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Conference Paper

Secular Stagnation? Growth, Asset Returns and Welfare in the Next Decades

Beiträge zur Jahrestagung des Vereins für Socialpolitik 2016: Demographischer Wandel - Session: Demographic Change and the Macroeconomy I, No. B12-V2

Provided in Cooperation with:
Verein für Socialpolitik / German Economic Association

Suggested Citation: Ludwig, Alexander; Geppert, Christian; Abiry, Raphael (2016) : Secular Stagnation? Growth, Asset Returns and Welfare in the Next Decades, Beiträge zur Jahrestagung des Vereins für Socialpolitik 2016: Demographischer Wandel - Session: Demographic Change and the Macroeconomy I, No. B12-V2, ZBW - Deutsche Zentralbibliothek für Wirtschaftswissenschaften, Leibniz-Informationszentrum Wirtschaft, Kiel und Hamburg

This Version is available at:
https://hdl.handle.net/10419/145764

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Secular Stagnation? Growth, Asset Returns and Welfare in the Next Decades*

Preliminary and Incomplete

Abstract

Ongoing demographic change will lead to a relative scarcity of raw labor to the effect that output growth will be decreasing in the next decades, a secular stagnation. As physical capital will be relatively abundant, this decrease of output will be accompanied by reductions of asset returns. We quantify these effects for the US economy by developing an overlapping generations model in which we explicitly model risky and risk-free asset returns. This enables us to predict how the natural interest rate is affected by the demography induced secular stagnation. Without adjustments of human capital, risky returns decrease until 2035 by about 0.7 percentage point, and the risk-free rate by about one percentage point, leading to severe welfare losses for asset rich households. Endogenous human capital adjustments strongly mitigate these effects. We conclude that human capital policies will be crucial in the context of labor shortages.

JEL classification: E17, C68, G12

Keywords: secular stagnation; demographic change; overlapping generations; natural rate; equity premium; growth; welfare; human capital

*We thank Martin Barbie, Axel Börsch-Supan, Helge Braun, Wouter den Haan, Daniel Harenberg, Philip Jung, Tom Krebs, Wolfgang Kuhle, and Edgar Vogel for helpful discussions and various seminar/conference participants at the IMF (Washington), the 2015 REDg Conference (Barcelona), the 2013 DFG SPP Conference (Cologne), SED 2011 (Ghent), ASU, IHS (Vienna), Deutsche Bundesbank, MEA (Mannheim), and University of Cologne for valuable comments. We also gratefully acknowledge financial support by the German National Research Foundation (SPP 1578) and by the Research Center SAFE, funded by the State of Hessen initiative for research LOEWE.

1 Introduction

In his famous keynote addresses at the IMF in 2013 and later at the NABE Policy conference in 2014, Summers (2013, 2014) argued in terms of three propositions as follows: in consequence of "secular stagnation"—a long-run or trend reduction of growth[1]—(i) it will be "increasingly difficult" to achieve "adequate growth" in the US and other industrialized economies; (ii) that these developments will likely be accompanied by "a substantial decline in the equilibrium or natural rate of interest"; and (iii) that "addressing these challenges requires different policy approaches than are represented by the current conventional wisdom".[2]

The present paper addresses all three propositions by analyzing the role of demographic change for the secular stagnation. First, we ask if—and if so how strongly and for how long—demographic change will effect growth of the US economy in the next decades. We do so by developing a structural model of an economy with a production sector and a household sector in order to capture how the relative demands and supplies of production factors change in consequence of the demographic evolution over the next 50 years. In order to realistically capture these demographic developments, we develop an overlapping generations model featuring a very detailed description of the aging process[3] and its role for household decisions.

Second, we investigate whether the potential reduction of growth is accompanied by a decline of real rates of return. To do so we explicitly distinguish between the equilibrium returns of productive capital and the returns of a (one period ahead) risk-free bond. While standard predictions suggest that demographic change leads to a shortage of labor and an abundance of physical capital to the effect that marginal productivities and hence capital returns decline, these effects may be very different across asset classes. Older households may have a higher preference for relatively risk-free investments which would increase relative demand for bonds and therefore bond returns may decrease by more than risky

[1]The term secular is used in contrast to cyclical or short-term, and suggests a change of fundamental dynamics. The term was originally coined by Hansen (1938).

[2]See Summers (2014, p. 66). As Summers (2014) further points out, one reason for secular stagnation might be the demographic development over the next decades leading to an increasing shortage of labor thereby depressing output.

[3]We use the terms "demographic change" and "aging" interchangeably.

asset returns. Also, our distinction between different assets enables us to be explicit about the notion of the "natural rate" which provides important guidance for central banks for their interest rate policies over the next decades. According to Wicksell (1898), the natural rate of interest is the interest rate which is compatible with a stable price level. Hence, an appropriate model of the development of the natural rate is required for fine tuning monetary policy instruments as directly follows from the Taylor rule (Taylor 1993). We deliver such a forecasting instrument by modeling the supply side of assets through a production sector with a capital structure with exogenous leverage, i.e., we assume that an exogenous share of total firms' assets is held in equity, the rest in bonds.[4] The demand side for differential assets stems from the household sector where intra-generational heterogeneity—induced by stochastic income processes—and intergenerational heterogeneity (by age) gives rise to trade across households in both assets.

Third, in terms of policy implications, we emphasize that human capital policies may play an important role in the next decades. While we do not model these policies explicitly, we consider two polar human capital scenarios. In our first scenario households react to increasing life-expectancy and increasing wages as well as falling real asset returns by increasing their human capital investments. According accumulated output losses in 2035—resulting from the demography induced reduction of growth rates—stand at 2 percent relative to a long-run path with constant growth rates. Risky asset returns decrease by roughly 0.16 and risk-free returns by 0.24 percentage points. These quantitative effects are rather mild. In our second scenario we restrict human capital adjustments. Then, output losses stand at roughly 6 percent, risky asset returns decrease by 0.7 and bond returns by 1 percentage point until 2035. This would lead to quite strong welfare losses for middle aged and old households who hold substantial physical and financial wealth. In light of real world frictions on markets for human capital (which we do not model)[5] we argue that our two scenarios bracket the evolution of future asset returns, growth and welfare. Furthermore, we show that social

[4]Leverage is frequently modeled this way in the finance literature to increase the volatility of stock returns, cf., e.g., Boldrin, Christiano, and Fisher (1995) and Croce (2014).

[5]These aspects are not included in the model for technical reasons.

security policies that hold constant contribution rates may be welfare improving.[6] Only then the aforementioned dampening effects on aggregate results and on welfare through human capital adjustments are at work.

These results are driven by demographic developments which we take as exogenous. According to our demographic projections, the working age population ratio[7] will decrease by roughly 10 percentage points. The ensuing shortage of the quantity of raw labor and the accompanying relative abundance of physical capital leads to decreasing output, increasing gross wages and falling asset returns. These relative price movements—in combination with the increasing lifetime horizon resulting from increasing life expectancies—will lead households to increase their human capital investments because relative human capital returns increase and real financial (and physical) capital returns decrease in absolute and relative terms. As a result, aggregate efficiency increases and equilibrium declines in output and asset returns will be mitigated. Furthermore, our model predicts that average portfolio shares of investments in risky assets are decreasing over the life-cycle. Because demographic change shifts population shares so that there are relatively more elderly households—the old-age dependency ratio[8] increases by more than 15 percentage points—the demand for bonds relative to equities increases. This reduces bond returns more strongly, hence the equity premium increases.

Our work relates to a relatively large literature that employs variants of Auerbach and Kotlikoff (1987) to quantitatively evaluate the consequences of demographic change for growth and welfare, cf., e.g., Börsch-Supan, Ludwig, and Winter (2006), Attanasio, Kitao, and Violante (2007), Krüger and Ludwig (2007), Ludwig, Schelkle, and Vogel (2012) and the literature cited therein. Relative to this literature, the main novel aspect in our work is to explicitly model differential asset returns. We thereby relate to a literature on aging and the equity premium (Bakshi and Chen 1994; Brooks 2004; Börsch-Supan, Ludwig, and Sommer 2003; Geanakoplos, Magill, and Quinzii 2004; Kuhle 2008) which has not reached

[6]These results are not yet contained in the current version of the paper.

[7]I.e., the fraction of the population in working age (age 20-64) relative to the total adult population.

[8]I.e., the population in retirement age (age 65 and older) as a fraction of the working age population.

a consensus on the quantitative effects of demographic change on differential asset returns. While Brooks (2004) reports substantial increases in the equity premium, the approximate calculations in Börsch-Supan, Ludwig, and Sommer (2003) rather suggest a small increase. Geanakoplos, Magill, and Quinzii (2004) conclude that "the equity premium is smaller when the population of savers is older" which the authors interpret as a contradiction to the findings of Bakshi and Chen (1994) and Brooks (2004). These papers all employ relatively stylized overlapping generations models with few generations. Such a periodicity severely restricts households to re-balance their portfolios. We avoid such restrictions by employing a large scale overlapping generations model that runs at an annual frequency. To significantly reduce computational costs we adopt the risky human capital framework developed in Krebs (2003) and Krebs and Wilson (2004). This setup gives rise to closed form solutions of households' policy functions for consumption and total saving, conditional on the law of motion of the aggregate state of the economy and the solution for optimal portfolio shares.[9]

This paper proceeds as follows. Section 2 develops simple models to provide the intuition why human capital investments may be more attractive in aging societies and why we may expect the equity premium to increase. Section 3 builds on these simple models to develop the large scale quantitative overlapping generations model. Section 4 describes our approach to numerically solve this model as well as the model's calibration. Section 5 presents the results and, finally, Section 6 concludes. Separate appendices provide proofs of propositions and additional results.

2 Simple Models

[TBC]

[9]Human capital is modeled as an asset and in suitable transformation of our model reduces to a standard portfolio choice model. Once portfolio shares are computed, policy functions of savings and consumption are linear in wealth, as in the seminal work by Merton (1969) and Samuelson (1969).

3 Quantitative Model

We extend the classical Diamond (1965) economy to a multi-period setup as in Auerbach and Kotlikoff (1987) with idiosyncratic and aggregate risk. On the household side, the novelty in this paper is to assume that labor income is a choice variable of households rather than being exogenously given. This feature is implemented by adopting the human capital framework developed in Krebs (2003) and Krebs and Wilson (2004) in an overlapping generations setup. In each period, a household of a given age chooses to invest a fraction of her overall wealth in human capital, respectively financial capital. As for the fraction of wealth invested in financial assets, the household solves a standard portfolio allocation problem by choosing how much to invest into risky physical capital and risk-free bonds. Consequently, there are three assets in the economy: risky human capital, risky physical capital and risk-free bonds. In this setup, once portfolio allocation decisions are made, household consumption and savings policies are linear functions of wealth, cf. Merton (1969) and Samuelson (1969). Therefore, conditional on expectations on the evolution of aggregate prices, the household problem is easy to solve. This feature of the model is particularly useful because it enables us to solve a large-scale OLG model with rather complex dynamics without incurring tremendous computational costs. On the firm side, the model is standard.

3.1 Risk and Time

Time is discrete and runs from $t = 0, \ldots, \infty$. Aggregate uncertainty is represented by an event tree. The economy starts with some fixed event λ_0, and each node of the tree is a history of exogenous shocks $\lambda^t = (\lambda_0, \lambda_1, \ldots, \lambda_t)$. The shocks are assumed to follow a Markov chain with finite support L and strictly positive transition matrix Π. For notational convenience, we will only index variables by time thereby suppressing the dependency of variables on λ^t but it is understood that all choice variables are functions of history.

3.2 Production

Production takes place with a standard Cobb-Douglas production function with total output at time t given by

$$Y_t = z_t \cdot K_t^{\alpha} \cdot (\Upsilon_t \cdot H_t)^{1-\alpha} \tag{1}$$

where K_t is the aggregate stock of physical capital, H_t is the aggregate stock of human capital, and z_t is a stochastic shock to total factor productivity. Υ_t is a human capital augmenting productivity parameter which grows at the exogenous constant rate g which captures the observed trend in GDP in the data.

Profit maximization of firms leads to the standard first order conditions stating that marginal products equal returns minus depreciations:

$$r_t^K = \alpha \cdot z_t \cdot \left(\frac{K_t}{\Upsilon_t \cdot H_t} \right)^{\alpha-1} - \delta_t^K \tag{2a}$$

$$r_t^H = \Upsilon_t \cdot (1-\alpha) \cdot z_t \cdot \left(\frac{K_t}{\Upsilon_t \cdot H_t} \right)^{\alpha}. \tag{2b}$$

Note that r^H grows along with Υ over time while r^K is trend-stationary. Following Krüger and Kübler (2006), Storesletten, Telmer, and Yaron (2007), Gomes and Michaelides (2008) and others we assume that the depreciation rate of physical capital, δ^K, is stochastic.

3.3 Demographics

The economy is populated with $J+1$ overlapping generations and the underlying population dynamics is the exogenous driving force of the model. Households enter the model at the age of 20 ($j = 0$) and live at most until turning 101 ($j = J + 1 = 81$). Population of age j in time period t is given recursively as

$$N_{t,j} = \begin{cases} N_{t-1,j-1} \cdot \varsigma_{t-1,j-1} & \text{for } j = 1, \ldots, J \\ \sum_{l=0}^{j_f} f_{t-20,l} \cdot N_{t-20,l} & \text{for } j = 0 \end{cases} \tag{3}$$

where $\varsigma_{t,j}$ and $f_{t,j}$ denote time and age-specific survival rates and fertility rates respectively. j_f is the age of menopause. Processes governing mortality and fertility are assumed to be non-stochastic.

3.4 Preferences

We take Epstein-Zin preferences. Let θ be the coefficient of relative risk-aversion and ξ denote the elasticity of inter-temporal substitution.

$$u_{t,j} = \left[c_{t,j}^{\frac{1-\theta}{\gamma}} + \beta \cdot \left(\mathbb{E}_{t,j}[\varsigma_{t,j} \cdot u_{t+1,j+1}^{1-\theta}] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \tag{4}$$

where $\gamma := (1-\theta)/(1-1/\xi)$. $0 < \beta < 1$ is the standard discount factor. For $\theta = 1/\xi$ we have $\gamma = 1$ and are back at standard CRRA preferences. β is the raw time discount factor and $c_{t,j}$ is consumption at time t, age j. $\mathbb{E}_{t,j}$ is the expectations operator and expectations are taken with respect to idiosyncratic shocks to human capital and aggregate shocks to productivity and physical capital depreciation conditional on information at time t, age j. As $\varsigma_{t,J}$ equals 0 for all t, equation (4) implies that $u_J = c_J$.

3.5 Endowments

When entering the economy at age $j = 0$, households are endowed with an initial level of human capital, $h_{t,0} = h_0$ for all t and financial wealth $k_{t,0}$ which is set to zero for all t for convenience. Summing financial assets and human capital makes up households' total wealth. Each period, households choose to invest a fraction of their total wealth in financial assets and in human capital respectively. Let $i_{t,j}^h$ denote the amount of wealth invested in human capital.

Human capital earns a gross rate of return of r_t^H which is the marginal product of human capital. The term $r_t^H \cdot h_{t,j}$ can be understood as gross earnings of a household at age j in period t.
We assume that human capital depreciates at the individual level by the age-specific deterministic rate δ_j^h. The age-profile of $\{\delta_j^h\}_{j=1}^{j_r}$ enables us to calibrate

the model such that it mimics decreasing returns to human capital accumulation as assumed elsewhere in the literature (e.g., Huggett, Ventura, and Yaron (2011)). We assume the following functional form

$$\delta_j^h = -\chi_0 + \exp(\chi_1 \cdot j), \quad \chi_0 > 0, \chi_1 \geq 0, \tag{5}$$

which is monotonically increasing in j such that $1 - \chi_0 \leq \delta_j^h \leq \delta_{j+1}^h$ for all j. χ_1 is the rate at which the household's human capital depreciation accelerates when getting older.

After the return to human capital is paid the household is hit by an additive idiosyncratic shock to its human capital holdings:

$$\eta_t \sim \mathcal{D}(0, \sigma^2(\lambda_t)) \tag{6}$$

where $\mathcal{D}$ is some distribution with mean zero further specified in the quantitative section 4. Although the shock is idiosyncratic, it depends on the current state of the economy, λ_t, because, as further discussed below, the variance of the idiosyncratic human capital shock, σ_t^2, depends on the current state of the economy.

Collecting all these elements, the human capital accumulation equation in period t, age j, is given by

$$h_{t+1,j+1} = h_{t,j} \cdot (1 - \delta_j^h + \eta_t) + \tilde{i}_{t,j}^h, \qquad h_{t,j} \geq 0 \ \ \forall\, t, j, \tag{7}$$

where $\tilde{i}_{t,j}^h := i_{t,j}^h / \Upsilon_t$.[10] Note that all variables in (7) are trend-stationary, i.e., they do not exhibit exogenous growth along with human capital productivity in production Υ_t[11].

The household faces a portfolio decision between risky and one period ahead risk-free financial assets, which we denote in the following by *risky equity* and *risk-free bonds* respectively.[12] Let $\alpha_{t,j}^s$ be the fraction of financial assets invested

[10] We assume that costs for human capital investment, i_t^h, grow with the same rate as Υ_t.

[11] As the return to human capital r_t^H already exhibits a trend growth along with Υ_t, human capital must be trend stationary in order to assure that gross human capital earnings, $h_{t,j} \cdot r_t^H$, grow at the same rate as Υ_t over time.

[12] The assumption of risk-free bonds is not innocent and relies on the assumption that debtors

in risky equity in period t, age j. The dynamic financial asset accumulation equation in period t, age j, is then given by

$$k_{t+1,j+1} = k_{t,j} \cdot (1 + r_t^f + \alpha_{t,j}^s \cdot (r_t^s - r_t^f)) + h_{t,j} \cdot r_t^H - i_{t,j}^h - c_{t,j} \tag{8}$$

3.6 Recursive Household Problem

We hereafter define recursively the household problem conditional on a law of motion of the aggregate state of the economy. We present a de-trended version of the household problem derived in section A.1 of the appendix. The symbol $\tilde{}$ indicates that a variable has been transformed, e.g. $\tilde{k} := k/\Upsilon$, thereby removing the trend from (exogenous) labor-augmenting technological progress. It is convenient to express next period's values with symbol $'$, irrespective of whether they are only time-dependent or both, age- and time-dependent. The states of the household problem are the exogenous states j, t, and λ, the endogenous idiosyncratic state of (de-trended) cash-on-hand, $\tilde{x}$, further specified below as well as the distribution of (de-trended) wealth, Ω, which is the endogenous aggregate state of the economy. The associated law of motion is $\Omega' = \Phi(\Omega, \lambda, \lambda', N')$. The existence of aggregate shocks implies that Ω evolves stochastically over time. Notice that λ' is a determinant of Ω' because it specifies $r^{K\prime}$ and $r^{H\prime}$. A change in demography, N', induces a transition of the economy from an initial stationary equilibrium to another. The (de-trended) household problem at age j in period t is then given by

$$v(\tilde{x}, \lambda, \Omega) = \max_{\tilde{c}, \tilde{x}', \widehat{\alpha}^{s\prime}, \widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta} \cdot (\mathbb{E}[v'(\tilde{x}', \lambda', \Omega')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \tag{9}$$

$$\text{s.t. } \tilde{x}' = \frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}) \cdot (1 + \widehat{r}'), \ \tilde{x}_0 > 0 \text{ given} \tag{10}$$

$$\Omega' = \Phi(\Omega, \lambda, \lambda', N'), \ N' \text{ given} \tag{11}$$

$$\pi(\lambda' \mid \lambda), \ \lambda_0 \text{ given} \tag{12}$$

$$\eta \sim \mathcal{D}(0, \sigma^2(\lambda)).$$

(which can be households and firms) always repay their debts. This is especially not subject to neither the aggregate state of the economy nor, in case of a household, the idiosyncratic state of the debtor.

where $\tilde{x} := (\tilde{k} + h/(1+g)) \cdot (1+\widehat{r})$, $\widehat{r}' := r^{f\prime} + \widehat{\alpha}^{s\prime} \cdot (r^{s\prime} - r^{f\prime}) + \widehat{\alpha}^{h\prime} \cdot (\widehat{r}^{h\prime} - r^{f\prime})$, $\widehat{r}^{h\prime} := (1+g) \cdot (\tilde{r}^{H\prime} + 1 - \delta^{h\prime} + \eta') - 1$, $\widehat{\alpha}^{s\prime} := \alpha^{s\prime} \cdot \tilde{k}'/(\tilde{k}' + h'/(1+g))$, $\widehat{\alpha}^{h\prime} := h'/(1+g)/(\tilde{k}' + h'/(1+g))$, and $\widehat{\beta} := \beta \cdot \varsigma^{\frac{1}{\gamma}} \cdot (1+g)^{\frac{1-\theta}{\gamma}}$.
The expectation $\mathbb{E}$ above is taken with respect to the realization of tomorrow's aggregate state λ' conditional on state λ today and the realization of tomorrow's idiosyncratic shock, η'. Note that λ' determines the shock to technology, z', the shock to physical capital depreciation, $\delta^{K\prime}$, and the variance of the idiosyncratic human capital depreciation shock, $\sigma^{2\prime}$.

Using results derived in Samuelson (1969) we can next state the following properties of the optimal policy functions.

Proposition 1. *Denote by $\widehat{\alpha}^{s*\prime}$ and $\widehat{\alpha}^{h*\prime}$ the optimal portfolio decisions that are the solutions to*

$$\mathbb{E}[(m')^{1-\theta-\gamma}(1+\widehat{r}')^{-\theta}(r^{s\prime} - r^{f\prime})] = 0 \tag{13}$$
$$\mathbb{E}[(m')^{1-\theta-\gamma}(1+\widehat{r}')^{-\theta}(\widehat{r}^{h\prime} - r^{f\prime})] = 0 \tag{14}$$

where m denotes the marginal propensity to consume out of (de-trended) cash-on-hand, $\tilde{x}$. Then the optimal (de-trended) consumption function is linear in cash-on-hand,

$$\tilde{c} = m \cdot \tilde{x}. \tag{15}$$

The marginal propensity to consume out of cash-on-hand is given by

$$m := \frac{[\beta^{\gamma} \cdot \varsigma \cdot \wp]^{\frac{1}{1-\theta-\gamma}}}{1 + [\beta^{\gamma} \cdot \varsigma \cdot \wp]^{\frac{1}{1-\theta-\gamma}}} \quad \text{, where} \quad \wp := \mathbb{E}[(m')^{1-\theta-\gamma} \cdot (1+\widehat{r}')^{1-\theta}].$$

Proof. Please see section A.1 in the appendix. □

Note that portfolio decisions do not dependent on and consumption is linear in current cash-on-hand. These features are due to the homotheticity of preferences and are particularly useful in the numerical solution of the simulation model. They transfer the results in Krebs (2003) and Krebs and Wilson (2004) which were derived in an infinite horizon model to a (finite) life cycle household problem. Furthermore, it implies that we do not need to break down the wealth distribution

into idiosyncratic characteristics other than age which we impose in the remainder of this paper for the sake of easier presentation.

3.7 Government

The government taxes accidentally bequeathed financial wealth of departed households and uses it for government consumption. For simplicity we assume that the tax rate on bequests is 100%. Note that the government receives bequeathed wealth including associated interests in period $t+1$.[13]

$$\tilde{G}_{t+1} = \frac{1}{1+g} \cdot \sum_{j=0}^{J} N_{t,j} \cdot (1-\varsigma_{t,j}) \cdot \tilde{x}_{t,j} \cdot (1-m_{t,j}) \cdot (1 + r^f_{t+1} + \alpha^s_{t+1,j+1} \cdot (r^s_{t+1} - r^f_{t+1})) \tag{16}$$

3.8 Equilibrium

Equilibrium in the economy is defined recursively and presented in de-trended form, cf. section 3.6. It requires market clearing in all periods while optimal decisions and aggregation conditions have to hold. In the following, $'$ ($_l$) indicates next (last) period's variables while we make the dependency on age, j, explicit.

A recursive competitive equilibrium is a value function $v(j, \tilde{x}, \lambda, \Omega)$ and policy functions, $\widehat{\alpha}^{s\prime}(j, \lambda, \lambda', \Omega')$, $\widehat{\alpha}^{h\prime}(j, \lambda, \lambda', \Omega')$, $m(j, \lambda, \lambda', \Omega')$, for the household, policy functions for the firm, $\tilde{K}(\lambda, \Omega)$, $H(\lambda, \Omega)$, pricing functions $r^s(\lambda, \Omega)$, $r^H(\lambda, \Omega)$, $r^{f\prime}(\lambda, \Omega)$, the demographic distribution, N, the wealth distribution, Ω, and its associated (aggregate) law of motion, $\Phi(\Omega, \lambda, \lambda', N')$, such that for all (λ, Ω)

1. $v(\cdot)$, $\tilde{x}(j, \lambda, \Omega)$, $\tilde{c}(j, \tilde{x}, \lambda, \Omega)$, $\widehat{\alpha}^{s\prime}(\cdot)$, $\widehat{\alpha}^{h\prime}(\cdot)$, $m(\cdot)$ are measurable, $v(\cdot)$ satisfies the household's recursive problem, and $\widehat{\alpha}^{s\prime}(\cdot)$, $\widehat{\alpha}^{h\prime}(\cdot)$, $m(\cdot)$ are the associated policy functions following from the conditions in proposition 1, given $\mathbb{E}[r^{s\prime}(\lambda', \Phi(\Omega, \lambda, \lambda', N'))]$, $\mathbb{E}[\widehat{r}^{h\prime}(j+1, \eta', \lambda', \Phi(\Omega, \lambda, \lambda', N'))]$, $r^{f\prime}(\cdot)$ and $\tilde{x}(\cdot)$,

2. firms behave optimally as according to equations (2),

[13] Again, an aggregation over age only suffices due to the independence of policy funtions of $\tilde{x}$ as explained at the end of section 3.6.

3. government consumption financed by accidental bequests fulfills equation (16),

4. market clearing on bond, stock, human capital, and final good markets as according to equations (17), (18), (19), and (20) respectively:

$$\tilde{K}'(\cdot) \cdot \frac{\ell}{1+\ell} = \frac{1}{1+g} \cdot \sum_{j=0}^{J} N(j) \cdot \tilde{x}(j,\cdot) \cdot (1 - m(j,\cdot)) \cdot (1 - \widehat{\alpha}^{s\prime}(j,\cdot) - \widehat{\alpha}^{h\prime}(j,\cdot)) \tag{17}$$

$$\tilde{K}'(\cdot) \cdot \frac{1}{1+\ell} = \frac{1}{1+g} \cdot \sum_{j=0}^{J} N(j) \cdot \tilde{x}(j,\cdot) \cdot (1 - m(j,\cdot)) \cdot \widehat{\alpha}^{s\prime}(j,\cdot) \tag{18}$$

$$H(\cdot) = \sum_{j=0}^{J} N(j) \cdot \frac{\tilde{x}(j,\cdot)}{1+\widehat{r}(j,\cdot)} \cdot \widehat{\alpha}^{h}(j,\cdot) \tag{19}$$

$$\tilde{Y}(\lambda,\Omega) = \tilde{C}(\lambda,\Omega) + \tilde{G}(\lambda,\Omega) + \tilde{I}^{K}(\lambda,\Omega) + I^{h}(\lambda,\Omega) \tag{20}$$

   where $\ell := \frac{B}{S}$ is the leverage ratio of bonds over stocks of the representative firm which is exogenous and fix by assumption, cf. section 4. The bond price $q^f(\lambda,\Omega) := (1 + r^{f\prime}(\lambda,\Omega))^{-1}$ is determined such that it clears the bond market in period t. (20) is the aggregate resource constraint which is derived in section A.2 of the appendix,

5. the aggregate law of motion Φ satisfies

$$\Omega' = \Phi(\Omega, \lambda, \lambda', N'). \tag{21}$$

   It is generated by the exogenous population dynamics, the exogenous stochastic processes and the endogenous asset accumulation decisions as captured by the policy functions,

6. the initial wealth distribution, Ω_0;

7. the transition matrices for the exogenous processes.

Definition 1. *A stationary recursive competitive equilibrium is a special case of the equilibrium described above. It is characterized by time-constant individual*

policy functions $m(\cdot)$, $\widehat{\alpha}^{s\prime}(\cdot)$, $\widehat{\alpha}^{h\prime}(\cdot)$, *and a time-constant aggregate law of motion* $\Phi(\cdot)$. *This requires a time-constant demographic distribution,* N.

4 Calibration and Numerical Solution

In terms of expectations, we solve an approximate rational expectations equilibrium by applying a variant of the method of Krusell and Smith(1997, 1998) as further described in appendix B.1.

Calibration of the model is in part by reference to other studies and in part by informal matching of moments procedures. The period length is one year. Table 1 summarizes structural model parameters where target values refer to year 2010. The additional parameters governing stochastic and demographic processes are only described in the text.

Table 1: First and Second Stage Parameters

Parameter	Value	Target	Target Source, Comment
Firm sector			
Capital share: α	0.36	1^{st} stage	wage share (NIPA)
Technological progress: g	0.018	1^{st} stage	TFP growth (NIPA)
Leverage ratio: ℓ	0.67	1^{st} stage	RZ
Mean depreciation rate K: δ_0^K	0.086	$r^f = 0.013$	PST, Shiller (2015)
Households			
Life cycle: $j = \{0, j_r, J\}$	$\{0, 45, 80\}$	1^{st} stage	biological age: $\{20, 65, 100\}$
Elasticity inter-temp. substit., ξ	1.5	1^{st} stage	Bansal and Yaron (2004)
Endowment: $\{h_0, k_0\}$	$\{1.0, 0.0\}$	1^{st} stage	normalization
Time discount factor: β	0.936	$K/Y = 2.65$	NIPA
Relative risk aversion: θ	8.4	$r^s - r^f = 0.062$	PST
Depreciation rate h: $\{\chi_0, \chi_1\}$	$\{0.976, 0.0007\}$	$\{r^H \cdot h_j\}_{j=20}^{64}$	PSID

Source: Baseline model: The target year is 2010. *Notes*: RZ $\hat{=}$ Rajan and Zingales (1995). PST $\hat{=}$ Piazzesi, Schneider, and Tuzel (2007). We target the average of the post-Second World War risk-free rates of PST and Shiller (2015).

The time- and age-specific demographic data for the population dynamics in (3) are based on (Human Mortality Database 2008) and the United Nations' population projections (United Nations 2007).

We assume that aggregate risk is driven by a four state Markov chain with support $L = \{\lambda_1, \ldots, \lambda_4\}$ and transition matrix $\Pi = (\pi_{ik})$. Each aggregate state maps into a combination of low or high technology shocks and low or high physical capital depreciation. Precisely, we assume that

$$z_t = z(\lambda_t) = \begin{cases} z_0(1+\bar{z}) \text{ for } \lambda \in \lambda_1, \lambda_2 \\ z_0(1-\bar{z}) \text{ for } \lambda \in \lambda_3, \lambda_4 \end{cases}, \delta_t^k = \delta^k(\lambda_t) = \begin{cases} \delta_0^k + \overline{\delta}^k \text{ for } \lambda \in \lambda_1, \lambda_3 \\ \delta_0^k - \overline{\delta}^k \text{ for } \lambda \in \lambda_2, \lambda_4. \end{cases} \tag{22}$$

One feature specific to the model is that the endogenous fluctuations generated by the financial savings and human capital accumulation channels are higher than in the standard model with exogenous labor income. Therefore, the auto-correlation of the exogenous technology shock process, ρ^z, and the probability of a high (low) depreciation state conditional on being in a low (high) technology state, ρ^δ, must be lower than in the standard model. We assume $\rho^\delta = 0.6$ and $\rho^z = 0.7$ which comes close to Gomes and Michaelides (2008) who use 0.5 and 0.67 respectively. $\bar{z}$ is set to 0.02 which results in a standard deviation of GPD growth of 4% and a standard deviation of consumption growth of 3.8%. This is slightly higher than the 3% measured usually in the data. The standard deviation of the shock to the depreciation rate of physical capital is set to $\overline{\delta}^k = 0.1$ such that the model matches the standard deviation of the stock return in the data of about 16.7% (cf., e.g., Shiller 2015).

The value of the capital share parameter, $\alpha = 0.36$, is based on an estimation of the aggregate production function for the US, cf. Krüger and Ludwig (2007), and lies in the usual range considered in the literature. The value of the mean depreciation rate of physical capital, $\delta_0^K = 0.086$, lies at the upper end of the range of empirical estimates and leads to a risk-free interest rate of 1.3%. We assume that the representative firm keeps an exogenous fixed leverage ratio, $\ell := \frac{B}{S}$, which is set to the empirically observed value, 0.67 (cf. Rajan and Zingales 1995). Thereby, corporate bonds are in positive net supply.

The value of households' raw time discount factor, $\beta = 0.936$, is at the lower range of values considered in the literature. It yields the in NIPA data observed capital-output ratio of 2.65. The elasticity of inter-temporal substitution, ξ, equals 1.5. It lies in the range considered in the asset pricing literature (cf. the

discussion in Bansal and Yaron (2004, pp. 1492-93)) and results in a hump-shaped consumption profile which is in line with the data, cf. Fernández-Villaverde and Krüger (2006). While being mostly flat between 45 and 65 the peak lies at around the age of 55. The value of the coefficient of relative risk aversion, $\theta = 8.4$, must be considered high relative to the literature. However, Mehra and Prescott (1985) argue that the upper bound of reasonable values of the parameter of risk aversion is 10. With this value, the model is able to generate an empirically observed equity premium of about 6.2%.

Due to the homotheticity of preferences, the initial level of human capital h_0 is irrelevant and we normalize human capital by setting $h_0 = 1$. We calibrate the human capital depreciation rate, δ^h, by setting the corresponding parameters, χ_0 and χ_1, such that the model matches observed wage profiles based on PSID data provided by Huggett, Ventura, and Yaron (2011).[14] Idiosyncratic shocks to human capital, η, are uncorrelated but the variance of η depends on the current state of the economy which has been documented in the data and used in the asset pricing literature (cf. Storesletten, Telmer, and Yaron (2004) and Constantinides and Duffie (1996) respectively). We follow the approach of Storesletten, Telmer, and Yaron (2007) and set the standard deviation σ_t to

$$\sigma_t = \sigma(\lambda) = \begin{cases} 0.2 & \text{for } \lambda \in (\lambda_1, \lambda_2) \\ 0.1 & \text{for } \lambda \in (\lambda_3, \lambda_4) \end{cases} \tag{23}$$

which is within the range considered in Krebs and Wilson (2004).

5 Results

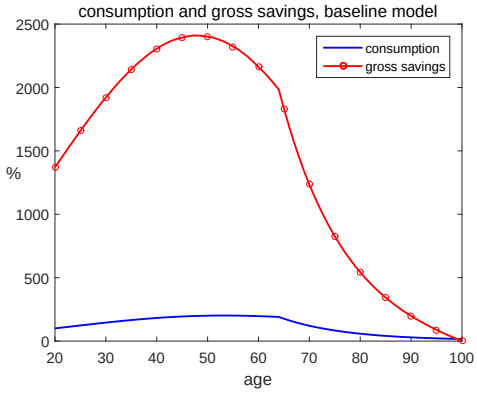
5.1 Cross-Sectional Profiles in 2010

Figure 1 shows resulting key cross-sectional age profiles of the model economy in year 2010. The top left panel shows consumption and gross savings by age. Consumption is hump-shaped as in the data (cf., e.g., Fernández-Villaverde and Krüger (2006)) and remains at its maximum level between 48 and 64. Gross sav-

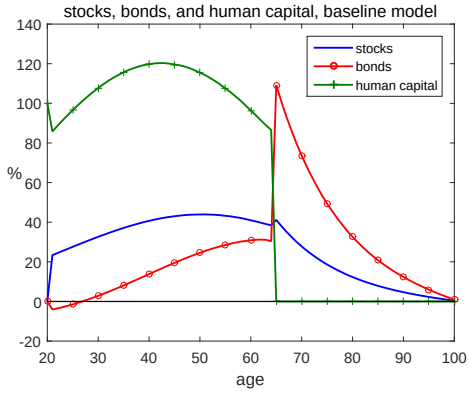
[14] We thank Mark Huggett for sending us the data.

Figure 1: Cross-Sectional Profiles in 2010

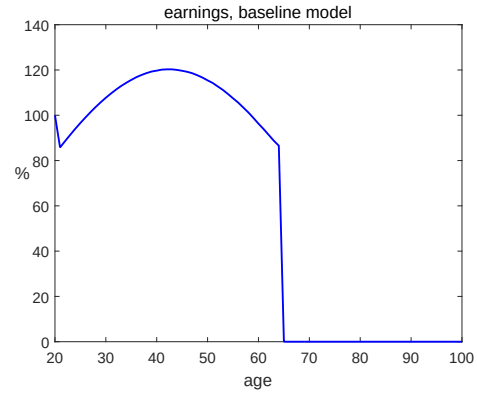
(a) Consumption and Gross Savings



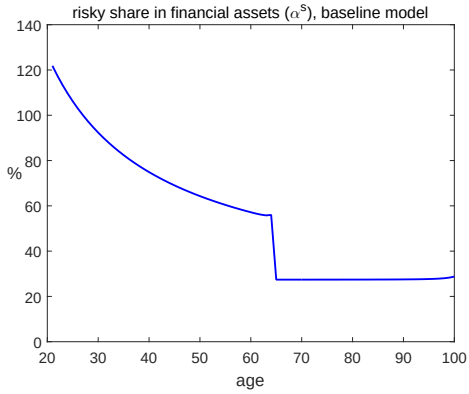
(b) Portfolio Allocation



(c) Earnings



(d) Risky Share in Financial Assets



Source: Baseline model in year 2010: Selected average cross-sectional age profiles.

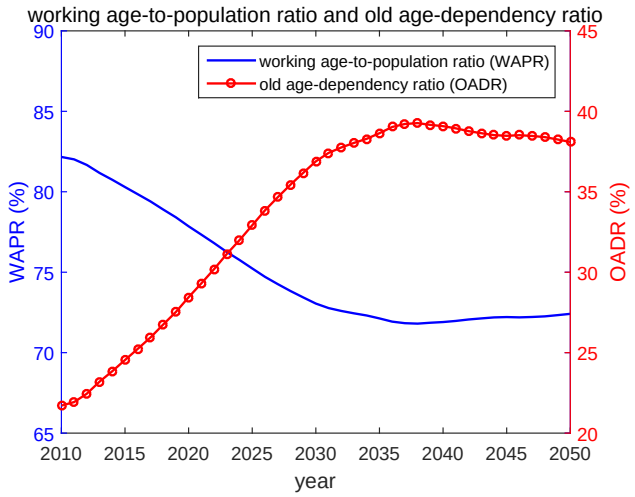
ings exhibit the typical saving-dis-saving pattern as in standard life-cycle models. The top right panel depicts the portfolio allocation of households by age. It shows the following pattern. Households enter their economically relevant lifetime with zero financial assets but positive human capital. Subsequently, the latter follows a hump-shaped pattern over the working life which results in a corresponding pattern in the age-earnings profile (bottom left panel). This is a target in the calibration. The pattern of financial asset holdings shows in the bottom right panel. It depicts the share of risky assets in the financial portfolio and follows a declining pattern over the working life cycle. Note that this stems from the co-incident decrease of human wealth which is defined as the present value of expected remaining lifetime earnings. Despite its riskiness, human wealth shows to resemble rather the holdings of one period ahead risk-free bonds than of stocks. Consequently, households re-allocate their portfolio toward bonds when human wealth decreases (cf., e.g., Campbell and Viceira (2002, ch. 6 et seqq.)). As human wealth equals zero as of the retirement of a household the risky financial portfolio share remains approximately constant over the retirement spell.

5.2 Macroeconomic Aggregates and Asset Returns

In the following, we show resulting time paths of key variables in the model induced by the demographic transition. These are macroeconomic aggregates and the returns to the different kinds of assets as well as the resulting equity premium. In order to reveal the role of human capital in the demographic transition we show results of two variants of the model for the main period of projection, i.e., 2010 to 2050. Left sub-figures depict results for the baseline case while right sub-figures belong to an auxiliary variant of the model in which all human capital shares in total wealth are held fix at the level of 1960. By the latter, we approximate a model without human capital adjustments and are able to show their mitigating effect for dynamics of aggregate measures and asset returns in the demographic transition.

Figure 2 summarizes the demographic transition of the U.S. economy which is the exogenous driving force in the model. It reflects the aging process in the U.S. economy by depicting the evolution of its working age-to-population ratio.

Figure 2: Working Age-to-Population Ratio



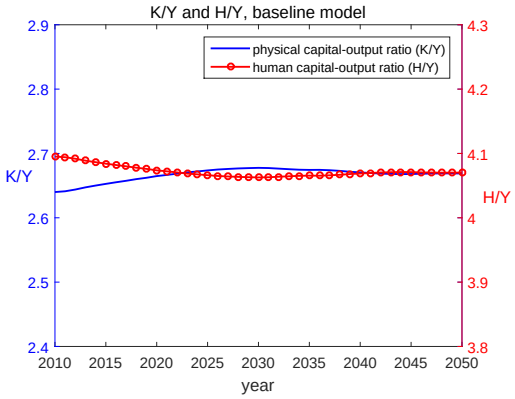
Source: Own calculations based on United Nations (2007) and Human Mortality Database (2008). *Notes:* The working age-to-population ratio equals the population at ages $20-64$ over the total population of the model economy. The retirement age of 65 is held fix over time.

However, note that the demographic structure of the model is much richer than that summary statistic featuring the entire distribution of the population and its survival probabilities over age. The most severe change in the age structure of the economy is expected to evolve until around 2030. Subsequently, there is almost no further change in the working age-to-population ratio until 2050. However, although this is not captured by the summary statistics, the age distribution still changes albeit with a lower degree of severity.

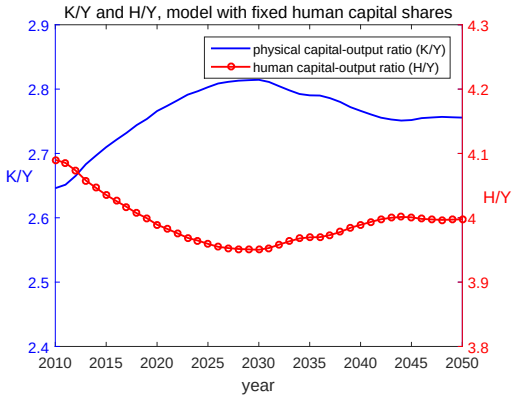
The blue solid lines (left scales) in figure 3 and the green dashed lines (right scales) show the resulting paths of human capital-output ratio, H/Y, and physical capital-output ratio, K/Y, respectively. Conventional analyses suggest that aging induces a relative shortage of labor and a relative abundance of physical capital in the economy. This corresponds to an increase in the physical capital-output ratio and a decrease in the human capital-output ratio. Both applies in the baseline case depicted by the left panel of figure 3. It implies that endogenous human capital adjustments do not offset the shortage of labor arising from the aging process. However, the right panel reveals that labor scarcity is much more

Figure 3: Macroeconomic Aggregates: Physical and Human Capital

(a) Baseline



(b) Fixed Human Capital Shares



Source: Baseline model (panel (a)) and auxiliary model with fixed human capital shares (panel (b)) in the main period of projection. *Notes:* Human capital shares in the auxiliary model are held fix at the level of the initial stationary equilibrium in 1960.

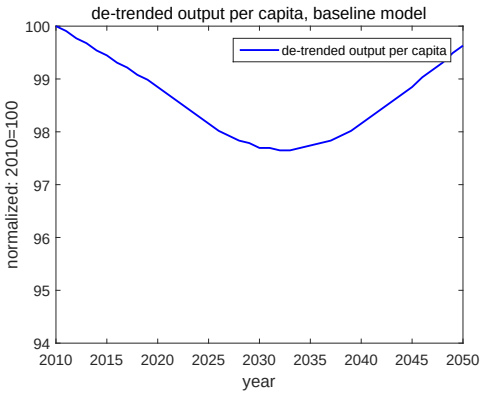
pronounced if human capital shares remained constant. This underscores the importance of endogenous individual human capital adjustments for the macroeconomic composition of capital and labor in the demographic transition. Note that dynamics are less pronounced as of around 2030 which re-emerges in subsequent figures. As explained above, demographic shifts are less severe as of 2030 leading to smaller movements in aggregates and prices.

Figure 4 shows the corresponding reaction of aggregate output. [TBC]

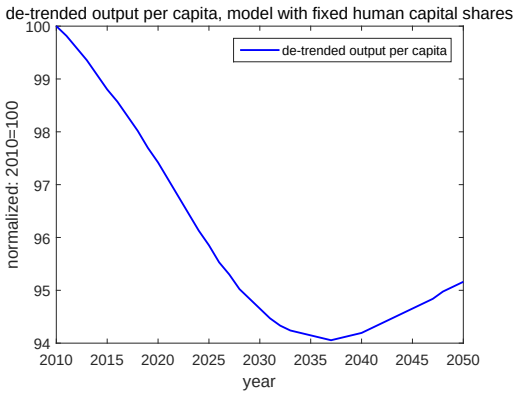
What does the change in the aggregate measures imply for the returns to physical and human capital? Figure 5 plots the corresponding time paths. It shows that rates of return to physical capital (blue solid lines) decline which is the mirror image of the increasing capital-output ratios in figure 3. Again, this is consistent with conventional analyses. Correspondingly, human capital returns follow an increasing pattern which reflects the augmenting relative scarcity of labor in the demographic transition. Note that the effects are quantitatively small in the baseline model but sizeable when we hold human capital shares constant. This confirms the findings of Ludwig, Schelkle, and Vogel (2012). In the latter case the change in the risky interest rate amounts to a substantial one percentage point decrease until 2030.

Figure 4: Aggregate Output

(a) Baseline



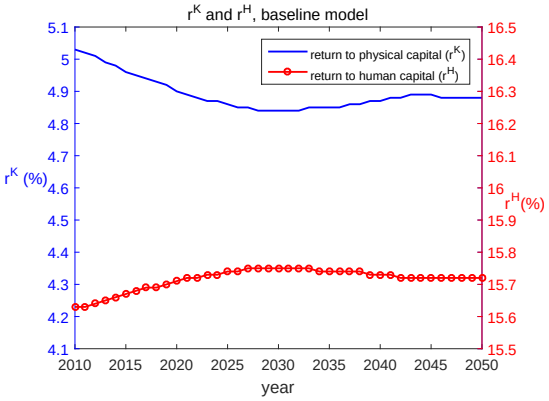
(b) Fixed Human Capital Shares



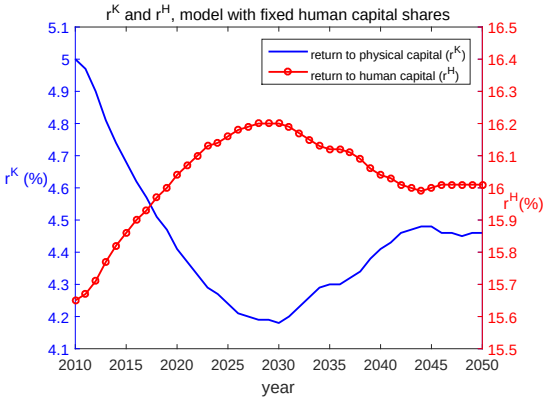
Source: Baseline model (panel (a)) and auxiliary model with fixed human capital shares (panel (b)) in the main period of projection. *Notes:* Human capital shares in the auxiliary model are held fix at the level of the initial stationary equilibrium in 1960.

Figure 5: Risky Asset Returns

(a) Baseline



(b) Fixed Human Capital Shares

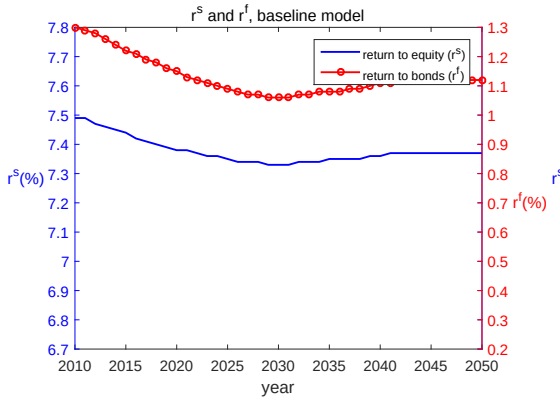


Source: Baseline model (panel (a)) and auxiliary model with fixed human capital shares (panel (b)) in the main period of projection. *Notes:* Human capital shares in the auxiliary model are held fix at the level of the initial stationary equilibrium in 1960.

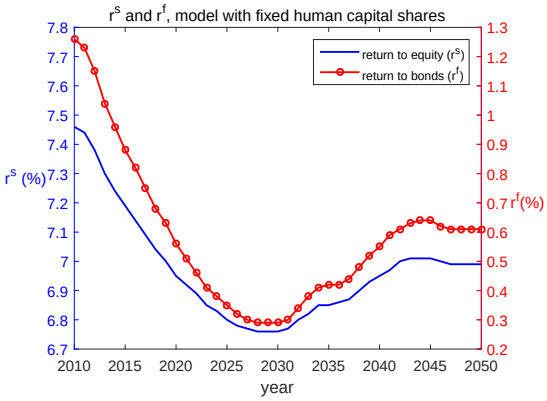
Let us now turn to the key concern of this paper, i.e., how the effect of demographic change on physical asset returns differs by the risk nature of assets. While returns to equity exhibit business cycle risk, returns to bonds are risk-free for a time horizon of one period. Figure 6 plots resulting time paths for returns to equity (blue solid lines) and returns to bonds (green dashed lines). Both re-

Figure 6: Return to Equity and Risk-Free Rate

(a) Baseline



(b) Fixed Human Capital Shares



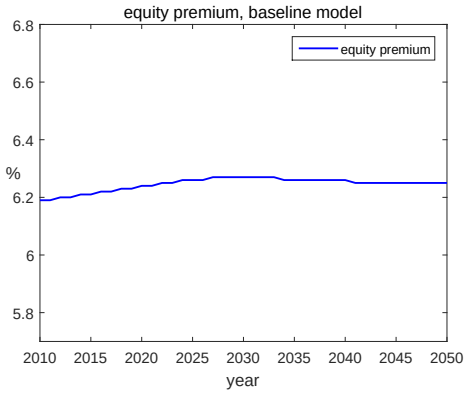
Source: Baseline model (panel (a)) and auxiliary model with fixed human capital shares (panel (b)) in the main period of projection. *Notes:* Human capital shares in the auxiliary model are held fix at the level of the initial stationary equilibrium in 1960.

turns decline over time which corresponds to the result of a declining return to physical capital described above. Again effects are much larger in the model with constant human capital shares. The development shows that we can expect low returns for (many) decades to come, irrespective of the risk nature of the asset. Moreover, figure 7 reveals the relative size of those declines by plotting the evolution of the equity premium, defined as the difference between the return to equity and the return to bonds. While effects in the baseline case are rather negligible, the equity premium increases by about 30 basis points until 2030 in the model with fixed human capital shares. This comes from the fact that households hold relatively more bonds in their financial portfolio as was shown in panel (d) of figure 1.[15] In the course of aging, that implies a higher relative demand

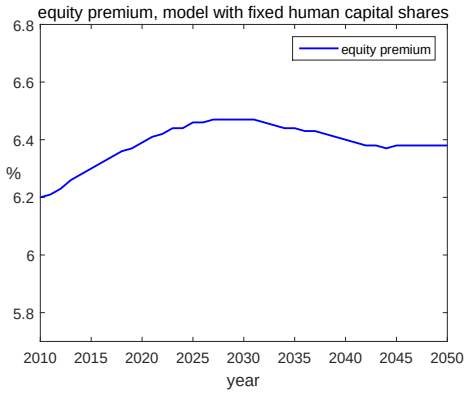
[15]In fact, this includes a counteracting portfolio adjustment effect isolated in Kuhle (2008) which arises if the absolute return level drops which is the case here.

Figure 7: Equity Premium

(a) Baseline



(b) Fixed Human Capital Shares



Source: Baseline model (panel (a)) and auxiliary model with fixed human capital shares (panel (b)) in the main period of projection. *Notes:* The lines show the equity premium defined as the differential return between equity and bonds. Human capital shares in the auxiliary model are held fix at the level of the initial stationary equilibrium in 1960.

for bonds in the economy which drives down the return to bonds more strongly than the return to equity. The comparison of the two panels shows that human capital adjustments work as an opposing force to that boosting effect of aging on the equity premium. Higher human capital investments, in particular, by young households imply higher earnings along their working life, enforced by the co-incident increase in the return to human capital. This implies higher human wealth of a household. As was discussed in section 5.1 and shown elsewhere in the literature (cf., e.g., Campbell and Viceira (2002, ch. 6 et seqq.)) rising human wealth elevates the risky share in the financial portfolio of a household as long as the (positive) correlation of their returns is not too high.[16]

Tables 2 and 3 summarize the results described above by reporting the corresponding numbers for the years 2010, 2030, and 2050.

[16]The correlation in the model equals about 0.1.

Table 2: Summary of Baseline Results

	$WAPR$	K/Y	r^f	r^s	$\{\widehat{r}_j^h\}_{avrg.}$	EP
2010 (in %)	82.00	265.00	1.30	7.49	11.64	6.19
2030 (in %)	73.00	268.00	1.06	7.33	11.81	6.27
2050 (in %)	72.00	267.00	1.12	7.37	11.80	6.25
$\Delta_{\{2030-2010\}}$ *(in %p)*	-9.00	+3.00	-0.24	-0.16	+0.17	+0.08
$\Delta_{\{2050-2010\}}$ *(in %p)*	-10.00	+2.00	-0.18	-0.12	+0.16	+0.06

Source: Baseline model in the main period of projection. *Notes:* $WAPR\widehat{=}$ working age-to-population ratio. $\{\widehat{r}_j^h\}_{avrg.}\widehat{=}$ average return to human capital of all agents alive. $EP := r^s - r^f$. The top three lines show the values of the considered variables for the year 2010, 2030, and 2050 in percent. The bottom two rows show the percentage point ($\%p$) change of the considered variables from 2010 to 2030 and 2010 to 2050.

Table 3: Effect of Endogenous Human Capital

	$WAPR$	k/y	r^f	r^s	$\{\widehat{r}_j^h\}_{avrg.}$	EP
Baseline						
$\Delta_{\{2030-2010\}}$ (in %p)	-9.00	+3.00	-0.24	-0.16	+0.17	+0.08
$\Delta_{\{2050-2010\}}$ (in %p)	-10.00	+2.00	-0.18	-0.12	+0.16	+0.06
Holding human capital shares constant						
$\Delta_{\{2030-2010\}}$ (in %p)	-9.00	+16.00	-0.97	-0.70	+0.60	+0.27
$\Delta_{\{2050-2010\}}$ (in %p)	-10.00	+11.00	-0.65	-0.47	+0.42	+0.18

Source: Baseline model (top two rows) and auxiliary model with fixed human capital shares (bottom two rows) in the main period of projection. *Notes:* $WAPR\widehat{=}$ working age-to-population ratio. $\{\widehat{r}_j^h\}_{avrg.}\widehat{=}$ average return to human capital of all agents alive. $EP := r^s - r^f$. The numbers show the percentage point ($\%p$) change of the considered variables from 2010 to 2030 and 2010 to 2050.

5.3 Consequences for Welfare

What are the welfare consequences of the price dynamics described in the previous section? How do they differ across generations? In order to answer these questions, we follow Davila, Hong, Krusell, and Rios-Rull (2012), Harenberg and Ludwig (2015), and others, and measure welfare by ex-ante expected utility at the beginning of a household's life, $\mathbb{E}[v_{\cdot,0}]$. All households of a given cohort are ex-ante identical and turn heterogeneous along the life cycle due to idiosyncratic shocks to human capital. The welfare concept of ex-ante expected utility is the natural objective of a social planner who is behind the veil of ignorance (cf. Davila, Hong, Krusell, and Rios-Rull (2012, p. 2439)). From this it follows that it provides also the natural perspective on the consequences that arise from the exogenous force in this model.

Demographic change affects a household's welfare in two ways. The first effect arises from changes in survival probabilities which are exogenous in this model. The second effect stems from changing good allocations induced by changes in wages and asset returns. We want to isolate the second effect and, therefore, conduct the following auxiliary computation. We compute welfare of households which face on the individual level the time- and age-dependent survival probabilities of the demographic transition while living in the aggregate environment of the initial stationary equilibrium in which no demographic change takes place.[17] Given that, households decide fully rational as in the baseline case. We then measure welfare effects of demographic change in terms of consumption equivalent variation, i.e., how much compensation in percent of consumption a household must receive in all periods of lifetime in order to be indifferent between the worlds with and without demographic change on the aggregate level.[18] Based on the homotheticity of the value function, consumption equivalent variation in period t can be measured as

$$cev_t := \frac{\mathbb{E}[v_{t,0}]}{\mathbb{E}[v^A_{t,0}]} - 1 \tag{24}$$

[17] This implies that both, the coefficients in the aggregate law of motion and the age distribution of the economy equal the corresponding values of the initial stationary equilibrium.

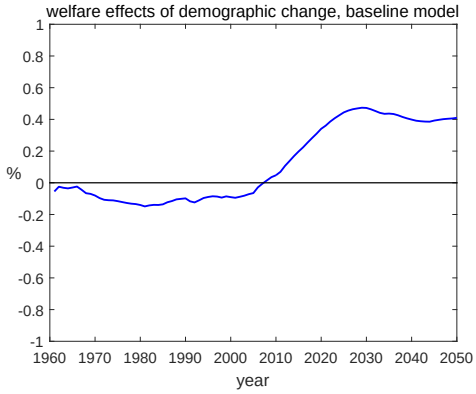
[18] We simulate the two model variants using both, identical initial conditions and the identical 50000 time series of aggregate shock realizations.

where $v^A_{\cdot,0}$ is the lifetime value of a newborn household in the auxiliary world without demographic change on the aggregate level.

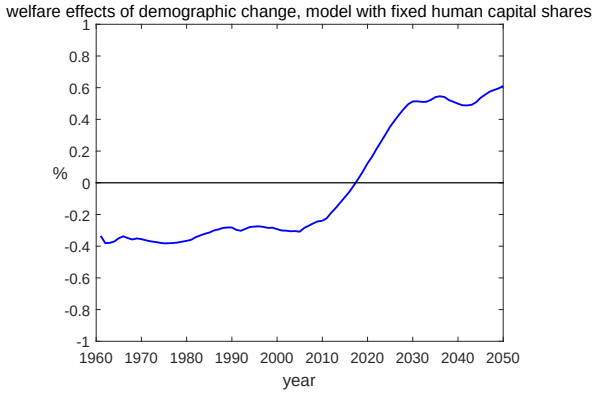
Figure 8 shows resulting welfare effects of demographic change for all generations born as of 1960. Again, the *birth* of a generation occurs when individuals turn 20. As before, the left sub-figure depicts the result for the baseline case while the right sub-figure belongs to the auxiliary variant of the model in which all human capital shares in total wealth are held fix at the level of 1960.

Figure 8: Welfare Effects of Demographic change

(a) Baseline



(b) Fixed Human Capital Shares



Source: Welfare evaluation in the baseline model (panel (a)) and in the auxiliary model with fixed human capital shares (panel (b)) in the main period of projection. *Notes:* Welfare effects of the generation born in the indicated year measured as consumption equivalent variation. *birth* of a generation occurs when individuals turn 20. Negative values indicate welfare losses from demographic change. Human capital shares in the auxiliary model are held fix at the level of the initial stationary equilibrium in 1960.

The left panel shows that welfare effects of demographic change differ considerably across generations. While generations born after 2005 benefit from the price effects induced by demographic change early generations, i.e., those born still in the 20$^{\text{th}}$ century, lose. This can be traced back to the timing of the dynamics of the returns to physical and human capital which were described in the previous section. Note that assets play a changing role along the life cycle. While young households rather own little assets and hold negative bonds for financing human capital accumulation old households rely on assets as the only source of

income. Consequently, early generations lose from the decline in the physical capital return as major return changes evolve in the period 2010 to 2030. At the same time, those generations benefit rather little from rising returns to human capital in the aforementioned period. This is because they have spent a significant part of their working life in a period with comparatively low returns to human capital. As a result, early generations, in particular the baby boomers, suffer from welfare losses due to demographic change. On the contrary, future generations benefit from demographic change as the aforementioned developments hit them at another point in the life cycle. A declining risk-free rate at young ages and increasing returns to human capital, in particular, at middle ages when human capital is at a high level lead to their welfare gains. Moreover, asset returns start to rise slightly again as of 2030 which affects their capital income in retirement positively.

The right panel of figure 8 shows the corresponding welfare effects for the auxiliary model variant with fixed human capital shares. Also in this auxiliary model, future generations benefit from demographic change while early generations lose. The comparison to the baseline model in the left panel shows that human capital adjustments mitigate the welfare effects. This mirrors the mitigating effect of human capital on return dynamics which was discussed above. Note that the mitigating effect works in both directions. While early generations suffer from stronger welfare losses if human capital is fixed, future generations benefit from higher welfare gains. This is an effect already observed by Ludwig, Schelkle, and Vogel (2012). It shows that, when investing into human capital, each household faces a trade-off between changes in individual productivity on the one hand and induced movements in equilibrium market returns on the other. Apparently, there is a positive net effect for early generations while the opposite is true for future generations. Look exemplarily at the generations born in the $1960s$. They are already retired by the year 2010 when human capital returns start to rise significantly. Consequently, their individual human capital stocks are irrelevant in that period and, moreover, they are not affected by changes in returns to human capital. However, they finance consumption from savings and capital income, and suffer from declining asset returns. Hence, they are heavily interested in a high human capital stock of working generations in that period.

Again, to state it clearly, the increase in the equity premium in the auxiliary model with fixed human capital shares observed in figure 7(b) induces positive welfare effects for young compared to old households in that period. This is due to the declining pattern in the risky financial portfolio share over age as implied by figure 1(d).

5.4 Sensitivity Analyses

[TBC]

6 Conclusion

This paper develops a quantitative overlapping generations model to evaluate the effects of demographic change for the secular stagnation of the US economy in the next decades. We show that cumulative output losses from reductions of raw labor may stand at minus 6% in 2035, relative to a constant growth path. Endogenous human capital adjustments may strongly reduce these losses to about minus 2%. This relative reduction in long-run output will be accompanied by falling asset returns. Risky returns are projected to decrease by 0.16, respectively 0.7, percentage points and risk-free returns by 0.24, respectively 1, percentage point with flexible, respectively without, human capital adjustments; hence, the equity premium is projected to increase. Such rate of return predictions will imply profound welfare losses for asset rich households. They, and in particular our predictions of the risk-free rate, are also crucial for providing central banks with the necessary information for fine tuning monetary policy instruments in light of the demographic transition.

The differences across human capital scenarios in our model are large. While we do not model frictions to human capital adjustments, these differences lead us to conclude that human capital policies will play a crucial role in aging societies. Extending our framework by an explicit model of such frictions (e.g., borrowing and time constraints to human capital formation) is a key relevant step for future research.

A Theoretical Appendix

A.1 Solution of the Household Problem

We hereafter define recursively the household problem conditional on a law of motion of the aggregate state of the economy. It is convenient to express next period's values with symbol $'$, irrespective of whether they are only time-dependent or both, age- and time-dependent. The states of the household problem are the exogenous states j, t, and λ, the endogenous idiosyncratic states k and h, as well as the endogenous aggregate state of the economy, Γ, which is the distribution of physical capital, k, and human capital, h, among households. The associated law of motion is $\Gamma' = \Phi(\Gamma, \lambda, \lambda', N')$. The household problem at age j in period t is then given by

$$
\begin{aligned}
v(k,h,\lambda;\Gamma) = \max_{c,\alpha^{s\prime},i^h,k'} & \left\{ c^{\frac{1-\theta}{\gamma}} + \beta \cdot \left(\mathbb{E}[\varsigma \cdot v(k',h';\lambda';\Gamma')^{1-\theta}] \right)^{\frac{1}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}} \\
\text{subject to} & \qquad (25) \\
& k' = k \cdot (1 + r^f + \alpha^s \cdot (r^s - r^f)) + h \cdot r^H - i^h - c \\
& h' = h \cdot (1 - \delta^h + \eta) + \tilde{i}^h \\
& h_0 > 0, \;\; k_0 = 0 \text{ given} \\
& \Gamma' = \Phi(\Gamma, \lambda, \lambda', N'), \;\; N' \text{ given} \\
& \pi(\lambda' \mid \lambda), \;\; \lambda_0 \text{ given.} \\
& \eta \sim \mathcal{D}(0, \sigma^2(\lambda)).
\end{aligned}
$$

The expectation $\mathbb{E}$ above is taken with respect to the realization of tomorrow's aggregate state λ' conditional on state λ today and the realization of tomorrow's idiosyncratic shock, η'. Note that λ' determines the shock to technology, z', the shock to physical capital depreciation, $\delta^{k\prime}$, and the variance of the idiosyncratic human capital depreciation shock, $\sigma^{2\prime}$.

In the following, we transform the household problem and derive the first-order conditions of its solution. We start with de-trending the accumulation

equation for financial assets, (8), leading to:

$$\tilde{k}' = \frac{1}{1+g} \cdot \Big(\tilde{k} \cdot (1 + r^f + \alpha^s \cdot (r^s - r^f)) + h \cdot \tilde{r}^H - \tilde{i}^h - \tilde{c} \Big)$$

where, e.g., $\tilde{r}^H = r^H / \Upsilon$.
Let's combine the de-trended accumulation equation for financial assets with the human capital production technology given by equation (7).[19] Thereby we get:

$$\begin{aligned}
\tilde{k}' &= \frac{1}{1+g} \cdot \Big(\tilde{k} \cdot (1 + r^f + \alpha^s \cdot (r^s - r^f)) + h \cdot \tilde{r}^H + h \cdot (1 - \delta^h + \eta) - h' - \tilde{c} \Big) \\
\tilde{k}' + \frac{1}{1+g} \cdot h' &= \frac{1}{1+g} \cdot \Big((\tilde{k} + \frac{h}{1+g}) \cdot (1 + r^f) + \tilde{k} \cdot \alpha^s \cdot (r^s - r^f) \\
&\quad + \frac{h}{1+g} \cdot ((1+g) \cdot (\tilde{r}^H + 1 - \delta^h + \eta) - (1 + r^f)) - \tilde{c} \Big)
\end{aligned}$$

Define $\tilde{w} := \tilde{k} + \frac{1}{1+g} \cdot h$ as total (de-trended) wealth, we get:

$$\tilde{w}' = \frac{1}{1+g} \cdot (\tilde{w} \cdot (1 + \widehat{r}) - \tilde{c})$$

where $\widehat{r} := r^f + \widehat{\alpha}^s \cdot (r^s - r^f) + \widehat{\alpha}^h \cdot (\widehat{r}^h - r^f)$ is the transformed net return on the total portfolio in period t, age j, and $\widehat{\alpha}^s := \tilde{k} \cdot \alpha^s / \tilde{w}$, $\widehat{\alpha}^h := \frac{h}{1+g} / \tilde{w}$, and $\widehat{r}^h := (1+g) \cdot (\tilde{r}^H + 1 - \delta^h + \eta) - 1$.
Let $\tilde{x} := \tilde{w} \cdot (1 + \widehat{r})$ be total resources, or, alternatively, "cash-on-hand" (Deaton 1991). It follows that

$$\tilde{x}' = \frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}) \cdot (1 + \widehat{r}').$$

Next, we transform the utility function into a de-trended version

$$\begin{aligned}
u &= [c^{\frac{1-\theta}{\gamma}} + \beta \cdot (\mathbb{E}[\varsigma \cdot u'^{1-\theta}])^{\frac{1}{\gamma}}]^{\frac{\gamma}{1-\theta}} \\
\Upsilon \cdot \tilde{u} &= [\Upsilon^{\frac{1-\theta}{\gamma}} \cdot \tilde{c}^{\frac{1-\theta}{\gamma}} + \beta \cdot (\mathbb{E}[\varsigma \cdot \Upsilon'^{1-\theta} \cdot \tilde{u}'^{1-\theta}])^{\frac{1}{\gamma}}]^{\frac{\gamma}{1-\theta}} \\
\tilde{u} &= [\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta} \cdot (\mathbb{E}[\tilde{u}'^{1-\theta}])^{\frac{1}{\gamma}}]^{\frac{\gamma}{1-\theta}} \text{ where } \widehat{\beta} = \beta \cdot \varsigma^{\frac{1}{\gamma}} \cdot (1+g)^{\frac{1-\theta}{\gamma}},
\end{aligned}$$

[19] Again, human capital must be trend stationary in order to assure that gross human capital earnings, $h_{t,j} \cdot r_t^H$, grow at the same rate as Υ_t over time.

and finally state the de-trended household problem in period t, age j as:

$$\begin{aligned} v(\tilde{x},\lambda,\Omega) = &\max_{\tilde{c},\tilde{x}',\widehat{\alpha}^{s\prime},\widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta}\cdot(\mathbb{E}[v'(\tilde{x}',\lambda',\Omega')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \\ \text{s.t. } &\tilde{x}' = \frac{1}{1+g}\cdot(\tilde{x}-\tilde{c})\cdot(1+\widehat{r}'),\ \tilde{x}_0>0 \text{ given} \\ &\Omega' = \Phi(\Omega,\lambda,\lambda',N'),\ N' \text{ given} \\ &\pi(\lambda'\mid\lambda),\ \lambda_0 \text{ given} \\ &\eta\sim\mathcal{D}(0,\sigma^2(\lambda)). \end{aligned}$$

Note that Ω is the distribution of (de-trended) wealth, $\tilde{w}$, among households with associated law of motion $\Phi(\cdot)$. Ω follows directly from the transformation of Γ.

In what follows, we prove that the optimal household policy functions are given by proposition 1.

Proof. We guess that $v = m^l\cdot\tilde{x}$ where l is some parameter to be determined below and m is the marginal propensity to consume out of $\tilde{x}$ and show below that this is indeed true. From the guess it follows that

$$\begin{aligned} v &= \max_{\tilde{c},\tilde{x}',\widehat{\alpha}^{s\prime},\widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta}\cdot(\mathbb{E}[(m'^l\cdot\tilde{x}')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \text{ s.t. } \tilde{x}' = \frac{1}{1+g}\cdot(\tilde{x}-\tilde{c})\cdot(1+\widehat{r}') \\ v &= \max_{\tilde{c},\widehat{\alpha}^{s\prime},\widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta}\cdot(\mathbb{E}[m'^{l\cdot(1-\theta)}\cdot(\frac{1}{1+g}\cdot(\tilde{x}-\tilde{c})\cdot(1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \\ v &= \max_{\tilde{c},\widehat{\alpha}^{s\prime},\widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g}\cdot(\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}}\cdot\widehat{\beta}\cdot(\mathbb{E}[(m'^l\cdot(1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \end{aligned}$$

Next, we compute the first-order conditions (FOCs) with respect to $\tilde{c},\widehat{\alpha}^{s\prime},\widehat{\alpha}^{h\prime}$:

- FOC with respect to consumption:

$$\begin{aligned} 0 =& \frac{\gamma}{1-\theta}\cdot\{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g}\cdot(\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}}\cdot\widehat{\beta}\cdot(\mathbb{E}[(m'^l\cdot(1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}-1} \\ &\cdot\{\frac{1-\theta}{\gamma}\cdot\tilde{c}^{\frac{1-\theta-\gamma}{\gamma}} - \frac{1-\theta}{\gamma\cdot(1+g)}\cdot(\frac{1}{1+g}\cdot(\tilde{x}-\tilde{c}))^{\frac{1-\theta-\gamma}{\gamma}}\cdot\widehat{\beta}\cdot(\mathbb{E}[(m'^l\cdot(1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\} \\ \tilde{c} =& (\tilde{x}-\tilde{c})\cdot(\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}}\cdot\widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}}\cdot(\mathbb{E}[(m'^l\cdot(1+\widehat{r}'))^{1-\theta}])^{\frac{1}{1-\theta-\gamma}} \end{aligned}$$

Defining $n := \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}$, $o := (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}}$, and $m := \frac{o \cdot n}{1+o \cdot n}$, we get

$$\tilde{c} = m \cdot \tilde{x}.$$

- FOC with respect to stock portfolio share:

$$\begin{aligned}
0 =& \frac{\gamma}{1-\theta} \cdot \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}-1} \\
& \cdot (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot \frac{1}{\gamma} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}-1} \\
& \cdot \mathbb{E}[m'^{l \cdot (1-\theta)} \cdot (1-\theta) \cdot (1+\widehat{r}')^{-\theta} \cdot (r^{s\prime} - r^{f\prime})] \\
0 =& \mathbb{E}[m'^{l \cdot (1-\theta)} \cdot (1+\widehat{r}')^{-\theta} \cdot (r^{s\prime} - r^{f\prime})]
\end{aligned}$$

- FOC with respect to human capital portfolio share:

$$\begin{aligned}
0 =& \frac{\gamma}{1-\theta} \cdot \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}-1} \\
& \cdot (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot \frac{1}{\gamma} \cdot (\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{\gamma}-1} \\
& \cdot \mathbb{E}[m'^{l \cdot (1-\theta)} \cdot (1-\theta) \cdot (1+\widehat{r}')^{-\theta} \cdot (\widehat{r}^{h\prime} - r^{f\prime})] \\
0 =& \mathbb{E}[m'^{l \cdot (1-\theta)} \cdot (1+\widehat{r}')^{-\theta} \cdot (\widehat{r}^{h\prime} - r^{f\prime})]
\end{aligned}$$

What is left is to show that indeed $v = m^{l} \cdot \tilde{x}$. Using $\tilde{c} = m \cdot \tilde{x}$, $n = \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot$

$(\mathbb{E}[(m'^{l} \cdot (1+\widehat{r}'))^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}$, $m = \frac{o \cdot n}{1+o \cdot n}$, and $o = (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}}$ in u we get:

$$
\begin{aligned}
v &= \left\{ (m \cdot \tilde{x})^{\frac{1-\theta}{\gamma}} + \left(\frac{1}{1+g} \cdot (\tilde{x} - m \cdot \tilde{x}) \right)^{\frac{1-\theta}{\gamma}} \cdot n^{\frac{1-\theta-\gamma}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}} \\
&= \tilde{x} \cdot \left\{ m^{\frac{1-\theta}{\gamma}} + (1-m)^{\frac{1-\theta}{\gamma}} \cdot (\frac{1}{1+g})^{\frac{1-\theta}{\gamma}} \cdot n^{\frac{1-\theta-\gamma}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}} \\
&= \tilde{x} \cdot \left\{ (\frac{o \cdot n}{1+o \cdot n})^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+o \cdot n})^{\frac{1-\theta}{\gamma}} \cdot o^{\frac{1-\theta-\gamma}{\gamma}} \cdot n^{\frac{1-\theta-\gamma}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}} \\
&= \tilde{x} \cdot \left\{ \frac{(o \cdot n)^{\frac{1-\theta}{\gamma}} + (o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}}{(1+o \cdot n)^{\frac{1-\theta}{\gamma}}} \right\}^{\frac{\gamma}{1-\theta}} = \tilde{x} \cdot \left\{ \frac{(o \cdot n + 1) \cdot (o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}}{(1+o \cdot n)^{\frac{1-\theta}{\gamma}}} \right\}^{\frac{\gamma}{1-\theta}} \\
&= \tilde{x} \cdot \left\{ \frac{(o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}}{(1+o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}} \right\}^{\frac{\gamma}{1-\theta}} = \tilde{x} \cdot m^{\frac{1-\theta-\gamma}{1-\theta}}
\end{aligned}
$$

Hence, $v = m^{l} \cdot \tilde{x}$ where $l = \frac{1-\theta-\gamma}{1-\theta}$. Plugging this into the FOCs yields:

$$
\begin{aligned}
\tilde{c} &= m \cdot \tilde{x} \\
0 &= \mathbb{E}[(m')^{1-\theta-\gamma} \cdot (1+\widehat{r}')^{-\theta} \cdot (r^{s\prime} - r^{f\prime})] \\
0 &= \mathbb{E}[(m')^{1-\theta-\gamma} \cdot (1+\widehat{r}')^{-\theta} \cdot (\widehat{r}^{h\prime} - r^{f\prime})]
\end{aligned}
$$

Defining $\wp := \mathbb{E}[(m'^{\frac{1-\theta-\gamma}{1-\theta}} \cdot (1+\widehat{r}'))^{1-\theta}]$, the marginal propensity to consume equals:

$$
\begin{aligned}
m = \frac{o \cdot n}{1+o \cdot n} &= \frac{(\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}}{1 + (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}} \\
&= \frac{(\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot (\beta \cdot \varsigma^{\frac{1}{\gamma}} \cdot (1+g)^{\frac{1-\theta}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}}{1 + (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot (\beta \cdot \varsigma^{\frac{1}{\gamma}} \cdot (1+g)^{\frac{1-\theta}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}} \\
&= \frac{(\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot (\beta \cdot \varsigma^{\frac{1}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot (1+g)^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}}{1 + (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot (\beta \cdot \varsigma^{\frac{1}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot (1+g)^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}} \\
&= \frac{(\beta \cdot \varsigma^{\frac{1}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}}{1 + (\beta \cdot \varsigma^{\frac{1}{\gamma}})^{\frac{\gamma}{1-\theta-\gamma}} \cdot \wp^{\frac{1}{1-\theta-\gamma}}} = \frac{(\beta^{\gamma} \cdot \varsigma \cdot \wp)^{\frac{1}{1-\theta-\gamma}}}{1 + (\beta^{\gamma} \cdot \varsigma \cdot \wp)^{\frac{1}{1-\theta-\gamma}}}
\end{aligned}
$$

□

A.2 Derivation of the Aggregate Resource Constraint

Deriving the aggregate resource constraint, first we take the population weighted sums of the (de-trended) individual budget constraints and the individual human capital accumulation constraints in period t (cf. equations (8) and (7)) and add them up. Note that it is understood that we sum over all individuals of each age bin characterized by the idiosyncratic mean zero-shock η without making this explicit. We then get

$$
\begin{aligned}
&(1+g)\cdot\sum_{j=0}^{J} N_{t,j}\cdot\tilde{k}_{t+1,j+1}+\sum_{j=0}^{J} N_{t,j}\cdot h_{t+1,j+1}\\
&\quad=\sum_{j=0}^{J} N_{t,j}\cdot\tilde{k}_{t,j}\cdot(1+r_t^f+\alpha_{t,j}^s\cdot(r_t^s-r_t^f))+\sum_{j=0}^{J} N_{t,j}\cdot h_{t,j}\cdot\tilde{r}_t^H\\
&\qquad-\sum_{j=0}^{J} N_{t,j}\cdot\tilde{c}_{t,j}+\sum_{j=0}^{J} N_{t,j}\cdot h_{t,j}\cdot(1-\delta_j^h)\\
&(1+g)\cdot\sum_{j=0}^{J} N_{t,j}\cdot\tilde{k}_{t+1,j+1}+\sum_{j=0}^{J} N_{t,j}\cdot\varsigma_{t,j}\cdot h_{t+1,j+1}+\sum_{j=0}^{J} N_{t,j}\cdot(1-\varsigma_{t,j})\cdot h_{t+1,j+1}\\
&+N_{t+1,0}\cdot h_{t+1,0}-N_{t+1,0}\cdot h_{t+1,0}\\
&\quad=\sum_{j=1}^{J} N_{t-1,j-1}\cdot\varsigma_{t-1,j-1}\cdot\tilde{k}_{t,j}\cdot(1+r_t^f+\alpha_{t,j}^s\cdot(r_t^s-r_t^f))\\
&\qquad+\sum_{j=1}^{J} N_{t-1,j-1}\cdot(1-\varsigma_{t-1,j-1})\cdot\tilde{k}_{t,j}\cdot(1+r_t^f+\alpha_{t,j}^s\cdot(r_t^s-r_t^f))\\
&\qquad-\sum_{j=1}^{J} N_{t-1,j-1}\cdot(1-\varsigma_{t-1,j-1})\cdot\tilde{k}_{t,j}\cdot(1+r_t^f+\alpha_{t,j}^s\cdot(r_t^s-r_t^f))\\
&\qquad+\sum_{j=0}^{J} N_{t,j}\cdot h_{t,j}\cdot(1+\tilde{r}_t^H-\delta_j^h)-\sum_{j=0}^{J} N_{t,j}\cdot\tilde{c}_{t,j}\\
&(1+g)\cdot\sum_{j=0}^{J} N_{t,j}\cdot\tilde{k}_{t+1,j+1}+\sum_{j=0}^{J} N_{t+1,j}\cdot h_{t+1,j}+\sum_{j=0}^{J} N_{t,j}\cdot(1-\varsigma_{t,j})\cdot h_{t+1,j+1}\\
&\quad=\sum_{j=1}^{J} N_{t-1,j-1}\cdot\tilde{k}_{t,j}+\sum_{j=1}^{J} N_{t-1,j-1}\cdot\tilde{k}_{t,j}\cdot(r_t^f+\alpha_{t,j}^s\cdot(r_t^s-r_t^f))\\
&\qquad-\sum_{j=1}^{J} N_{t-1,j-1}\cdot(1-\varsigma_{t-1,j-1})\cdot\tilde{k}_{t,j}\cdot(1+r_t^f+\alpha_{t,j}^s\cdot(r_t^s-r_t^f))\\
&\qquad+\sum_{j=0}^{J} N_{t,j}\cdot h_{t,j}\cdot(1+\tilde{r}_t^H-\delta_j^h)-\sum_{j=0}^{J} N_{t,j}\cdot\tilde{c}_{t,j}+N_{t+1,0}\cdot h_{t+1,0}\\
&(1+g)\cdot\tilde{K}_{t+1}+H_{t+1}\\
&\quad=\tilde{K}_t+\tilde{K}_t\cdot r_t^K+H_t+H_t\cdot\tilde{r}_t^H-\tilde{G}_t-\tilde{C}_t\\
&\qquad-\sum_{j=0}^{J} N_{t,j}\cdot(1-\varsigma_{t,j})\cdot h_{t+1,j+1}-N_{t+1,0}\cdot h_{t+1,0}-\sum_{j=0}^{J} N_{t,j}\cdot h_{t,j}\cdot\delta_j^h
\end{aligned}
$$

or, finally,

$$\tilde{Y}_t = \tilde{C}_t + \tilde{G}_t + \tilde{I}_t^K + I_t^h \tag{26}$$

where we used equilibrium conditions summarized in 3.8 as well as $K_{t+1} = K_t \cdot (1 - \delta_t^K) + I_t^K$ and $I_{t+1}^H = H_t + I_t^H$. Note that I_t^h is the aggregate of gross human capital investments defined as: $I_t^h := I_t^H - N_{t+1,0} \cdot h_{t+1,0} + \sum_{j=0}^{J} N_{t,j} \cdot (1 - \varsigma_{t,j}) \cdot h_{t+1,j+1} + \sum_{j=0}^{J} N_{t,j} \cdot h_{t,j} \cdot \delta_j^h$.

B Computational Appendix

Numerical computations are implemented in Fortran 90 using routines which are partly based on Press, Teukolsky, Vetterling, and Flannery (1996). If not otherwise stated the convergence criterion of a root finding algorithm is set to 10^{-6} and the weight on resulting variables in the updating step of a Gauss-Seidel algorithm (cf., e.g., Ludwig 2007) is set to 10%.

B.1 Numerical Solution

We solve an approximate rational expectations equilibrium by adapting the computational method developed in Krusell and Smith (1997, 1998) to the case of transitional dynamics with time-varying aggregate laws of motion due to a time-varying demographic distribution, N. Therefore, we follow the approach in Geppert and Ludwig (2015).

The solution of the model begins in year 1960 ($t = 0$) in which we assume a fix demographic distribution leading to an artificial initial stationary equilibrium. We redo the exercise in year 2500 ($t = T$) with a fix demographic distribution of year 2100. Aggregate laws of motion (ALOM) in those stationary equilibria are assumed to be linear functions of a small number of moments of the endogenous aggregate state. They are specified as

$$\ln \tilde{K}' = \phi_{t,\lambda}^{K,0} + \phi_{t,\lambda}^{K,1} \cdot \ln \tilde{K} + \phi_{t,\lambda}^{K,2} \cdot \ln \kappa + \phi_{t,\lambda}^{K,3} \cdot \mu \tag{27a}$$

$$\ln \kappa' = \phi_{t,\lambda}^{\kappa,0} + \phi_{t,\lambda}^{\kappa,1} \cdot \ln \tilde{K} + \phi_{t,\lambda}^{\kappa,2} \cdot \ln \kappa + \phi_{t,\lambda}^{\kappa,3} \cdot \mu \tag{27b}$$

$$\mu' = \phi_{t,\lambda'}^{\mu,0} + \phi_{t,\lambda'}^{\mu,1} \cdot \ln \tilde{K}' + \phi_{t,\lambda'}^{\mu,2} \cdot \ln \kappa' + \phi_{t,\lambda'}^{\mu,3} \cdot \mu \tag{27c}$$

where $\kappa := \tilde{K}/H$ is the (de-trended) capital-human capital ratio, $\mu := E[r^{s\prime}] - r^{f\prime}$ is the ex-ante equity premium, and $'$ indicates the next simulation step. Note that the ALOMs depend on both, time and the exogenous aggregate state. We determine the coefficients in the ALOMs by Monte Carlo simulations, further described below.[20]

In the main step of the procedure, we compute the (stochastic) transition of the economy from the initial to the final stationary equilibrium which is induced by the exogenous deterministic dynamics of the demographic distribution between 1960 and 2100. The standard *brute force* approach would be to assume a separate law of motion (27) for each time period in the transition.[21] Instead, we follow Geppert and Ludwig (2015) and specify parameterized laws of motion for the transition by multiplying the coefficients of the stationary equilibria with time

[20]Note that, in a narrow definition, the ALOMs consist only of (27b). μ does not belong to the aggregate state variables because it can be derived contemporaneously from Ω. We follow Krusell and Smith (1997), Storesletten, Telmer, and Yaron (2007), Harenberg and Ludwig (2015), and others and assume a law of motion for μ in order to avoid solving for the bond market equilibrium in all future states of the world in the determination of policy functions. Basically, the latter would be feasible but causing tremendous computational costs.

[21]The advantage of this approach compared to the brute force approach is discussed at the end of this section.

polynomials using ideas from Judd (2002):

$$\ln \tilde{K}_{t+1} = \exp(-\nu^K \cdot t) \cdot \sum_{l=0}^{3} \phi_{0,\lambda}^{K,l} \cdot P_{\lambda}^{K,l}(t) \cdot y_t^l + \left(1 - \exp(-\nu^K \cdot t)\right) \cdot \sum_{l=0}^{3} \phi_{T,\lambda}^{K,l} \cdot y_t^l \tag{28a}$$

$$\ln \kappa_{t+1} = \exp(-\nu^\kappa \cdot t) \cdot \sum_{l=0}^{3} \phi_{0,\lambda}^{\kappa,l} \cdot P_{\lambda}^{\kappa,l}(t) \cdot y_t^l + (1 - \exp(-\nu^\kappa \cdot t)) \cdot \sum_{l=0}^{3} \phi_{T,\lambda}^{\kappa,l} \cdot y_t^l \tag{28b}$$

$$\ln \mu_{t+1} = \exp(-\nu^\mu \cdot t) \cdot \sum_{l=0}^{3} \phi_{0,\lambda'}^{\mu,l} \cdot P_{\lambda'}^{\mu,l}(t) \cdot y_t^l + (1 - \exp(-\nu^\mu \cdot t)) \cdot \sum_{l=0}^{3} \phi_{T,\lambda'}^{\mu,l} \cdot y_t^l \tag{28c}$$

where $y_t^l \in \{1, \ln \tilde{K}_t, \ln \kappa_t, \mu_t\}$ for (28a-28b) and $y_t^l \in \{1, \ln \tilde{K}_{t+1}, \ln \kappa_{t+1}, \mu_t\}$ for (28c).

Here, ν^i is the coefficient that determines the speed of convergence of the law of motion of variable $i \in \{\tilde{K}, \kappa, \mu\}$ to the corresponding law of motion in the final stationary equilibrium of the economy. $P_{\lambda}^{i,l}(t)$ for all $i \in \{\tilde{K}, \kappa, \mu\}$, $l \in \{0, 1, 2, 3\}$, and $\lambda \in L$, are flexible global time polynomials of Chebyshev's first kind writing as:

$$P_{\lambda}^{i,l}(t) = \sum_{q=0}^{n_q} \psi_{\lambda}^{i,l,q} \cdot \mathcal{T}^q(t) \tag{29}$$

where $\mathcal{T}^0(t) = 1$, $\mathcal{T}^1(t) = t$, $\mathcal{T}^q(t) = 2 \cdot t \cdot \mathcal{T}^{q-1}(t) - \mathcal{T}^{q-2}(t)$ for all $q \geq 2$, and n_q is the order of the polynomial. We determine the coefficients in the time polynomials by Monte Carlo simulations, further described next.

In analogy to Krusell and Smith (1997, p. 404) and Gomes and Michaelides (2008), we employ the following algorithm for the determination of the coefficient vector Φ.[22] Note that Φ contains the coefficients of the ALOM (27) in t in the case of solving for the stationary equilibrium in period t (i.e., $t_1 = t_2 = t$). Meanwhile

[22]Note that, for the sake of simplicity, we use the same symbol, Φ, for denoting the actual ALOM and its approximating coefficient vector.

Φ contains the coefficients of the time polynomials in (28) given by (29) in the case of the transition (i.e., $t_1 = 1$ and $t_2 = T - 1$):

1. Build grids for $\tilde{K}$, κ, and μ.[23]

2. Draw M^A series of M^B aggregate shock realizations where $M^A = 1$ in the case of a stationary equilibrium while $M^B = T - 1$ in case of the transition.

3. Iterate on the vector of coefficients Φ until convergence (fixed point iteration).

   (a) Choose an initial guess for Φ.[24]

   (b) Solve the household problem (9) for policy functions $(m, \widehat{\alpha}^h, \widehat{\alpha}^s)$ from $t = t_2, ..., t_1$ at all (λ, j) and $(\tilde{K}, \kappa, \mu)$ of the respective grid. Therefore, use (27) and (28) respectively as well as the exogenous law of motion for λ in order to make expectations on $(\tilde{K}', \kappa', \mu')$ and to determine $(r^{s\prime}, r^{f\prime}, r^{h\prime})$. Store the policy functions.

   (c) Simulate the economy $M^A \cdot M^B$ times.

      i. Determine λ_{m^A, m^B}.

      ii. If $m^B = 1$ choose initial $(\tilde{K}, \kappa, \mu, r^f)$[25] otherwise use $(\hat{\tilde{K}}', \hat{\kappa}', \hat{\mu}, \hat{r^{f\prime}})_{m^B - 1}$ of the previous iteration step.

      iii. Iterate on μ until the bond market clears.

         A. Interpolate on the policy functions with respect to $(\tilde{K}, \kappa, \mu)$.

         B. Determine r^s and $\tilde{r}^H$ using (2).

         C. Aggregate and determine $(\hat{\tilde{K}}', \hat{\kappa}')$ as well as the aggregate excess demand on the bond market.

      iv. Store $(\hat{\tilde{K}}', \hat{\kappa}', \hat{\mu})$.

[23] We build grids around the solution of the mean shock equilibrium (MSE) which assumes aggregate uncertainty to realize at its unconditional mean while otherwise fully accounting for the stochastic feature of the model. Please see section B.2 for more detailed information.

[24] We use the solutions of the MSE in order to choose the initial coefficient guesses. For the initialization of the transition we run non-linear regressions of (28) given the mean shock path, $\{\Omega^{\mathcal{M}}\}_{t1}^{t2}$. This yields rates of convergence speed (ν^K, ν^κ, and ν^μ) and initial guesses for the coefficients in (29).

[25] We use the corresponding MSE values in case of a stationary equilibrium respectively a random realization of the initial stationary equilibrium in case of the transition.

(d) Discard the first M^D of M^B periods whereof $M^D = 0$ in case of the transition. Determine $\hat{\Phi}$ by running regressions using $\{\{\hat{\tilde{K}}, \hat{\kappa}, \hat{\mu}\}_{m^B=M^D+1}^{M^B}\}_{m^A=1}^{M^A}$ together with (27) respectively (28).

(e) Update the coefficient vector according to $\Phi^{new} = \vartheta \cdot \Phi^{old} + (1-\vartheta) \cdot \hat{\Phi}$ where $0 < \vartheta < 1$ is an arbitrary adjustment factor.

We select second order time polynomials in (29) and use $M^B = 8250$ and $M^D = 750$ for the stationary equilibria as well as $M^A = 50$ for the transition. Note that while the coefficients of the laws of motion of the alternative brute force approach would be identified solely by cross-sectional variation, those of the time polynomials are identified also by time variation. Accordingly, this method requires a much smaller number of simulations M^A compared to the brute force approach.
After convergence, the Euler equation errors of households are small with a maximum error of $5 \cdot e^{-4}$ while R^2 of all regressions are higher than 0.982.

B.2 Mean Shock Equilibrium

As an initialization step, we solve for a degenerate path of the economy where the realizations of all aggregate shocks are at their respective means. We accordingly set $\lambda = (z, \delta^K)$ to $\bar{\lambda} = (\bar{z}, \bar{\delta^K}) = (\mathbb{E}[z], \mathbb{E}[\delta^K])$. We assume that households accurately solve their forecasting problem for each realization of the aggregate state. This means that we approximate the above approximate law of motion as

$$(\tilde{K}', \kappa', \mu') = \hat{\tilde{\Phi}}(t; \tilde{K}, \kappa, \mu, \bar{\lambda}, \bar{\lambda}') \tag{30}$$

Observe that in the two stationary equilibria of our model, that is in periods $t = 0$ and $t = T$, respectively, we have the fixed point relation

$$(\tilde{K}', \kappa', \mu') = \hat{\tilde{\Phi}}(t; \tilde{K}, \kappa, \mu, \bar{\lambda}, \bar{\lambda}') = (\tilde{K}, \kappa, \mu) \tag{31}$$

With these assumptions, we can solve the mean shock path by standard Gauss-Seidel iterations as, e.g., described in Ludwig (2007). That is, we first solve for the steady state equilibria in periods $t = 0$ and $t = T$, respectively, and

then compute the transitional dynamics between those steady states. While the numerical methods are the same as in the solution to a deterministic economy, the actual behavior of households fully takes into account the stochastic nature of the model. The fixed-point computed in this auxiliary equilibrium yields the aggregate moments $\{\tilde{K}_t^{\mathcal{M}}, \kappa_t^{\mathcal{M}}, \mu_t^{\mathcal{M}}\}_{t_1}^{t_2}$ as well as the corresponding cross-sectional distributions of agents denoted by $\{\Omega_t^{\mathcal{M}}\}_{t_1}^{t_2}$.

The employed algorithm of the MSE determination iterates on the vector of aggregate (state) variables $\{\tilde{K}, \kappa, \mu\}_{t_1}^{t_2}$ until convergence (fixed point iteration) as follows. Again, $t_1 = t_2 = t$ in the case of a stationary equilibrium while $t_1 = 1$ and $t_2 = T - 1$ in the case of the transition:

1. Choose an initial guess for $\{\tilde{K}, \kappa, \mu\}_{t_1}^{t_2}$.
2. Solve the household problem (9) for policy functions $(m, \widehat{\alpha}^h, \widehat{\alpha}^s)$ from $t = t_2, ..., t_1$ for all (λ, j) of the respective grid.[26]
3. Iterate forward in time for $t = t_1, ..., t_2$ and iterate in each t on μ until the bond market in t clears:
   (a) Choose an initial guess for μ.
   (b) Update the policy functions $(m, \widehat{\alpha}^h, \widehat{\alpha}^s)$ at t for all (λ, j) of the respective grid.
   (c) Aggregate over all households in the mean shock state by interpolating on the policy functions and returns. Determine $\hat{\tilde{K}}'$, $\hat{\kappa}'$, and the aggregate excess demand on the bond market.
4. Update $\{\tilde{K}, \kappa, \mu\}_{t_1}^{t_2}$ according to $\{\tilde{K}, \kappa, \mu\}_{t_1}^{t_2 new} = \vartheta \cdot \{\tilde{K}, \kappa, \mu\}_{t_1}^{t_2 old} + (1 - \vartheta) \cdot \{\hat{\tilde{K}}, \hat{\kappa}, \hat{\mu}\}_{t_1}^{t_2}$ where $0 < \vartheta < 1$ is an arbitrary adjustment factor.

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[26] This step is not executed in case of a stationary equilibrium as step 3b solves for policy functions, anyway.

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