frac{L^Y}{L} = \frac{1}{1-v} \left(1 - v \frac{\tilde{D}^Y(p, \mathbf{B})}{D} \right). \quad (104)$$

According to (26) and (83), we can write for the long run:

$$q^N N = \frac{(1-\gamma)pS}{r}. \quad (105)$$

Moreover, combining (10) with (26), we have $R^X N x = \gamma p S$. Combining $X = N x$, (85)

³³It is easy to show that $v < 1$ is equivalent to

$$\frac{\beta}{1-\alpha} - \frac{\gamma(1-\eta)}{1-\gamma\eta} \underbrace{\frac{\delta^X}{\frac{(1-\gamma)r}{1-\gamma\eta} + \delta^X}}_{<1} > 0.$$

and (86) we thus find that for the long run:

$$q^X X = \frac{R^X N x}{r + \delta^X} = \frac{\gamma p S}{r + \delta^X}. \quad (106)$$

Using (72), (105) and (106) in (8) leads to long run total asset value:

$$A = \frac{\alpha}{\beta} \frac{w L^Y}{r + \delta^K} + \frac{(1 - \gamma) p S}{r} + \frac{\gamma p S}{r + \delta^X}. \quad (107)$$

Next, substituting (86) into condition (26) and using (89) gives us

$$p S = N (B^h p^*)^{\frac{1}{1-\gamma\eta}} (B^X)^{\frac{\gamma}{1-\gamma\eta}} \left(\frac{L^X}{N} \frac{1}{\delta^X} \right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{\gamma\eta B^M}{r + \delta^X} \right)^{\frac{\gamma\eta}{1-\gamma\eta}}. \quad (108)$$

The total tax revenue that is redistributed to households reads as

$$T = \tau_r r A + \tau_r R^Z Z^Y + \tau_w w L. \quad (109)$$

Combining (79) and (9) with $\dot{A} = 0$, we obtain

$$\left(\frac{1}{\theta} + 1 \right) p S = r A + w L + R^Z Z^Y. \quad (110)$$

Substituting (107) into (110) and using (109) leads to

$$\left(\frac{1}{\theta} + \frac{\gamma \delta^X}{r + \delta^X} \right) p S = w \left(\frac{\alpha}{\beta} \frac{r L^Y}{r + \delta^K} + L \right) + R^Z Z^Y. \quad (111)$$

Inserting (73) into (74) leads to

$$R^Z = (1 - \alpha - \beta) \left(\frac{\alpha}{r + \delta^K} \right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left(\frac{L^Y}{Z^Y} \right)^{\frac{\beta}{1-\alpha}}. \quad (112)$$

By substituting (73), (108) and (112) into (111) we obtain

$$\begin{aligned} & \left(\frac{1}{\theta} + \frac{\gamma\delta^X}{r + \delta^X} \right) \frac{N}{Z^Y} (B^h p)^{\frac{1}{1-\gamma\eta}} (B^X)^{\frac{\gamma}{1-\gamma\eta}} \left(\frac{L^X}{N} \frac{1}{\delta^X} \right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{\gamma\eta B^M}{r + \delta^X} \right)^{\frac{\gamma\eta}{1-\gamma\eta}} \left(\frac{L^Y}{Z^Y} \right)^{-\frac{\beta}{1-\alpha}} \\ &= \left(\frac{\alpha}{r + \delta^K} \right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left[\frac{\alpha r}{r + \delta^K} + \beta \left(\frac{L}{L^Y} - 1 \right) + 1 - \alpha \right]. \end{aligned} \quad (113)$$

Substituting $L^Y/Z^Y = \tilde{D}^Y(p, \mathbf{B})$, (99) and (102) into (113) and recalling the definition of ϱ in (A1) implies, for the long run,

$$\begin{aligned} & \frac{\left(\frac{1}{\theta} + \frac{\gamma\delta^X}{r^* + \delta^X} \right) \left(1 - \frac{\tilde{D}^Y(p^*, \mathbf{B})}{D} \right) (B^h p^*)^{\frac{1}{1-\gamma\eta}} (B^X)^{\frac{\gamma}{1-\gamma\eta}} \left(\frac{v}{\delta^X} \right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{\gamma\eta B^M}{r^* + \delta^X} \right)^{\frac{\gamma\eta}{1-\gamma\eta}}}{\psi^{\frac{1-\gamma}{1-\gamma\eta}} \left(v \frac{\tilde{D}^Y(p^*, \mathbf{B})}{D} - 1 \right) \tilde{D}^Y(p^*, \mathbf{B})^e} \\ &= \left(\frac{\alpha}{r^* + \delta^K} \right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left[\frac{\alpha r^*}{r^* + \delta^K} + \beta v \left(\frac{1 - \frac{\tilde{D}^Y(p^*, \mathbf{B})}{D}}{v \frac{\tilde{D}^Y(p^*, \mathbf{B})}{D} - 1} \right) + 1 - \alpha \right]. \end{aligned} \quad (114)$$

Substituting (81), (67), (98) in (114) we find that long run the long run price for housing services, $p^* \equiv \tilde{p}^*(\mathbf{B}, D)$, is implicitly given by $0 = \Theta(p^*, \mathbf{B}, D)$, where

$$\Theta(p, \mathbf{B}, D) \equiv \frac{1 - \alpha - \beta}{1 - \gamma} \left(\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{\frac{\rho}{1-\tau_r} + \delta^X} \right) \frac{\frac{\tilde{D}^Y(p, \mathbf{B})}{D} - 1}{1 - v \frac{\tilde{D}^Y(p, \mathbf{B})}{D}} - \frac{\frac{\rho}{1-\tau_r} + (1 - \alpha)\delta^K}{\frac{\rho}{1-\tau_r} + \delta^K}. \quad (115)$$

Note that $\Theta_p(p, \mathbf{B}, D) > 0$,³⁴ according to (115). Hence, if $p^* > 0$ exists, it is unique. Moreover, we have $\tilde{D}^Y(0, \mathbf{B}) = 0$ and $\lim_{p \rightarrow \infty} \tilde{D}^Y(p, \mathbf{B}) \rightarrow \infty$, according to (67). Thus, $\lim_{p \rightarrow 0} \Theta(p, \mathbf{B}, D) < 0$ and $\lim_{p \rightarrow \infty} \Theta(p, \mathbf{B}, D) \rightarrow -\infty$. Hence, p^* is unique.

Setting $\Theta(p^*, \mathbf{B}, D) = 0$ implies that

$$\frac{D}{\tilde{D}^Y(\tilde{p}^*(\mathbf{B}, D), \mathbf{B})} = \frac{\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{\frac{\rho}{1-\tau_r} + \delta^X} + \frac{\frac{\rho}{1-\tau_r} + (1-\alpha)\delta^K}{\frac{\rho}{1-\tau_r} + \delta^K} \frac{(1-\eta)\gamma\delta^X}{\beta \left(\frac{\rho}{1-\tau_r} + \delta^X \right)}}{\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{\frac{\rho}{1-\tau_r} + \delta^X} + \frac{1-\gamma}{1-\alpha-\beta} \frac{\frac{\rho}{1-\tau_r} + (1-\alpha)\delta^K}{\frac{\rho}{1-\tau_r} + \delta^K}} \equiv \mu. \quad (116)$$

It is easy to show that $\mu \in (v, 1)$ if and only if $v < 1$, which holds according to (A1). Thus, also (35) holds under (A1). This concludes the proof. ■

Proof of Proposition 2. Applying the implicit function theorem to $0 = \Theta(p^*, \mathbf{B}, D)$

³⁴Subscripts on functions denote partial derivatives throughout.

and using Lemma A.1 confirms part (i). To prove parts (ii) and (iii), use (73), (81) and (112) to find

$$w^* = \beta \left(\frac{\alpha}{\frac{\rho}{1-\tau_r} + \delta^K} \right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left(\frac{\mu}{D} \right)^{\frac{1-\alpha-\beta}{1-\alpha}}, \quad (117)$$

$$R^{Z^*} = (1 - \alpha - \beta) \left(\frac{\alpha}{\frac{\rho}{1-\tau_r} + \delta^K} \right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left(\frac{D}{\mu} \right)^{\frac{\beta}{1-\alpha}}, \quad (118)$$

respectively. Using (84) and again using that μ is independent of \mathbf{B} and D confirms parts (ii) and (iii). Part (iv) just restates (81). This concludes the proof. ■

Proof of Proposition 3. First, use (101) and (103), to find

$$\mathfrak{z}^{Y^*} = \frac{Z^{Y^*}}{Z} = \frac{\mu - v}{1 - v} [= 1 - \mathfrak{z}^{N^*}], \quad (119)$$

$$l^{X^*} = \frac{L^{X^*}}{L} = \frac{v(1 - \mu)}{(1 - v)\mu} [= 1 - l^{Y^*}], \quad (120)$$

respectively. Next, define the "house-price-to-rent ratio" as the ratio of the price of one house with x_t units of structures put on ψ units of land to the cost of renting h_t units of housing services produced with the same amount of structures, x_t . Formally,

$$\mathfrak{p}_t \equiv \frac{P_t^H}{p_t h_t} = \frac{q_t^N + q_t^X x_t}{p_t h_t}. \quad (121)$$

Using (4), (10), (81), (83) and (85) in (121) implies a long run value

$$\mathfrak{p}^* = \frac{1 + \frac{(1-\gamma)(1-\tau_r)\delta^X}{\rho}}{\frac{\rho}{1-\tau_r} + \delta^X}. \quad (122)$$

Moreover, according to (100) and (116),

$$\frac{N^*}{Z} = \frac{1 - \mu}{\psi(1 - v)}. \quad (123)$$

By using (29), we can write for the housing wealth-to-GDP ratio, the non-residential-land

to GDP ratio, and the non-residential wealth-to-GDP ratio as

$$\mathfrak{H}_t = \frac{P_t^H N_t}{GDP_t} = \frac{\mathbf{p}_t}{1 + \frac{Y_t}{p_t N_t h_t} + \frac{w_t L_t^X}{p_t h_t N_t}}, \quad (124)$$

$$\mathfrak{K}_t = \frac{K_t}{GDP_t} = \frac{\frac{K_t}{Y_t}}{1 + \frac{p_t N_t h_t}{Y_t} + \frac{w_t L_t}{Y_t} \frac{L_t^X}{L_t}}. \quad (125)$$

$$\mathfrak{Z}_t = \frac{P_t^Z Z_t^Y}{GDP_t} = \frac{\frac{P_t^Z Z_t^Y}{Y_t}}{1 + \frac{p_t N_t h_t}{Y_t} + \frac{w_t L_t}{Y_t} \frac{L_t^X}{L_t}}. \quad (126)$$

According to (81), (82) and (84), in the long run,

$$q^{N*} = \frac{\psi R^{Z*}}{r^*}. \quad (127)$$

Combining (83) and (127) and using (4) yields

$$p^* h^* = \frac{\psi R^{Z*}}{1 - \gamma}. \quad (128)$$

Substituting (118) into (128) leads to

$$p^* h^* = \frac{\psi(1 - \alpha - \beta)}{1 - \gamma} \left(\frac{\alpha}{\frac{\rho}{1 - \tau_r} + \delta^K} \right)^{\frac{\alpha}{1 - \alpha}} (B^Y)^{\frac{1}{1 - \alpha}} \left(\frac{D}{\mu} \right)^{\frac{\beta}{1 - \alpha}}. \quad (129)$$

According to (1), (4) and (72), we have

$$\frac{Y}{p N h} = \frac{B^Y \left(\frac{\alpha w}{\beta r + \delta^K} \right)^\alpha \left(\frac{L^Y}{Z^Y} \right)^{\alpha + \beta} \frac{Z^Y}{Z}}{p h \frac{N}{Z}}. \quad (130)$$

Moreover, using $\tilde{D}^Y(\tilde{p}^*(\mathbf{B}, D), \mathbf{B})/D = 1/\mu$ from (116) in (115), we have

$$\frac{\mu - v}{1 - \mu} = \frac{1 - \alpha - \beta}{1 - \gamma} \frac{\left(\frac{1}{\theta} + \frac{\eta \gamma \delta^X}{\frac{\rho}{1 - \tau_r} + \delta^X} \right) \left(\frac{\rho}{1 - \tau_r} + \delta^K \right)}{\frac{\rho}{1 - \tau_r} + (1 - \alpha) \delta^K} \quad (131)$$

Substituting (123), (119), (117), (129) and $L^{Y*}/Z^{Y*} = \tilde{D}^Y(p^*, \mathbf{B}) = D/\mu$ (recall (116))

into (130), and using (131), we get

$$\frac{Y^*}{p^*N^*h^*} = \frac{\left(\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{\frac{\rho}{1-\tau_r} + \delta^X}\right) \left(\frac{\rho}{1-\tau_r} + \delta^K\right)}{\frac{\rho}{1-\tau_r} + (1-\alpha)\delta^K} \quad (132)$$

in long run equilibrium. Moreover,

$$\frac{wL^X}{phN} = \frac{w}{ph} l^X \frac{L/Z}{N/Z}. \quad (133)$$

Substituting (120), (117) and (123) into (133) and using (81), $D = L/Z$ and v as given by (98), we obtain

$$\frac{w^*L^{X^*}}{p^*N^*h^*} = \frac{(1-\eta)\gamma\delta^X}{\frac{\rho}{1-\tau_r} + \delta^X}. \quad (134)$$

Using (122), (132) and (134) in (124) confirms (37).

According to (69), (70), (71) and $L^Y = L(1 - l^X)$, we have

$$\frac{K}{Y} = \frac{\alpha}{r + \delta^K}, \quad (135)$$

$$\frac{wL}{Y} = \frac{\beta}{1 - l^X}, \quad (136)$$

$$\frac{R^Z Z^Y}{Y} = 1 - \alpha - \beta. \quad (137)$$

Using that (81) and (84), in long run equilibrium,

$$\frac{P^{Z^*} Z^{Y^*}}{Y^*} = \frac{(1 - \alpha - \beta)(1 - \tau_r)}{\rho}. \quad (138)$$

Substituting (132), (135), (136) and (138) in (125) and (126), respectively, and using (81), (98), (120) and (131) confirm (38) and (39). This concludes the proof. ■

Proof of Corollary 1. The result immediately follows from (33) and (34). ■

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Online-Appendix A: New Model

In Online-Appendix A, we first summarize the dynamical system of our new model. We then invoke Walras' law as consistency check of the analytical derivation of the long run equilibrium. Finally, we present an additional result for the long run equilibrium that characterizes the amount of structures and housing services.

Dynamic System

Recall $\pi = (1 - \gamma)pB^h x^\gamma$ from (11) and $x = X/N$. For given K_0, N_0, x_0 , the dynamic system reads as:

$$\frac{\dot{C}}{C} = (1 - \tau_r)r - \rho, \quad (139)$$

$$\frac{\dot{q}^N}{q^N} + \frac{\pi}{q^N} = r, \quad (140)$$

$$\frac{\dot{P}^Z}{P^Z} + \frac{R^Z}{P^Z} = r, \quad (141)$$

$$\frac{\dot{q}^X}{q^X} + \frac{R^X}{q^X} - \delta^X = r, \quad (142)$$

$$\dot{X} = B^X M^\eta (L^X)^{1-\eta} - \delta^X X, \quad (143)$$

$$\dot{A} = rA + R^Z Z^Y + wL - C - pS, \quad (144)$$

$$\dot{N} = \frac{q^N - \psi P^Z}{\xi}, \quad (145)$$

$$\left(\frac{r + \delta^K}{\alpha}\right)^\alpha \left(\frac{w}{\beta}\right)^{1-\alpha} = B^Y \left(\frac{Z^Y}{L^Y}\right)^{1-\alpha-\beta}, \quad (146)$$

$$R^Z = B^Y (1 - \alpha - \beta) \left(\frac{\alpha w}{\beta r + \delta^K}\right)^\alpha \left(\frac{L^Y}{Z^Y}\right)^{\alpha+\beta}. \quad (147)$$

$$R^X = pB^h \gamma x^{\gamma-1}, \quad (148)$$

$$\frac{L^X}{M} = \left(\frac{1}{q^X \eta B^M B^X}\right)^{\frac{1}{1-\eta}}, \quad (149)$$

$$w = (1 - \eta) (\eta B^M)^{\frac{\eta}{1-\eta}} (B^X q^X)^{\frac{1}{1-\eta}}, \quad (150)$$

$$Z^Y = Z - \psi N, \quad (151)$$

$$L^X + L^Y = L, \quad (152)$$

$$\theta C = pS, \quad (153)$$

$$S = NB^h x^\gamma, \quad (154)$$

$$A = K + q^N N + q^X X, \quad (155)$$

$$K = \frac{\alpha}{\beta} \frac{w}{r + \delta^K} L^Y. \quad (156)$$

These are 18 equations and 18 endogenous variables: $X, N, A, C, q^N, P^Z, q^X, K, Z^Y, L^X, L^Y, M, S, r, w, R^X, R^Z, p$.

Consistency Check Using Walras' Law

We show that the market for the numeraire good clears in long run equilibrium, i.e. $Y = C + \delta^K K + p^M M$ holds in steady state (market for numeraire good clears, according to equilibrium condition 9 in Definition 1, and $\dot{K} = \dot{N} = \dot{X} = 0$). Recall the following definitions hold.

$$\varrho \equiv \frac{\beta}{1 - \alpha} - \frac{\gamma(1 - \eta)}{1 - \gamma\eta}, \quad (157)$$

$$\chi \equiv \frac{\eta^\eta \left(\frac{\beta}{1 - \eta}\right)^{\frac{\gamma\eta(1 - \eta)}{1 - \gamma\eta}} \left(\frac{\alpha}{r + \delta^K}\right)^{\frac{\alpha\eta}{1 - \alpha}} \left(\frac{\nu}{\delta^X}\right)^{\frac{1 - \eta}{1 - \gamma\eta}}}{1 - \nu} \frac{1 - \mu}{\mu^{1 - \eta + \frac{\beta\eta}{1 - \alpha}}} \left(\frac{(1 - \alpha - \beta)\gamma}{(1 - \gamma)(r + \delta^X)}\right)^{\frac{\eta(1 - \gamma)}{1 - \gamma\eta}}, \quad (158)$$

$$v \equiv \frac{(1 - \alpha - \beta)(1 - \eta)\gamma\delta^X}{\beta(1 - \gamma)(r + \delta^X)}, \quad (159)$$

$$D \equiv \frac{L}{Z}. \quad (160)$$

In steady state, the following relationships hold (we suppress superscript (*)):

$$K = \frac{\alpha}{\beta} \frac{w}{r + \delta^K} L^Y, \quad (161)$$

$$\frac{K}{Y} = \frac{\alpha}{r + \delta^K}, \quad (162)$$

$$w = \beta \left(\frac{\alpha}{r + \delta^K} \right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left(\frac{\mu}{D} \right)^{\frac{1-\alpha-\beta}{1-\alpha}}, \quad (163)$$

$$C = \frac{pS}{\theta}, \quad (164)$$

$$\frac{L^X}{L} = \frac{v(1-\mu)}{(1-v)\mu} = 1 - \frac{L^Y}{L} \quad (165)$$

$$\Rightarrow \frac{L^Y}{L} = \frac{1}{1-v} - \frac{v}{1-v} \frac{1}{\mu}, \quad (166)$$

$$\frac{S}{L} = \chi^\gamma \left(\frac{1-\mu}{\psi(1-v)} \right)^{1-\gamma} \frac{B^h}{D^{1-\gamma+\frac{(1-\alpha-\beta)\gamma\eta}{1-\alpha}}} (B^M)^{\gamma\eta} (B^X)^\gamma (B^Y)^{\frac{\gamma\eta}{1-\alpha}}, \quad (167)$$

$$B^h p = \left(\frac{D}{\mu} \right)^{\varrho(1-\gamma\eta)} \frac{\left(\frac{\alpha}{r+\delta^K} \right)^{\frac{\alpha(1-\gamma\eta)}{1-\alpha}} (B^Y)^{\frac{1-\gamma\eta}{1-\alpha}} \left(\frac{\psi(1-\alpha-\beta)}{1-\gamma} \right)^{1-\gamma}}{(\eta B^M)^{\gamma\eta} \left(\frac{B^X \gamma}{r+\delta^X} \right)^\gamma \left(\frac{1-\eta}{\beta} \right)^{\gamma(1-\eta)}}, \quad (168)$$

$$\frac{N}{Z} = \frac{1-\mu}{\psi(1-\nu)}, \quad (169)$$

$$M = (L^X)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{\gamma\eta p B^M B^h (\delta^X N)^{1-\gamma} (B^X)^\gamma}{r + \delta^X} \right)^{\frac{1}{1-\gamma\eta}}. \quad (170)$$

We first derive

$$\begin{aligned}
\frac{Y - C - \delta^K K}{K} &= \frac{Y}{K} - \frac{C}{K} - \delta^K \\
&= \frac{r + \delta^K}{\alpha} - \frac{\frac{pS}{\theta}}{\frac{1}{\beta} \frac{\alpha}{r + \delta^K} \beta \left(\frac{\alpha}{r + \delta^K} \right)^{\frac{1}{1-\alpha}} (BY)^{\frac{1}{1-\alpha}} \left(\frac{\mu}{D} \right)^{\frac{1-\alpha-\beta}{1-\alpha}} LY} - \delta^K \\
&= \frac{\chi^\gamma \left(\frac{1-\mu}{1-\nu} \right)^{1-\gamma} \left(\frac{D}{\mu} \right)^{\varrho(1-\gamma\eta)} \left(\frac{\alpha}{r + \delta^K} \right)^{\frac{\alpha(1-\gamma\eta)}{1-\alpha}} \left(\frac{1-\alpha-\beta}{1-\gamma} \right)^{1-\gamma} \left(\frac{\beta}{1-\eta} \right)^{\gamma(1-\eta)}}{\theta D^{1-\gamma + \frac{(1-\alpha-\beta)\gamma\eta}{1-\alpha}} \left(\frac{\alpha}{r + \delta^K} \right)^{\frac{1}{1-\alpha}} \left(\frac{\mu}{D} \right)^{\frac{1-\alpha-\beta}{1-\alpha}} \left(\frac{1}{1-\nu} - \frac{\nu}{1-\nu} \frac{1}{\mu} \right) \eta^{\gamma\eta} \left(\frac{\gamma}{r + \delta^X} \right)^\gamma} + \\
&\quad \frac{r + (1-\alpha)\delta^K}{\alpha} \\
&= \frac{\left(\eta^\eta \left(\frac{\nu}{\delta^X} \right)^{\frac{1-\eta}{1-\gamma\eta}} \left(\frac{1-\alpha-\beta}{1-\gamma} \frac{\gamma}{r + \delta^X} \right)^{\frac{\eta(1-\gamma)}{1-\gamma\eta}} \left(\frac{\beta}{1-\eta} \right)^{\frac{\gamma\eta(1-\eta)}{1-\gamma\eta}} \right)^\gamma \frac{1-\mu}{1-\nu} \times}{\mu^{(1-\eta + \frac{\beta\eta}{1-\alpha})\gamma} \theta D^{1-\gamma + \frac{(1-\alpha-\beta)\gamma\eta}{1-\alpha}} \frac{\alpha}{r + \delta^K} \left(\frac{1}{1-\nu} - \frac{\nu}{1-\nu} \frac{1}{\mu} \right) \eta^{\gamma\eta} \left(\frac{\gamma}{r + \delta^X} \right)^\gamma} \\
&\quad \left(\frac{D}{\mu} \right)^{1 - \frac{\beta\gamma\eta}{1-\alpha} - \gamma(1-\eta)} \left(\frac{1-\alpha-\beta}{1-\gamma} \right)^{1-\gamma} \left(\frac{\beta}{1-\eta} \right)^{\gamma(1-\eta)} + \frac{r + (1-\alpha)\delta^K}{\alpha} \\
&= \frac{\left(\frac{\nu}{\delta^X} \right)^{\frac{(1-\eta)\gamma}{1-\gamma\eta}} \left(\frac{D}{\mu} \right)^{1 - \frac{\beta\gamma\eta}{1-\alpha} - \gamma(1-\eta)} \left(\frac{1-\alpha-\beta}{1-\gamma} \right)^{\frac{1-\gamma}{1-\gamma\eta}} \left(\frac{\beta}{1-\eta} \right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \frac{1-\mu}{\mu}}{\mu^{(1-\eta + \frac{\beta\eta}{1-\alpha})\gamma} \theta D^{1-\gamma + \frac{(1-\alpha-\beta)\gamma\eta}{1-\alpha}} \frac{\alpha}{r + \delta^K} \left(\frac{\gamma}{r + \delta^X} \right)^{\frac{(1-\eta)\gamma}{1-\gamma\eta}} \frac{\mu-v}{\mu}} + \\
&\quad \frac{r + (1-\alpha)\delta^K}{\alpha} \\
&= \frac{r + (1-\alpha)\delta^K}{\alpha} - \frac{r + \delta^K}{\alpha} \frac{1}{\theta} \left(\frac{1-\alpha-\beta}{1-\gamma} \right)^{\frac{1-\gamma}{1-\gamma\eta}} \times \\
&\quad \left(\frac{\beta\nu}{1-\eta} \right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{r + \delta^X}{\gamma\delta^X} \right)^{\frac{(1-\eta)\gamma}{1-\gamma\eta}} \frac{1-\mu}{\mu-v} \tag{171}
\end{aligned}$$

For $Y = C + \delta^K K + p^M M$ to hold, this must be equal to

$$\begin{aligned}
\frac{p^M M}{K} &= \frac{(L^X)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{\gamma\eta \left(\frac{D}{\mu}\right)^{\theta(1-\gamma\eta)} (B^Y)^{\frac{1-\gamma\eta}{1-\alpha}} \left(\frac{\psi(1-\alpha-\beta)}{1-\gamma}\right)^{1-\gamma} (\delta^X N)^{1-\gamma} (B^X)^\gamma}{(r+\delta^X)^{1-\gamma\eta} \gamma^\eta (B^X)^\gamma \left(\frac{1-\eta}{\beta}\right)^{\gamma(1-\eta)}} \right)^{\frac{1}{1-\gamma\eta}}}{\frac{\alpha}{r+\delta^K} (B^Y)^{\frac{1}{1-\alpha}} \left(\frac{\mu}{D}\right)^{\frac{1-\alpha-\beta}{1-\alpha}} L^Y} \\
&= \frac{\left(\frac{v}{1-v} \left[\frac{1}{\mu} - 1\right]\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \eta \left(\frac{(B^Y)^{\frac{1-\gamma\eta}{1-\alpha}} (1-\alpha-\beta)^{1-\gamma} (\delta^X \frac{1-\mu}{1-v})^{1-\gamma}}{(r+\delta^X)^{1-\gamma} \left(\frac{1-\eta}{\beta}\right)^{\gamma(1-\eta)}} \right)^{\frac{1}{1-\gamma\eta}} \left(\frac{\gamma}{1-\gamma}\right)^{\frac{1-\gamma}{1-\gamma\eta}}}{\frac{\alpha}{r+\delta^K} (B^Y)^{\frac{1}{1-\alpha}} \left(\frac{1}{1-v} - \frac{v}{1-v} \frac{1}{\mu}\right) \mu^{\frac{1-\gamma}{1-\gamma\eta}}} \\
&= \left(\frac{\gamma(1-\alpha-\beta)\delta^X}{(1-\gamma)(r+\delta^X)}\right)^{\frac{1-\gamma}{1-\gamma\eta}} \left(\frac{\beta v}{1-\eta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \frac{1-\mu}{\mu-v} \frac{r+\delta^K}{\alpha} \eta. \tag{172}
\end{aligned}$$

Recall that μ is implicitly defined by

$$\frac{1-\alpha-\beta}{1-\gamma} \left(\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{r+\delta^X}\right) \frac{1-\mu}{\mu-v} = \frac{r+(1-\alpha)\delta^K}{r+\delta^K}. \tag{173}$$

The market for the numeraire good thus clears if

$$\begin{aligned}
&\frac{r+(1-\alpha)\delta^K}{r+\delta^K} \\
&= \left(\frac{\gamma(1-\alpha-\beta)\delta^X}{(1-\gamma)(r+\delta^X)}\right)^{\frac{1-\gamma}{1-\gamma\eta}} \left(\frac{\beta v}{1-\eta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \frac{1-\mu}{\mu-v} \eta + \\
&\frac{1}{\theta} \left(\frac{1-\alpha-\beta}{1-\gamma}\right)^{\frac{1-\gamma}{1-\gamma\eta}} \left(\frac{\beta v}{1-\eta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{r+\delta^X}{\gamma\delta^X}\right)^{\frac{(1-\eta)\gamma}{1-\gamma\eta}} \frac{1-\mu}{\mu-v} \\
&\iff \left(\frac{1-\alpha-\beta}{1-\gamma}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} = \left(\frac{\beta v}{1-\eta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{r+\delta^X}{\gamma\delta^X}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \\
&\iff \left(\frac{1-\alpha-\beta}{1-\gamma}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} = \left(\frac{(1-\alpha-\beta)\gamma\delta^X}{(1-\gamma)(r+\delta^X)}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{r+\delta^X}{\gamma\delta^X}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}}, \tag{174}
\end{aligned}$$

which holds. ■

Structures and Housing Services in Long Run Equilibrium

Proposition A.1. (Long run "height" of structures and housing consumption). *Under (A1), in long run equilibrium,*

(i) *the amount of structures per unit of land, X^*/Z , is increasing in D , B^Y , B^X , B^h ,*

B^M , and independent of ψ ;

(ii) the amount of housing services per capita, S^*/L , is increasing in B^Y , B^X , B^h , B^M , and decreasing in both D and ψ .

Proof. Inserting (89) into (86) and using (81), (99), (123) and (116) implies that the long run equilibrium amount of structure per unit of land is given by

$$\frac{X^*}{Z} = \frac{1-\mu}{1-\nu} \left(\frac{\nu}{\delta^X} \frac{D}{\mu} \right)^{\frac{1-\eta}{1-\gamma\eta}} \left(\frac{\gamma\eta B^M B^h p^*}{\left(\frac{\rho}{1-\tau_r} + \delta^X\right)\psi^{1-\gamma}} \right)^{\frac{\eta}{1-\gamma\eta}} (B^X)^{\frac{1}{1-\gamma\eta}}. \quad (175)$$

According to (67) and (116), we have

$$B^h p^* = \left(\frac{D}{\mu} \right)^{\varrho(1-\gamma\eta)} \frac{\left(\frac{\alpha}{\frac{\rho}{1-\tau_r} + \delta^K} \right)^{\frac{\alpha(1-\gamma\eta)}{1-\alpha}} (B^Y)^{\frac{1-\gamma\eta}{1-\alpha}} \left(\frac{\psi(1-\alpha-\beta)}{1-\gamma} \right)^{1-\gamma}}{(\eta B^M)^{\gamma\eta} \left(\frac{B^X \gamma}{\frac{\rho}{1-\tau_r} + \delta^X} \right)^{\gamma} \left(\frac{1-\eta}{\beta} \right)^{\gamma(1-\eta)}} \quad (176)$$

Substituting (176) into (175) and using the definition of ϱ in (A1) yields

$$\begin{aligned} \frac{X^*}{Z} &= \chi D^{1-\eta+\frac{\beta\eta}{1-\alpha}} (B^M)^\eta B^X (B^Y)^{\frac{\eta}{1-\alpha}}, \text{ where} \quad (177) \\ \chi &\equiv \frac{\eta^\eta \left(\frac{\beta}{1-\eta} \right)^{\frac{\gamma\eta(1-\eta)}{1-\gamma\eta}} \left(\frac{\alpha}{\frac{\rho}{1-\tau_r} + \delta^K} \right)^{\frac{\alpha\eta}{1-\alpha}} \left(\frac{\nu}{\delta^X} \right)^{\frac{1-\eta}{1-\gamma\eta}}}{1-\nu} \frac{1-\mu}{\mu^{1-\eta+\frac{\beta\eta}{1-\alpha}}} \left(\frac{1-\alpha-\beta}{1-\gamma} \frac{\gamma}{\frac{\rho}{1-\tau_r} + \delta^X} \right)^{\frac{\eta(1-\gamma)}{1-\gamma\eta}} \end{aligned} \quad (178)$$

According to (26) and $X = Nx$, the consumption of housing services per capita is given by

$$\frac{S}{L} = \frac{N}{L} B^h \left(\frac{X}{N} \right)^\gamma = \left(\frac{N}{Z} \right)^{1-\gamma} \frac{B^h}{D} \left(\frac{X}{Z} \right)^\gamma. \quad (179)$$

Using (123) and (175) in (179), we obtain

$$\frac{S^*}{L} = \chi^\gamma \left(\frac{1-\mu}{\psi(1-\nu)} \right)^{1-\gamma} \frac{B^h}{D^{1-\gamma+\frac{(1-\alpha-\beta)\gamma\eta}{1-\alpha}}} (B^M)^{\gamma\eta} (B^X)^\gamma (B^Y)^{\frac{\gamma\eta}{1-\alpha}}. \quad (180)$$

Parts (i) and (ii) follow from (177) and (180), respectively, and using that μ is independent of D and \mathbf{B} . This concludes the proof. ■

Since $\mathfrak{z}^{N^*} = \psi N^*/Z$ is independent of population density, D , and productivity parameters, \mathbf{B} , according to (119), the number of houses per unit of land, N^*/Z , is inde-

pendent of D and \mathbf{B} as well. Thus, an increase in D or \mathbf{B} means that more structures are built per unit of land, e.g., houses become "higher" rather than more numerous in more densely populated and in more advanced regions, reflecting the opportunity costs of land in its alternative use in the Y -sector. Technological progress therefore implies that the amount of housing services increases as well. That would also hold if on each unit of land more houses could be built (decrease in ψ). Finally, because of decreasing returns in transforming structures on a piece of land into housing services, the per capita amount of housing services is decreasing in population density.

Factor Income Shares and Investment Rates in Long Run Equilibrium

We finally consider the labor share in total income, $\mathfrak{L}_t \equiv w_t L_t / GDP_t$, and the investment (and savings) rate of the economy can be decomposed into a non-residential and residential (housing) investment rate, denoted by $\mathfrak{s}_t^K \equiv I_t^K / GDP_t$ and $\mathfrak{s}_t^H \equiv (I_t^N + q_t^X I_t^X) / GDP_t$, respectively, i.e.,

$$\mathfrak{s}_t \equiv \frac{I_t^K + I_t^N + q_t^X I_t^X}{GDP_t} = \mathfrak{s}_t^K + \mathfrak{s}_t^H. \quad (181)$$

Proposition A.2. (Long run factor income shares and investment rates).

Under (A1), in long run equilibrium, the labor share in total income is

$$\mathfrak{L}^* = \frac{w^* L}{GDP^*} = \frac{\beta + \frac{\frac{\rho}{1-\tau_r} + (1-\alpha)\delta^K}{\frac{\rho}{1-\tau_r} + \delta^K} \frac{(1-\eta)\gamma\delta^X}{\left(\frac{\frac{\rho}{1-\tau_r} + \delta^X}{\theta} + \eta\gamma\delta^X\right)}}{1 + \frac{\frac{\rho}{1-\tau_r} + (1-\alpha)\delta^K}{\frac{\rho}{1-\tau_r} + \delta^K} \frac{\left(\frac{\rho}{1-\tau_r} + \delta^X(1+(1-\eta)\gamma)\right)}{\left(\frac{\frac{\rho}{1-\tau_r} + \delta^X}{\theta} + \eta\gamma\delta^X\right)}}, \quad (182)$$

and the economy's total investment rate is

$$\mathfrak{s}^* = \frac{\left(\frac{\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{\frac{\rho}{1-\tau_r} + \delta^X}\right) \left(\frac{\rho}{1-\tau_r} + \delta^K\right)}{\frac{\rho}{1-\tau_r} + (1-\alpha)\delta^K} + \frac{(1-\eta)\gamma\delta^X}{\frac{\rho}{1-\tau_r} + \delta^X} - \frac{1}{\theta}}{1 + \frac{\left(\frac{\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{\frac{\rho}{1-\tau_r} + \delta^X}\right) \left(\frac{\rho}{1-\tau_r} + \delta^K\right)}{\frac{\rho}{1-\tau_r} + (1-\alpha)\delta^K} + \frac{(1-\eta)\gamma\delta^X}{\frac{\rho}{1-\tau_r} + \delta^X}}}. \quad (183)$$

Thus, \mathfrak{L}^* , and \mathfrak{s}^* are independent of D , \mathbf{B} , ψ . Also net capital income shares $r^* \mathfrak{S}^*$ and

$r^*\mathfrak{N}^*$, the non-residential investment rate, \mathfrak{s}^{K^*} , and the residential investment rate, \mathfrak{s}^{H^*} , are independent of D , \mathbf{B} , ψ .

Proof. Using (29), we have

$$\mathfrak{L}_t = \frac{\frac{w_t L_t}{Y_t}}{1 + \frac{p_t N_t h_t}{Y_t} + \frac{w_t L_t}{Y_t} \frac{L_t^X}{L_t}}. \quad (184)$$

Using (120) and (136) together with (81), (98) and (116) in (184) confirms (182). Moreover, by using (28) and (181),

$$\mathfrak{s}_t = 1 - \frac{C_t + p_t S_t}{GDP_t} = 1 - \frac{\frac{C_t}{p_t S_t} + 1}{1 + \frac{Y_t}{p_t N_t h_t} + \frac{w_t L_t^X}{p_t N_t h_t}}. \quad (185)$$

Using (37), (79) and (122) in (185) confirms (183). With respect to capital income shares, the result follows from part (iv) of Proposition 2 and Proposition 3. Also note that $\dot{K} = 0$ implies $I^{K^*} = \delta^K K^*$; thus, $\mathfrak{s}^{K^*} = I^{K^*}/GDP^* = \delta^K \mathfrak{K}^*$, confirming its claimed properties by recalling Proposition 3. Finally, recall $\mathfrak{s}^{H^*} = \mathfrak{s}^* - \mathfrak{s}^{K^*}$ and (183). This concludes the proof. ■

ρ	τ_r	\mathfrak{L}^*	$r^*\mathfrak{H}^*$	$r^*\mathfrak{N}^*$	l^{X^*}	\mathfrak{z}^{N^*}	\mathfrak{s}^*	\mathfrak{s}^{K^*}	\mathfrak{s}^{H^*}
0.02	0.3	59.6	10.3	14.0	6.6	17.0	16.9	12.2	4.7
0.03	0.3	59.0	11.8	14.8	5.3	17.1	14.3	10.7	3.6
0.02	0.26	59.6	10.1	13.0	6.5	16.7	17.2	12.4	4.8
0.03	0.26	59.1	11.6	14.6	5.2	16.8	14.7	10.9	3.8
0.02	0.44	59.3	11.1	14.1	7.2	18.2	15.5	11.4	4.1
0.03	0.44	58.8	12.6	15.7	5.6	18.0	12.9	9.8	3.1

Table A.1. Long run implications for labor income share, net capital income shares, allocation variables, and investment rates.

Notes: All endogenous values in percent. In addition to the displayed values for ρ and τ_r , results are based on calibration $\alpha = 0.21$, $\beta = 0.69$, $\gamma = 0.9$, $\eta = 0.38$, $\theta = 0.22$, $\delta^X = 0.015$, $\delta^K = 0.07$. Recall $\mathfrak{L}^* = \frac{w^* L}{GDP^*}$, $\mathfrak{K}^* = \frac{K^*}{GDP^*}$, $\mathfrak{N}^* = \frac{K^* + P^{Z^*} Z^{Y^*}}{GDP^*}$, $l^{X^*} = \frac{L^{X^*}}{L}$, $\mathfrak{z}^{N^*} = \frac{Z^{N^*}}{Z}$, $\mathfrak{s}^* = \frac{I^{K^*} + I^{N^*} + q^{X^*} I^{X^*}}{GDP^*}$, $\mathfrak{s}^{K^*} = \frac{I^{K^*}}{GDP^*}$, $\mathfrak{s}^{H^*} = \frac{I^{N^*} + q^{X^*} I^{X^*}}{GDP^*}$.

According to Table A.1, net income shares, the factor allocation and investment rates are not critically depending on ρ . The implied long-run labor share in income, \mathfrak{L}^* , is about 59 percent. Looking at the latest available year, the labor income share was 57.2 percent in the year 2012 in the US and 62.1 percent in 2011 in the UK (Karabarbounis and Neiman, 2014, "CLS KN merged"). For $\rho = 0.03$, the capital shares with respect to housing and non-residential capital are 11.8 and 14.8 percent, respectively, and somewhat lower for $\rho = 0.02$. Ronglie (2015) suggests that the G7-countries' average for these respective capital shares are about 10 and 20 percent in recent times, with only the housing capital share rising over time. The implied employment fraction in residential construction, l^{X*} , is 5.3 and 6.6 percent for ρ equal to three and two percent, respectively. It is relatively high compared with the more recent US value of 2.7 percent used for calibrating the model. The result is associated with a higher stock of structures, X , in the long run compared to the current (off steady state) one, in line with our prediction of a rising \mathfrak{H} . A high construction labor employment share copes with the constant depreciation rate of structures, δ^X , given a high value of X . The implied long run fraction of land devoted to the housing sector in the economy, \mathfrak{J}^{N*} , is about 17 percent.

The long run investment rate for the baseline calibration, \mathfrak{s}^* , is 14.3 and 16.9 percent for ρ equal to three and two percent, respectively, and may be compared to the US gross private domestic investment rate of 16.5 percent for the year 2014 (Bureau of Economic Analysis, 2015b). The long run fraction of non-residential investment in GDP, \mathfrak{s}^{K*} , that we used for calibrating the model, is 10.7 and 12.2 percent for ρ equal to three and two percent, respectively, and may be compared to the longer term US average of 12.6 percent. That is, the implied long run residential investment rate, $\mathfrak{s}^{H*} = \mathfrak{s}^* - \mathfrak{s}^{K*}$, is in the range of 3.6 to 4.7 percent.

Online-Appendix B: Canonical Model

In Online-Appendix B, we first characterize analytically the long run equilibrium of the canonical model. We then invoke Walras' law as a consistency check of the analytical derivation of the long run equilibrium. Finally, we summarize the dynamical system of the canonical model.

Long Run Equilibrium

Define $l^X \equiv L^X/L$, $l^Y \equiv L^Y/L$, $k^X \equiv K^X/L$, $k^Y \equiv K^Y/L$, $k \equiv K/L$, $h \equiv H/L$ and $\bar{z} \equiv \bar{Z}/L$. Thus, (56) and (57) can be written as $k^X + k^Y = k$ and $l^X + l^Y = 1$, respectively.

Proposition B.1. *In the canonical model, (i) the long run allocation of labor, (l^{X*}, l^{Y*}) , is independent of both new land per capita, \bar{z} , and productivity parameters B^X, B^Y, B^H ; (ii) the long run capital-labor ratio (k^*) is independent of \bar{z}, B^X, B^H , and increasing in B^Y .*

Proof of Proposition B.1. Denote by p_t^X the price of structures at time t . Moreover, define $k^X \equiv K^X/L$, $k^Y \equiv K^Y/L$. Profit maximization in the X - and Y -sector then implies

$$\alpha B^Y \left(\frac{l^Y}{k^Y} \right)^{1-\alpha} = \gamma p^X B^X \left(\frac{l^X}{k^X} \right)^{1-\gamma} = r + \delta^K, \quad (186)$$

$$(1 - \alpha) B^Y \left(\frac{k^Y}{l^Y} \right)^\alpha = (1 - \gamma) p^X B^X \left(\frac{k^X}{l^X} \right)^\gamma = w. \quad (187)$$

Combining (186) and (187) implies

$$k^X = \frac{\gamma}{1 - \gamma} \frac{w l^X}{r + \delta^K}, \quad (188)$$

$$k^Y = \frac{\alpha}{1 - \alpha} \frac{w l^Y}{r + \delta^K}. \quad (189)$$

Thus, using resource constraints $k^X + k^Y = k$ and $l^X + l^Y = 1$, we obtain

$$k = \frac{w}{r + \delta^K} \left(\frac{\gamma}{1 - \gamma} l^X + \frac{\alpha}{1 - \alpha} (1 - l^X) \right). \quad (190)$$

Moreover, according to (186), (187) and resource constraints,

$$p^X = \frac{(1 - \alpha)B^Y \left(\frac{k - k^X}{1 - l^X}\right)^\alpha}{(1 - \gamma)B^X \left(\frac{k^X}{l^X}\right)^\gamma}, \quad (191)$$

$$w = (1 - \alpha)B^Y \left(\frac{k - k^X}{1 - l^X}\right)^\alpha. \quad (192)$$

The current-value Hamiltonian of the representative housing firm associated with its profit maximization problem together with the necessary first-order conditions can be expressed as

$$\mathcal{H}^H = pH - p^X X - R^Z Z^H + q^H [B^H X^\beta \bar{Z}^{1-\beta} - \delta^H H], \quad (193)$$

$$\left[\frac{\partial \mathcal{H}^H}{\partial X} = \right] - p^X + \beta q^H B^H \left(\frac{\bar{Z}}{X}\right)^{1-\beta} = 0, \quad (194)$$

$$\left[\frac{\partial \mathcal{H}^H}{\partial \bar{Z}} = \right] - R^Z + (1 - \beta)q^H B^H \left(\frac{X}{\bar{Z}}\right)^\beta = 0, \quad (195)$$

$$\left[-\frac{\partial \mathcal{H}^H}{\partial H} = \right] - p + \delta^H q^H = \dot{q}^H - r q^H. \quad (196)$$

Substituting (52) into (194) and (195), we have

$$p^X = \beta q^H \frac{\tilde{B}^H}{B^X} \left(\frac{\bar{z}}{(k^X)^\gamma (l^X)^{1-\gamma}}\right)^{1-\beta}, \quad (197)$$

$$R = (1 - \beta)q^H \tilde{B}^H \left(\frac{(k^X)^\gamma (l^X)^{1-\gamma}}{\bar{z}}\right)^\beta, \quad (198)$$

respectively, where we used $\tilde{B}^H = B^H (B^X)^\beta$. Combining (191) and (197) leads to

$$q^H = \frac{(1 - \alpha)B^Y \left(\frac{k - k^X}{1 - l^X}\right)^\alpha (l^X)^{1-\beta(1-\gamma)}}{(1 - \gamma)\beta \tilde{B}^H \bar{z}^{1-\beta} (k^X)^{\beta\gamma}}. \quad (199)$$

Setting $\dot{H} = 0$ in (53) we get

$$h = \frac{\tilde{B}^H}{\delta^H} (k^X)^{\beta\gamma} (l^X)^{\beta(1-\gamma)} \bar{z}^{1-\beta}. \quad (200)$$

Using $S = H$, analogously to (79) and (80), utility maximization of the representative household yields

$$C = \frac{p^H}{\theta}, \quad (201)$$

$$\frac{\dot{C}}{C} = r - \rho. \quad (202)$$

In long run equilibrium, again, $\dot{C} = 0$ implies $r^* = \rho$. Using $\dot{q}^H = 0$ in (196) we find

$$p = (r + \delta^H)q^H. \quad (203)$$

Moreover, setting $\dot{A} = 0$ in (55) and using (201), $S = H$, (54) and (203) leads to

$$rk + w + R^Z \bar{z} = \left(\frac{r + (1 + \theta)\delta^H}{\theta} \right) q^H h. \quad (204)$$

Substituting (190), (192), (198), (199) and (200) into (204), we get

$$l^X = \frac{[r + (1 - \alpha)\delta^K] \beta \theta \delta^H (1 - \gamma)}{(1 - \alpha)(r + \delta^K) [r + (1 + \beta\theta)\delta^H] + \beta \theta \delta^H r (\alpha - \gamma)} \equiv l^{X*}. \quad (205)$$

This confirms part (i) of Proposition B.1. Substituting (192) into (190) leads to

$$k = \frac{(1 - \alpha)B^Y}{r + \delta^K} \left(\frac{\gamma}{1 - \gamma} l^X + \frac{\alpha}{1 - \alpha} (1 - l^X) \right) \left(\frac{k - k^X}{1 - l^X} \right)^\alpha, \quad (206)$$

Substituting $w = (1 - \alpha)B^Y (k^Y/l^Y)^\alpha$ from (187) into (189) and using $l^X = l^{X*}$ we obtain

$$k^Y = k - k^X = \left(\frac{\alpha B^Y}{r + \delta^K} \right)^{\frac{1}{1-\alpha}} (1 - l^X) \equiv k^{Y*}. \quad (207)$$

Substituting (207) into (206) and using $l^X = l^{X*}$ yields

$$k = \left(\frac{\alpha B^Y}{r + \delta^K} \right)^{\frac{1}{1-\alpha}} \left(\frac{(\gamma - \alpha)l^{X*}}{\alpha(1 - \gamma)} + 1 \right) \equiv k^*. \quad (208)$$

Thus, in view of (205), also part (ii) of Proposition B.1 is confirmed. ■

Proposition B.2. *In the canonical model, (i) the long run house price, q^{H*} , is*

decreasing in new land per capita, \bar{z} , increasing in B^Y , and decreasing in B^X and B^H ;
(ii) the long run stock of houses per capita, h^* , is increasing in B^X , B^H , B^Y , and \bar{z} .

Proof of Proposition B.2. Substituting (192) into (188), we obtain

$$k^X = \frac{\gamma}{1-\gamma} \frac{(1-\alpha)B^Y l^X}{r + \delta^K} \left(\frac{k - k^X}{1 - l^X} \right)^\alpha. \quad (209)$$

Next, substitute (209) into (199) and (200) and use both (207) and $l^X = l^{X*}$ to find

$$q^H = \frac{(1-\alpha)^{1-\beta\gamma} (B^Y)^{\frac{1-\beta\gamma}{1-\alpha}} \alpha^{\frac{\alpha(1-\beta\gamma)}{1-\alpha}}}{\gamma^{\beta\gamma} (1-\gamma)^{1-\beta\gamma} \beta \tilde{B}^H} \left(\frac{1}{r + \delta^K} \right)^{\frac{\alpha-\beta\gamma}{1-\alpha}} \left(\frac{l^{X*}}{\bar{z}} \right)^{1-\beta} \equiv q^{H*}, \quad (210)$$

$$h = \frac{\tilde{B}^H}{\delta^H} \left(\frac{\gamma}{1-\gamma} \right)^{\beta\gamma} (1-\alpha)^{\beta\gamma} \alpha^{\frac{\alpha\beta\gamma}{1-\alpha}} \left(\frac{B^Y}{r + \delta^K} \right)^{\frac{\beta\gamma}{1-\alpha}} (l^{X*})^\beta \bar{z}^{1-\beta} \equiv h^*, \quad (211)$$

respectively. Parts (i) and (ii) are confirmed by (210) and (211), respectively. ■

Proposition B.3. Let housing-wealth-to-GDP ratio be defined by $\mathfrak{H}_t = \frac{q_t^H H_t}{Y_t + p_t H_t + q_t^H I_t^H}$. In the canonical model, a change in B^X , B^H , B^Y , and \bar{z} do not affect (i) the long run housing-wealth-to-GDP ratio, (ii) the capital to GDP ratio, (iii) the labor share in GDP, and (iv) the savings rate.

Proof of Proposition B.3. The "house-price-to-rent ratio" in the canonical model is $\mathfrak{p}_t = q_t^H / p_t$. Using $\dot{q}^H = 0$ in (196), the long run value of the house-price-to-rent ratio is given by

$$\mathfrak{p}^* = \frac{1}{r + \delta^H}. \quad (212)$$

Using $p^*/q^{H*} = r + \delta^H$, the definition of GDP in (59), $Y/L = B^Y (k^Y)^\alpha (l^Y)^{1-\alpha}$ from (50), and long run relationship $I^H/L = \delta^H h$ in $\mathfrak{H}_t = \frac{q_t^H H_t}{Y_t + p_t H_t + q_t^H I_t^H}$, we obtain

$$\mathfrak{H}^* = \frac{q^{H*} h^*}{\left(\frac{Y}{L}\right)^* + p^* h^* + q^{H*} \left(\frac{I^H}{L}\right)^*} = \frac{1}{\frac{B^Y (k^{Y*})^\alpha (l^{Y*})^{1-\alpha}}{q^{H*} h^*} + r + 2\delta^H}. \quad (213)$$

Using (210) and (211), we have

$$q^{H^*}h^* = \frac{(1-\alpha)(B^Y)^{\frac{1}{1-\alpha}}}{(1-\gamma)\beta\delta^H} \left(\frac{\alpha}{r+\delta^K} \right)^{\frac{\alpha}{1-\alpha}} l^{X^*}. \quad (214)$$

According to (207), (214) and $l^{Y^*} = 1 - l^{X^*}$, we obtain

$$\frac{B^Y (k^{Y^*})^\alpha (l^{Y^*})^{1-\alpha}}{q^{H^*}h^*} = \frac{(1-\gamma)\beta\delta^H}{1-\alpha} \frac{1-l^{X^*}}{l^{X^*}}. \quad (215)$$

Using (215) in (213) and recalling (205) confirms the part (i).

Using (59), the long run capital to GDP ratio can be written as

$$\mathfrak{N}^* = \frac{k^*}{\left(\frac{Y}{L}\right)^* + p^*h^* + q^{H^*} \left(\frac{I^H}{L}\right)^*} = \frac{\frac{k^*}{q^{H^*}h^*}}{\frac{B^Y (k^{Y^*})^\alpha (l^{Y^*})^{1-\alpha}}{q^{H^*}h^*} + r + 2\delta^H}. \quad (216)$$

Combining (208) and (214) implies

$$\frac{k^*}{q^{H^*}h^*} = \frac{\beta\delta^H}{(1-\alpha)(r+\delta^K)} \left(\gamma - \alpha + \frac{\alpha(1-\gamma)}{l^{X^*}} \right). \quad (217)$$

Using (215) and (217) in (216) and recalling (205) confirms the part (ii). Using (59), the long run labor share in GDP can be written as

$$\mathfrak{L}^* = \frac{w^*}{\left(\frac{Y}{L}\right)^* + p^*h^* + q^{H^*} \left(\frac{I^H}{L}\right)^*} = \frac{\frac{w^*}{q^{H^*}h^*}}{\frac{B^Y (k^{Y^*})^\alpha (l^{Y^*})^{1-\alpha}}{q^{H^*}h^*} + r + 2\delta^H}. \quad (218)$$

Using $l^Y = 1 - l^X$ in (187) and (207), we have

$$w^* = (1-\alpha)(B^Y)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{r+\delta^K} \right)^{\frac{\alpha}{1-\alpha}}. \quad (219)$$

Combining (219) and (214) implies

$$\frac{w^*}{q^{H^*}h^*} = \frac{(1-\gamma)\beta\delta^H}{l^{X^*}}. \quad (220)$$

Using (215) and (220) in (218) and recalling (205) confirms the part (iii).

Finally, to prove part (iv), according to (59), the long run savings rate can be rewritten as

$$\mathbf{s}^* = 1 - \frac{\left(\frac{C}{L}\right)^* + p^* \left(\frac{S}{L}\right)^*}{\left(\frac{Y}{L}\right)^* + p^* h^* + q^{H^*} \left(\frac{I^H}{L}\right)^*} = 1 - \frac{\left(\frac{1}{\theta} + 1\right) (r + \delta^H)}{\frac{B^Y (k^{Y^*})^\alpha (l^{Y^*})^{1-\alpha}}{q^{H^*} h^*} + r + 2\delta^H}, \quad (221)$$

where we divided both nominator and denominator by $q^{H^*} h^*$ and used (201), $(S/L)^* = h^*$ and $p^*/q^{H^*} = r + \delta$ to derive the second equation. Using (215) in (221) and recalling (205) concludes the proof. ■

Consistency Check Using Walras' Law

We show that the market for the numeraire good clears in long run equilibrium, where $\dot{K} = \dot{H} = 0$. That is, it has to hold that

$$\frac{Y^*}{L} = \frac{C^*}{L} + \delta^K \frac{K^*}{L}. \quad (222)$$

Since $Y/L = B^Y (k^Y)^\alpha (l^Y)^{1-\alpha}$, $C/L = ph/\theta$ (recall (201) and $S/L = h$) and $K/L = k$, we have to check if $B^Y (k^{Y^*})^\alpha (l^{Y^*})^{1-\alpha} = p^* h^*/\theta + \delta^K k^*$, i.e.,

$$\frac{B^Y (k^{Y^*})^\alpha (l^{Y^*})^{1-\alpha}}{q^{H^*} h^*} = \frac{p^*}{\theta q^{H^*}} + \frac{\delta^K k^*}{q^{H^*} h^*}. \quad (223)$$

Substituting (208), (214), (215) and $p^*/q^{H^*} = r + \delta^H$ into (223) it should hold that:

$$\frac{(1-\gamma)\beta\delta^H (1-l^{X^*})}{(1-\alpha)l^{X^*}} = \frac{r + \delta^H}{\theta} + \frac{\delta^K \left(\frac{\alpha B^Y}{r + \delta^K}\right)^{\frac{1}{1-\alpha}} \left(\frac{(\gamma-\alpha)l^{X^*}}{\alpha(1-\gamma)} + 1\right)}{\frac{(1-\alpha)\alpha^{\frac{1-\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}}}{(1-\gamma)\beta\delta^H} \left(\frac{1}{r + \delta^K}\right)^{\frac{1-\alpha}{1-\alpha}} l^{X^*}}, \quad (224)$$

being equivalent to

$$l^{X^*} = \frac{[r + (1-\alpha)\delta^K] (1-\gamma)\beta\theta\delta^H}{(r + \delta^K) [(1-\alpha)(r + \delta^H) + (1-\gamma)\beta\theta\delta^H] + \beta\theta\delta^H \delta^K (\gamma - \alpha)}. \quad (225)$$

It is easy to show that

$$\begin{aligned} & (r + \delta^K) [(1-\alpha)(r + \delta^H) + (1-\gamma)\beta\theta\delta^H] + \beta\theta\delta^H \delta^K (\gamma - \alpha) \\ &= (1-\alpha)(r + \delta^K) [r + (1 + \beta\theta)\delta^H] + \beta\theta\delta^H r (\alpha - \gamma). \end{aligned} \quad (226)$$

Thus, (205) and (225) coincide. ■

Dynamic System

$$\frac{\dot{C}}{C} = r - \rho. \quad (227)$$

$$\dot{A} = rA + wL + R\bar{Z} - pH - C, \quad (228)$$

$$\dot{H} = \tilde{B}^H (K^X)^{\beta\gamma} (L^X)^{\beta(1-\gamma)} \bar{Z}^{1-\beta} - \delta^H H, \quad (229)$$

$$\frac{\dot{q}^H}{q^H} + \frac{p}{q^H} = r + \delta^H. \quad (230)$$

$$C = \frac{pH}{\theta}, \quad (231)$$

$$A = K + q^H H, \quad (232)$$

$$K^X + K^Y = K, \quad (233)$$

$$L^X + L^Y = L, \quad (234)$$

$$K^X = \frac{\gamma}{1-\gamma} \frac{wL^X}{r + \delta^K}, \quad (235)$$

$$K = \frac{w}{r + \delta^K} \left(\frac{\gamma}{1-\gamma} L^X + \frac{\alpha}{1-\alpha} (L - L^X) \right), \quad (236)$$

$$w = (1-\alpha) B^Y \left(\frac{K - K^X}{L - L^X} \right)^\alpha, \quad (237)$$

$$q^H = \frac{(1-\alpha) B^Y \left(\frac{K - K^X}{L - L^X} \right)^\alpha (L^X)^{1-\beta(1-\gamma)}}{(1-\gamma) \beta \tilde{B}^H \bar{Z}^{1-\beta} (K^X)^{\beta\gamma}}, \quad (238)$$

$$R^Z = (1-\beta) q^H \tilde{B}^H \left(\frac{(K^X)^\gamma (L^X)^{1-\gamma}}{\bar{Z}} \right)^\beta. \quad (239)$$

Recall that $\tilde{B}^H = B^H (B^X)^\beta$. Initial state variables are $K_0 = \text{given}$ and $H_0 = \text{given}$ and $A_0 = K_0 + q_0^H H_0$ (notice that A_0 is a jump variable). These are 13 equations and 13 endogenous variables: $A, C, L^X, L^Y, K, K^X, K^Y, H, r, w, R^Z, p, q^H$.