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# Contracting frictions and inefficient layoffs of older workers

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## Abstract

In light of the low re-employment opportunities that workers above age 55 face in continental Europe, inefficient separations of older workers may generate a significant welfare loss for the economy. The economic literature therefore suggests eliminating any governmental policies that distort firing and retirement decisions. In this paper, I argue that even without any governmental distortions, there may still be inefficient destruction of older workers' jobs due to market-inherent contracting frictions that may arise due to asymmetric information. I demonstrate this in a two period model of the labor market where risk-averse workers choose their preferred wage contract from a restricted subset of the contract space. I find that an equilibrium with bilaterally inefficient layoffs of older workers emerges under certain conditions. These conditions are satisfied more easily if high outside options, such as early retirement benefits, become available for the elderly. This prediction is consistent with empirical studies postulating that generous social security systems increase the incidence of being 'pushed' into early retirement by the employer. On the other hand, training programs, wage subsidies, and employment protection are found to decrease the incidence of bilaterally inefficient layoffs.

## 1 Introduction

For its *Employment Outlook 2013*, the OECD analyzed the incidence of job displacement and its economic consequences for different groups of workers. A "job displacement" was defined as an "involuntary job separation due to economic or technological reasons or as a result of structural change" (p.194). The report concludes on pages 225–226 that

*"[S]ome workers are more prone to job displacement, and to negative consequences after displacement, than others. In particular, older workers and those with low education levels have a higher displacement risk, take longer to get back into work and suffer greater (and more persistent) earnings losses in most countries examined."*

Yet, there is substantial cross-country variation in how strong the disadvantage of older workers is relative to their younger counterparts. This is evident from Table A.1 in Appendix A,

which summarizes the most important figures of the OECD report.<sup>1</sup> Column 1 relates the displacement rate of older workers (55-64 years) to those of prime-age workers (35-44 years). Column 2 shows the respective ratio of finding re-employment within one year after displacement. Labor market conditions for older workers seem to be particularly bad in continental Europe, where old-age displacement rates are high, re-employment rates are low and a large share of old individuals becomes inactive within one year of displacement. Since early exits from the labor force increase the financial pressure on the social welfare system, various measures have been proposed and were already implemented by national governments in order to facilitate re-integration of unemployed older workers into the labor market.<sup>2</sup> However, picking up the pieces might not be as (cost-)effective as policies that aim at reducing separations of older workers in the first place. Indeed, it is not at all clear that the incidence of job separations of older workers that is found in the data is near a level that can be considered as “socially efficient”. In fact, they need not even be *bilaterally* efficient.

Bilateral efficiency means that apart from exogenous reasons, an employment spell ends if and only if the joint surplus of the firm-worker match falls to zero. At this point, parting ways is optimal for both the firm and the worker, and also optimal from a planner’s perspective. This property is common in labor market models and arises from bilaterally efficient wage determination mechanisms such as generalized Nash bargaining or directed search (cf. Mortensen and Pissarides, 1999, Section 4). It remains valid when these models are put into a life-cycle context (Chéron et al., 2011, 2013).

For older workers, however, bilateral efficiency seems hard to align with a number of empirical observations. Bilateral efficiency implies that observed job separations should to a large extent be considered optimal by both parties. Individual survey data calls this view into question. Using ISSP data, Dorn and Sousa-Poza (2010) report that a substantial amount of transitions to early retirement happens “not by choice” of the worker. In Germany, for instance, every second transition to early retirement is reported as “involuntary”. The share is much higher in Europe (France 41%, UK 28.9%) than in North America (Canada 12.2%, US 9.4%).<sup>3</sup> Bilateral efficiency also implies a specific relation between the wage profile and the timing of a separation. If workers are risk neutral, wages should not affect the timing of a separation at all. In fact, the only role for wages is the determination of the present discounted value for firms, which influences job creation (Hornstein et al., 2005; Shimer, 2005). With risk averse workers, higher wages increase joint surplus and should therefore lead to later retirement.<sup>4</sup> Neither of this is consistent with the empirical findings of Frimmel et al. (2015), who report that workers tend to be separated earlier if their employer pays high rewards to older employees, or if wage levels in

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<sup>1</sup>The lack of a unified data source made it necessary to use different databases for each country. Because of imperfect comparability, it is formally distinguished whether the displacements were identified from employee- or employer-reported data. The particular data sources and definitions can be found on pages 232–233 in OECD (2013).

<sup>2</sup>Table 5.2 in OECD (2006) provides an overview of the measures taken.

<sup>3</sup>The authors control for cultural attitudes and different health levels across countries. A similar figure for the UK has been reported by Marmot et al. (2003), using the English Longitudinal Study of Ageing.

<sup>4</sup>This is because the worker’s contribution to joint surplus is weighted by the inverse of marginal utility, see Menzio and Moen (2010) and Danthine and Navarro (2013).

the firm are strongly increasing with tenure.

Consequently, it is not all clear that labor market transitions of older workers follow the principles of bilateral efficiency. Our model therefore allows for potential inefficiencies in job separations. In particular, we consider that the (otherwise efficient) wage determination mechanism is impeded by a contracting friction that the parties cannot overcome. This can be motivated by the presence of asymmetric information about the current productivity of the match. It turns out that although all workers are subject to this friction, only old and long-tenured workers suffer in the form of inefficient layoffs. Prime-age individuals are affected only indirectly through lower job-finding probabilities.

Our comparative static analysis reveals that the incidence of bilaterally inefficient separations increases if the government provides attractive outside options to older workers, such as generous early retirement benefits or high unemployment provisions. The detrimental effects of contracting frictions are therefore reinforced by inappropriate governmental policies which lower the value of employment for older individuals. On the other hand, discouraging early retirement, subsidizing older workers' wages, taxing layoffs, or providing employer incentives for continued training initiatives are found to reduce inefficient layoffs in the economy.

The paper is structured as follows. Section 2 briefly summarizes the literature on inefficient layoffs and introduces the particular contracting friction considered in this paper. Section 3 lays out the model. The labor market equilibrium is characterized in Section 4. The first part of that section is devoted to the optimal search behavior of old unemployed individuals. The second part focuses on the prime-age unemployed. To get a sense of magnitude, the theoretical results are numerically illustrated in Section 5 using a sample parameterization. Section 6 derives policy implications and provides links to the empirical literature. Section 7 discusses the robustness of our findings, before Section 8 concludes. All proofs and additional lemmas are provided in Appendix B.

## 2 What causes inefficient layoffs?

Labor market outcomes arise from the interaction of workers' labor supply and firms' labor demand. Both margins may be severely distorted by governmental policies and/or market-inherent frictions, thereby resulting in an inefficient allocation of labor. The relation between *public policy* and the labor market exit of older workers has been intensively studied in the literature during the last decade. Fisher and Keuschnigg (2008), Jaag et al. (2010), and Hairault et al. (2012) argue that the social welfare system distorts individual behavior by introducing implicit taxes in the labor participation and retirement decision, unless the pension formula is actuarially fair at the optimal retirement age. Because wages are determined by generalized Nash bargaining, however, job separations are nevertheless bilaterally efficient.

By contrast, the findings of Winter-Ebmer (2003) and Dorn and Sousa-Poza (2010) suggest that public policy may also distort bilateral efficiency. They demonstrate that generous unemployment insurance systems and early retirement possibilities do not only lower the retirement age that is considered optimal by the individuals. Instead, these systems are also used by em-

ployers as a means to conveniently get rid of older workers, which are often the most expensive part of their workforce.

Apart from distortionary governmental policies, there may also exist more fundamental, *market-inherent frictions* that lead to bilaterally inefficient job destruction of older workers. Mechanisms that have been proposed in this regard include asymmetric information (Hashimoto, 1981; Hall and Lazear, 1984), adverse selection (Weiss, 1980), and moral hazard (Lazear, 1979; Ramey and Watson, 1997). In these papers, inefficient layoffs arise due to contracting problems, since the presence of market failures restricts the set of feasible remuneration schemes that can be implemented in equilibrium. However, the role of worker's age remains unclear from this literature, and hence the question of why any of these frictions should be particularly detrimental for older workers is left open.

This paper intends to bridge this gap. We build a stylized model of the labor market comprising prime-age and old-age workers. We assume that the wage determination mechanism is impeded by a market-inherent contracting friction that can be motivated by the presence of asymmetric information. Our model can therefore be regarded as a life-cycle extension of the ideas of Hall and Lazear (1984). We assume that the contracting friction manifests itself in a restriction of the space of feasible wage contracts. In the main text of the paper, we impose the following assumptions:

- (i) the productivity of a worker is stochastic in each period,
- (ii) wage contracts are written before productivity realizes and may not be contingent on productivity,
- (iii) wage renegotiation is not possible.

This implies that in some situations the pre-negotiated wage levels may *ex-post* be inappropriate to sustain the match, because one of the parties would suffer a loss and unilaterally terminates the match. In our model it will be the firm that in some cases finds the contracted wage levels too high to keep up employment. The worker is then laid off, which is inefficient because contracting a lower wage would have been superior for both parties from an *ex-post* point of view.

The above set of assumptions is not arbitrarily chosen, but borrowed from the literature on severance pay, where the starting point is typically an inefficient labor market (Alvarez and Veracierto, 2001; Boeri et al., 2013). Nevertheless, a few remarks are in order. Assumption (i) is in itself innocuous as it would still lead to a bilaterally efficient labor market allocation if firms announced productivity-dependent wage schedules. However, even if productivity can be observed by both the firm and the worker, it may not be verifiable by a third party, such as a court. Therefore, a contract that specifies productivity-contingent wages may not be enforceable in practice, which motivates assumption (ii). Still, the parties could attain bilateral efficiency by renegotiating the wage after productivity has been revealed. This is, however, ruled out by assumption (iii), which implies that wage renegotiation is not even allowed in situations when this would be beneficial for both parties. While this seems very restrictive at first sight, we demonstrate in Section 7.5 that our model gives rise to the same equilibria

if we drop assumptions (ii)–(iii) and instead allow for wage renegotiation under asymmetric information, assuming that the current productivity state is private knowledge of the firm. We consider such an informational friction plausible because the employer typically has better insight on how much an individual worker is contributing to the company’s output (cf. Hall and Lazear, 1984). It may also be the case that worker and firm have different perceptions about the worker’s performance, in which case it is the firm’s valuation that determines whether or not the match continues. The equivalence between no renegotiation and renegotiation under asymmetric information stems from firm incentives. An employer can always increase her own surplus by making the worker agree on a wage cut. This creates an innate incentive to cheat on the worker and always pretend that a wage cut is required to prevent a layoff, even when this is not the case. Rational workers anticipate this and refuse to renegotiate under certain conditions. The equilibria of the model derived under assumptions (i)–(iii) turn out to satisfy exactly these conditions. See Section 7.5 for further details. For the remainder of the paper we directly impose assumptions (i)–(iii) to simplify the analysis, but keep the equivalence to a game with asymmetric information in the back of our head.

It is important to note that assumptions (i)–(iii) only create a possibility for inefficient layoffs, but do *not* by themselves imply that inefficient separations will actually occur in equilibrium. Workers still have the possibility to choose a wage contract that completely rules out endogenous separations. Assuming directed search, job-seekers demand contracts that bear the risk of a bilaterally inefficient separation if and only if they consider this *ex-ante* optimal. The purpose of this paper is to work out which conditions are required for such a situation to emerge, and which economic factors facilitate this.

### 3 Model setup

#### 3.1 Workers, productivity and wage contracts

Time is discrete and the world lasts for two periods. At  $t = 0$ , a continuum of individuals is born. There is no mortality risk, such that everybody lives until  $t = 2$ . In each period, an individual is either employed or unemployed. Unemployed individuals receive an income  $b$  that comprises home production and unemployment benefits. Among the employed individuals, we differentiate three worker states: prime-age ( $m$ ), senior ( $s$ ), and old ( $o$ ). In period 1, any working individual is considered a *prime-age worker*. In period 2, senior workers and old workers co-exist. A *senior worker* was already working at the same firm in period 1, whereas an *old worker* was either unemployed or working for a different firm in period 1.

The period output  $Y_i$  that a worker generates is stochastic and drawn from a distribution that depends on her current state  $i \in \{m, s, o\}$ . The draws are independent across workers and states. For simplicity, we assume a Bernoulli process that can take only two values which are identical across workers, i.e.  $Y_i \in \{y^H, y^L\}$  where  $y^H > y^L > b$ . Heterogeneity enters the process via the probability with which these two productivity levels are attained. For a prime-age worker, the probability of exhibiting high productivity equals  $\phi_m \in (0, 1)$ . For senior

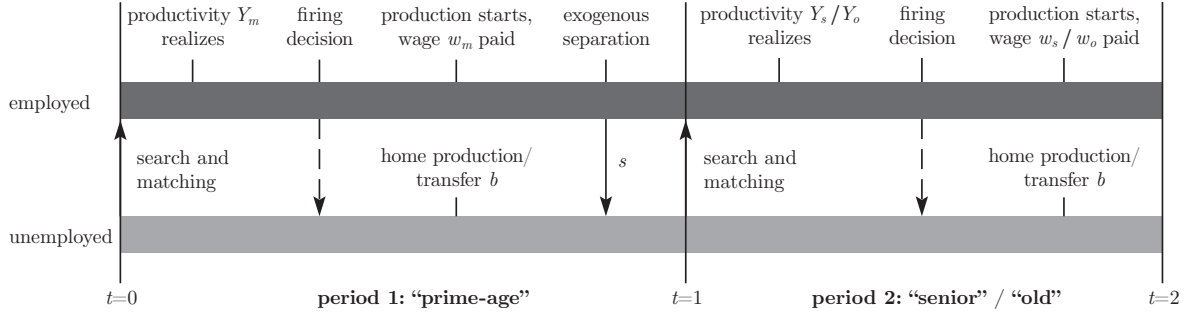


Figure 1: Model timing

and older workers, the probability for the high draw is denoted by  $\phi_s \in (0, 1)$  and  $\phi_o \in (0, 1)$ , respectively. Hence workers know in advance from which distribution their productivity in state  $i$  will be drawn, but they do not know the value that will realize.<sup>5</sup>

The timing of the model is illustrated in Figure 1. Initially, all individuals are unemployed and apply to a firm that offers a wage contract  $(w_m, w_s)$ . The contract specifies a wage  $w_m$  that is due in period 1 and a senior wage  $w_s$  due in period 2. With probability  $p(\theta_m)$ , the application is successful and a match is formed.<sup>6</sup> Firm and worker then commit to the wage contract but not to actual employment. That is, either party can terminate the match at any time. If employment takes place, remuneration must be according to the contract. There is no possibility to renegotiate. Additionally, productivity-contingent wage contracts are ruled out because of a lack of enforceability. These two restrictions, together with the fact that the worker's productivity profile  $(Y_m, Y_s)$  is a random variable when contracts are written, are the source of inefficient separations in the model. If either of these three features (stochastic productivity, lack of renegotiation, and lack of enforceability) is given up, job separations are bilaterally efficient as in Acemoglu and Shimer (1999).<sup>7</sup>

If a firm meets a worker, the period 1 productivity draw  $Y_m$  is revealed. A bad draw might lead to a situation where the firm realizes a loss by honoring the contract. This happens if the wage stream promised to the worker exceeds the sum of today's output and expected future output. To prevent a financial loss, the worker is laid off *before* production starts and remains unemployed until  $t = 1$ . Otherwise production takes place and the worker receives the wage  $w_m$  specified in the contract.

At the end of period 1, the match might end for exogenous reasons with probability  $s \in [0, 1]$ . If neither exogenous nor endogenous separations happen, the worker is still with the firm at

<sup>5</sup>The discrete productivity process might seem unhandy because it introduces non-differentiabilities into the model. Assuming a continuous distribution would indeed simplify our analysis, provided that the lowest productivity draw is less than or equal to  $b$ . In this case, however, it is quite trivial that inefficient layoffs occur in equilibrium, because the maximum wage that workers can demand without taking the risk of an inefficient layoff equals their outside option. We instead want to allow workers to choose contracts that feature no layoff risk and are still superior to non-employment. To this end, the lower boundary of the productivity distribution must exceed  $b$ . This, however, would introduce non-differentiabilities also with a continuous distribution function. Furthermore, the equivalence result of Section 7.5 is straightforward to show for a binary process, but may lead to further technical complications if the underlying productivity distribution is continuous.

<sup>6</sup>The determination of  $(w_m, w_s)$  and  $p(\theta_m)$  is described in detail in the next section.

<sup>7</sup>Arguments that motivate and justify these assumptions are presented in Section 2.

$t = 1$  and according to our terminology becomes a senior worker. Next, the productivity draw  $Y_s$  realizes and, again, the worker is laid off before the production stage if productivity is too low relative to the contracted wage  $w_s$ . Otherwise the match continues until  $t = 2$ , production takes place, and the worker earns  $w_s$ .

All prime-age workers who are not matched to a vacancy at  $t = 0$  or lose their jobs during period 1 can (re-)enter employment at  $t = 1$ . At this date, they search for a one-period contract  $w_o$ . If the application is successful, the value of  $Y_o$  is revealed and the firm chooses between firing or retaining the worker. All individuals laid off during period 2 as well as those who could not find a job at  $t = 1$  have no further opportunity to enter employment.

At this stage, it is important to note that in our model any endogenous layoff that occurs in equilibrium is inefficient from a social planner's perspective. Since  $y^L > b$ , the value that a low productivity worker contributes to total output in the economy is higher in market production than in home production. The layoff occurs because the rent that would accrue to the firm is negative, which could in principle be avoided by the worker demanding a different wage contract at the search stage.

### 3.2 Firms, search, and matching

The economy is populated by a continuum of identical firms. Each firm consists of a single job and uses a constant returns to scale production technology. Firms can freely enter the labor market, but posting a vacancy is involved with a period cost  $c > 0$ . To capture the search and matching of unemployed workers to vacancies, we use the notion of directed search introduced by Shimer (1996) and Moen (1997). One of the merits of this approach is that if firms are able to age-direct their hiring process, such that individuals with different age operate in separate labor markets. The labor market equilibrium is hence independent of the age distribution in the economy.

For prime-age workers, the labor market is segmented by  $(\theta_m, w_m, w_s)$ . Firms can open vacancies in any of these submarkets, and unemployed of prime-age apply to the submarket where an application yields the highest expected present discounted surplus. In equilibrium,  $\theta_m = \theta_m(w_m, w_s)$  denotes the labor market tightness of the submarket where the wage profile  $(w_m, w_s)$  is paid. This is pinned down by the free entry condition of the firm sector. Typically, job applicants face a trade-off between the contracted wage stream and the number of vacancies posted by firms, as measured by the labor market tightness  $\theta_m$ . The latter determines the probability that a job application is successful. Within each submarket,  $v_m$  vacancies and  $u_m$  applicants are randomly matched by a constant returns to scale matching technology  $M(u_m, v_m)$ . The labor market tightness is defined as the ratio between vacancies and applicants,  $\theta_m := v_m/u_m$ . The associated probability of filling a vacancy is  $q(\theta_m) = M(u_m, v_m)/v_m = M(1/\theta_m, 1)$ , and the probability of an application to be accepted is  $p(\theta_m) = M(u_m, v_m)/u_m = \theta_m q(\theta_m)$ .<sup>8</sup>

For the old unemployed, the situation is similar. Their labor market is segmented by  $(\theta_o, w_o)$

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<sup>8</sup>We omit dependence of  $v_m$  and  $u_m$  on  $(w_m, w_s)$  for the sake of brevity. The matching function  $M$  is the same in all submarkets.



and they also direct their application to the submarket that yields the highest expected surplus. In equilibrium, the labor market tightness  $\theta_o = \theta_o(w_o)$  is pinned down by the firms' free entry condition. The matching technology is the same as for prime-age individuals.

## 4 Model solution

To determine the labor market equilibrium, we first characterize optimal search behavior of the prime-age and old unemployed. We will see that depending on the model parameters, particularly  $\phi_o$  and  $\phi_s$ , individuals either prefer a safe wage contract, or a risky contract that promises higher wages but in period 2 leads to a layoff if productivity is low.

The model can be solved recursively. The search problem faced by unemployed workers in period 2 is independent from their behavior in the first period. In the following Section 4.1 we thus start our analysis with the problem of a representative old unemployed individual. The problem of prime-aged unemployed is treated in Section 4.2. We proceed under the following functional restrictions:

**Assumption 4.1.** *Firms are risk neutral. Workers are risk averse, their utility function  $u(\cdot)$  is twice differentiable with  $u' > 0$  and  $u'' < 0$ . The matching function is such that  $p'(\theta) > 0$  and  $q'(\theta) < 0$  for all  $\theta \geq 0$  together with*

$$\begin{aligned}\lim_{\theta \rightarrow 0} p(\theta) &= \lim_{\theta \rightarrow \infty} q(\theta) = 0, \\ \lim_{\theta \rightarrow \infty} p(\theta) &= \lim_{\theta \rightarrow 0} q(\theta) = 1.\end{aligned}$$

The elasticity of the vacancy filling probability,  $\varepsilon(\theta) := -\frac{q'(\theta)\theta}{q(\theta)}$ , satisfies  $\lim_{\theta \rightarrow 0} \varepsilon(\theta) = 0$  and<sup>9</sup>

$$0 \leq \frac{\varepsilon'(\theta)\theta}{\varepsilon(\theta)(1 - \varepsilon(\theta))} \leq 1.$$

### 4.1 Old unemployed

An old unemployed individual seeks to maximize her expected surplus from applying to a vacancy with characteristics  $(\theta_o, w_o)$ , which is  $p(\theta_o)\mathbb{E}G_o(w_o)$ . With probability  $p(\theta_o)$ , the application is successful and generates an expected worker surplus of  $\mathbb{E}G_o(w_o)$ . Otherwise, the individual remains unemployed and her surplus is zero by definition.

Firms, on the other hand, only operate in segments of the labor market where their expected surplus of posting a vacancy is non-negative. In equilibrium, any expected profit from vacancy creation is exploited by entering new firms, such that the expected firm surplus of posting a vacancy just makes up for the posting cost  $c$ . This gives rise to the so-called *free entry curve*  $q(\theta_o)\mathbb{E}J_o(w_o) = c$ , where  $q(\theta_o)$  is the probability that the vacancy turns into a match, and  $\mathbb{E}J_o(w_o)$  denotes the expected firm surplus of this match. Workers take into account this

<sup>9</sup>In the special case of the matching function proposed by den Haan et al. (2000),  $M(u, v) = uv/(u^l + v^l)^{1/l}$ , this unwieldy condition simplifies to  $l \in [0, 1]$ .

constraint of the firm in their application decision. Formally, they solve

$$V_o := \max_{(w_o, \theta_o)} p(\theta_o) \mathbb{E}G_o(w_o) \quad \text{s.t.} \quad q(\theta_o) \mathbb{E}J_o(w_o) = c. \quad (\text{O})$$

The next section derives the expected surplus functions of firm and worker,  $\mathbb{E}J_o(w_o)$  and  $\mathbb{E}G_o(w_o)$ .

#### 4.1.1 Value functions

The firm surplus of a match with wage  $w_o$  and productivity draw  $y_o$  is  $\tilde{J}_o(w_o | Y_o = y_o) = y_o - w_o$ , while the worker surplus is  $\tilde{G}_o(w_o) = u(w_o) - u(b)$ . However, a vacancy with  $w_o < b$  will not be filled because workers are better off by engaging in home production. If  $w_o > y_o$ , on the other hand, the firm is better off with a vacant job (which has a value of zero), and fires the worker before the production stage. Therefore, a vacancy only turns into a *productive match* if (i) the worker is willing to work for the given wage and (ii) the worker is profitable for the firm.<sup>10</sup> Hence the *effective surplus functions* are

$$J_o(w_o | Y_o) := \begin{cases} \tilde{J}_o(w_o | Y_o) & \text{if } \tilde{J}_o(w_o | Y_o) \geq 0 \text{ and } \tilde{G}_o(w_o) \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad (1)$$

$$G_o(w_o | Y_o) := \begin{cases} \tilde{G}_o(w_o) & \text{if } \tilde{J}_o(w_o | Y_o) \geq 0 \text{ and } \tilde{G}_o(w_o) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

For the problem at hand, the set of wages  $w_o$  with  $\tilde{J}_o(w_o | Y_o) \geq 0$  and  $\tilde{G}_o(w_o) \geq 0$  simply corresponds to the interval  $[b, Y_o]$ . Yet, we are willing to take this notational overhead because the search problem of the prime-age agents in Section 4.2 follows the same logic, yet resolving the particular expressions is more involved.

From (1) it is straightforward to derive unconditional expected firm surplus as

$$\mathbb{E}J_o(w_o) = \mathbb{E}[\mathbb{E}J_o(w_o | Y_o)] = \begin{cases} \bar{y}_o - w_o & \text{if } b \leq w_o \leq y^L \\ \phi_o(y^H - w_o) & \text{if } y^L < w_o \leq y^H \end{cases}, \quad (3)$$

and  $\mathbb{E}J_o(w_o) = 0$  otherwise, where  $\bar{y}_o := \phi_o y^H + (1 - \phi_o)y^L$ . If the wage does not exceed the lowest productivity level,  $w_o \leq y^L$ , expected firm surplus is non-negative irrespective of the actual productivity draw and no layoffs are initiated. If instead  $w_o \in (y^L, y^H]$ , continuing a low productive match is not profitable for the firm, and only matches with high productivity are sustained. If the wage exceeds the maximum output level,  $w_o > y^H$ , the match is terminated with probability 1. If  $w_o < b$ , the worker will not apply to the vacancy.

<sup>10</sup>A wage contract with layoff risk implies that the firm cannot recoup the posting cost in certain states of the world. Therefore, we implicitly consider firm owners to have “deep pockets”, such that they only need to break even in expected value terms.

Similarly, equation 2 can be used to obtain expected worker surplus

$$\mathbb{E}G_o(w_o) = \mathbb{E}[\mathbb{E}G_o(w_o | Y_o)] = \begin{cases} u(w_o) - u(b) & \text{if } b \leq w_o \leq y^L \\ \phi_o(u(w_o) - u(b)) & \text{if } y^L < w_o \leq y^H \end{cases}, \quad (4)$$

and  $\mathbb{E}G_o(w_o) = 0$  otherwise. By the same argumentation as above, the layoff probability is zero if  $w_o \leq y^L$ . For  $w_o \in (y^L, y^H]$ , the individual receives the wage payment only in case of high productivity, i.e. with probability  $\phi_o$ .

Substituting (3)–(4) into the maximization problem (O) reveals that special attention is required at  $w_o = y^L$ . Expected firm surplus (3) has a kink at this point, while expected worker surplus (4) exhibits a jump. It is therefore *not* sufficient to focus on interior optima. The corner  $w_o = y^L$  has to be considered as a potential solution candidate as well—and indeed will be optimal for a large set of parameter combinations.

Before characterizing the full set of solution candidates, we build some intuition by illustrating the firms' free entry curve and the workers' indifference curves in the  $(w_o, \theta_o)$ -plane.

**Lemma 4.1.** *The free entry curve of firms, implicitly given by  $q(\theta_o)\mathbb{E}J_o(w_o) = c$ , is continuous, convex and strictly decreasing in the  $(w_o, \theta_o)$ -space, with a kink at  $w_o = y^L$ . It is well-defined if  $\mathbb{E}J_o(w_o) \geq c$ . Iso-utility curves are discontinuous at  $w_o = y^L$ , where they jump to a higher level of  $\theta_o$ . Both arcs of a curve are strictly convex and strictly decreasing in the  $(w_o, \theta_o)$ -space.*

Figure 2 illustrates the properties stated by Lemma 4.1. The solid line is the free entry curve of the firm sector. Since  $q(\theta) \in [0, 1]$  for all  $\theta \geq 0$ , the curve is well-defined only if  $\mathbb{E}J_o(w_o) \geq c$ . The highest feasible wage is denoted by  $\bar{w}_o$  in the figure, whereas the lowest feasible wage is  $b$ . The dashed, dotted and dash-dotted curves are the worker's indifference curves for three different utility levels. As indicated above, a structural break occurs at  $w_o = y^L$ . The free entry curve flattens abruptly, and iso-utility curves jump upwards. This implies that, for a given labor market tightness  $\theta_o$ , there are often two ways to achieve the same utility level: either by demanding a low wage,  $w_o \leq y^L$ , or by demanding a high wage,  $w_o > y^L$ . The wage spread between the two wage levels exactly compensates the individual for the difference in layoff probabilities.

#### 4.1.2 Solution candidates and optimality

To find the optimal solution to problem (O), we employ the following two step strategy. Because  $w_o = y^L$  is the only problematic point, we first split the original problem into two subproblems, one for  $w_o \in [b, y^L]$  and one for  $w_o \in (y^L, y^H]$ , and then compare utility levels. For the two subproblems, the Karush–Kuhn–Tucker (KKT) theorem provides necessary optimality conditions since both the objective function and the free entry curve are continuously differentiable in the interior of these regions. All points that satisfy the KKT conditions of one of the subproblems need to be considered as solution candidates for the original problem (O). Among these points, any contract that attains the highest value of the objective function is an optimal

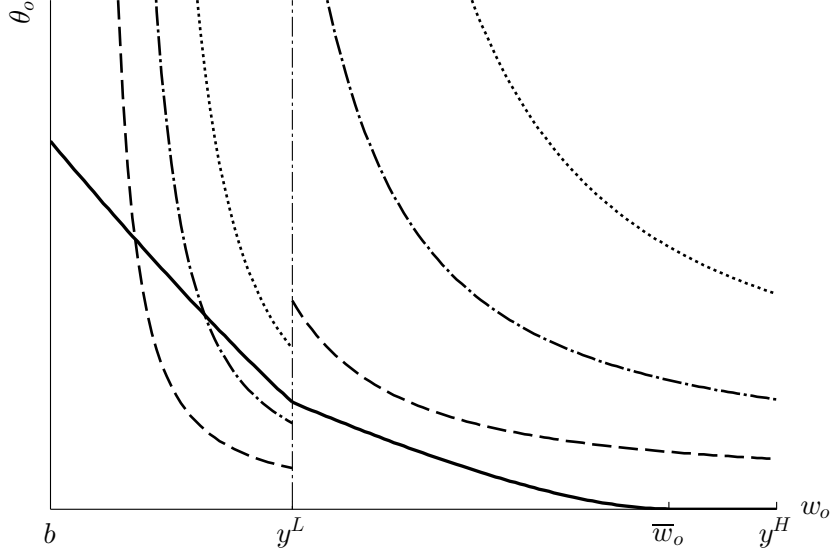


Figure 2: Free entry curve (solid) and indifference curves of the old unemployed

solution to (O). Some attention is warranted if one of the subproblems does not possess a solution. To characterize existence, uniqueness and optimality properties of solution candidates, the following function is central.

**Definition 4.1.** Let  $\phi_o \in [0, 1]$ . Define

$$\Phi(w; \phi_o) := \frac{\varepsilon(\theta_o(w; \phi_o))}{1 - \varepsilon(\theta_o(w; \phi_o))} \mathbb{E}J_o(w; \phi_o)u'(w) - \mathbb{E}G_o(w; \phi_o)$$

where  $\mathbb{E}J_o(w; \phi_o)$  and  $\mathbb{E}G_o(w; \phi_o)$  are given in (3)–(4), and  $\theta_o(w; \phi_o)$  is the free entry curve, implicitly defined by

$$q(\theta_o(w; \phi_o))\mathbb{E}J_o(w; \phi_o) = c. \quad (5)$$

Where not necessary, we omit the dependence on  $\phi_o$  and just write  $\Phi(w)$ . As will be shown in Proposition 4.1, the set of interior solution candidates for problem (O) corresponds to the roots of the map  $w \rightarrow \Phi(w; \phi_o)$ . Furthermore, the monotonicity properties summarized in Lemma B.1 in the appendix are used to establish existence and uniqueness of solution candidates.

We first determine all possible solution candidates of (O) by solving the two subproblems separated by  $w_o = y^L$ . To simplify matters we proceed under the following assumption on model parameters.

**Assumption 4.2.** Assume that  $\phi_o(y^H - y^L) > c$ .

This restriction ensures that there exists a  $w_o > y^L$  such that  $(w_o, \theta_o(w_o))$  is a feasible pair for problem (O).<sup>11</sup> Otherwise firms never offer wage contracts with layoff risk because expected profit would be negative.

<sup>11</sup>To see this, recall Lemma 4.1 and note that Assumption 4.2 is equivalent to  $\mathbb{E}J_o(y^L) > c$ . The existence of a feasible pair with  $w_o > y^L$  follows by continuity of  $\mathbb{E}J_o$ .

For ease of exposition, we from now on identify a feasible pair  $(w_o, \theta_o)$  with its wage level  $w_o$  since the corresponding  $\theta_o$  is uniquely determined by the free entry curve (5). Our first observation is that there are up to two contracts that qualify for the solution to (O), one in each of the intervals separated by  $w_o = y^L$ . The first solution candidate features no separations and is either an interior point,  $w_o \in (b, y^L)$ , or at the corner  $w_o = y^L$ . For certain parameter combinations, a second solution candidate exists with wage  $w_o \in (y^L, y^H)$ . The latter promises a higher wage, but at the price of being laid off and falling back to  $b$  if productivity turns out to be low.

**Proposition 4.1.** *Restricted to the interval  $w_o \in [b, y^L]$ , problem (O) has a unique solution  $w_o^s$ . If  $\Phi(y^L) < 0$ , it is interior and determined by  $\Phi(w_o^s) = 0$ . Otherwise it is the corner solution  $w_o^s = y^L$ .*

*Restricted to the interval  $w_o \in (y^L, y^H]$ , problem (O) has a solution if and only if  $\Phi^+(y^L) > 0$ . In this case it is interior and uniquely determined by  $\Phi(w_o^r) = 0$ .<sup>12</sup>*

Intuitively, we label  $w_o^s$  as “safe” and  $w_o^r$  as “risky”. The safe contract  $w_o^s$  is the unique maximizer of (O) subject to the additional constraint  $w_o \in [b, y^L]$ , which is the set of wages for which firm surplus is always non-negative. The risky contract solves (O) subject to  $w_o \in (y^L, y^H]$ . Because this is not a closed set, a maximizing point need not exist. If it exists, Proposition 4.1 guarantees its uniqueness, and we denote this point by  $w_o^r$ .

In the remainder of this section, we derive the particular conditions under which  $w_o^r$  is the individual’s optimal choice, and thus inefficient layoffs emerge in equilibrium. First, it turns out that the risky contract can only be optimal if the best safe contract is at the boundary, i.e.  $w_o^s = y^L$ . In this case, without the contracting friction the individual would have picked a wage level above  $y^L$ . The friction, however, causes the layoff probability to rise from zero to  $\phi_o > 0$  when crossing  $y^L$ . The worker then either stays at the boundary and has a safe job. Or she is willing to take the layoff risk and jumps to  $w_o^r > y^L$ .

**Proposition 4.2.** *If the safe contract  $w_o^s$  is interior, it is the unique optimal solution to problem (O). Otherwise the risky contract  $w_o^r$  co-exists with  $w_o^s = y^L$ , and both contracts have to be considered for the optimum.*

Which of the two contracts is preferred in the latter case crucially depends on  $\phi_o$ , the probability of receiving the high draw. We show in Theorem 4.1 that a sufficiently high probability will induce the worker to choose a risky contract, while for lower levels a safe contract is optimal. The intuition behind this result is straight-forward: Demanding  $w_o^r$  is risky because it leads to a layoff in case of a bad productivity draw. Because this happens with probability  $1 - \phi_o$ , a higher  $\phi_o$  lowers the layoff risk associated with a risky contract. In the limit  $\phi_o \rightarrow 1$  the layoff probability approaches zero, such that the “risky” contract becomes as safe as the “safe” contract but offers a higher wage. Under a mild condition, workers in this case find it optimal to demand  $w_o^r$ . By continuity, the risky contract is then preferred if  $\phi_o$  is sufficiently

<sup>12</sup>Throughout the paper we use  $f^+(x_0)$  as a short-hand notation for the right-sided limit,  $\lim_{x \rightarrow x_0^+} f(x)$ .

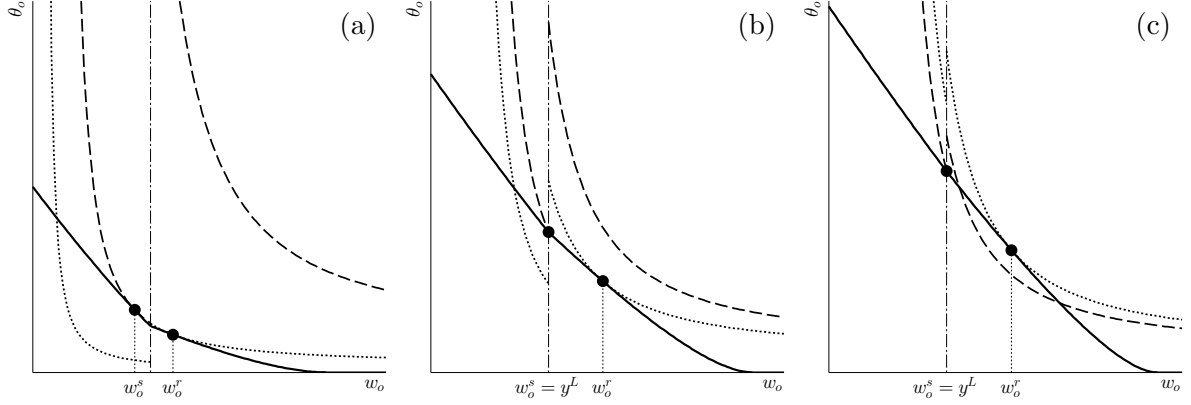


Figure 3: Location of solution candidates and indifference curves for three levels of  $\phi_o$ , assuming  $\Phi(y^L; 1) > 0$ . (a)  $\phi_o < \phi'_o \wedge \Phi(y^L; \phi_o) < 0$ , (b)  $\phi_o < \phi'_o \wedge \Phi(y^L; \phi_o) \geq 0$ , (c)  $\phi_o > \phi'_o$ .

close to 1. In fact, there exists a threshold  $\phi'_o$  such that the risky contract is optimal if and only if  $\phi_o \in (\phi'_o, 1)$ :

**Theorem 4.1.** *If  $\Phi(y^L; 1) \leq 0$ , the safe contract  $w_o^s$  is optimal regardless of the value of  $\phi_o \in (0, 1)$ . Otherwise there exists a threshold  $\phi'_o < 1$  such that the safe contract is optimal if  $\phi_o < \phi'_o$ , and the risky contract  $w_o^r$  is optimal if  $\phi_o > \phi'_o$ .*

Theorem 4.1 is the central result of this section. If  $\Phi(y^L; 1) > 0$ , the individual optimally chooses a safe contract although no layoff risk is involved with demanding  $w_o > y^L$ . This can emerge because applications for high paying jobs have lower chances to be accepted. The utility gains due to a higher wage might therefore be fully offset by the associated decrease in  $p(\theta_o(w_o))$ . If this is the case for  $\phi_o = 1$ , the worker will prefer the safe contract all the more if  $\phi_o < 1$ , because then a contract with  $w_o > y^L$  additionally features a positive layoff probability.

If  $\Phi(y^L; 1) > 0$ , on the other hand, the risky contract is optimal for  $\phi_o = 1$ . Continuity and monotonicity of the value functions in  $\phi_o$  ensure existence of a threshold  $\phi'_o < 1$  such that for  $\phi_o > \phi'_o$ , old unemployed optimally search for jobs that have a positive probability of being inefficiently destroyed.

The statements of Proposition 4.2 and Theorem 4.1 are visualized in Figure 3, assuming  $\Phi(y^L; 1) > 0$ . In each of the three panels, solution candidates are represented by a black dot. The dashed (dotted) line is the indifference curve corresponding to the utility level of the safe (risky) contract. Panel (a) illustrates the situation for  $\phi_o < \phi'_o$  and  $\Phi(y^L; \phi_o) < 0$ , in which case  $w_o^s$  is an interior point. Although the risky candidate  $w_o^r$  exists, the safe contract lies on a higher iso-utility curve and is therefore optimal. In Panel (b), the best safe contract is no longer interior because  $\Phi(y^L; \phi_o)$  switches sign with increasing  $\phi_o$ . Nevertheless,  $w_o^s = y^L$  still gives higher utility than  $w_o^r$ . If  $\phi_o > \phi'_o$ , this relationship reverses and the risky contract becomes optimal, as shown in panel (c).<sup>13</sup>

<sup>13</sup>From Lemma B.1(iii) it is easy to verify that there is a critical value of  $\phi_o$  above which  $w_o^s = y^L$ . Furthermore, the figure reveals that with increasing  $\phi_o$ , the free entry curve (solid line) moves upwards. For given wage, a higher  $\phi_o$  increases expected firm surplus such that firms post more vacancies, which raises the labor market tightness  $\theta_o$ .

### 4.1.3 Comparative statics

Naturally, the particular level of the threshold  $\phi'_o$  depends on the remaining parameters of the model, and it proves insightful to determine their marginal effects:

**Theorem 4.2.** *An increase in the outside option  $b$  or the high productivity state  $y^H$  decreases the threshold  $\phi'_o$ . An increase in the posting cost  $c$  or the low productivity state  $y^L$  raises the threshold.*

If a worker is laid off, she loses her wage claim  $w_o^r$  and instead receives an income of  $b$ . The higher this outside option, the less harmful is it to lose the job. Therefore, a high level of  $b$  increases the incentive to take risk, such that the worker prefers the risky contract already for lower levels of  $\phi_o$ . An increase of  $y^H$  has a similar effect because it raises  $w_o^r$  and thus the premium that can be earned by risk-taking. A higher level of  $y^L$ , on the other hand, increases the safe wage, while the optimal risky contract remains unaffected. Hence  $\phi'_o$  increases. The same is true for a higher posting cost. Because the firm has to recoup these costs, in equilibrium the number of vacancies posted for high-paying jobs decreases relative to lower paying jobs. This depresses a worker's chance to find employment at wage  $w_o^r$  compared to finding employment at wage  $w_o^s < w_o^r$ .

## 4.2 Prime-age unemployed

After this detailed analysis of old job-seekers, we turn to the problem that prime-aged individuals face in period 1. The main result of this section is an analogue of Theorem 4.1, where the parameter  $\phi_s$  acts as the critical parameter. Yet, with potentially two periods of employment ahead, prime-aged individuals have much more scope to write contracts that provide high wage levels and low layoff risk at the same time. At the senior stage (when these workers are in their second period at the firm), the situation is very similar to before. No layoffs occur if  $w_s \leq y^L$ , while the layoff probability is  $1 - \phi_s$  if  $w_s > y^L$ . However, if the worker anticipates at the search stage that the constraint  $w_s = y^L$  will be binding in the second period, she can still increase her expected lifetime income by demanding a higher  $w_m$  in the first period. Clearly, this feature of long-run employment contracts was not available to the old job-seekers described in Section 4.1.

Formally, a prime-aged job-seeker searches for a long-run contract  $(w_m, w_s)$  that maximizes her expected surplus from applying subject to the firms' zero expected profit condition,

$$V_m := \max_{(w_m, w_s, \theta_m)} p(\theta_m) \mathbb{E}G_m(w_m, w_s) \quad \text{s.t.} \quad q(\theta_m) \mathbb{E}J_m(w_m, w_s) = c. \quad (\text{P})$$

The expected surplus functions  $\mathbb{E}J_m$  and  $\mathbb{E}G_m$  are more complex than their counterparts in (3)–(4). Since both parties form rational expectations, they anticipate at the search stage that the match may be dissolved at some point in time. The firm can terminate the match in either of the two periods. Additionally, there may be an exogenous separation event. The expected surplus functions have to reflect timing and probability of separations, which in turn depends on the location of the wage contract  $(w_m, w_s)$  in the contract space.

### 4.2.1 Value functions

To obtain the expected surplus functions, it is convenient to proceed backwards in time. Assume a worker with contract  $(w_m, w_s)$  is still employed at  $t = 1$ . After observing the productivity draw  $Y_s$ , the firm decides to retain or to fire the worker. As in Section 4.1, the worker is laid off if and only if productivity falls short of the promised wage, i.e.  $w_s < Y_s$ . On the other hand, senior workers are only willing to work if  $w_s \geq b$ . Therefore, the expected surplus functions have the same form as in (3)–(4), only with productivity parameter  $\phi_s$  instead of  $\phi_o$ :

$$\mathbb{E}J_s(w_s) = \begin{cases} \bar{y}_s - w_s & b \leq w_s \leq y^L \\ \phi_s(y^H - w_s) & y^L < w_s \leq y^H \end{cases}, \quad \mathbb{E}G_s(w_s) = \begin{cases} u(w_s) - u(b) & b \leq w_s \leq y^L \\ \phi_s(u(w_s) - u(b)) & y^L < w_s \leq y^H \end{cases}, \quad (6)$$

and  $\mathbb{E}J_s(w_s) = \mathbb{E}G_s(w_s) = 0$  otherwise.

Next we turn to period 1. After realization of period 1 productivity,  $Y_m$ , the worker is retained if and only if the discounted expected firm surplus from the match is non-negative,  $\tilde{J}_m(w_m, w_s | Y_m = y_m) = y_m - w_m + \delta \mathbb{E}J_s(w_s) \geq 0$  where  $\delta := \beta(1 - s)$  is the effective discount factor, consisting of a pure time discount factor  $\beta \in [0, 1]$ , and the probability of not being separated for exogenous reasons,  $1 - s$ , where  $s \in [0, 1]$ . Since productivity is revealed sequentially,  $Y_s$  is still a random variable in period 1, and the continuation value of the match is the expected period 2 surplus  $\mathbb{E}J_s(w_s)$  given in (6). If the firm is willing to start production, expected worker surplus is  $\tilde{G}_m(w_m, w_s) = u(w_m) - u(b) + \delta(\mathbb{E}G_s(w_s) - V_o)$ . However, workers only apply to jobs for which this expression is non-negative. Unconditional expected period 1 surplus therefore equals

$$\mathbb{E}J_m(w_m, w_s) = \begin{cases} \bar{y}_m - w_m + \delta \mathbb{E}J_s(w_s) & \text{if } \underline{w}_m(w_s) \leq w_m \leq y^L + \delta \mathbb{E}J_s(w_s) \\ \phi_m[y^H - w_m + \delta \mathbb{E}J_s(w_s)] & \text{if } y^L + \delta \mathbb{E}J_s(w_s) < w_m \leq y^H + \delta \mathbb{E}J_s(w_s) \end{cases} \quad (7)$$

for the firm, and

$$\mathbb{E}G_m(w_m, w_s) = \begin{cases} u(w_m) - u(b) + \delta(\mathbb{E}G_s(w_s) - V_o) & \text{if } \underline{w}_m(w_s) \leq w_m \leq y^L + \delta \mathbb{E}J_s(w_s) \\ \phi_m[u(w_m) - u(b) + \delta(\mathbb{E}G_s(w_s) - V_o)] & \text{if } y^L + \delta \mathbb{E}J_s(w_s) < w_m \leq y^H + \delta \mathbb{E}J_s(w_s) \end{cases} \quad (8)$$

for the worker. For  $w_m$  outside these intervals, both expected surplus functions are identically zero. At the lower bound  $\underline{w}_m(w_s) := u^{-1}(u(b) - \delta(\mathbb{E}G_s(w_s) - V_o))$ , the prime-aged individual is indifferent between market and home production.

Combining the case distinctions of (7)–(8) with (6) gives rise to four feasible regions of the wage space, each characterized by a different separation policy.<sup>14</sup> By contrast, we only had to differentiate two regions in Section 4.1. There are two reasons for this multiplication of cases. First, the firm takes the worker's age into account in the firing decision. If a worker performs

<sup>14</sup>Contracts for which  $w_m$  lies outside the bounds specified in (7)–(8) violate the free entry condition and are therefore not feasible for problem (P). Two additional regions with certain destruction of the job in period 2 ( $w_s < b$  and  $w_s > y^H$ ), can be ruled out by the assumption  $\phi_s \geq \phi_o$ .



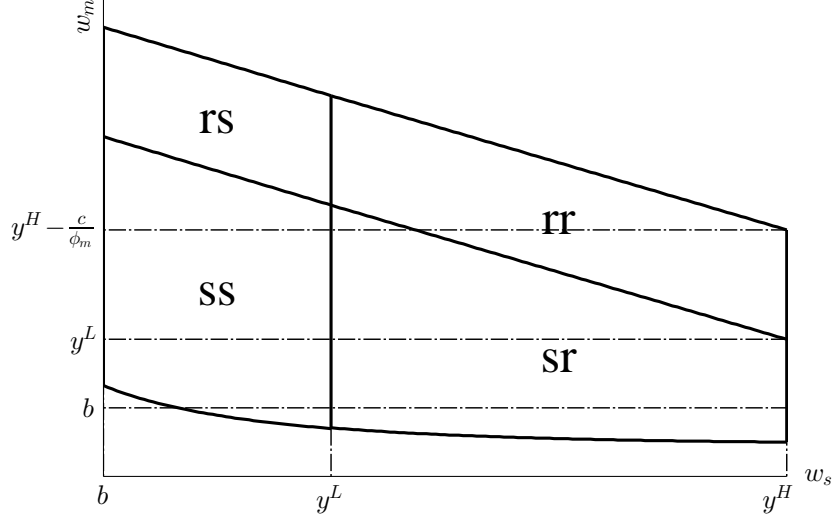


Figure 4: The four regions of the wage space that may contain the optimal contract of prime-age job-seekers

poorly in period 1 such that  $w_m > Y_m$ , the firm will not fire her in case expected period 2 surplus can make up for this loss. By contrast, if the same situation occurs in the second period, i.e.  $w_s > Y_s$ , the firm will not hesitate to terminate the match.

**Lemma 4.2.** *The feasible wage space for problem (P) consists of four adjacent regions, each characterized by a different layoff regime. If the contract satisfies*

$$b \leq w_s \leq y^L \wedge u^{-1}(u(b) - \delta(u(w_s) - u(b) - V_o)) \leq w_m \leq y^L + \delta(\bar{y}_s - w_s), \quad (\text{ss})$$

there are no endogenous layoffs. If

$$y^L < w_s \leq y^H \wedge u^{-1}(u(b) - \delta(\phi_s(u(w_s) - u(b)) - V_o)) \leq w_m \leq y^L + \delta\phi_s(y^H - w_s), \quad (\text{sr})$$

layoffs only occur in period 2, when all low productive senior workers are fired. If

$$b \leq w_s \leq y^L \wedge y^L + \delta(\bar{y}_s - w_s) \leq w_m \leq y^H + \delta(\bar{y}_s - w_s) - c/\phi_m, \quad (\text{rs})$$

layoffs only emerge in period 1, when all low productive prime-age workers are fired. Receiving a bad productivity draw results in a layoff irrespective of the worker's age if

$$y^L < w_s \leq y^H \wedge y^L + \delta\phi_s(y^H - w_s) < w_m \leq y^H + \delta\phi_s(y^H - w_s) - c/\phi_m. \quad (\text{rr})$$

The expressions for  $\mathbb{E}J_m(w_m, w_s)$  and  $\mathbb{E}G_m(w_m, w_s)$  valid on these regions are reported in Table B.1 in the appendix.

These four regions of the wage space are illustrated in Figure 4. The period 2 wage  $w_s$  is plotted on the x-axis while the y-axis shows the period 1 wage  $w_m$ . All regions are individually rational for the worker and only differ with respect to the firm's layoff policy. If the contract

falls into region (ss), the firm experiences a non-negative surplus in both periods irrespective of the actual productivity realizations. Hence the job is safe in the sense that the match lasts until the end of period 2, unless it ends for exogenous reasons. If the senior wage is too high,  $w_s > y^L$ , the job is risky in period 2 because the firm would make a loss on low productive senior workers and dismisses them instead. This is the case in the two regions to the right. In region (sr), poorly performing prime-age individuals are retained. Whereas in region (rr) also prime-age workers with a low productivity draw are fired, because their expected period 2 surplus cannot make up for the financial loss that the firm incurs in period 1. Finally, in region (rs) the firm only fires poorly performing prime-age workers, while senior workers are retained because of their low wage. Hence the job is risky in the first period, but safe once the worker has made it to the senior stage.

#### 4.2.2 Solution candidates and optimality

The functional form of the expected surplus functions  $\mathbb{E}J_m(w_m, w_s)$  and  $\mathbb{E}G_m(w_m, w_s)$  depends on the region in which the considered wage contract  $(w_m, w_s)$  is located. It can be seen from the expressions in Table B.1 in the appendix that  $\mathbb{E}J_m$  is continuous but not differentiable at the boundaries between the regions, while  $\mathbb{E}G_m$  is not even continuous. In solving the search problem of the prime-aged, we therefore face a whole continuum of non-differentiabilities, while only a single point ( $w_o = y^L$ ) required special treatment in the search problem of the old unemployed. Nevertheless, we can employ the same solution strategy. First, we solve problem (P) separately on each of the four regions. For these subproblems, the KKT theorem provides necessary optimality conditions. All points satisfying these conditions are considered as solution candidates for the original problem (P), and any point that attains the highest level of the value function is an optimal solution. We solve the problem under the assumption that all of the four regions are non-empty.

**Assumption 4.3.** *Assume that  $\phi_m(y^H - y^L) > c$ .*

Section B.1 in the appendix shows that, restricted to any of the four regions, problem (P) either has no solution or a unique solution. Hence there are at most four solution candidates for problem (P). The solution candidate in region (ss) always exists, either in the interior or at the right boundary. The set of candidates is therefore non-empty. The existence of the other three candidates depends on the model parameters. It is verified in Proposition B.5 in the appendix that the optimal contract has to be one of the solution candidates.

The solid black line in Figure 5 shows the potential locations of these points, as established in Propositions B.1–B.4 in the appendix. Because workers have concave utility, they want to smooth consumption across periods. Since they cannot access the capital market, however, consumption smoothing can only be achieved via the wage contract. Therefore, workers favor contracts with a flat wage profile,  $w_m = w_s$ , among which they pick the one that implements the desired worker’s share in total job surplus. Any such contract is located on the 45 degree line in Figure 5. However, the optimal worker surplus might not always be achievable by a

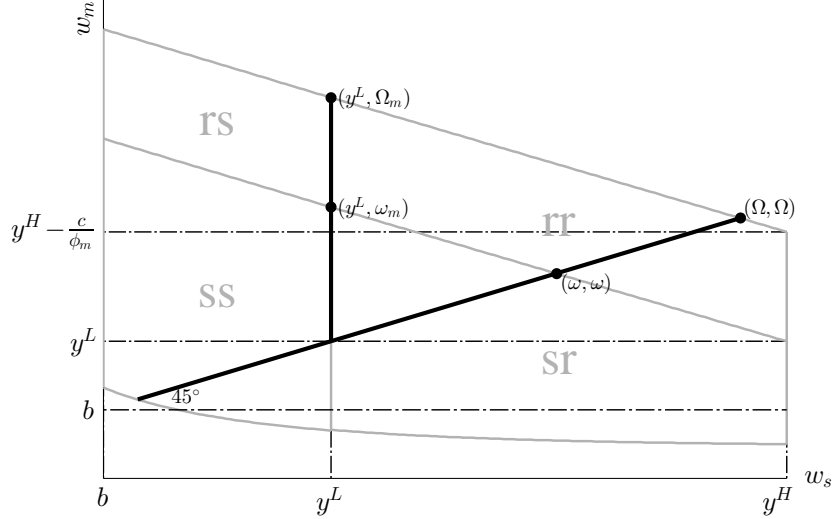


Figure 5: Location of solution candidates of problem (P)

flat profile, because only a senior wage up to  $w_s = y^L$  can be demanded safely. If the worker does not want to take the risk of a layoff, a front-loaded profile with  $w_m > w_s = y^L$  might be considered instead. Any such contract is located at the vertical section in Figure 5.

In the remainder, we derive the particular conditions under which the equilibrium contract leads to separations when old. The solution candidates of regions (sr) and (rr) feature a flat profile and hence potentially allow to smooth consumption, but at the risk of a layoff. Intuitively, one of these contracts can only be optimal if no flat contract *without* layoff risk is among the set of solution candidates. If no such contract exists, a contract in region (sr) might be desirable. This is the first part of the upcoming proposition. The second part postulates that if for the candidate in region (sr) no constraints are binding in period 1 ( $w < \omega$  in Figure 5), a contract in region (rs) or (rr) cannot be optimal. The reason is that taking the risk of a layoff in some period  $t$  is only sensible if for all remaining solution candidates the firing constraint is binding in period  $t$ .

**Proposition 4.3.** *If the candidate in region (ss) is interior, this is the optimal solution to problem (P). Otherwise the candidate in region (sr) exists and, provided it is interior, dominates any contract in regions (rs) and (rr).*

For the remaining section let us therefore assume that the candidate in region (ss) is at the boundary,  $w_s = y^L$ , and that the candidate in region (sr) is interior,  $w_s \in (y^L, \omega)$ . In this case, Proposition 4.3 implies that the solution to the search problem of the prime-aged must be one of these two contracts. These reflect a trade-off between old-age job security and intertemporal consumption smoothing: The candidate in (ss) is associated with no layoff risk but a front-loaded income profile,  $w_m > w_s = y^L$ , which translates into an unbalanced time profile of consumption. The candidate in (sr), on the other hand, in principle allows individuals to smooth consumption across periods,  $w_m = w_s > y^L$ , but guarantees employment only in the first period. In period 2, the worker is laid off and falls back to her outside option  $b$  in case of a bad productivity draw,

i.e. with probability  $1 - \phi_s$ . A higher  $\phi_s$  lowers the layoff risk associated with this contract, which makes it more attractive compared to the safe alternative in region (ss). The following theorem provides a generalization of Theorem 4.1 to long-run contracts. The function  $\Psi_{\text{id}}$  is the two-period counterpart of  $\Phi$  and formally defined in Definition B.1 in the appendix.

**Theorem 4.3.** *If  $\Psi_{\text{id}}(y^L; 1) \leq 0$ , the safe candidate  $w^{ss}$  solves problem (P) irrespective of  $\phi_s \in (0, 1)$ . Otherwise there exists a threshold  $\phi'_s \in (0, 1)$  such that  $w^{ss}$  dominates any contract of region (sr) for  $\phi_s < \phi'_s$ , while the risky  $w^{sr}$  dominates any contract of region (ss) if  $\phi_s > \phi'_s$ .*

The assumption  $\Psi_{\text{id}}(y^L; 1) \leq 0$  means that the candidate in region (ss) is interior for  $\phi_s = 1$ . By monotonicity, it will then also be interior for  $\phi_s \in (0, 1)$  and thus, by Proposition 4.3, the optimal solution of (P). If  $\Psi_{\text{id}}(y^L; 1) > 0$ , on the other hand, the risky contract is optimal for  $\phi_s = 1$ . Continuity and monotonicity of the value functions in  $\phi_s$  ensure existence of a threshold  $\phi'_s < 1$  such that for  $\phi_s > \phi'_s$ , prime-age workers prefer a job with positive probability of being turned down when old over completely safe job.

Notice that the second part of Theorem 4.3 only establishes a relation between the candidates of region (ss) and (sr). It does not address whether for  $\phi_s > \phi'_s$  the risky contract  $w^{sr}$  is actually the optimal solution to problem (P). In general, this contract could still be dominated by the solution candidates of the remaining two regions (rs) or (rr). According to Proposition 4.3, however, we can rule this out if the candidate in (sr) is interior. This allows us to formulate the last result of this chapter.

**Theorem 4.4.** *Let  $\phi'_s$  be defined as in Theorem 4.3 and assume that  $w^{sr} < \omega$  for  $\phi_s = \phi'_s$ . Then  $w^{rs}$  is the optimal solution to problem (P) for all  $\phi_s > \phi'_s$ .*

### 4.2.3 Comparative statics

Unlike in Section 4.2, we refrain from a thorough comparative static analysis of the threshold  $\phi'_s$  as the signs of the marginal effects of most parameters are ambiguous. We only highlight the effect of an increase in the level of the outside option,  $b$ , and the productivity parameters of newly hired older workers,  $\phi_o$ . For the remaining parameters, we conduct a sensitivity analysis in Section 5.3 using a sample parameterization.

**Theorem 4.5.** *An increase in the outside option  $b$  or the probability of being highly productive as an old worker  $\phi_o$  lowers the threshold  $\phi'_s$ .*

Since laid off workers fall back to the outside option  $b$ , an increase in this value makes a layoff less costly and thus increases the incentive to write risky contracts. The same is true if the expected productivity of the worker on an alternative job is higher.

## 4.3 Labor market equilibrium

The labor market equilibrium consists of a pair  $(\theta_o, w_o)$  and a triple  $(\theta_m, w_m, w_s)$  that solve the directed search problems (O) and (P), respectively. As argued in the respective sections, the

specified assumptions guarantee that an equilibrium exists, and that it is generically unique. Theorems 4.1 and 4.4 showed that whether or not inefficient layoffs occur in equilibrium, depends in a crucial way on the parameters  $\phi_o$  and  $\phi_s$ , which govern the probability of being highly productive as an old or senior worker. Taking risk in wage contracting only pays off if the success probability is sufficiently high.

What has not been stressed so far is that risky contracts are not only associated with higher layoff risk but also with lower job-acceptance probabilities. This is due to the downward sloping free entry curve of firm, which open less vacancies for high-paying jobs. That  $\theta_o(w_o^r) < \theta_o(w_o^s)$  follows immediately from  $w_o^s \leq y^L < w_o^r$ . For prime-age workers, this is not so clear because the relation between  $w_y^{sr}$  and  $w^{sr}$  seems to be ambiguous in general. However, Lemma B.4 in the appendix demonstrates that if the worker is indifferent,  $\phi_s = \phi'_s$ , the firm surplus and therefore the matching probability is higher for the safe contract. As a result, any reform that reduces inefficient layoffs in the economy will, as a side-effect, also increase hiring rates.

Having said this, in our economy parametric reforms shift the threshold levels  $\phi'_o$  or  $\phi'_s$ , but in general do not alter the actual layoff rates. Among old workers, for instance, the layoff rate is either zero or  $1 - \phi_o$ , depending on whether  $\phi_o \leq \phi'_o$  or  $\phi_o > \phi'_o$ . A parametric reform affects layoffs only if it is so large that  $\phi'_o$  jumps across  $\phi_o$ . The same observation holds concerning  $\phi_s$ . This is clearly an artifact of the discrete productivity process combined with a homogenous level of  $\phi_o$  and  $\phi_s$ .

More realistic predictions could be reached if individuals differ in their ability to maintain high productivity levels at the old and senior stage as captured by  $\phi = (\phi_o, \phi_s)$ . In this case, all theoretical statements of the previous sections remain valid. The reason is that due to directed search, workers with different  $\phi$  search in separate labor markets, provided  $\phi$  is public information at the search stage. The distribution of worker types in the economy is then independent of the labor market equilibrium, and conditional on the level of  $\phi$  all results of Section 4.1 and Section 4.2 directly apply to this modified economy.

It can then be shown that a reform that decreases the threshold levels indeed increases layoff rates and vice versa. It is insightful to show this for  $\phi'_s$ . Assume to this purpose that  $\phi_o$  is identical across workers whereas  $\phi_s \in [\phi_o, 1)$  is governed by a cdf  $F$  with density  $f$ . The aggregate layoff rate of senior workers can then be expressed as

$$P[Lay|s] = \int_{\phi'_s}^1 (1 - \phi_s) dF(\phi_s|s) = \int_{\phi'_s}^1 (1 - \phi_s) f(\phi_s|s) d\phi_s.$$

where  $F(\phi_s|s)$  denotes the distribution of  $\phi_s$  among the workers that enter the senior stage and  $f(\phi_s|s)$  is the associated density. These differ from the unconditional distribution of  $\phi_s$  because the job-finding probability of prime-age workers depend on  $\phi_s$ . Assuming that all workers prefer contracts without layoff risk in prime-age, the conditional density is

$$f(\phi_s|s) = \frac{p(\theta_m^*(\phi_s))f(\phi_s)}{\int_0^1 p(\theta_m^*(\psi))f(\psi)d\psi},$$

which is discontinuous at  $\phi'_s$  because the higher probability of job destruction is reflected in a lower hiring rate. Therefore,  $f(\phi'_s|s) - f^+(\phi'_s|s) > 0$ .

Some algebra reveals that a change in any model parameter  $x$  affects the layoff rate by

$$\begin{aligned} \frac{P[ Lay|s ]}{dx} &= -\frac{d\phi'_s}{dx}(1 - \phi'_s)f(\phi'_s|s) + \int_{\phi'_s}^1 (1 - \phi_s) \frac{df(\phi_s|s)}{dx} d\phi_s \\ &= -\frac{d\phi'_s}{dx} \left[ (1 - \phi'_s)f(\phi'_s|s) + (f(\phi'_s|s) - f^+(\phi'_s|s))P[ Lay|s ] \right] \\ &\quad + \int_{\phi'_s}^1 (1 - \phi_s)f(\phi_s|s) \left\{ \frac{dp(\theta_m(\phi_s))}{dx} - \frac{\int_0^1 \frac{dp(\theta_m(\psi))}{dx} f(\psi) d\psi}{\int_0^1 p(\theta_m(\psi))f(\psi) d\psi} \right\} d\phi_s \end{aligned}$$

The second line captures the two effects that  $x$  exerts on the layoff rate via the threshold  $\phi'_s$ . Assume that the threshold increases. Then workers who were previously located at the threshold now opt for a safe contract (first term in square brackets). Additionally, these marginal workers experience an abrupt increase in their hiring probability, which increases their relative share among the senior workforce (second term in square brackets). Additionally,  $x$  may change the distribution of  $\phi_s$  through effects unrelated to the change in  $\phi'_s$ . This is captured by the third line. Assuming that  $x$  affects hiring rates of all workers roughly proportionally, i.e.  $\frac{dp(\theta_m^*(\phi_s))}{dx} \approx \lambda p(\theta_m^*(\phi_s))$  for  $\phi_s \neq \phi'_s$ , the integral term becomes negligible, such that

$$\text{sgn} \left[ \frac{P[ Lay|s ]}{dx} \right] = -\text{sgn} \left[ \frac{d\phi'_s}{dx} \right].$$

Intuitively, the layoff rate and the threshold  $\phi'_s$  are inversely related. The higher the latter, the lower the mass of individuals that prefer the risky contract, and the lower their share in the labor force because of comparatively lower hiring rates. Hence a reform that increases the threshold lowers the layoff rate of senior workers and vice versa.<sup>15</sup>

## 5 Numerical illustration and discussion

One natural question arising from Theorems 4.1 and 4.4 is the minimum level of  $\phi_o$  and  $\phi_s$  necessary to obtain an equilibrium with layoffs of old or senior workers. The following numerical assessment shall shed some light on this. We do, however, not attempt to calibrate the model to an existing economy since we do not believe that the current specification is rich enough for such a purpose. Rather we want to stress the comparative static effects that parameter changes exert on the thresholds  $\phi'_o$  and  $\phi'_s$ .

<sup>15</sup>These findings generalize to the case where  $\phi_o$  is also heterogeneous. However, the analysis is then complicated by the assumption that  $\phi_o \geq \phi_s$ . This implies that  $\phi_o$  and  $\phi_s$  are not independent from each other, such that  $\phi'_s$  varies with  $\phi_o$ . Assuming a functional dependence  $\phi_s = g(\phi_o)$  with  $\phi_s \geq \phi_o$  and  $g'(\phi_o) > 0$  allows to establish a new threshold value  $\tilde{\phi}'_s$  that does not depend on  $\phi_o$  such that any worker with  $\phi_s > \tilde{\phi}'_s$  prefers the risky contract. This works because  $\phi'_s$  is a decreasing function of  $\phi_o$  (Theorem 4.5), while  $\phi_s = g(\phi_o)$  is increasing in  $\phi_o$ . Hence there is a unique intersection  $\tilde{\phi}_o$  which defines  $\tilde{\phi}'_s = g(\tilde{\phi}_o)$ .

$y^H$	1.5	$\phi_m$	0.90	$\beta$	1.0	$l$	0.5
$y^L$	1.0	$\phi_s$	0.75	$s$	0.0	$\kappa$	2.0
$b$	0.7	$\phi_o$	0.75	$c$	0.04		

Table 1: Benchmark parameterization

<i>Old unemployed</i>				<i>Prime-age unemployed</i>				
	$w_o$	$p(\theta_o)$	$V_o$		$w_m$	$w_s$	$p(\theta_m)$	$V_m$
region (s)	1.000	0.453	<b>0.0252</b>	region (ss)	1.277	1.000	0.532	<b>0.0612</b>
region (r)	1.116	0.394	0.0206	region (sr)	1.194	1.194	0.508	0.0560

Table 2: Solution candidates and optimum for the benchmark parameterization.

## 5.1 Parameter values and functional forms

Assume that the utility function exhibits constant absolute risk aversion,  $u(w) = (1 - e^{-\kappa w})/\kappa$ , where  $\kappa > 0$ . This is convenient since we do not model savings and the CARA specification rules out wealth effects. Furthermore, by letting  $\kappa \rightarrow 0$  we can approach linear utility which is the typical benchmark in the labor market literature. Concerning the matching function, we use the specification of den Haan, Ramey, and Watson (2000),

$$m(u, v) = \frac{uv}{(u^l + v^l)^{1/l}}.$$

This gives rise to the vacancy filling probability  $q(\theta) = 1/(1 + \theta^l)^{1/l}$ , the job finding probability  $p(\theta) = \theta/(1 + \theta^l)^{1/l}$ , and the elasticity  $\varepsilon(\theta) = \theta^l/(1 + \theta^l)$ . For  $l \in (0, 1]$ , the matching function satisfies all conditions of Assumption 4.1.<sup>16</sup>

Furthermore, we use the parameter values given in Table 1 as a benchmark. The low productivity level,  $y^L$ , is normalized to 1. The high output level,  $y^H$ , is assumed to be 50% higher. The income of unemployed is set to  $b = 0.7$ , which in equilibrium amounts to a replacement rate of 61.5% of the average period wage. In the benchmark parameterization, we assume prime-age workers on average to be more productive than old-age workers, thus  $\phi_m > \phi_s$  and  $\phi_m > \phi_o$ . For prime-age workers, the bad productivity draw has a probability of only 10%, while this is 25% for old-age workers. The low productivity state therefore constitutes a relatively rare event that might become more likely if job-relevant skills depreciate towards the end of a career. For the moment, we assume that senior and older workers are equally productive,  $\phi_m = \phi_o$ .

In our benchmark parameterization, we abstract from discounting ( $\beta = 1$ ) and exogenous separations ( $s = 0$ ). The posting cost is taken as  $c = 0.04$ , which amounts to about 3 percent of first period expected productivity. The parameter of the matching function is set to the intermediate value of  $l = 0.5$ , and we assume a slight degree of risk aversion equal to  $\kappa = 2$ .

<sup>16</sup>Although a Cobb-Douglas matching function  $m(u, v) = u^\alpha v^{1-\alpha}$  with truncated vacancy filling and job finding probabilities violates the strict monotonicity properties in Assumption 4.1, the basic insights are similar.

## 5.2 Equilibrium

Solution candidates and optimal individual choices for the benchmark parameterization are reported in Table 2. The left panel reveals that older unemployed workers choose a wage level that prevents layoffs. Since  $w_o^* = y^L$ , the firing constraint is binding in equilibrium. A probability of  $\phi_o = 0.75$  for the high draw is not sufficient to opt for a risky contract. The situation therefore looks as in panel (b) of Figure 3. As postulated by Theorem 4.1, the risky contract will become optimal provided  $\phi_o$  is sufficiently high. Numerical analysis reveals that this regime switch occurs at  $\phi_o' = 0.902$ , which means that newly hired old workers would have to be more productive than prime-age workers – a scenario that does not seem very realistic.

Concerning prime-age job-seekers, only two of the potentially four solution candidates exist for the benchmark parameterization. These are reported in the right part of Table 2. For the safe contract, the layoff constraint is binding in period 2,  $w_s^{ss} = y^L$ . It involves a substantial wage cut after period 1 as the senior wage is only 80% of the prime-age wage. This contrasts with the second solution candidate, which offers a smooth wage profile but at the risk of being laid off in period 2. This contract, however, is dominated by the safe candidate in utility terms, such that the benchmark parameterization gives rise to an equilibrium without any layoffs.<sup>17</sup>

Since our parameterization satisfies all assumptions of Theorem 4.4, layoffs of senior workers will ultimately occur in equilibrium if  $\phi_s$  is sufficiently high, keeping the remaining model parameters unchanged. The respective threshold value turns out to be  $\phi_s' = 0.900$ , which (by chance) coincides with the benchmark value of  $\phi_m$ . Hence if senior workers are on average at least as productive as during prime-age, it is optimal for prime-aged unemployed to search for contracts that involve a positive layoff risk in period 2. This suggests that the incentive to take risk in wage contracting might differ by occupation. Some occupations require mainly manual skills that deteriorate rather quickly with age. Other occupations rely more on cognitive skills. While some cognitive abilities such as speed of learning and problem-solving capabilities also deteriorate, Skirbekk (2008) reports that “procedural knowledge used to solve everyday problems tends not to decrease at older ages.” Our theory therefore suggests that inefficient layoffs may be more of an issue for high-skilled workers.

The high numerical values of  $\phi_o'$  and  $\phi_s'$  should not come as a surprise since our discrete setup is very restrictive. Depending on the wage contract, individuals are laid off either with probability 0 or  $\phi_s > 0$ . With a continuous productivity process, the individual could instead implement any layoff probability in  $[0, 1]$  that she considers optimal – and we conjecture a positive, but probably small layoff probability in equilibrium.

## 5.3 Comparative statics

We next ask how the threshold values  $\phi_o'$  and  $\phi_s'$  depend on the remaining model parameters, which informs our assessment of policy implications in Section 6. We have already determined the sign of some of these marginal effects in Theorems 4.2 and 4.5. The numerical exercise

<sup>17</sup>The layoff constraint is not binding in period 1 for any of the two contracts since  $w_m^{ss} < \omega_m = 1.45$  and  $w^{sr} < \omega = 1.236$ .



		$b$		$y^L$		$y^H$		$\phi_m$		$\phi_o$	
value of ...	BM	0.6	0.8	0.9	1.1	1.4	1.6	0.8	1.0	0.5	1.0
$\phi'_s$	.900	.946	.817	.768	.975	.944	.857	.912	.889	.904	.896
$\phi'_o$	.902	.961	.791	.733	.990	.952	.856	—	—	—	—

		$\beta$		$s$		$c$		$l$		$\kappa$	
value of ...	BM	0.8	0.9	0.1	0.2	0.02	0.06	0.4	0.6	1	3
$\phi'_s$	.900	.913	.907	.907	.913	.866	.926	.937	.871	.904	.918
$\phi'_o$	.902	—	—	—	—	.844	.939	.945	.865	.856	.938

Table 3: The threshold value  $\phi'_s$  for different parameter values. BM: benchmark, all parameters as in Table 1.  $\phi'_o$  does not depend on  $\phi_m$ ,  $\phi_o$ ,  $\beta$ , and  $s$ .

Elasticities of $\phi'_s$					Elasticities of $\phi'_o$				
$b$	-0.5225	$\phi_m$	-0.1301	$c$	0.0742	$b$	-0.6893	$c$	0.1091
$y^L$	1.2861	$\phi_o$	-0.0096	$l$	-0.2064	$y^L$	1.5903	$l$	-0.2466
$y^H$	-0.8090	$\beta$	-0.0737	$\kappa$	0.0282	$y^H$	-0.9086	$\kappa$	0.1015

Table 4: Elasticity of  $\phi'_s$  and  $\phi'_o$  with respect to the model parameters. Values indicate the percentage change in  $\phi'_s$  and  $\phi'_o$  for a 1% deviation from the benchmark level of the parameter. Remaining parameters as in Table 1.

additionally gives us an idea about the quantitative importance of each of the parameters. Table 3 reports the level of  $\phi'_s$  and  $\phi'_o$  for different parameter values. This is supplemented by the respective elasticities in Table 4, which indicate the relative change of the threshold to a 1% deviation from the parameter's benchmark level. In both tables, only one parameter is varied at a time, with the remaining ones kept at the benchmark reported in Table 1. We mainly discuss the results concerning  $\phi'_s$  at this point, because the intuition behind the marginal changes in  $\phi'_o$  was already given after Theorem 4.2.

**Outside options.** In line with the analytical predictions, Tables 3 and 4 reveal a negative relation between the outside option,  $b$ , and the threshold values. An increase from  $b = 0.7$  to 0.8 brings down the threshold  $\phi'_s$  to 0.817, while a cut to  $b = 0.6$  lifts the threshold to 0.946. The impact of  $b$  on  $\phi'_o$  is even more pronounced. Clearly, a higher outside option makes losing the job less detrimental and thereby increases the willingness of risk-taking. Since  $b$  in reality captures both the individual value of leisure and public transfers, this variable can in part be controlled by the policymaker. Our results thus indicate that generous public welfare systems may promote bilaterally inefficient layoffs in an economy. This is an important insight. Usually, public transfers to the workers only affect the *social optimality* of the labor market equilibrium (Acemoglu and Shimer, 1999), while job destruction is still *bilaterally* efficient. In our setup, however, unemployment insurance also affects the efficiency of bilateral work agreements via its effect on  $\phi'_o$  and  $\phi'_s$ . In particular, we observe that  $\phi'_s \rightarrow 0$  and  $\phi'_o \rightarrow 0$  as  $b \rightarrow y^L$ , because there is no point in applying to a safe job if this does not yield any income gain over unemployment.

$b_m$	0.5	0.6	0.7	0.8	0.9
$b_o$	0.9	0.8	0.7	0.6	0.5
$\phi'_s$	.791	.864	.900	.925	.945

Table 5: Value of  $\phi'_s$  for different age patterns of  $b$ .

$y_m^L$	.8	.9	1.0	1.1	1.2
$y_o^L$	1.2	1.1	1.0	.9	.8
$\phi'_s$	.999	.980	.937	.864	.759

Table 6: Value of  $\phi'_s$  for different profiles of  $(y_m^L, y_o^H)$ .

Apart from the level of  $b$ , the age pattern matters a lot. To study this, we allow different levels of  $b$  for prime-age individuals ( $b_m$ ) and old-age individuals ( $b_o$ ). We find that  $\phi'_s$  is particularly sensitive to changes in  $b_o$ . The elasticity with respect to  $b_o$  is  $-0.3871$ , while the elasticity with respect to  $b_m$  is only a third of this value, namely  $-0.1353$ . This is intuitive since  $b_o$  is the value the individual receives in case of a layoff in period 2. The higher this value, the less painful is losing the job when old, which makes a contract in region (sr) more attractive. By contrast, a higher  $b_m$  decreases the expected surplus of safe and risky contracts to a similar extent, which implies a weaker effect on  $\phi'_s$ . This can also be seen from Table 5 where we keep the average benefit across ages constant at 0.7 and only vary the age profile.<sup>18</sup>

**Productivity process.** Table 3 also shows that taking the layoff risk becomes more attractive if  $y^H$  or  $\phi_m$  are high. The reason is that both parameters raise the expected productivity of the match. To capture part of this gain, workers demand a higher wage stream. With a risky contract, they can uniformly increase  $w_y$  and  $w_s$ . With the safe contract, however, they are constrained in period 2 by  $w_s = y^L$ , such that the desired increase in life-time wage income must be achieved by raising  $w_y$ . This makes the consumption profile that is associated with the safe contract even more unbalanced. Therefore, while the utility of both contracts increases in  $y^H$  and  $\phi_m$ , the increase is higher for the risky contract because of the consumption smoothing motive.

The same intuition holds for an increase in  $y^L$ , which also raises expected match productivity. Yet, an increase in  $y^L$  is found to *increase* the threshold  $\phi'_s$ . The reason is that additional to higher expected productivity of both contracts,  $y^L$  affects the maximum wage that can be safely demanded in period 2. Hence the higher  $y^L$ , the more intertemporal consumption smoothing is achieved with the safe contract. Quantitatively, this direct effect dominates the effect highlighted above, such that the overall effect of  $y^L$  on  $\phi'_s$  is positive. In fact, Table 4 reveals that  $y^L$  exerts the highest elasticity among all the model parameters.

In Table 6 we show the effect of different age-patterns of  $y^L$ , keeping the average at 1. The threshold value seems to be most sensible to the base productivity of old workers. Alluding to

<sup>18</sup>Trivially, the threshold value of old workers,  $\phi'_o$ , only depends on  $b_o$ , but not on  $b_m$ . The effect of a change in  $b_o$  on  $\phi'_o$  therefore coincides with the effects reported for  $b$  in Tables 3 and 4.

the argument from above, it is  $y_o^L$  and not  $y_m^L$  which governs how much workers can safely claim in period 2.

**Discounting and exogenous separations.** The discount factor,  $\beta$ , and the probability of exogenous layoffs,  $s$ , enter the model only via the product  $\delta = \beta(1 - s)$ . Therefore the figures reported in Table 3 are identical for these two variables. The lower  $\beta$ , the stronger the future is discounted. Hence a high wage payment in the second period becomes less valuable, while a high period 1 wage becomes more important. This makes the safe candidate more attractive and the threshold for risk-taking increases. The same is true for a higher  $s$ , which decreases the *ex-ante* probability of the match surviving until period 2 and thus lowers the probability that the high period 2 wage that a risky contract promises can actually be collected.

These observations imply that the preference for long-run contracts with a *positive* probability of being laid off in period 2 is highest if prime-age workers are (i) very certain to stay with the current firm until the beginning of period 2 and (ii) very concerned about their period 2 level of consumption. The reason for this somewhat counter-intuitive finding is that we do not model savings and therefore expected period 2 consumption can only be increased by writing a contract with a high period 2 wage, which is associated with a layoff risk.

**Posting costs and matching technology.** We know from Lemma B.4 that at the threshold, expected firm surplus is lower with a risky contract. Hence if the cost of opening a vacancy,  $c$ , increases, it becomes relatively harder for individuals to get accepted for a risky job than for a safe job. This raises the threshold  $\phi'_s$ . The opposite applies to an increase in the  $l$  parameter of the matching function. This increases the vacancy filling probability  $q(\theta)$  for any  $\theta > 0$ , but more for lower values of  $\theta$ .<sup>19</sup> Since  $\theta^{sr} < \theta^{ss}$  at the threshold, firms increase vacancy posting for risky jobs relatively more and individuals find these jobs more easily.

**Preferences.** While all parameters discussed above seem to have a monotonic impact on  $\phi'_s$  according to Table 3, this is not true for the risk aversion parameter  $\kappa$  in the worker's utility function. Our benchmark case,  $\kappa = 2$ , gives rise to a lower value of  $\phi'_s$  than both  $\kappa = 1$  and  $\kappa = 3$ . Figure 6 plots  $\phi'_s$  as a function of  $\kappa$  and reveals a V shaped relation. The curve attains a global minimum at  $\kappa = 1.55$  and approaches  $\phi'_s = 1$  at both ends of its domain.<sup>20</sup>

The intuition for this non-monotonic relation is the following: Prime-age individuals effectively face a trade-off between intertemporal consumption-smoothing and job security. With standard utility functions (such as CARA or CRRA) the intertemporal elasticity of substitution and the degree of risk aversion are closely related and in our case captured by the single parameter  $\kappa$ . For low levels of  $\kappa$ , individuals are less concerned about layoff risk, but at the same time their preference to balance consumption across periods is low. In the limit  $\kappa \rightarrow 0$ , CARA utility approaches linear utility  $u(w) = w$ . While agents become risk neutral, they only

<sup>19</sup>This is true as long as  $\theta^{sr} > l^{1/l}$ .

<sup>20</sup>The curve is defined on the interval  $(0, \bar{\kappa})$  where  $\bar{\kappa} = 10.86$  is the level of  $\kappa$  at which the candidate in region (sr) ceases to exist. The existence of an upper bound  $\bar{\kappa}$  follows from noting that  $w^{sr}$  is defined if and only if  $\tilde{\Psi} := \Psi_{id}^+(y^L)/u'(y^L) > 0$ . It is easy to verify that  $d\tilde{\Psi}/d\kappa < 0$  and  $\lim_{\kappa \rightarrow \infty} \tilde{\Psi} = -\infty$ .

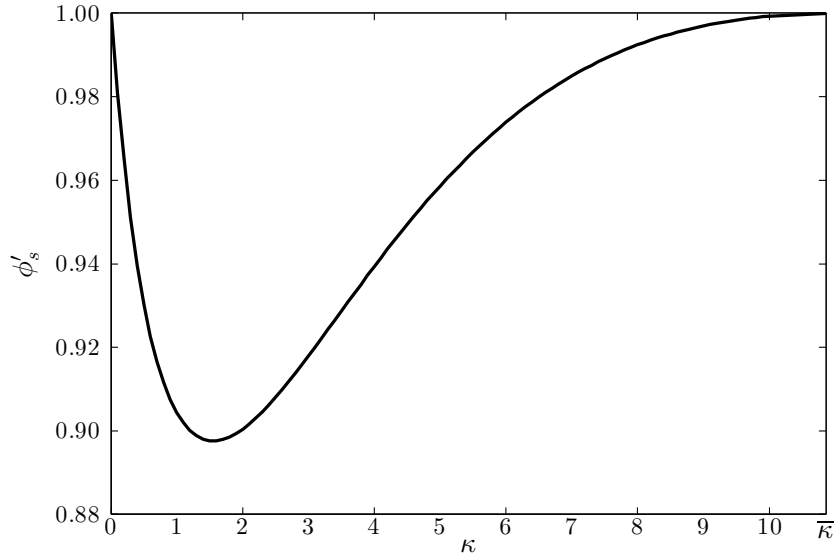


Figure 6: Threshold value  $\phi'_s$  for different levels of risk aversion  $\kappa$

care about the present discounted value of an income stream and are indifferent between any profile that achieves this value. Hence there is no preference for a balanced income profile and the rationale for picking a contract with a positive layoff probability vanishes. As a result, for  $\kappa \rightarrow 0$ , a safe contract is preferred for any  $\phi_s < 1$ .<sup>21</sup> If  $\kappa$  is high, on the other hand, there is a large preference for intertemporal consumption smoothing. At the same time, however, individuals are very risk-averse and not willing to put their job at stake. Thus, for separations of senior workers to emerge in equilibrium, a moderate level of  $\kappa$  is required, together with a sufficiently high chance to retain the job,  $\phi_s > \phi'_s$ .<sup>22</sup>

By contrast, the threshold probability of older workers,  $\phi'_o$ , is monotonically increasing in  $\kappa$ . With only one period of employment, inter-temporal consumption smoothing plays no role and  $\kappa$  solely captures risk aversion.

## 6 Policy implications and relation to the empirical literature

### 6.1 Unemployment insurance and early retirement benefits

Our model predicts that the incidence for layoffs of old-age workers should be particularly high in economies which provide attractive outside options to them, such as generous early retirement provisions or extensive unemployment insurance. This is confirmed by a number of empirical studies. Dorn and Sousa-Poza (2010) analyze survey data that incorporates individual information on “voluntary” and “involuntary” early retirement in selected OECD countries.<sup>23</sup>

<sup>21</sup>This is formally shown in Theorem 7.1.

<sup>22</sup>Notably, one could disentangle the two effects captured by  $\kappa$  by using Epstein–Zin preferences. While the did not carry out this exercise, we expect a lower coefficient of risk aversion or a higher intertemporal elasticity of substitution to decrease the threshold  $\phi'_s$ .

<sup>23</sup>The distinction between “voluntary” and “involuntary” is based on a question of the 1997 International Social Survey Program to early retirees whether they retired early “by choice” or “not by choice”.

The authors find that generous early retirement provisions in a country not only promote “voluntary” transitions to early retirement, but that also “involuntary” early retirement occurs more frequently. In particular, the share of involuntary in total transitions to early retirement is found to be constant across countries with different early retirement schemes. Hence the authors conclude that such schemes induce firms to “push more employees to retire early”. Hakola and Uusitalo (2005) reach a similar conclusion by investigating the employment effects of a pension reform in Finland. Using a change in the Austrian unemployment insurance system as a natural experiment, Winter-Ebmer (2003) shows that extending unemployment benefit duration promotes separations of senior workers, and that this can mainly be attributed to an increase in layoffs rather than in voluntary quits. He concludes that “firms take advantage of the extended benefit duration to get rid of high-tenured and therefore expensive workers”.<sup>24</sup>

The views expressed in these studies suggest that low labor market participation among older workers is partly due to rent-seeking firms which have an interest to turn down their older workforce. According to the insights provided by our model, this should not be blamed on greedy employers, but may be a necessary—and ultimately foreseen—implication of the worker’s optimal choice about the wage contract. Whereas in traditional models of the labor market literature, such as Pissarides (1990), wages always adjust to make separations bilaterally efficient, this is no longer true with contracting frictions. As our analysis shows, the detrimental effect of such a market failure on the employment of the elderly is reinforced by a welfare state which provides inappropriate incentives in bilateral wage negotiations. Discouraging early job exit by decreasing benefit generosity for older individuals or tightening eligibility criteria may therefore improve efficiency. In this sense, the pension reforms recently implemented in many countries may have a double benefit.

Notably, the empirical studies cited above do not take a stance on efficiency. Separations that are employer-initiated or perceived “involuntary” by workers could nevertheless be bilaterally efficient. Nevertheless, we believe that particularly the finding of Dorn and Sousa-Poza (2010) can better be described by a model with inefficiencies. Theoretically, a uniformly higher outside option for older workers increases the number of voluntary quits relative to the number of involuntary layoffs, also because some layoff events become perceived as voluntary. Additionally, in economies with a dense social net of benefits, we should expect older workers to earn higher wages. If bargaining is efficient, these higher wages do not affect separation probabilities, such that the ratio of involuntary in total separations falls. With contracting frictions, however, the wage response will increase layoff rates. In our model, a higher  $b$  lets more individuals choose a high-paying, yet more risky, contract. Provided that this effect is sufficiently large, higher outside options do not increase the share of voluntary separations in total separations in the

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<sup>24</sup>In the model framework such a reform would constitute a change in  $b$  at  $t = 1$ . As wages are pre-determined, however, it would not affect the layoff rates of senior workers. If instead the wage is renegotiated at the beginning of each period (for instance by Nash bargaining about the expected period surplus), an increase in  $b$  will indeed raise the layoff rate of senior workers due to higher risk-taking incentives. In practice, additional factors may contribute to the empirical findings that are beyond the scope of our model, such as reputational or moral concerns about the welfare of dismissed workers that become less important when their value of unemployment increases.

economy, which is in accordance with Dorn and Sousa-Poza (2010).<sup>25</sup>

## 6.2 Training and life-long learning

According to Table 6, an increase in the base productivity  $y^L$  of older or senior workers can bring down the layoff thresholds  $\phi'_o$  and  $\phi'_s$  substantially. Measures that may shift the productivity distribution upwards include age-friendly work environments, employee health programs and continued training for older workers (Göbel and Zwick, 2013).

In practice, however, firms are reluctant to invest into training of senior workers because they expect them to leave soon, see Figure 3.8 in OECD (2006). This, in turn, may become a self-fulfilling prophecy when some older employees are not able to effectively handle new technologies installed in the firm and fall behind in their productivity. Indeed, Figure 3.9 in the same study reveals a significant correlation of 0.65 between the incidence of training among older workers and their five-year retention rates. Using European household data, Fouarge and Schils (2009) calculate the probability that workers aged 55-64 participate in training to be on average 5 percentage points lower compared to workers aged 25-44.

The causal effect of training on retention rates, however, is less clear. Part of this ambiguity may be due to different notions of training that are used in different data sets. Kristensen (2012) and Berg et al. (2015), for instance, find only little effects of training practices on retention rates of older employees in Danish and German establishments, respectively. Although both studies rely on matched employer-employee data, Berg et al. (2015) can only estimate establishment-level effects since the training information cannot be linked to the treated employees. By contrast, Singer and Toomet (2013) can match this data set with individual-specific information on participation in the German WeGebAU program. This provides government-subsidized training to low-skilled workers and to employed workers who are over 45 years old. The authors find that for workers above age 50, participation in the program increases the probability of remaining in paid employment by 4.0 percentage points in the two-year period following treatment. Using data from the Netherlands, Picchio and van Ours (2013) also conclude that “firm-provided training may be an important instrument to retain older workers at work.”

## 6.3 Wage subsidies

In our model, wage subsidies for older workers have an effect that is very similar to training programs. Conceptually, the worker receives  $w_o$  (or  $w_s$ ) but only costs the employer  $w_o - S$  (or  $w_s - S$ ), where  $S > 0$  denotes the subsidy that the employer receives from the government. This is analytically equivalent to increasing both productivity states by  $S$ . As a numerical example, a subsidy of  $S = 0.05$  increases both thresholds  $\phi'_o$  and  $\phi'_s$  from the benchmark levels 0.902 and 0.900 to 0.936. The elasticities reported in Table 4 reveal that an even larger effect could be

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<sup>25</sup>These observations are based on a model extension with heterogeneous outside options. With a homogeneous outside option, only the latter channel is present. As long as  $b \leq y^L$ , there are no quits in the economy and an increase in  $b$  will only increase layoffs. If  $b > y^L$ , on the other hand, any separation is perceived as a quit.

achieved by limiting the subsidy to low productivity workers. However, because we assumed productivity to be not verifiable for third parties, such a policy is not feasible.

As a remedy, the subsidy could be targeted to low wage earners, which can be even more effective. For illustration, assume that the wage ceiling for receiving the subsidy  $S$  is  $y^L$ . This increases the value of safe contracts because of a higher firm surplus. By contrast, the value of risky contracts is unaffected. Implementing this policy with  $S = 0.05$  increases the thresholds to  $\phi'_o = 0.936$  and  $\phi'_s = 0.952$ . This suggests that wage subsidies targeted to low-wage contracts might reduce inefficient layoffs substantially. The relatively large effect is due to the fact that a low-wage subsidy specifically raises the value of safe contracts. By contrast, the policies discussed earlier (decreasing unemployment benefits or productivity-enhancing measures) also increase the value of risky contracts, albeit to a lower extent.

Empirical studies that study employment effects of wage cost subsidy programs targeted at older workers include Huttunen et al. (2013) and Albanese and Cockx (2015). Huttunen et al. (2013) studies a Finnish program that addresses low-wage earners. The authors find no significant causal effect of the subsidy on employment rates, but a slight increase in working hours for employed workers above age 58. This is the age beyond which workers are eligible for part-time pension arrangements. Since the hourly wage did not change, the increased working time seems to capture a labor demand effect. Since the Finish program only subsidized full-time employment, part-time retirement became relatively more costly for employers, which allowed workers to engage in full-time work for a longer period of time. Albanese and Cockx (2015) assess the effects of a wage subsidy program in Belgium that covered all workers above age 58. In contrast to Huttunen et al. (2013), the authors find a significant positive effect also at the extensive margin, albeit only for certain subpopulations. For employees who are at high risk of leaving to early retirement, they estimate a modest causal effect of the subsidy of a 2.2 percentage points higher short-run employment rate.<sup>26</sup> There is again no indication that the subsidy increased hourly wages (apart from the low wage group), suggesting that workers benefited from the program mainly through higher retention rates.

Both studies only assess the short-term effects of wage subsidies. The substantial reduction in layoffs that our model predicts is mainly driven by higher job-finding rates associated with a subsidized wage contract, which requires a longer perspective. For example, with a wage ceiling for the subsidy of  $y^L$ , the period 2 wage level of the best safe contract remains at  $w_o = y^L$  (or  $w_s = y^L$ ). Therefore, for individuals who are already employed at the time when the subsidy is introduced, the subsidy provides no incentive to switch from a risky to a safe contract. For unemployed individuals, by contrast, the lower wage cost increases the job-finding probability of the best safe contract from 0.453 to 0.481 for the old unemployed, and from 0.532 to 0.540 for prime-age workers, which makes looking for safe jobs more attractive. Hence the long-run effect of wage subsidies on employment may be well above the estimates of Huttunen et al. (2013) and Albanese and Cockx (2015).

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<sup>26</sup>Employees at high risk of leaving to early retirement are identified as working in subsectors in which the exit rate from employment was above the median rate of 18% for employees aged 57.75 within the next two quarters.

## 6.4 Layoff taxes and severance pay

Also layoff taxes seem to be very effective in reducing inefficient separations. Punishing layoffs of older workers with a fine of  $t = 0.05$  leads to substantial increase in the thresholds to  $\phi'_o = 0.958$  and  $\phi'_s = 0.946$  relative to the benchmark. The tax discourages risky contracts because firm surplus and therefore the job-finding probability decreases. Safe contracts, on the other hand, become more attractive since wages up to  $y^L + t$  can be demanded. Hence the presence of a layoff tax encourages firms and workers to reduce the layoff risk inherent in wage contracts.

This prediction is, for example, in line with Schnalzenberger and Winter-Ebmer (2009), who analyze the employment effects of a 1996 reform in Austria. This introduced a tax on employer-initiated separations of workers older than 50 years with a tenure of at least 10 years. The authors report find significant effects on layoff rates only after the layoff tax had been raised by about a third in 2000, amounting to up to 1.6 times the worker's monthly gross salary. The highest effects are found for tax-eligible male workers above age 55, for which layoff probabilities decreased by 0.27 percentage points (55–75.5 years) and 1.32 percentage points (75.5–60 years). In light of the fact that the overall displacement rates in these age groups are 1.19% and 2.13%, respectively, these are economically sizable effects. Large employment effects are also found by Hakola and Uusitalo (2005) following the introduction of experience rating of early retirement pension benefits in Finland.

However, layoff taxes can also have unfavorable side-effects. Behaghel et al. (2008) report that hiring rates of over 50 year olds were oppressed substantially by the layoff tax in France. This effect is also present in our model. For risky contracts, firms anticipate higher costs when a layoff is necessary and therefore reduce vacancy posting. In the model, however, this is accompanied by a composition effect that increases hiring rates of old unemployed in the aggregate. The reason is that the tax encourages some workers to redirect their search activities from risky to safe jobs. Since the acceptance rate at safe jobs is in general higher, the aggregate hiring rate increases.

A related policy tool that also discourages layoffs is mandated severance pay. In this case the fine  $t$  does not accrue to the government but is redistributed to the worker. Since this increases the worker's income after a layoff from  $b_o$  to  $b_o + t$ , the effect on layoffs turns out to be slightly lower compared to a conventional layoff tax. In particular, a severance pay of  $t = 0.05$  leads to  $\phi'_o = 0.950$  and  $\phi'_s = 0.936$ .

## 7 Robustness

In the last section of this paper we discuss the robustness of our results with respect to alternative specifications of the model. As evident from the comparative static analysis with respect to  $\kappa$ , individual preferences play an important role whether or not layoffs occur in equilibrium. Most importantly, risk neutral agents will never choose a contract with a positive probability of being laid off in period 2. This is verified as a first result. Next we prove that the main theoretical results of the previous chapters (Theorems 4.1 and 4.4) can be restored by extending the model



to on-the-job search. We also provide some numerical results for the model with on-the-job search when agents are risk averse and/or have access to a saving technology. Finally, we prove that our results remain valid if wage renegotiation is allowed, but subject to information asymmetries.

## 7.1 Risk neutrality

Although risk neutrality of workers might be an unrealistic assumption by itself, it is used as a benchmark in the bulk of the labor market literature. We introduce the following modification of Assumption 4.1 to capture the case of linear utility.

**Assumption 4.1'.** *Let all conditions of Assumption 4.1 prevail, but assume linear utility,  $u(w) = w$ .*

All of the above results were derived under the assumption of curved utility, which implies that the individual has the preference to smooth consumption across periods. Theorems 4.3 and 4.4 hinge on the trade-off between job security and intertemporal consumption smoothing. With risk neutrality, the latter effect is absent, because the individual only cares about the present discounted value of consumption and not about its time pattern. Therefore, neither of the two theorems can be sustained with risk neutral agents, such that risk neutrality rules out layoffs of senior workers completely.<sup>27</sup>

**Theorem 7.1.** *Under Assumption 4.1', the equilibrium is one without endogenous layoffs of senior workers for any  $\phi_s \in (0, 1)$ . In particular, the optimal contract is either located in region (ss) or (rs).*

In the knife-edge case  $\phi_s = 1$  the individual is indifferent between a contract in region (ss) and (sr). For any  $\phi_s < 1$ , however, she is strictly better off with a contract in the safe region (ss) because it promises a higher present discounted income stream. Similarly, any contract in region (rr) is strictly dominated by a contract in region (rs) for  $\phi_s \in (0, 1)$ .

This finding is also true if we maintain curved utility, but allow for an individual saving technology instead. This allows the individual to transform an arbitrary income stream into a smooth consumption stream such that there is again no rationale to look for a risky job. Imposing borrowing constraints would not alter this result, because with a contract at the boundary of region (ss) workers actually want to *save* part of their period 1 income. We discuss the possibility of savings in conjunction with on-the-job search in Section 7.4.

## 7.2 Risk neutrality and on-the-job search

If the optimal wage contract is located at the boundary of region (ss) or in (sr), it may feature a substantial decrease in wages before retirement. But this is not what we observe in practice. First, such a contract promotes opportunistic behavior of the worker during prime-age. Second,

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<sup>27</sup>By contrast, Theorem 4.1 remains valid with linear utility. With only one period ahead, intertemporal consumption smoothing motives obviously play no role.

workers have an incentive to search for other job opportunities which promise a higher period 2 wage. While we refrain from introducing moral hazard into the model, we in the following allow individuals to search on the job. In particular, employed workers may at the end of period 1 apply to a firm in some submarket  $(w_q, \theta_q)$  for employment in period 2. With probability  $p(\theta_q)$ , their application is successful and they switch to the new employer. Otherwise they continue working at the current firm.<sup>28</sup> For simplicity we assume that the productivity draw at the new job is independent of previous realizations and that the high productivity state is attained with probability  $\phi_o$ . Hence the expected period 2 surplus changes from  $\mathbb{E}G_s(w_s)$  to  $\mathbb{E}G_s(w_s) + V_q(w_s)$ , where  $V_q(w_s)$  captures the additional surplus that can be achieved by quitting,

$$V_q(w_s) := \max_{(w_q, \theta_q)} p(\theta_q) [\mathbb{E}G_o(w_q) - \mathbb{E}G_s(w_s)] \quad \text{s.t.} \quad q(\theta_q) \mathbb{E}J_o(w_q) = c. \quad (9)$$

$\mathbb{E}J_o$  and  $\mathbb{E}G_o$  have been defined in equations (3)–(4), respectively. In the on-the-job search process, the worker's outside option is  $\mathbb{E}G_s(w_s)$ , which is the value of staying on the current contract defined in (6).

### 7.2.1 Optimal search on the job

As evident from (9), the optimal sub-market to apply to as well as the value that can be achieved by quitting depend on  $w_s$ , the period 2 wage specified in the worker's current employment contract. Intuitively, the higher the senior wage negotiated with the current employer, the higher the worker's threat point in wage negotiations with other firms and the higher the optimal wage  $w_q$  that is demanded. Since high paying jobs are harder to get, a high senior wage  $w_s$  at the current firm therefore decreases the probability of a job-to-job transition. This is formalized in the following proposition, in which the optimal solution to (9) is denoted by  $(w_q(w_s), \theta_q(w_s))$ .<sup>29</sup>

**Proposition 7.1.** *The optimal outside wage,  $w_q(w_s)$ , is weakly increasing in  $w_s$  on  $[b, y^L]$  and  $(y^L, y^H]$ , while the optimal probability of quitting to another job,  $p(\theta_q(w_s))$ , is weakly decreasing in these regions.*

Figure 7 illustrates the above result and also gives the expected value that can be gained from searching on the job. The solid and dashed lines correspond to two different parameterizations, which differ only in the level of  $\phi_o$ . The solid (dashed) profiles correspond to a relatively high (low) level of  $\phi_o$ . To already lay the grounds for Theorem 7.2, the values of  $\phi_o$  were chosen to generate  $w_q(y^L) > y^L$  and  $w_q(y^L) = y^L$ , respectively.

Panel (d) shows the additional surplus  $V_q(w_s)$  that can be expected from participating in on-the-job search, and panel (c) shows the surplus  $\mathbb{E}G_q^*(w_s) := \mathbb{E}G_o(w_q(w_s)) - \mathbb{E}G_s(w_s)$  in case of a successful job search. Naturally, the value that can be gained from a job-to-job transition

<sup>28</sup>The job-to-job transition takes place before period 2 productivity has been realized. Therefore, also quitting individuals face a layoff risk in case  $w_q > y^L$ .

<sup>29</sup>The results of Section 4.1 can easily be generalized to see that for given  $w_s$ , problem (9) has at most two solution candidates. If the solution is not unique, the worker is indifferent between  $w_q = y^L$  and  $w_q > y^L$ . In this case, let us define  $w_q(w_s) = y^L$ .

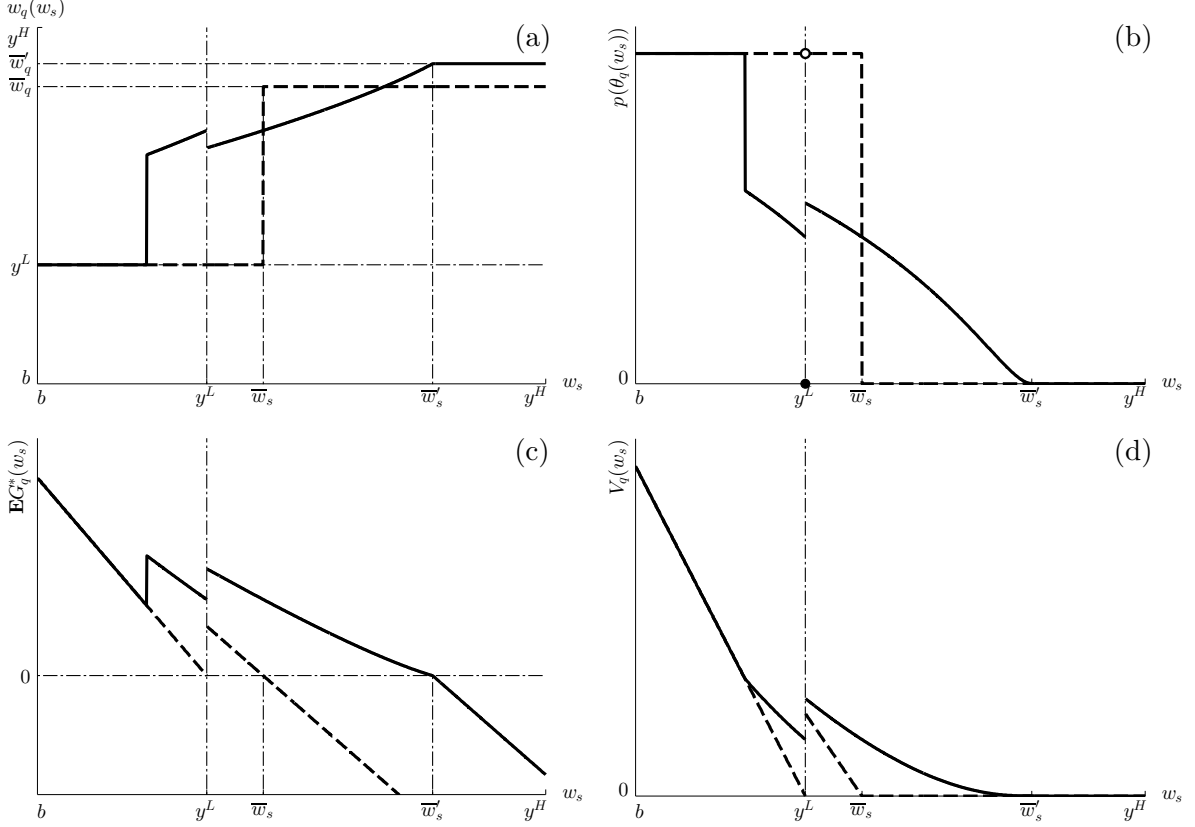


Figure 7: Optimal wage  $w_q(w_s)$ , quitting probability  $p(\theta_q(w_s))$ , surplus  $\mathbb{E}G_q^*(w_s)$ , and value  $V_q(w_s)$  of a job-to-job transition for a high (solid line) and low (dashed line) value of  $\phi_o$ .

positively depends on the expected productivity in the alternative job, which is captured by the parameter  $\phi_o$ . Hence the solid line (high  $\phi_o$ ) never falls below the dashed line (low  $\phi_o$ ) in panels (c) and (d). It can also be verified that the additional surplus that can be expected from switching the job decreases with the wage  $w_s$  that is earned at the current firm. The higher  $w_s$ , the less likely to find a job that promises an even higher wage.

There exists an upper bound on  $w_s$  (denoted by  $\bar{w}_s$  and  $\bar{w}'_s$  in the figure) above which nothing can be gained by quitting,  $V_q(w_s) = 0$ . The reason is that the optimal wage demand  $w_q(w_s)$  of the individual reaches the maximum wage that outside firms are able to afford, indicated by  $\bar{w}_q$  and  $\bar{w}'_q$  in panel (a). This upper limit is due to the vacancy posting cost  $c$  that the new employer has to recoup. The incumbent firm, by contrast, can pay up to  $w_s = y^H$  and finance the posting cost from period 1 revenue. The upper bound on the wage is determined by  $\mathbb{E}J_o(\bar{w}_q) = c$ , which yields  $\bar{w}_q = y^H - c/\phi_o \in (y^L, y^H)$ . The lower  $\phi_o$ , the lower expected firm surplus and the lower the maximum affordable wage. As a result, there exists a whole range of contracts from which workers never quit voluntarily, namely any  $w_s$  with  $\mathbb{E}G_s(w_s) \geq \mathbb{E}G_o(\bar{w}_q)$ . In the figure, this corresponds to the interval  $[\bar{w}_s, y^H]$  for the scenario with low  $\phi_o$ , and to  $[\bar{w}'_s, y^H]$  for scenario with high  $\phi_o$ .

Although the pattern of  $V_q(w_s)$  looks qualitatively similar in the two scenarios, the underlying optimal wage and quitting probability depicted in panels (a) and (b) are quite different. Let

us first discuss the case with a low  $\phi_o$ , which is represented by the dashed lines. If the worker's current job is safe in period 2,  $w_s \leq y^L$ , it is optimal to apply to a job that is also safe but pays a higher wage,  $w_q = y^L$ . In the special case  $w_s = y^L$ , the current and the optimal alternative job pay exactly the same wage, and the individual stays with the current employer.<sup>30</sup> Among the workers on a risky job, the ones with  $w_s \in [y^L, \bar{w}_s]$  also try to switch to safe jobs with wage  $w_q = y^L$ . There is no quitting from jobs with  $w_s > \bar{w}_s$ , because the wage demand reaches the upper bound  $\bar{w}_q$ . Altogether, we see that if  $\phi_o$  is low, the worker either quits to a safe job or, in case this is not profitable, stays with the current employer.

For a higher level of  $\phi_o$ , optimal job search behavior can be quite different as evident from the solid lines in Figure 7. Workers who are on a job that is safe in period 2 and earn a wage  $w_s \in (w'_s, y^L]$  find it optimal to apply to a risky job. The expected wage increase more than compensates for the higher layoff risk. Workers who are already on a risky job search for another risky job that promises a higher wage. There is again a threshold wage,  $\bar{w}'_s$ , above which no quits occur. Note that in contrast to the first scenario, workers at the boundary  $w_s = y^L$  have a positive probability to leave,  $p(\theta_q(y^L)) > 0$ , because  $w_q(y^L) > y^L$ .

### 7.2.2 Optimal long-run wage contracts

Having discussed individual optimal behavior in period 2 conditional on  $w_s$ , we can now ask how the optimal contract  $(w_m, w_s)$  for problem (P) looks with on-the-job search. The firm and worker surplus functions have to be modified accordingly and are reported in Table B.2 in the appendix. It turns out that the possibility to search on the job has two opposing effects on the worker's objective function,  $p(\theta_m)\mathbb{E}G_m(w_m, w_s)$ . On the one hand, there is a positive direct effect from the additional value  $V_q(w_s)$  that can be captured. This increases expected worker surplus  $\mathbb{E}G_m(w_m, w_s)$  for any given wage profile. At the same time, however, the shorter expected duration of the employment spell reduces the firm surplus  $\mathbb{E}J_m(w_m, w_s)$  of the incumbent firm. This discourages vacancy posting, reduces  $p(\theta_m)$ , and exerts a negative indirect effect on the worker's objective function. Conceptually, it is not clear which of the two effects dominates. At least with linear utility, it can be verified that the latter, negative effect is stronger, and the worker therefore seeks to minimize her probability of leaving to a different job.

**Proposition 7.2.** *Under Assumption 4.1', the wage profile of solution candidates is no longer undetermined. If  $\phi_s \geq \frac{y^L - b}{y^H - b}$ , the senior wage  $w_s$  is chosen such as to minimize the probability of quitting,  $p(\theta_q(w_s))$ , in the respective region. Hence allowing workers to search on the job never makes them better off.*

*Ex ante*, workers would ultimately want to “tie their hands” and commit to stay with the employer. But when period 2 arrives, quitting brings an extra surplus in general (cf. Figure 7) and the worker gives in to this incentive. In practice, workers cannot legally bind themselves to the employer such that no full commitment technology is available. By Proposition 7.1,

<sup>30</sup>Proposition 7.2 states that in case of being indifferent between quitting or staying in period 2, it is optimal from the viewpoint of period 1 for the individual not to leave.

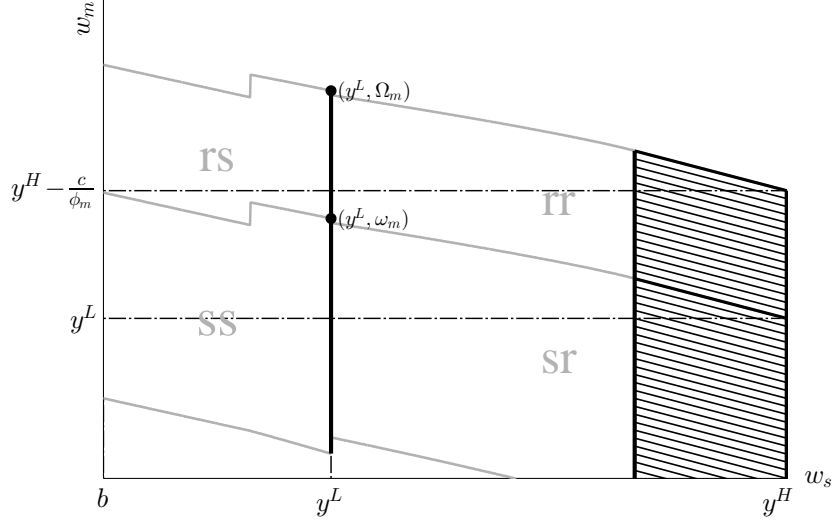


Figure 8: Location of solution candidates of problem (P)

however, demanding a high senior wage  $w_s$  can be used as a *partial* commitment device because it decreases the probability of getting an even better job. It is this mechanism that allows to restore the results of Theorem 4.4 also with risk neutral agents.

**Theorem 7.2.** *Let Assumption 4.1' prevail and assume a  $\underline{\phi} \geq \frac{y^L - b}{y^H - b}$  such that  $\Psi_{y^L}(\omega_m(\underline{\phi}); \underline{\phi}) < 0$  and  $\Psi_{y^H}(y^L; \underline{\phi}) < 0$ . If  $w_q(y^L) = y^L$ , no senior workers are laid off in equilibrium. If  $w_q(y^L) > y^L$ , there exists a unique  $\phi'_s \in [\underline{\phi}, 1)$  such that an equilibrium with layoffs of senior workers emerges if  $\phi_s \in (\phi'_s, 1)$ , but not if  $\phi_s \in [\underline{\phi}, \phi'_s)$ .*

The two latter assumptions on  $\underline{\phi}$  let us focus on regions (ss) and (sr). While the driving force behind Theorem 4.4 was the preference for smooth consumption, here it is exclusively the incentive to restrict one's opportunities from on-the-job search. As discussed earlier, there are always contracts in region (sr) where the probability of quitting is zero. This is indicated by the hatched area in Figure 8. With linear utility, the solution candidate in region (sr) is not a single point anymore, but corresponds to one of the downward sloping lines. These connect all contracts that yield the same present discounted wage income.

If  $w(y^L) = y^L$ , there is no reason to quit from contracts with  $w_s = y^L$ , such that also the solution candidate of region (ss) is associated with a zero probability of leaving. Therefore the solution to the problem with on-the-job search is identical to the solution of the basic model without this possibility, and Theorem 7.1 applies.

If  $w(y^L) > y^L$ , however, the best safe contract is involved with a trade-off. On the one hand, there is no risk of being laid off in period 2. On the other hand, the worker finds it optimal to apply to alternative firms and quits the current job with some probability. The solution candidate in region (sr) has exactly the opposite properties. For high enough  $\phi_s$ , the difference in layoff risk is fully compensated by the higher wage stream that the firm is willing to promise to workers who can commit not to leave. Hence, with  $w_q(y^L) > y^L$ , we can restore the results of Theorem 4.4.

The effect of on-the-job search on the threshold  $\phi'_s$  is illustrated in Figure 9, which shows  $\phi'_s$  as a function of  $\phi_o$ . All parameters apart from  $\phi_o$  and  $\kappa$  are fixed at their benchmark levels reported in Table 1. According to Theorem 7.1, with linear utility and no search on the job the threshold is identically equal to one (solid line). Including on-the-job search decreases the threshold below  $\phi'_s = 1$  because  $w_q(y^L) > y^L$  for  $\phi_o > 0.4$  (dashed line). Furthermore, a higher  $\phi_o$  increases the incentive to quit from the safe candidate because the expected productivity in a new job increases. By contrast, the value of the optimal contract in region (sr) remains unaffected because the quitting probability is zero in any case. Hence the threshold  $\phi'_s$  decreases in the probability  $\phi_o$ .

### 7.3 Risk aversion and on-the-job search

The rationale for risky contracts brought forward in the last section remains valid if agents are risk averse. However, individuals will in general not be willing to drive their quitting probability to zero, as this would imply an extremely back-loaded wage profile. In fact, an optimal contract trades off between intertemporal consumption smoothing and a low quitting probability. The optimality condition (B.7) becomes

$$p(\theta_m) \left\{ [1 - p(\theta_q(w_s))] [u'(w_s) - u'(w_m)] - \frac{dp(\theta_q(w_s))}{dw_s} [Y(w_s) - w_s] u'(w_m) \right\} = \mu_1$$

where  $Y(w_s) = \bar{y}_s$  if  $w_s \in [b, y^L]$  and  $Y(w_s) = y^H$  if  $w_s \in (y^L, y^H]$ . Therefore, an interior optimal contract satisfies

$$u'(w_s) - u'(w_m) = \frac{dp(\theta_q(w_s))/dw_s}{1 - p(\theta_q(w_s))} (Y(w_s) - w_s) u'(w_m) \quad (10)$$

at all  $w_s$  where  $\theta_q(w_s)$  is continuously differentiable. The left hand side represents the direct effect from marginally increasing  $w_s$  at the expense of  $w_m$ . The right hand side is the indirect effect induced by a marginally lower quitting probability  $p(\theta_q(w_s))$ . The fraction captures the hazard rate of leaving the firm. Since  $p' > 0$  and  $\theta'_q \leq 0$  by Proposition 7.1, the wage profile of an interior solution candidate is either flat or back-loaded, i.e.  $w_m \leq w_s$ .

The analytical results of Section 3 can partially be generalized to the economy with on-the-job search with some adaptations. The jumps in  $p(\theta_q(w_s))$  add further discontinuities to problem (P). As a result, each of the previously four regions of the contract space breaks in potentially four subregions, based on whether  $p(\theta_q(w_s)) = 0$  or  $p(\theta_q(w_s)) > 0$ , in the latter case differentiating between  $w_q(w_s) \leq y^L$ . This gives a total of 16 regions that have to be considered for the optimum. Equation (10) defines a map  $h : w_m \mapsto w_s(w_m)$  that gives the optimal senior wage  $w_s$  as a function of the junior wage  $w_m$ . Provided this map is strictly decreasing,  $\Psi_h$  is monotonic, and each of the regions holds at most one solution candidate.<sup>31</sup> We believe that counterparts of the Theorems 4.3 and 4.4 can also be formulated, but this would be very

<sup>31</sup>More precisely,  $h$  is well-defined if  $1 - p(\theta_q(w_s))$  is not too elastic with respect to  $w_s$ , and strictly decreasing if  $1 - p(\theta_q(w_s))$  is sufficiently curved, i.e.  $\frac{d \ln(1 - p(\theta_q))}{d \ln w_s} < \frac{w_s}{Y(w_s) - w_s} \leq \frac{d \ln[d(1 - p(\theta_q))/dw_s]}{d \ln w_s}$ . The latter condition can be relaxed, but is satisfied in our benchmark parameterization.

*Prime-age unemployed*

	$w_m$	$w_s$	$p(\theta_m)$	$V_m$	$w_q$	$p(\theta_q)$	$V_q$
region (ss)	1.229	1.000	0.519	<b>0.0587</b>	1.308	0.224	0.0021
region (sr) {	1.144	1.259	0.496	0.0548	1.351	0.162	0.0008
	1.040	1.447	0.493	0.0529	1.447	0.000	0.0000

Table 7: Solution candidates and optimum for the benchmark parameterization with OTJS.

cumbersome in light of the many solution candidates that have to be considered. Therefore, we in this section restrict ourselves to numerical results.

For the benchmark parameterization given in Table 1, we find that optimal on-the-job search behavior is qualitatively identical to the solid lines in Figure 7. The safe candidate encourages to quit to another firm, since  $p(\theta_q(y^L)) > 0$ . The first row in Table 7 reveals that the quitting probability is 22.4%. Compared to the economy without on-the-job search, this reduces the value of the best safe contract by 4.1% from 0.0612 to 0.0587. Table 7 also shows that two solution candidates exist in region (sr). One with a positive quitting probability, and one that reduces the quitting probability to zero. Since an extremely back-loaded profile is necessary to fully commit to stay, the best risky contract features a moderate degree of back-loading together with a quitting probability of 16.2%. This is lower than for the safe contract. Therefore, the value of the optimal candidate in region (sr) decreases by only 2.1% relative to the benchmark of Table 2. This suggests that in case  $p(\theta_q(y^L)) > 0$ , on-the-job search makes contracts with *layoff* risk more attractive because the probability of *quitting* is lower. Hence the worker's decision problem with on-the-job search involves an additional trade-off beyond the consumption smoothing motive: a low-wage contract comes with no layoff risk but large incentives to leave voluntarily, while a high-paying contract binds the individual to the firm but makes layoffs necessary.

If  $p(\theta_q(y^L)) = 0$ , things are different. The individual then has no incentive to quit from the safe contract, whereas there might be a positive quitting probability from the best risky contract. In this case, on-the-job search either increases the threshold  $\phi'_s$  or leaves it unaffected relative to the basic model. This is confirmed by Figure 9. First, we note that  $p(\theta_q(y^L)) = 0$  for  $\phi_o < 0.59$  and  $p(\theta_q(y^L)) > 0$  otherwise. Second, we find that for  $\phi_o < 0.59$ , the individual also does not have an incentive to quit from the optimal candidate in region (sr) since  $w_q(w_s^{sr}) = \bar{w}_s$ . Therefore including on-the-job search does not alter the threshold for  $\phi_o < 0.59$ , and the dotted and dash-dotted line in Figure 9 overlap. For  $\phi_o \geq 0.59$ , however, the inclusion of on-the-job strictly lowers the threshold. This effect is most pronounced around  $\phi_o = 0.71$ , where  $\phi'_s$  falls to 0.831. For higher levels of  $\phi_o$  the threshold increases again because to keep the quitting probability low, more and more back-loading of wages is required, which interferes with consumption smoothing.

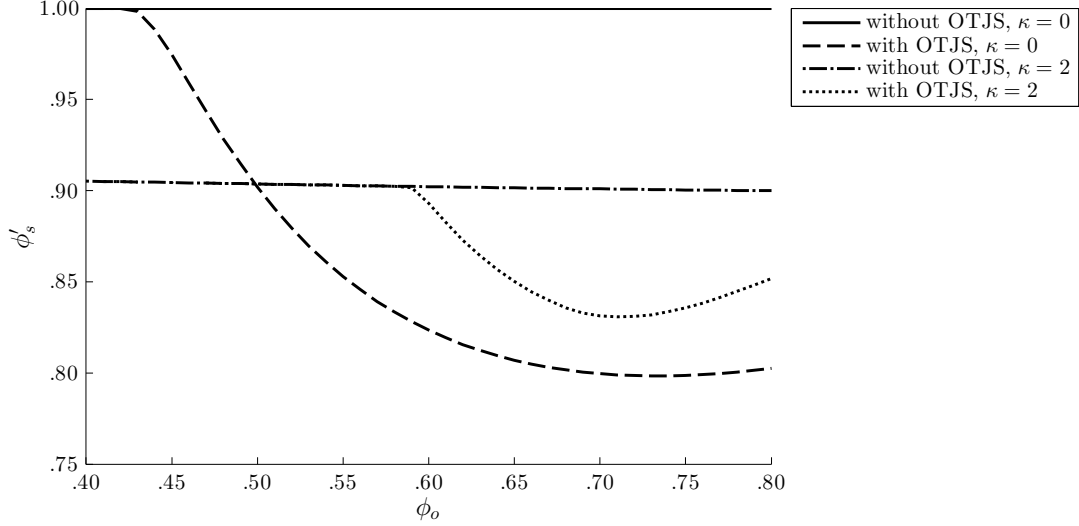


Figure 9: The threshold  $\phi'_s$  as a function of  $\phi_o$  under four different scenarios. Remaining parameters at their benchmark levels.

#### 7.4 Risk aversion, on-the-job search, and savings

So far individuals were completely cut off the capital market and could smooth consumption only by smoothing wage income. If individuals are allowed to accumulate savings, intertemporal consumption smoothing can also be achieved with a safe contract by using part of the large prime-age income to finance consumption when old. While this eliminates one virtue of risky contracts (consumption smoothing), a high period 2 wage is still associated with lower incentives to quit the job voluntarily. Therefore, contracts with inefficient separations of senior workers can emerge in equilibrium in spite of precautionary saving.

In the following, we illustrate how the inclusion of savings changes the optimality conditions, and afterwards again provide a numerical assessment. For the sake of exposition, we abstract from exogenous separations ( $s = 0$ ) in the following analysis. Assume that after period 1, the individual decides on how much of period 1 income to invest into a riskless asset that pays an exogenous gross interest rate  $R > 1$ . The savings decision is dependent on the worker's employment state. A worker who was unemployed during period 1 saves  $s^U$  of his income  $b$ . Whereas an employed worker saves  $s^E$ , which will depend on her current wage schedule  $(w_y, w_s)$ .<sup>32</sup> Altogether, the worker takes three decisions:

- (a) at  $t = 0$  the worker searches for a long-run contract with characteristics  $(w_y, w_s, \theta_y)$ ,
- (b) based on the employment status during period 1, she decides on saving  $s^U$  or  $s^E$ ,
- (c) if employed during period 1, she additionally decides in which segment  $(\theta_q, w_q)$  of the labor market to search for an alternative job.

The decision problem of the worker can be solved backwards, starting with the on-the-job search problem (c) of employed workers. Conditional on the period 2 wage contracted with the

<sup>32</sup>We do not consider borrowing constraints, since these are hardly binding in equilibrium.



incumbent firm,  $w_s$ , and the current stock of assets,  $a$ , the worker solves

$$V_q(w_s, a) = \max_{(w_q, \theta_q)} p(\theta_q) [\mathbb{E}G_o(w_q, a) - \mathbb{E}G_s(w_s, a)] \quad \text{s.t.} \quad q(\theta_q) \mathbb{E}J_o(w_q) = c, \quad (11)$$

where

$$\mathbb{E}G_o(w, a) = \begin{cases} u(w + Ra) - u(b + Ra) & \text{if } b \leq w \leq y^L, \\ \phi_o[u(w + Ra) - u(b + Ra)] & \text{if } y^L < w \leq y^H \end{cases}$$

and

$$\mathbb{E}G_s(w, a) = \begin{cases} u(w + Ra) - u(b + Ra) & \text{if } b \leq w \leq y^L, \\ \phi_s[u(w + Ra) - u(b + Ra)] & \text{if } y^L < w \leq y^H. \end{cases}$$

This resembles the search problem (9), but takes into account that period 2 wage income is augmented by the capital income  $Ra$ . In the following, we assume that the utility function is CARA as in our numerical implementation. This renders the problem extremely tractable since  $u(w + Ra) - u(b + Ra) = u'(Ra)[u(w) - u(b)]$ . Exploiting this property reveals that the optimal  $(w_q, \theta_q)$  is independent of accumulated savings  $a$ , and that

$$V_q(w_s, a) = u'(Ra) \max_{(w_q, \theta_q)} p(\theta_q) [\mathbb{E}G_o(w_q, 0) - \mathbb{E}G_s(w_s, 0)] = u'(Ra) V_q(w_s, 0).$$

Hence the stock of assets only affects the expected surplus from on-the-job search, but not worker's search behavior. This simplifies the savings decision (b) significantly, because the value of having a job with contract  $(w_m, w_s)$  can be written as  $\tilde{W}_m^*(w_m, w_s) := \max_{s^E} \tilde{W}_m(w_m, w_s, s^E)$ , where

$$\begin{aligned} \tilde{W}_m(w_m, w_s, s^E) &= u(w_m - s^E) + \beta [u(b + Rs^E) + \mathbb{E}G_s(w_s, s^E) + V_q(w_s, s^E)] \\ &= u(w_m - s^E) + \beta u(b + Rs^E) + \beta u'(Rs^E) [\mathbb{E}G_s(w_s, 0) + V_q(w_s, 0)]. \end{aligned}$$

Optimal savings equalizes the marginal utility of consumption today with the expected marginal utility of consumption tomorrow. With CARA utility, the problem admits a unique closed form solution,

$$s^E(w_m, w_s) = \frac{w_m}{1 + R} + \frac{\ln K(w_s)}{\kappa(1 + R)} \quad (12)$$

where  $K(w_s) := R\beta[u'(b) - \kappa\mathbb{E}G_s(w_s, 0) - \kappa V_q(w_s, 0)]$ . It is easy to verify that the optimal level of savings is increasing in  $w_m$  and decreasing in  $w_s$ . Similarly, unemployed workers maximize

$$U_m^* = \max_{s^U} U_m(s^U) = \max_{s^U} \left\{ u(b - s^U) + \beta [u(b + Rs^U) + u'(Rs^U) V_o] \right\},$$

which yields  $s^U = b/(1 + R) + \ln[R\delta(u'(b) - \kappa V_o)]/(\kappa(1 + R))$ .

Using these insights, we turn to (a). The optimal long-term contract is the solution to

$$\max_{w_m, w_s, \theta_m} p(\theta_m) \mathbb{E}G_m(w_m, w_s) \quad \text{s.t.} \quad q(\theta_m) \mathbb{E}J_m(w_m, w_s) = c$$

*Prime-age unemployed*

	$w_m$	$w_s$	$p(\theta_m)$	$V_m$	$w_q$	$p(\theta_q)$	$V_q$	
region (ss)	1.267	1.000	0.503	<b>0.0594</b>	1.308	0.224	0.0021	
region (sr)	{	1.202	1.151	0.493	0.0545	1.304	0.228	0.0023
		1.038	1.450	0.493	0.0523	1.447	0.000	0.0000

Table 8: Solution candidates and optimum for the benchmark parameterization with OTJS.

where  $\mathbb{E}G_m(w_m, w_s) = \mathbb{E}W_m^*(w_m, w_s) - U_m^*$ ,  $\mathbb{E}J_m(w_m, w_s)$  is given in Table B.2, and

$$\mathbb{E}W_m^*(w_m, w_s) = \begin{cases} \tilde{W}_m^*(w_m, w_s, s^E) & \text{if } \tilde{J}_m(w_m, w_s | Y_m = y^L) \leq 0 \\ \phi_m \tilde{W}_m^*(w_m, w_s, s^E) & \text{if } \tilde{J}_m(w_m, w_s | Y_m = y^H) \leq 0 < \tilde{J}_m(w_m, w_s | Y_m = y^L). \end{cases}$$

The first order conditions for an interior optimum are

$$\begin{aligned} \frac{1 - \varepsilon(\theta_m)}{\varepsilon(\theta_m)} \frac{\mathbb{E}G_m(w_m, w_s)}{\mathbb{E}J_m(w_m, w_s)} &= u'(w_y - s^E), \\ u'(w_s + R s^E) - u'(w_m - s^E) &= \frac{dp(\theta_q(w_s))/dw_s}{1 - p(\theta_q(w_s))} (Y(w_s) - w_s) u'(w_m - s^E) \end{aligned} \quad (13)$$

where  $s^E = s^E(w_m, w_s)$  satisfies (12), and  $Y(w_s) = \bar{y}_s$  if  $w_s \in [b, y^L]$  and  $Y(w_s) = y^H$  if  $w_s \in (y^L, y^H]$ . The first condition is familiar from (B.8), only  $u'(w_y)$  is replaced by  $u'(w_y - s^E)$ . The second condition is the equivalent of (10), again adjusted for savings. Interestingly, by substituting (12) into (13) a closed-form solution for  $w_s$  can be obtained. The senior wage of an optimal long-term contract either has to equal  $w_s = y^L$  or satisfies

$$u'(w_s + \ln K(w_s)/\kappa) = 1 + \frac{dp(\theta_q(w_s))/dw_s}{1 - p(\theta_q(w_s))} (Y(w_s) - w_s).$$

The left-hand side captures the effect of  $w_s$  on period 2 consumption, which stems from both direct wage income ( $w_s$ ) and savings ( $\ln K(w_s)/\kappa$ ). The right hand-side is the relative change in the quitting probability associated with a change in  $w_s$ . Notice the difference to (10), which can be written as

$$u'(w_s - w_m) = 1 + \frac{dp(\theta_q(w_s))/dw_s}{1 - p(\theta_q(w_s))} (Y(w_s) - w_s) \quad (14)$$

Including savings makes  $w_s$  independent from the first period wage  $w_m$ , and hence from any consumption smoothing motives. In particular, only period 2 consumption and separation probabilities matter for  $w_s$ , while consumption smoothing is achieved through an appropriate adjustment in savings  $s^E$ .

We conclude with a numerical example, building on the previous section. Comparing Table 8 and Table 7 shows that the value of the safe candidate increases if people are allowed to save, while the risky contracts slightly decrease in value. This decrease is due to the higher value of unemployment,  $U_m^*$ . For the safe candidate, this is more than compensated by the utility gain that is achieved by consumption smoothing. These results suggests that the possibility to save

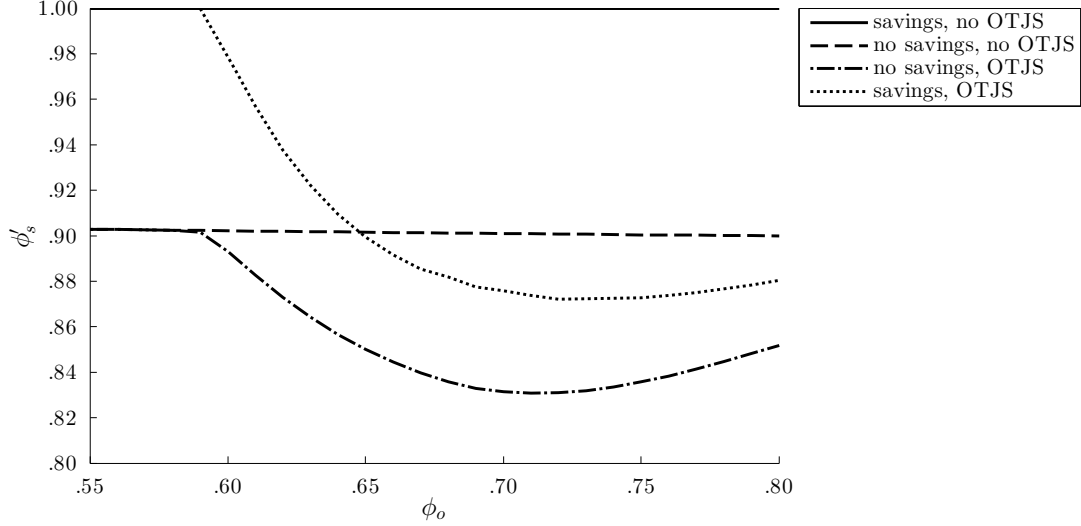


Figure 10: The threshold  $\phi'_s$  as a function of  $\phi_o$  under four different scenarios. Remaining parameters at their benchmark levels.

increases the value of the safe candidate relative to the best risky contract for given  $\phi_s$ . Hence we should expect a higher threshold value  $\phi'_s$  than in the economy without savings. Indeed,  $\phi'_s$  increases from  $\phi'_s = 0.836$  to  $\phi'_s = 0.872$ .

Figure 10 graphically compares the threshold values of four model specifications. With savings and no on-the-job search (solid line) there is no rationale to search for a risky contract, such that  $\phi'_s = 1$ . It is always superior to take a safe contract and smooth consumption using savings. The dotted line represents the economy with savings and on-the-job search. As in Section 7.3, the threshold  $\phi'_s$  is strictly less than one provided that the individual has an incentive to quit from the risky contract,  $p(\theta_q(y^L)) > 0$ , which is the case for  $\phi_o > 0.59$ . For  $\phi_o > 0.65$ , the dotted line even falls below the dashed line, which represents the threshold obtained from the basic model without on-the-job search and without savings.

## 7.5 Equivalence to Renegotiation under Asymmetric Information

It remains to convince the reader that the exogenously imposed renegotiation frictions under which the results of this paper were derived are less restrictive than they appear. Recall the three key assumptions (i)–(iii) introduced in Section 2. While assumption (ii) can be motivated by lack of enforceability, assumption (iii) is arguably very restrictive because it rules out wage renegotiations even if this would be beneficial for both parties. In this section we demonstrate that a generalization of the model where renegotiation is possible but subject to asymmetric information, gives rise to the same results. In particular, we give up assumptions (ii) and (iii) and instead impose

- (ii') the productivity draw of the current period is private knowledge of the firm,
- (iii') the current period wage can be renegotiated by mutual consent.

If no mutual consent is reached in the wage renegotiation, the wage level stated in the original contract prevails. The view that the firm has private information about the worker’s productivity is for instance adopted in Hashimoto (1981) and Hall and Lazear (1984). While the worker might have some general assessment about her performance, the employer typically has better insight how much the individual worker actually contributes to the company’s output.<sup>33</sup>

We consider a setting where the firm has two possible actions after observing the worker’s current productivity draw. It can either stick to the initial contract or propose a renegotiation to the worker. To keep things as simple as possible, we treat the wage that the firm proposes to the worker as exogenous and assume that it makes the firm just indifferent between terminating or continuing the match. This is the minimal adjustment that is necessary for the match to survive.<sup>34</sup>

In case the productivity draw turns out to be too low to maintain a positive firm surplus, the employer will approach the worker and propose to renegotiate. If the worker agrees, the layoff can be avoided and both parties are better off than by a dismissal. In case the worker does not agree, a layoff is inevitable. Importantly, the firm can only credibly reveal the worker’s productivity state if it is incentive-compatible to do so, i.e. if the firm only proposes when a layoff would occur otherwise. However, we find that the firm has an incentive to trigger a renegotiation even if the match would continue anyway. By making the worker agree on a lower wage level, the firm can increase its profit by “stealing” part of the worker’s surplus. Whether or not the worker agrees if the firm proposes to renegotiate depends the worker’s expectations about the true productivity state. For this reason, the presence of asymmetric information impedes efficient wage renegotiation, and bilaterally inefficient layoffs may occur in equilibrium despite the fact that renegotiation has not been ruled out exogenously.

### 7.5.1 The renegotiation game

Below we analyze the Perfect Bayesian Equilibria (PBE) of the game sketched above, that we will refer to as the “renegotiation game”. Since productivity realizations are independent across time, the model with asymmetric information can be interpreted as a multi-stage game where the renegotiation game is played in each period of the worker’s career. We should, however, keep in mind that the expected outcome of future renegotiations affects today’s behavior through the payoffs associated with each of the players’ actions. Figure 11 shows the tree of the renegotiation game of worker type  $i \in \{y, s, o\}$ . To make the game meaningful, we assume that the wage contract the parties have agreed on at the search stage is such that low productive workers would be laid off, i.e.  $\tilde{J}_i(w_i|Y_i = y^L) < 0$ . This is equivalent to  $w_i > \bar{w}_i$ , where  $\bar{w}_i$  is the wage level for which the firm is indifferent between firing and retaining a low productivity worker, implicitly defined by  $\tilde{J}_i(\bar{w}_i|Y_i = y^L) = 0$ . If  $w_i \leq \bar{w}_i$  it is common knowledge that the match

<sup>33</sup>Notably, in the “shirking” literature (e.g. Shapiro and Stiglitz, 1984) it is the worker who has private knowledge of her productivity because the firm cannot observe working effort. This would add a second source of information friction to the model, which we neglect at this stage. Instead we demonstrate that inefficient layoffs can arise *even if* workers are acting honestly.

<sup>34</sup>A more general setup where the firm also chooses a wage level when proposing renegotiation leads to the same conclusions and is covered in Kerndler (2015).

will continue anyway, which makes the game redundant.

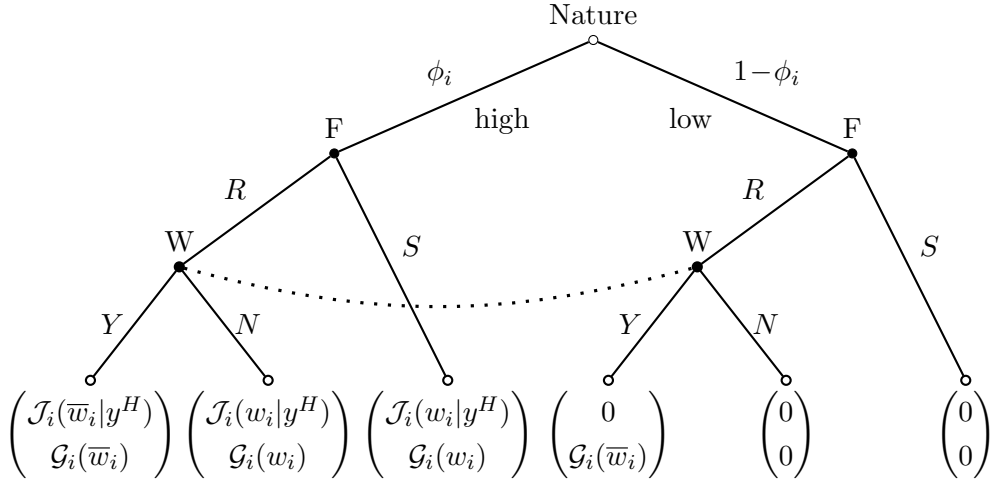


Figure 11: The renegotiation game for worker type  $i \in \{y, s, o\}$ .

To distinguish the payoffs of the game from the surplus functions defined in the main text of the paper,  $\mathcal{G}_i(w)$  denotes the worker payoff and  $\mathcal{J}_i(w|y) := \mathcal{J}_i(w|Y_i = y)$  denotes the firm payoff of a type  $i$  worker with current wage  $w$  and productivity  $y$ . To be precise, these are the *excess* payoffs that can be earned on top of the players' outside options.<sup>35</sup> The game works as follows: After observing the current productivity draw  $Y_i \in \{y^L, y^H\}$ , the firm may propose renegotiation (action  $R$ ) or stick to the contract (action  $S$ ). If productivity is low (right branch of the tree) and the firm chooses  $S$ , the match with the worker is dissolved to avoid a loss. The firm as well as the worker earn an excess payoff of zero in this case. Alternatively, if the firm proposes renegotiation  $R$ , the worker can reach a positive payoff by agreeing (action  $Y$ ). By our convention that renegotiation takes place to the wage that just suffices to sustain the match, firm payoff is also zero in this case,  $\mathcal{J}_i(\bar{w}_i|y^L) = 0$ .<sup>36</sup>

Whether or not the worker agrees to renegotiate depends on her perception about the true productivity state. In case of low productivity,  $Y_i = y^L$ , she always gains from renegotiation because  $\mathcal{G}_i(\bar{w}_i) > 0$ . Whereas if  $Y_i = y^H$ , renegotiating is harmful since  $\bar{w}_i < w_i$  implies  $\mathcal{G}_i(\bar{w}_i) < \mathcal{G}_i(w_i)$ . The worker would thus want the firm to reveal the current productivity state. This would require the firm playing  $R$  only if  $Y_i = y^L$ . However, it is easy to see from Figure 11 that for the firm  $R$  weakly dominates  $S$  irrespective of the productivity draw, such that a PBE equilibrium with truthful revelation cannot exist.

**Proposition 7.3.** *A firm strategy that stipulates  $S$  in case  $Y_i = y^H$  and assigns a positive probability to  $R$  in case  $Y_i = y^L$  cannot be part of a PBE.*

It is evident from Figure 11 that whenever there is a positive probability that a high productivity worker agrees on a lower wage, the firm must propose. A rational firm that observes

<sup>35</sup>For prime-age workers these payoffs also depend on the expected outcome of wage renegotiations at the senior stage. We can neglect this for the moment because future behavior is independent of today's actions.

<sup>36</sup>This convention is just to simplify matters and not crucial in any way.

a high productivity draw would therefore only stick to the contract if the worker refuses renegotiation with certainty. But this can only occur in a PBE if the firm also does not propose to low-productivity workers. Otherwise, the worker learns that she is low productive and would agree. This, however, would induce the firm to change its strategy and also propose to high productive workers.

Hence in any PBE, the firm either never proposes renegotiation, or it proposes renegotiation also to high productivity workers with positive probability. This gives rise to a very strong result, namely that there exists at most one PBE where layoffs can *always* be avoided by renegotiation. This is a PBE in pure strategies, where the firm always proposes and the worker always agrees. Moreover, perfect Bayesian equilibria in which renegotiation is successful in some cases exist if and only if the payoffs satisfy  $\mathcal{G}_i(\bar{w}_i) \geq \phi_i \mathcal{G}_i(w_i)$ .

**Theorem 7.3.** *A PBE with renegotiation exists if and only if  $\mathcal{G}_i(\bar{w}_i) \geq \phi_i \mathcal{G}_i(w_i)$ . The only PBE where layoffs can be completely avoided is given by  $\{(R, R), Y\}$ . It yields ex-ante expected firm payoff  $\mathbb{E}\mathcal{J}_i = \phi_i \mathcal{J}_i(\bar{w}_i|y^H)$  and worker payoff  $\mathbb{E}\mathcal{G}_i = \mathcal{G}_i(\bar{w}_i)$ .*

In case  $\mathcal{G}_i(\bar{w}_i) \geq \phi_i \mathcal{G}_i(w_i)$ , the worker agrees to renegotiate even if she cannot extract any information on her productivity state from the firm's strategy. If the firm proposes renegotiation irrespective of the particular draw, the worker will therefore always agree. In this scenario, layoffs can be averted completely. However, the game also has PBE in which the worker refuses whenever the firm proposes to renegotiate. Such an equilibrium exists independent of the structure of payoffs.

**Theorem 7.4.** *A PBE without renegotiation exists. Any such equilibrium yields ex-ante expected firm payoff  $\mathbb{E}\mathcal{J}_i = \phi \mathcal{J}_i(w_i|y^H)$  and worker payoff  $\mathbb{E}\mathcal{G}_i = \phi \mathcal{G}_i(w_i)$ . Low productive workers are always laid off.*

Taken together, the last two theorems imply that if the worker payoffs satisfy  $\phi_i \mathcal{G}_i(w_i) > \mathcal{G}_i(\bar{w}_i)$ , in any PBE of the game renegotiation fails. In the opposite case,  $\mathcal{G}_i(\bar{w}_i) \geq \phi_i \mathcal{G}_i(w_i)$ , we can show that an initial contract with wage  $w_i = \bar{w}_i$  would for both parties be weakly superior to the outcome of *any* PBE.

**Theorem 7.5.** *Let  $\mathcal{G}_i(\bar{w}_i) \geq \phi_i \mathcal{G}_i(w_i)$ . For any PBE of the game, an initial contract that specifies  $w_i = \bar{w}_i$  gives rise to (weakly) higher expected payoffs for both parties and avoids any need to renegotiate.*

A PBE where renegotiation always occurs is clearly equivalent to writing a contract with  $w_i = \bar{w}_i$  in the first place. On the other extreme, a PBE where renegotiation is never successful has a lower expected firm payoff than the alternative contract  $w_i = \bar{w}_i$ . At the same time, expected worker payoff is at least weakly lower under the above payoff restriction. This observation also holds for all "intermediate" PBE where renegotiation occurs with some probability between 0 and 1.

### 7.5.2 Equivalence of directed search equilibria

The above analysis verifies that in solving the directed search problem of the generalized model with renegotiation under asymmetric information it is sufficient to focus on wage contracts that are “renegotiation-proof”. This means that in each period of the employment relation there is either no need to renegotiate ( $w_i \leq \bar{w}_i$ ), or the worker always refuses to renegotiate ( $\phi_i \mathcal{G}_i(w_i) > \mathcal{G}_i(\bar{w}_i)$ ) and is laid off in case of low productivity since  $w_i > \bar{w}_i$ . In any other case, having an initial contract that pays  $\bar{w}_i$  in the current period is at least weakly better for both parties, and already a tiny renegotiation cost makes them strictly better off.

Concerning old unemployed searchers, a contract ( $w_o$ ) is renegotiation-proof whenever  $w_o \in \Omega_o = \{w_o \in [b, \bar{w}_o] : w_o \leq \bar{w}_o \vee \phi_o \mathcal{G}_o(w_o) > \mathcal{G}_o(\bar{w}_o)\}$ , where  $\bar{w}_o$  is the highest feasible wage,  $\mathcal{J}_o(\bar{w}_o|y^H) = 0$ . Moreover, we can see that for old workers, the payoffs of the game are just equal to the surplus functions given in (1)–(2),  $\mathcal{J}_o(w_o|Y) = \tilde{J}_o(w_o|Y)$  and  $\mathcal{G}_o(w_o) = \tilde{G}_o(w_o)$ . According to Theorem 7.4, the expected payoffs of the game are then

$$\begin{aligned} \mathbb{E}\mathcal{J}_o(w_o) &= \begin{cases} \bar{y}_o - w_o & \text{if } b \leq w_o \leq y^L \\ \phi_o(y^H - w_o) & \text{if } y^L < w_o \leq y^H \wedge \phi_o \tilde{G}_o(w_o) > \tilde{G}_o(y^L) \end{cases}, \\ \mathbb{E}\mathcal{G}_o(w_o) &= \begin{cases} u(w_o) - u(b) & \text{if } b \leq w_o \leq y^L \\ \phi_o(u(w_o) - u(b)) & \text{if } y^L < w_o \leq y^H \wedge \phi_o \tilde{G}_o(w_o) > \tilde{G}_o(y^L) \end{cases}, \end{aligned}$$

irrespective of the particular PBE that emerges. Apart from the additional constraint  $\phi_o \tilde{G}_o(w_o) > \tilde{G}_o(y^L)$  the expected payoffs of the renegotiation game coincide with the expected surplus functions (3)–(4), which were derived under the assumption that renegotiation is not possible at all. It is therefore not surprising that problem (O) and the more general problem where the payoffs are determined by the renegotiation game,

$$\max_{(w_o, \theta_o)} p(\theta_o) \mathbb{E}\mathcal{G}_o(w_o) \quad \text{s.t.} \quad q(\theta_o) \mathbb{E}\mathcal{J}_o(w_o) = c, \quad w_o \in \Omega_o, \quad (\text{O}')$$

have identical solutions. Since the proof is short and insightful, we state it at this place.

**Theorem 7.6.** *The solutions of (O) and (O') coincide.*

*Proof.* Because objective function and constraint of (O) and (O') coincide for  $w_o \in \Omega_o$ , the only property that needs to be verified is that any contract  $w_o^*$  that solves (O) is a renegotiation-proof contract of the game, i.e.  $w_o^* \in \Omega_o$ . This is trivial if  $w_o^* \in [b, y^L]$ . If  $w_o^* \in (y^L, y^H]$ , suppose that the contract were not renegotiation-proof,  $\phi_o \tilde{G}_o(w_o^*) \leq \tilde{G}_o(y^L)$ . In this situation the alternative contract  $w_o = y^L$  gives rise to higher expected surplus, because  $\mathbb{E}J_o(w_o^*) < \mathbb{E}J_o(y^L)$  by monotonicity and  $\mathbb{E}G_o(w_o^*) = \phi_o \tilde{G}_o(w_o^*) \leq \tilde{G}_o(y^L) = \mathbb{E}G_o(y^L)$  by assumption. As a result,  $(y^L, \theta_o(y^L))$  attains a *strictly* higher value in (O) compared to  $(w_o^*, \theta_o(w_o^*))$ , because the free-entry curve  $\theta_o(\cdot)$  is strictly decreasing according to Lemma 4.1. This is a contradiction to  $w_o^*$  being optimal. Hence any solution of (O) must be renegotiation-proof.  $\square$

Similarly, we can express the directed search problem for the prime-age job-seekers in the

generalized model with asymmetric information,<sup>37</sup>

$$\max_{(w_m, w_s, \theta_m)} p(\theta_m) \mathbb{E} \mathcal{G}_m(w_m, w_s) \quad \text{s.t.} \quad q(\theta_m) \mathbb{E} \mathcal{J}_m(w_m, w_s) = c, \quad (w_m, w_s) \in \Omega_m. \quad (\text{P}')$$

Slightly abusing notation,  $\Omega_m = \{w_s \in [b, \bar{w}_s], w_m \in [\underline{w}_m(w_s), \bar{w}_m(w_s)] : w_i \leq \bar{w}_i \vee \phi_i \mathcal{G}_i(w_i) > \mathcal{G}_i(\bar{w}_i), i = y, s\}$  is the set of contracts that are renegotiation-proof in both periods. Again it is easy to check that for any  $(w_m, w_s) \in \Omega_m$  in each period the expected payoffs of the game coincide with the expected surplus functions of the original model formulation. Analogous to Theorem 7.6 we can prove that precluding renegotiation right from the beginning does not affect the model solution.

**Theorem 7.7.** *The solutions of (P) and (P') coincide.*

If the optimal contract of (P) denoted by  $(w_m^*, w_s^*)$  in the context of the game leads to renegotiation in period 2, then the alternative contract  $(w_m^*, y^L)$  attains a strictly higher value of the objective function. Similarly, if the contract is not renegotiation-proof in the first period, then  $(\bar{w}_m(w_s^*), w_s^*)$  is strictly preferred *ex-ante*, where  $\bar{w}_m(w_s^*) = y^L + \delta \mathbb{E} J_s(w_s^*)$ .

As this section demonstrates, allowing for renegotiation under asymmetric information gives rise to exactly the same labor market outcomes as precluding renegotiation right from the beginning. This particularly implies that whenever the optimal contract characterized in the main text of this paper leads to layoffs of some types of workers, this is unchanged if renegotiation is possible but impeded by asymmetric information. Therefore, abstracting from renegotiation altogether does not limit the validity of our model predictions in any crucial way.

## 8 Conclusion

In this paper, we have analyzed a simple two-period model of the labor market, where the equilibrium is determined by directed search of risk-averse workers. The wage space is constrained by market frictions that can, for instance, be rationalized by information asymmetries. We find that *ex-ante* optimal wage contracts may turn out to exhibit too high wages *ex-post*, such that a layoff is necessary. These layoffs are bilaterally inefficient and could be prevented by the job-seekers demanding a lower wage in the first place.

Our first observation is that the detrimental effects of contracting frictions hit old workers much harder than their younger counterparts. The reason is that a forward looking firm is willing to retain unproductive prime-age workers even if it makes a loss on them, provided that expected future surplus sufficiently high. By contrast, there is no similar rationale for keeping an overpaid old worker who is close to retirement. Our second observation is that for a large part of the wage space, prime-age job-seekers effectively decide between two types of contracts. A safe contract that implies no layoff risk but unbalanced consumption across periods, and a contract that comes with a positive layoff risk when old but allows perfect intertemporal consumption

<sup>37</sup>A formal proof that also with multi-period contracts it is sufficient to focus on the set of renegotiation-proof contracts is omitted here and can be found in the accompanying note, Kerndler (2015).



smoothing in case of retention. We show that individuals prefer the latter, risky contract if they expect a sufficiently high productivity during old-age. Inefficient layoffs may therefore be particularly an issue of white-collar workers, who rely on cognitive skills that depreciate slowly.

We also discuss how public policy affects the incidence of inefficient layoffs. High outside options of older workers, such as generous unemployment provisions or attractive early retirement possibilities, decrease the value of a match when old and make a layoff less costly. According to our results, this may increase the employment risk inherent in long-term wage contracts. Recent reforms of social security systems might therefore as a by-product have increased efficiency of the labor market of older workers. By contrast, in those countries where labor force participation of older workers is still poor, disincentivizing early retirement might also reduce layoff rates of old and long-tenured workers. Our results also predict a higher layoff incidence if the productivity of older workers is generally low, because this decreases the wage level that can be demanded without being at the risk of a layoff. Therefore, public policy should also aim at keeping older individuals productive, for instance by fostering investment in age-friendly working conditions, employee health programs, and continued training for older workers. Similar effects on layoffs can be achieved by employment protection legislation and wage subsidies for older workers, especially if these are targeted to low income earners.

Our results remain valid for risk neutral workers if we at the same time allow them to search on the job. While the opportunity to switch to an alternative job *ceteris paribus* increases expected worker surplus, expected firm surplus decreases in light of a possible early quit of the worker, thereby reducing vacancy-posting in the first place. The total effect on expected worker surplus turns out to be negative, making workers worse-off than without on-the-job search. While a promise to stay with the employer is not time-consistent, workers can partly bind themselves to their employer by contracting a high period 2 wage, which reduces the probability to get into an even better paid job. The same intuition allows to establish inefficient layoffs in an economy where individuals can save and lend in the capital market.

Due to the simplistic setup that we have chosen to obtain meaningful analytical results, there are many possible directions for future research. Most importantly, the quantitative impact of the highlighted effects remains of particular interest. Since bilaterally inefficient separations are practically impossible to identify in the data, only a calibrated or structurally estimated model can shed light on the quantitative importance of inefficient layoffs. Since our analytical results are in line with a number of empirical observations, the model developed in this paper may indeed serve as a useful starting point for such an exercise. However, a more realistic life-cycle modeling seems necessary in this regard. This should include individual savings as an alternative way to smooth consumption, on-the-job search to generate realistic wage profiles and labor turnover, as well as a richer productivity process that can account for persistence and increasing uncertainty as workers age.

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## A Further tables

		Relative displacement rate	Relative re-employment rate	% of displaced workers out of labor force within one year of displacement	
country		ratio 55-64 years to 35-44 years		35-44 years	55-64 years
Self-defined disp.	Australia	1.27	0.65	53.2	74.1
	Canada	0.97	0.78	34.5	57.5
	France	1.83	0.26	22.4	78.9
	Japan	1.53	0.68	16.6	35.7
	Korea	1.30	0.50	51.3	68.1
	New Zealand	1.12	0.94	–	–
	Russia	1.63	0.84	52.7	89.4
Firm-identified disp.	Denmark	0.94	0.66	–	–
	Finland	0.96	0.77	–	–
	Germany	1.23	0.36	–	–
	Portugal	1.09	0.52	–	–
	Sweden	0.66	0.87	–	–
	United Kingdom	1.15	0.69	–	–
	United States	0.93	0.88	22.5	35.0

Table A.1: Selected characteristics from OECD (2013)

## B Mathematical appendix

### B.1 Solution candidates of the job-search problem of prime-aged workers

In analogy to Definition 4.1 it is useful to introduce a function  $\Psi_g$  that helps to characterize the solution candidates of problem (P).

**Definition B.1.** *Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $\phi_s \in [0, 1]$ . Define*

$$\Psi_g(w; \phi_s) := \frac{\varepsilon(\theta_g(w; \phi_s))}{1 - \varepsilon(\theta_g(w; \phi_s))} \mathbb{E}J_m(w, g(w); \phi_s)u'(w) - \mathbb{E}G_m(w, g(w); \phi_s)$$

where  $\theta_g(w; \phi_s)$  is the free entry curve, implicitly defined by

$$q(\theta_g(w; \phi_s))\mathbb{E}J_m(w, g(w); \phi_s) = c, \tag{B.1}$$

and the expected surplus functions  $\mathbb{E}G_m$  and  $\mathbb{E}J_m$  are given in Table B.1.

Where not necessary, we omit dependence on  $\phi_s$  and just write  $\Psi_g(w)$ . Some important properties of  $\Psi_g$  are summarized in Lemma B.3. The function  $g$  involved in the definition

captures the intertemporal wage profile and may be an arbitrary real-valued function. It is convenient to express the senior wage as a function of the period 1 wage because the optimal relation between these wage levels is easy to obtain from the first order conditions, while the actual levels can only be determined numerically.

The set of solution candidates for problem (P) is obtained by solving the problem separately on each of the four regions. It turns out that these subproblems each have either no or a unique solution. For the analysis, it is convenient to introduce symbols for four points of the wage space, cf. Figure 5.

**Definition B.2.** Denote the period 1 wage in the upper right corners of regions (ss) and (rs) as

$$\omega_m := y^L + \delta\phi_s(y^H - y^L) \quad \text{and} \quad \Omega_m := y^H + \delta\phi_s(y^H - y^L) - c/\phi_m,$$

respectively. Furthermore, we define

$$\omega := \frac{y^L + \delta\phi_s y^H}{1 + \delta\phi_s} \quad \text{and} \quad \Omega := y^H - \frac{c}{\phi_m(1 + \delta\phi_s)}$$

which corresponds to the wage at the intersection of the 45 degree line with the upper boundaries of region (sr) and (rr), respectively.

Constrained to region (ss), the directed search problem (P) has a unique solution. It is either an interior point with a constant wage profile, or located at the boundary formed by  $w_s = y^L$  and  $y^L \leq w_m \leq \omega_m$ .

**Proposition B.1.** Let  $g = \min\{\text{id}, y^L\}$ . Then the subproblem of region (ss) has a unique solution which belongs to one of three types:

- If  $\Psi_g(y^L) < 0$ , it is an interior solution with a constant wage profile,  $w_m = w_s \in (b, y^L)$ . It is characterized by  $\Psi_g(w_m) = 0$ .
- If  $\Psi_g(y^L) \geq 0$  and  $\Psi_g(\omega_m) < 0$ , it is a border solution with  $w_s = y^L$  and  $w_m \in [y^L, \omega_m)$ . It is characterized by  $\Psi_g(w_m) = 0$ .
- If  $\Psi_g(\omega_m) \geq 0$ , the corner point  $(w_m, w_s) = (\omega_m, y^L)$  is optimal.

Only points at the border  $w_s = y^L$  qualify as the optimal solution of the subproblem of region (rs). However, a solution does not necessarily exist because the constraint set is not closed and therefore need not contain a maximizing point. If a maximizer exists, it is unique.

**Proposition B.2.** A solution for the subproblem of region (rs) exists if and only if  $\Psi_{y^L}^+(\omega_m) > 0$ . In this case it is uniquely determined by  $\Psi_{y^L}(w_s) = 0$  and a boundary solution with  $w_s = y^L$  and  $w_m \in (\omega_m, \Omega_m)$ .

The situation is similar in the remaining two regions (sr) and (rr). Provided existence, each of the two subproblem has a unique solution. Because of the consumption smoothing motive, the optimal wage profiles are flat.

**Proposition B.3.** *A solution to the subproblem of region (sr) exists if and only if  $\Psi_{\text{id}}^+(y^L) > 0$ . In this case it is unique and belongs to one of two types:*

- *If  $\Psi_{\text{id}}(\omega) < 0$ , it is an interior solution with a constant wage profile,  $w_m = w_s \in (y^L, \omega)$ . It is characterized by  $\Psi_{\text{id}}(w_m) = 0$ .*
- *If  $\Psi_{\text{id}}(\omega) \geq 0$ , the boundary point  $(w_m, w_s) = (\omega, \omega)$  is optimal.*

**Proposition B.4.** *The subproblem of region (rr) has a solution if and only if  $\Psi_{\text{id}}^+(\omega) > 0$ . In this case it is uniquely determined  $\Psi_{\text{id}}(w_m) = 0$  and features a constant wage profile  $w_m = w_s \in (\omega, \Omega)$ .*

It is not obvious that the global optimum to problem (P) is indeed one of the (potentially) four solution candidates characterized by Propositions B.1–B.4. Attention is required if the solution candidate of a certain region does not exist because the respective subproblem does not have a solution. However, it can be verified that in this case any contract in this region is dominated by a solution candidate of some other region. Since the candidate in region (ss) always exists, the existence of an optimal solution is granted.

**Proposition B.5.** *An optimal solution to problem (P) exists and is among the four contracts characterized by Propositions B.1–B.4.*

## B.2 Additional lemmas

The function  $\Phi$  introduced in Definition 4.1 is crucial in the analysis of the maximization problem for the old agents. The following lemma summarizes useful properties.

**Lemma B.1.** *Let  $\Phi$  be given as in Definition 4.1 and Assumption 4.2 be satisfied.*

- (i) *The domain of  $\Phi(w; \phi)$  is  $\{(w, \phi) : \phi \in (0, 1) \wedge w \in [b, \bar{w}_o(\phi)]\}$  where  $\bar{w}_o(\phi) = y^H - c/\phi$ .*
- (ii) *For any  $\phi_o \in (0, 1)$ , the function  $w \mapsto \Phi(w; \phi_o)$  is strictly decreasing and continuous on the intervals  $[b, y^L]$  and  $(y^L, \bar{w}_o(\phi_o)]$ . At  $w = y^L$ , there is an upwards jump, i.e.  $\Phi(y^L; \phi_o) < \Phi^+(y^L; \phi_o)$ .*
- (iii) *The function  $\phi \mapsto \Phi(w_o; \phi)$  is continuous and strictly increasing for any  $w_o \in [b, y^L]$ .*

*Proof.* (i) Let  $\phi \in (0, 1)$ . By Lemma 4.1, the free entry curve  $\theta_o(w; \phi)$  is defined on the set  $\Omega_\phi = \{w : \mathbb{E}J_o(w; \phi) \leq c\}$ . Since expected firm surplus is zero for  $w_o < b$  and otherwise continuous and decreasing in the wage,  $\Omega_\phi$  is an interval of the form  $[b, \bar{w}_\phi]$  where  $\bar{w}_\phi := \max \Omega_\phi$ . By Assumption 4.2,  $\bar{w}_\phi > y^L$  and therefore simple algebra reveals  $\bar{w}_\phi = y^H - c/\phi$ . (ii) Again, let  $\phi_o \in (0, 1)$ . By Lemma 4.1 and (i), the free entry curve  $\theta_o(w)$  is continuous and decreasing in  $w$  on its entire domain  $[b, \bar{w}_o(\phi_o)]$ . Furthermore,  $\mathbb{E}J_o(w)$  is strictly decreasing and continuous for  $w \leq y^H$ .  $\mathbb{E}G_o(w)$  features a jump at  $w = y^L$ , but is continuous and strictly increasing on the two subintervals  $w \in [b, y^L]$  and  $w \in (y^L, y^H]$ . Together with  $\varepsilon' > 0$  and  $u'' < 0$ , the monotonicity of  $\Phi$  on these intervals can be directly seen from its definition. At  $w = y^L$ ,  $\mathbb{E}G_o(w)$  jumps downwards ( $\phi_o < 1$ ). Since all remaining terms are continuous,  $\Phi$  exhibits an upwards

jump at  $w = y^L$ . (iii) Let  $w_o \in [b, y^L]$ . In this case  $\mathbb{E}G_o$  does not depend on  $\phi_o$  whereas  $\frac{\partial \mathbb{E}J_o(w_o; \phi_o)}{\partial \phi_o} = y^H - y^L > 0$ . Furthermore, we find  $\frac{\partial \theta_o(w_o; \phi_o)}{\partial \phi_o} = -\frac{q(\theta_o)(y^H - y^L)}{q'(\theta_o)(\bar{y}_o - w_o)} > 0$  by implicit differentiation. Since  $\varepsilon' > 0$ , this implies  $\partial \Phi(w_o; \phi)/\partial \phi > 0$ .  $\square$

The function  $\Psi_g$  introduced in Definition B.1 is crucial in the analysis of the maximization problem for the prime-age agents. First we show how the choice of  $g$  affects the shape of the free-entry curve.

**Lemma B.2.** *Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be non-decreasing and  $\mathbb{E}J_m(w_m, w_s)$  be as specified in Table B.1. Then the function  $\theta_g$  implicitly defined by*

$$q(\theta_g(w))\mathbb{E}J_m(w, g(w)) = c \quad (\text{B.2})$$

*is strictly decreasing in  $w$  on its domain  $\Theta_g \subseteq (-\infty, \bar{w}_g]$  where  $\mathbb{E}J_m(\bar{w}_g, g(\bar{w}_g)) = c$ . If  $g$  is continuous,  $\theta_g$  is also continuous.*

*Proof.* By Assumption 4.1,  $\mathbb{E}J_m(w, g(w)) \geq c$  must be satisfied for  $\theta_g(w)$  to be well-defined. For a non-decreasing function  $g$ , the map  $w \rightarrow \mathbb{E}J_m^*(w, g(w))$  is strictly decreasing. Therefore, a unique  $\bar{w}_g$  exists such that  $\mathbb{E}J_m^*(w, g(w)) \geq c$  if and only if  $w \leq \bar{w}_g$ . Since  $\mathbb{E}J_m^* \geq \mathbb{E}J_m$ , the function  $\theta_g(w)$  is well-defined only if  $w \leq \bar{w}_g$ . Furthermore, because  $\mathbb{E}J_m^*(w, g(w))$  is strictly decreasing,  $q(\theta(w))$  must be strictly increasing in  $w$ . As  $q' < 0$ , this is equivalent to  $\theta_g$  being strictly decreasing on its domain. If  $g$  is continuous, all functions involved in (B.2) are continuous and therefore also  $\theta_g$  itself.  $\square$

Based on the above, the following lemma derive some important properties of  $\Psi_g$ .

**Lemma B.3.** *Let  $\Psi_g$  be given as in Definition B.1. The following properties hold:*

- (i) *For any continuous and non-decreasing  $g : \mathbb{R} \rightarrow \mathbb{R}$ , the function  $w \mapsto \Psi_g(w; \phi_s)$  is continuous and strictly decreasing when constrained to either of the four regions.*
- (ii) *The function  $\phi \mapsto \Psi_{\text{id}}(w; \phi)$  is continuous and strictly decreasing for  $(w, w) \in (ss)$ .*
- (iii) *The function  $\phi \mapsto \Psi_{y^L}(\omega_m(\phi); \phi)$  is continuous and strictly decreasing, where  $\omega_m(\phi) = y^L + \delta\phi(y^H - y^L)$*
- (iv) *The function  $\phi \mapsto \Psi_{\text{id}}(\omega(\phi); \phi)$  is continuous and strictly decreasing, where  $\omega(\phi) = \frac{y^L + \delta\phi y^H}{1 + \delta\phi}$ .*

*Proof.* Since  $g$  is continuous,  $w \mapsto \mathbb{E}J_m(w, g(w))$  and  $w \mapsto \mathbb{E}G_m(w, g(w))$  are continuous functions on each of the four regions. While the first function is strictly decreasing, the latter is strictly increasing. Furthermore, Lemma B.2 states that  $\theta_g$  is continuous and strictly decreasing whereas  $\varepsilon' > 0$  and  $u'' < 0$  by Assumption 4.1. Putting things together verifies (i). Concerning (ii), notice that on region (ss),  $\partial \mathbb{E}G_m/\partial \phi_s = 0$  and  $\partial \mathbb{E}J_m/\partial \phi_s = \delta(y^H - y^L)$ . Furthermore, implicit differentiation of (B.1) reveals  $\partial \theta_{\text{id}}/\partial \phi_s = -\frac{q(\theta_{\text{id}})\partial \mathbb{E}J_m/\partial \phi_s}{q'(\theta_{\text{id}})\mathbb{E}J_m} > 0$ . As a result,  $\partial \Psi_{\text{id}}/\partial \phi_s > 0$ . For (iii), notice that  $(\omega_m(\phi_s), y^L)$  is the upper-right corner of region (ss) for any



given  $\phi_s \in (0, 1)$ . It is therefore easily seen that  $\mathbb{E}J_m(\omega_m(\phi), y^L) = \phi_m(y^H - y^L)$  is independent from  $\phi$ , while  $\mathbb{E}G_m(\omega_m(\phi), y^L) = u(\omega_m(\phi)) - u(b) + \delta(u(y^L) - u(b) - V_o)$  is strictly increasing in  $\phi$  as  $\omega'_m(\phi) > 0$ . (iv) is shown in a similar way.  $\square$

### B.3 Proofs omitted in the text

*Proof of Lemma 4.1.* Since  $q'(\theta_o) < 0$  for  $\theta_o \geq 0$ , the free entry condition  $q(\theta_o)\mathbb{E}J_o(w_o) = c$  implicitly defines a function  $\theta_o(w_o) = q^{-1}(c/\mathbb{E}J_o(w_o))$ , which refer to as the ‘free entry curve’. By Assumption 4.1,  $q(\theta) \in [0, 1]$  and thus  $\mathbb{E}J_o(w_o) \geq c$  is necessary for  $\theta_o(w_o)$  to be well-defined.

Since  $\mathbb{E}J_o(w)$  is continuous at  $w = y^L$ , this also holds for  $\theta_o(w)$ . Implicit differentiation on the interior of the two subintervals yields

$$\theta'_o(w_o) = \frac{c \kappa(w_o)}{q'(\theta_o(w_o))\mathbb{E}J_o(w_o)^2}$$

where  $\kappa(w_o) = 1$  if  $w_o < y^L$  and  $\kappa(w_o) = \phi_o$  if  $w_o \in (y^L, y^H)$ . Since  $q' < 0$ , the curve is downwards sloping and becomes flatter at  $w_o = y^L$ . For  $w_o \neq y^L$ , the second derivative can be expressed as

$$\theta''_o(w_o) = -\frac{\kappa(w_o)\theta'_o(w_o)}{\mathbb{E}J_o(w_o)}Q(\theta_o(w_o))$$

where  $Q(\theta) = \frac{q''(\theta)q(\theta)}{q'(\theta)^2} - 2 = \frac{1-\varepsilon(\theta)}{\varepsilon(\theta)} \left[ 1 - \frac{\varepsilon'(\theta)\theta}{\varepsilon(\theta)(1-\varepsilon(\theta))} \right] \geq 0$  by Assumption 4.1. This together with  $\theta'(w_o) < 0$  shows that both parts of the curve are convex. Convexity is preserved by the kink since the curve is decreasing and flattens at  $w_o = y^L$ . Hence the free entry curve as a whole is convex.

The iso-utility curve for a utility level  $v > 0$  is implicitly defined by  $p(\theta_o)\mathbb{E}G_o(w_o) = v$ . Since  $p'(\theta_o) > 0$ , we can express it as  $\theta_o(w_o) = p^{-1}(v/\mathbb{E}G_o(w_o))$ . Because  $\mathbb{E}G_o(w)$  features as downwards jump at  $w = y^L$ , the iso-utility line has to jump upwards. For  $w_o \neq y^L$ , we find

$$\theta'_o(w_o) = -\frac{v \kappa(w_o)u'(w_o)}{p'(\theta_o(w_o))\mathbb{E}G_o(w_o)^2} < 0$$

and

$$\theta''_o(w_o) = \theta'_o(w_o) \frac{u''(w_o)}{u'(w_o)} + \frac{\kappa(w_o)\theta'_o(w_o)u'(w_o)}{\mathbb{E}G_o(w_o)} [2 + P(\theta_o(w_o))]$$

where  $P(\theta) = -\frac{p''(\theta)p(\theta)}{p'(\theta)^2} = \frac{\varepsilon(\theta)}{1-\varepsilon(\theta)} \left[ 1 + \frac{\varepsilon'(\theta)\theta}{\varepsilon(\theta)(1-\varepsilon(\theta))} \right] > 0$ . Therefore, both arcs of an iso-utility curve are strictly decreasing and strictly convex.  $\square$

*Proof of Proposition 4.1.* At the discontinuity point  $w_o = y^L$ , we break problem (O) into two subproblems such that we can work with the usual optimality conditions. First, consider problem (O) subject to  $w_o \in [b, y^L]$ . The according Lagrangian is

$$\mathcal{L} = p(\theta_o)(u(w_o) - u(b)) + \lambda[q(\theta_o)(\bar{y}_o - w_o) - c] + \mu(y^L - w_o),$$

where  $\lambda$  is the multiplier of the free entry condition and  $\mu$  is the multiplier of the constraint  $w_o \leq y^L$ . We do not explicitly include the second inequality  $w_o \geq b$ , but will verify afterwards that the obtained solution satisfies this constraint. The necessary Karush-Kuhn-Tucker conditions involve the first order conditions

$$\frac{\partial \mathcal{L}}{\partial \theta_o} = p'(\theta_o) \mathbb{E}G_o(w_o) + \lambda q'(\theta_o) \mathbb{E}J_o(w_o) = 0, \quad (\text{B.3})$$

$$\frac{\partial \mathcal{L}}{\partial w_o} = p(\theta_o) u'(w_o) - \lambda q(\theta_o) - \mu = 0, \quad (\text{B.4})$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = q(\theta_o) \mathbb{E}J_o(w_o) - c = 0, \quad (\text{B.5})$$

the complementary slackness condition  $\mu(y^L - w_o) = 0$  as well as non-negativity of the multiplier,  $\mu \geq 0$ . Expressing  $\lambda$  from (B.3) and substituting into (B.4) gives

$$\mu = p(\theta_o) \left[ u'(w_o) - \frac{1 - \varepsilon(\theta_o)}{\varepsilon(\theta_o)} \frac{\mathbb{E}G_o(w_o)}{\mathbb{E}J_o(w_o)} \right]. \quad (\text{B.6})$$

Any interior solution,  $w_o < y^L$ , requires that  $\mu = 0$  and hence that the term in square brackets equals zero. Substituting  $\theta_o(w_o)$  from the free entry condition (B.5), this becomes equivalent to the single optimality condition  $\Phi(w_o) = 0$  with the function  $\Phi$  from Definition 4.1. Similarly, the boundary point  $w_o = y^L$  can only be optimal if  $\mu \geq 0$  which, using (B.5) and (B.6), is equivalent to  $\Phi(y^L) \leq 0$ . Now, observe that  $\Phi(b) > 0$  because  $\mathbb{E}G_o(b) = 0$ . By Lemma B.1(ii),  $\Phi$  is continuous and strictly decreasing on  $[b, y^L]$ . According to above, any interior solution is a root of the  $\Phi$  function. If  $\Phi(y^L) < 0$ , there exists a unique  $w_o^s \in (b, y^L)$  satisfying  $\Phi(w_o^s) = 0$ . Hence the unique optimal safe contract is an interior one. If  $\Phi(y^L) \geq 0$ ,  $\Phi$  does not have a root on the considered interval, and the unique optimal safe contract is  $w_o^s = y^L$ .

Repeating the above analysis subject to the constraint  $w_o > y^L$  reveals that also any solution to this subproblem must to satisfy  $\Phi(w_o) = 0$ . By Lemma B.1,  $\Phi$  is continuous and strictly decreasing on  $(y^L, \bar{w}_o]$ , and  $\Phi(\bar{w}_o) < 0$  because  $\varepsilon(\theta(\bar{w}_o)) = 0$  by Assumption 4.1. Hence a root on  $(y^L, \bar{w}_o)$  exists if and only if  $\Phi^+(y^L) > 0$ . In this case it is unique and we denote it by  $w_o^r$ . Otherwise no maximizing point for this subproblem exists.  $\square$

*Proof of Proposition 4.2.* Write the objective function as  $V_o(w) := p(\theta_o(w)) \mathbb{E}G_o(w)$  where  $\theta_o(w)$  is the free entry curve. If the risky choice does not exist, it holds that  $V_o(w) \leq V_o^+(y^L)$  for any  $w \in (y^L, y^H)$  because the latter term is the supremum. Since  $\theta_o(w)$  is continuous at  $w = y^L$ , we have that  $V_o^+(y^L) = p(\theta_o(y^L)) \phi_o(u(y^L) - u(b)) < p(\theta_o(y^L))(u(y^L) - u(b)) = V_o(y^L) \leq V_o(w_o^s)$ . Therefore, if  $w_o^r$  does not exist, any feasible  $w > y^L$  is dominated by  $w_o^s$ . Now, suppose that  $w_o^r$  exists and that  $w_o^s$  in an interior point. As demonstrated in the proof of Proposition 4.1, in this case both satisfy  $\Phi(w_o^i) = 0$ ,  $i \in \{r, s\}$ . Using the definition of  $\Phi$ , the value achieved by such a point can be expressed as

$$V_o(w_o^i) = \frac{c \varepsilon(\theta_o(w_o^i))}{1 - \varepsilon(\theta_o(w_o^i))} \theta_o(w_o^i) u'(w_o^i).$$

According to Lemma 4.1 and Assumption 4.1, the right-hand side is strictly decreasing in  $w_o^i$ . Trivially,  $w_o^s < w_o^r$  and therefore  $V_o(w_o^s) > V_o(w_o^r)$ . Hence, if  $w_o^s$  is interior, it is the optimal solution to problem (O), regardless of whether a second (risky) candidate exists or not. This is the first part of the proposition. For the second, we show that  $w_o^s = y^L$  implies  $\Phi^+(y^L) > 0$ . According to Proposition 4.1,  $w_o^s = y^L$  is equivalent to  $\Phi(y^L) \geq 0$ . By Lemma B.1(ii), this implies  $\Phi^+(y^L) > \Phi(y^L) \geq 0$ .  $\square$

*Proof of Theorem 4.1.* Suppose  $\Phi(y^L; 1) \leq 0$ . According to Proposition 4.1 and Lemma B.1(iii), this implies that the safe choice  $w_o^s$  is interior for any  $\phi \in (0, 1)$ . By Proposition 4.2,  $w_o^s$  is optimal.

Suppose from now on that  $\Phi(y^L; 1) > 0$ . For  $w_o^r$  to be optimal, we require that  $w_o^s$  is at the boundary. Therefore, in a first step, we show that  $\phi \mapsto \Phi(y^L; \phi)$  has a unique root. The smallest  $\phi$  such that  $\theta_o(y^L; \phi)$  is feasible, is  $\phi = c/(y^H - y^L) < 1$  where the inequality stems from Assumption 4.2. By Assumption 4.1,  $\varepsilon(\theta_o(y^L; \underline{\phi})) = 0$  and thus  $\Phi(y^L; \underline{\phi}) < 0$ . Because  $\Phi(y^L; 1) > 0$  and  $\phi \mapsto \Phi(y^L; \phi)$  is continuous and strictly increasing (see Lemma B.1(iii)), there exists a unique  $\tilde{\phi} \in (\underline{\phi}, 1)$  satisfying  $\Phi(y^L; \tilde{\phi}) = 0$ . If  $\phi \geq \tilde{\phi}$ , the safe choice is at the corner  $w_o^s = y^L$  and, by Proposition 4.1, co-exists with the risky choice  $w_o^r > y^L$ .

In a next step we show that a second threshold  $\phi'_o \in (\tilde{\phi}, 1)$  exists above which the latter is optimal. As in the proof of Proposition 4.2, we express the objective function as  $V_o(w; \phi_o) := p(\theta_o(w; \phi_o))\mathbb{E}G_o(w; \phi_o)$  where  $\theta_o(w; \phi_o)$  is the free entry curve. As we have seen there,  $V_o(y^L; \phi_o) = \phi_o V_o^+(y^L; \phi_o)$ . Furthermore,  $w_o^r(\phi_o) > y^L$  for all  $\phi_o \in (\tilde{\phi}, 1]$ . Taking the limit  $\phi_o \rightarrow 1$  reveals  $V_o(y^L; 1) = V_o^+(y^L; 1) < V_o(w_o^r; 1)$  and therefore  $w_o^r$  dominates  $w_o^s = y^L$  in this case. To show existence of a threshold, we define the auxiliary function  $\Delta(\phi) := V(w_o^r(\phi); \phi) - V(y^L; \phi)$  with domain  $\phi \in [\tilde{\phi}, 1)$ . We have just seen that  $\lim_{\phi \rightarrow 1} \Delta(\phi) > 0$ , and we know from Proposition 4.2 that  $\Delta(\tilde{\phi}) < 0$ . Furthermore,  $\Delta$  is differentiable. The derivative of  $\Delta$  can be obtained by the Envelope Theorem and yields  $\phi \Delta'(\phi) = \Delta(\phi)/\varepsilon(\theta_o(w_o^r; \phi)) + V(y^L; \phi)$ . Therefore,  $\Delta'(\phi) > 0$  whenever  $\Delta \geq 0$ . Altogether, this proves that  $\Delta$  has a unique root  $\phi'_o \in (\tilde{\phi}, 1)$ , which satisfies the properties postulated by Theorem 4.1.  $\square$

*Proof of Theorem 4.2.* From the proof of Proposition 4.2,  $\phi'_o$  is implicitly defined by  $\Delta(\phi'_o) = 0$ , the implicit function theorem can be used to calculate comparative static effects. Assuming differentiability, for a general parameter  $a$  we have

$$\frac{d\phi'_o}{da} = -\frac{\partial\Delta/\partial a}{\partial\Delta/\partial\phi}$$

with the derivatives evaluated in  $\phi'_o$ . Since  $\partial\Delta/\partial\phi > 0$  at  $\phi = \phi'_o$ , the effect of a marginal change in  $a$  on  $\phi'_o$  has the opposite sign of  $\partial\Delta/\partial a$ . In calculating the latter expression, the Envelope Theorem can be used. Concerning a change in  $b$  we find

$$\left. \frac{\partial\Delta}{\partial b} \right|_{\Delta=0} = [p(\theta_o(w_o^r)) - \phi'_o p(\theta_o(y^L))] u'(b) > 0$$

region	expected firm surplus $\mathbb{E}J_m(w_m, w_s)$	expected worker surplus $\mathbb{E}G_m(w_m, w_s)$
(ss)	$\bar{y}_m - w_m + \delta(\bar{y}_s - w_s)$	$u(w_m) - u(b) + \delta(u(w_s) - u(b) - V_o)$
(sr)	$\bar{y}_m - w_m + \delta\phi_s(y^H - w_s)$	$u(w_m) - u(b) + \delta[\phi_s(u(w_s) - u(b)) - V_o]$
(rs)	$\phi_m[y^H - w_m + \delta(\bar{y}_s - w_s)]$	$\phi_m\{u(w_m) - u(b) + \delta(u(w_s) - u(b) - V_o)\}$
(rr)	$\phi_m[y^H - w_m + \delta\phi_s(y^H - w_s)]$	$\phi_m\{u(w_m) - u(b) + \delta[\phi_s(u(w_s) - u(b)) - V_o]\}$

Table B.1: Specification of the expected surplus functions on the four regions of the wage space;  $\bar{y}_i := \phi_i y^H + (1 - \phi_i)y^L$ .  $V_o$  is determined by (O).

since  $\theta_o(w)$  is downward sloping,  $p' > 0$ , and  $u' > 0$ . The marginal effect of  $y^L$  is

$$\left. \frac{\partial \Delta}{\partial y^L} \right|_{\Delta=0} = -(1 - \phi'_o)\lambda(y^L)q(\theta_o(y^L)) < 0,$$

where  $\lambda(y^L)$  denotes the Lagrange multiplier corresponding to the pair  $(y^L, \theta_o(y^L))$ . It can be shown that  $\lambda(w_o^r) > \lambda(y^L)$ , which implies

$$\left. \frac{\partial \Delta}{\partial y^H} \right|_{\Delta=0} = \phi'_o [\lambda(w_o^r)q(\theta_o(w_o^r)) - \lambda(y^L)q(\theta_o(y^L))] > 0,$$

$$\left. \frac{\partial \Delta}{\partial c} \right|_{\Delta=0} = \lambda(y^L) - \lambda(w_o^r) < 0.$$

□

*Proof of Lemma 4.2.* First, we show that contracts that imply a certain layoff in period 2 ( $w_s > y^H$ ) cannot be optimal for the worker. This is because for any  $\nu > 0$  the contract  $(w_m, y^H + \nu)$  is dominated by  $(w_m, y^H)$ . Expected firm surplus and hence the job finding probability are the same in both cases. Yet, the latter yields a higher expected worker surplus since the match is maintained in period 2 with a positive probability. Second, we show that, under Assumption 4.3, we can rule out contracts that induce the worker to quit after the first period. To this purpose we show that the contract  $(w_m, \underline{w}_s)$  dominates  $(w_m, \underline{w}_s - \nu)$  for any  $\nu > 0$ . In both cases,  $\mathbb{E}G_s = 0$  and thus  $\mathbb{E}G_m$  is the same. On the firm side,  $\mathbb{E}J_s(w_m, \underline{w}_s) = \bar{y}_s - \underline{w}_s$  while  $\mathbb{E}J_s(w_m, \underline{w}_s - \nu) = 0$ . By Assumption 4.3,  $\bar{y}_s > \underline{w}_s$  and thus the contract  $(w_m, \underline{w}_s)$  yields a higher firm surplus, which gives rise to a higher labor market tightness and a superior value of the objective function. As a result, an optimal contract must satisfy  $\mathbb{E}J_s(w_s) \geq 0$  and  $\mathbb{E}G_s(w_s) \geq 0$ , which rules out two of the six regions. The boundaries of the remaining four regions as well as the specific expressions for  $\mathbb{E}J_m(w_m, w_s)$  and  $\mathbb{E}G_m(w_m, w_s)$  are obtained by substituting (6) into (7) and (8). Assumption 4.3 also guarantees that  $\underline{w}_m(w_s) < y^L + \delta\mathbb{E}J_s(w_s)$  for  $w_s \in [\underline{w}_s, y^H]$ . □

*Proof of Proposition B.1.* The proof is a straightforward application of the KKT conditions to problem (P) subject to the additional constraints  $w_s \leq y^L$  and  $w_m \leq y^L + \delta(\bar{y}_s - w_s)$ . Notice that we ignore the lower bounds and afterwards verify that our solution indeed belongs to region (ss). The necessary conditions involve a set of first order conditions, the two complementary slackness

reg.	expected firm surplus $\mathbb{E}J_m(w_m, w_s)$	expected worker surplus $\mathbb{E}G_m(w_m, w_s)$
(ss)	$\bar{y}_m - w_m + \delta(1 - p(\theta_q(w_s)))(\bar{y}_s - w_s)$	$u(w_m) - u(b) + \delta(u(w_s) - u(b) + V_q(w_s) - V_o)$
(sr)	$\bar{y}_m - w_m + \delta(1 - p(\theta_q(w_s)))\phi_s(y^H - w_s)$	$u(w_m) - u(b) + \delta[\phi_s(u(w_s) - u(b)) + V_q(w_s) - V_o]$
(rs)	$\phi_m[y^H - w_m + \delta(1 - p(\theta_q(w_s)))(\bar{y}_s - w_s)]$	$\phi_m\{u(w_m) - u(b) + \delta(u(w_s) - u(b) + V_q(w_s) - V_o)\}$
(rr)	$\phi_m[y^H - w_m + \delta(1 - p(\theta_q(w_s)))\phi_s(y^H - w_s)]$	$\phi_m\{u(w_m) - u(b) + \delta[\phi_s(u(w_s) - u(b)) + V_q(w_s) - V_o]\}$

Table B.2: Specification of the expected surplus functions on the four regions of the wage space with on-the-job search.  $\theta_q(w_s)$  and  $V_q(w_s)$  are determined by (9).

conditions  $\mu_1(y^L - w_s) = 0$  and  $\mu_2[y^L - w_m + \delta(\bar{y}_s - w_s)] = 0$  as well as non-negativity of the multipliers  $\mu_1 \geq 0$  and  $\mu_2 \geq 0$ . The first order conditions can be reduced to the following three conditions,

$$\mu_1 = p(\theta_m)[u'(w_s) - u'(w_m)], \quad (\text{B.7})$$

$$\mu_2 = p(\theta_m) \left[ u'(w_m) - \frac{1 - \varepsilon(\theta_m)}{\varepsilon(\theta_m)} \frac{\mathbb{E}G_m(w_m, w_s)}{\mathbb{E}J_m(w_m, w_s)} \right], \quad (\text{B.8})$$

$$q(\theta_m)\mathbb{E}J_m(w_m, w_s) = c, \quad (\text{B.9})$$

Whereas (B.8) resembles condition (B.6) of the old individuals, condition (B.7) is novel and determines the wage profile. In an interior optimum,  $\mu_1 = \mu_2 = 0$  and hence Assumption 4.1 implies a flat profile  $w_m = w_s$ . Combining the two remaining conditions (B.8)–(B.9) and substituting  $w_m$  for  $w_s$  reveals that any interior optimum has to satisfy  $\Psi_{\text{id}}(w_m) = 0$  where  $\Psi$  is given in Definition B.1. A contract at the boundary  $w_s = y^L$ , can only be optimal if  $u'(w_s) \geq u'(w_m)$  and thus  $w_m \geq w_s = y^L$ , as can be seen from (B.7) and  $\mu_1 \geq 0$ . The remaining conditions imply that  $\Psi_{y^L}(w_m) = 0$  must hold for a contract with  $(w_m, y^L)$  to be optimal. By contrast, a contract at the upper boundary,  $w_m = y^L + \delta(\bar{y}_s - w_s)$ , can only be optimal if  $w_s = y^L$ . To see this, assume  $w_s < y^L$ . Then  $\mu_1 = 0$  and (B.7) implies  $w_m = w_s$ . Substituting this into  $w_m = y^L + \delta(\bar{y}_s - w_s)$  yields  $w_m = w_s = \frac{y^L + \delta\bar{y}_s}{1 + \delta} \geq y^L$ , which is a contradiction. Therefore along this boundary only the pair  $(\omega_m, y^L)$  needs to be considered, and it is a candidate for the optimal solution only if  $\Psi_{y^L}(\omega_m) \geq 0$ . To show that the subproblem of region (ss) has a unique solution, we define a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  by  $g(w) = w$  for  $w < y^L$  and  $g(w) = y^L$  for  $w = y^L$ . Any solution candidate, apart from the corner point, satisfies  $\Psi_g(w) = 0$ , whereas the corner  $(\omega_m, y^L)$  is a candidate if  $\Psi_g(\omega_m) \geq 0$ . Continuity and monotonicity of  $\Psi_g$  (see Lemma B.3(i)), together with the assumption  $\Psi_g(\underline{w}_s) > 0$  ensure that  $\Psi_g$  either has a unique on  $(\underline{w}_s, \omega_m)$ , or  $\Psi_g(\omega_m) \geq 0$ . This directly implies the three cases stipulated by the proposition. It remains to check that the obtained solution  $(w_m^*, g(w_m^*))$  also satisfies the so far neglected inequalities  $w_m \geq u^{-1}(u(b) - \delta[u(g(w_m)) - u(b) - V_o])$  and  $g(w_m) \geq \underline{w}_s$ . The latter is granted since  $g(w_m^*) \geq w_m^* > \underline{w}_s$  by definition of  $g$  and  $\Psi_g(\underline{w}_s) > 0$ . For the first inequality, notice that the left-hand side is strictly increasing in  $w_m$ , while the right-hand side is non-increasing. It therefore suffices to verify it for  $w_m = \underline{w}_s$ . Since  $\underline{w}_s < y^L$ , we know  $g(\underline{w}_s) = \underline{w}_s$  and, by definition of  $\underline{w}_s$ , the term in square brackets is zero. Hence the condition boils down to  $\underline{w}_s \geq b$ , which is true.  $\square$

*Proof of Proposition B.2.* Straight-forward application of the KKT conditions shows that any solution candidate is of the form  $(w_m, y^L)$  with  $w_m \in (\omega_m, \Omega_m]$  and satisfies  $\Psi_{y^L}(w_m) = 0$ . Since  $\mathbb{E}J_m(\Omega_m, y^L) = c$ , the free entry curve (B.1) reveals  $\theta_{y^L}(\Omega_m) = 0$  and thus  $\Psi_{y^L}(\Omega_m) < 0$ . Therefore, by Lemma B.3(i),  $\Psi_{y^L}$  has a unique root in  $(\omega_m, \Omega_m)$  if and only if  $\Psi_g^+(\omega_m) > 0$ . Assumption 4.3(i) ensures that indeed  $\omega_m < \Omega_m$ .  $\square$

*Proof of Proposition B.3.* The KKT conditions reveal that any optimal solution must satisfy  $w_m = w_s \in (y^L, \omega]$  and is therefore located along the diagonal. An interior solution additionally satisfies  $\Psi_{\text{id}}(w_m) = 0$ , whereas the boundary point  $(\omega, \omega)$  is a candidate if  $\Psi_{\text{id}}(\omega) \geq 0$ . By Lemma B.3(i), the subproblem can have a solution only if  $\Psi_{\text{id}}(y^L)^+ < 0$ . In this case, there is either a unique root on  $(y^L, \omega)$  or  $\Psi_{\text{id}}(\omega) \geq 0$ , giving rise to the two cases outlined in the proposition. Furthermore, the solution also individually rational since  $u(w_m^*) > u(b) - \delta[\phi_s(u(w_m^*) - u(b)) - V_o]$  because  $w_m^* > b$  and the term in square brackets is non-negative by Assumption 4.3(ii).  $\square$

*Proof of Proposition B.4.* Let  $\Omega := y^H - c/[\phi_m(1 + \delta\phi_s)]$ . By the KKT conditions, any optimal solution must specify a flat profile,  $w_m = w_s \in (\omega, \Omega]$  with  $\Psi_{\text{id}}(w_m) = 0$ . Since  $\mathbb{E}J_m(\Omega, \Omega) = c$  implies  $\Psi_{\text{id}}(\Omega) < 0$ ,  $\Psi_{\text{id}}$  has a unique root in  $(\omega, \Omega)$  if and only if  $\Psi_{\text{id}}^+(\omega) > 0$ . Otherwise there is no root in the specified interval and the subproblem does not have a solution. Assumption 4.3(i) ensures that indeed  $\omega < \Omega$ .  $\square$

*Proof of Proposition B.5.* We will demonstrate that whenever the subproblem of one region does not have a solution, any contract in this region is dominated by the existing candidate of some another region. According to Proposition B.1, the candidate in (ss) always exists and we denote it by  $w^{ss}$ . Express the objective function as  $V_g(w) := p(\theta_g(w))\mathbb{E}G_m(w, g(w))$  where  $\theta_g(w)$  is the free entry curve (B.1). First we consider the case that the candidate in (sr) does not exist. Then  $V_{\text{id}}^+(y^L) \geq p(\theta_m(w_m, w_s))\mathbb{E}G_m(w_m, w_s)$  for all  $(w_m, w_s) \in (sr)$  because  $w = (y^L, y^L)$  is the supremum for the subproblem of region (sr). Furthermore, we can easily verify that  $\mathbb{E}G_m(y^L, y^L) - \lim_{w \rightarrow y^L+} \mathbb{E}G_m^+(w, w) = \delta(1 - \phi_s)[u(y^L) - u(b)] > 0$  and thus  $V_{\text{id}}(y^L) > V_{\text{id}}^+(y^L)$ . Combining this with the above reveals  $V_{\text{id}}(y^L) > V_{\text{id}}^+(y^L) \geq p(\theta_m(w_m, w_s))\mathbb{E}G_m(w_m, w_s)$  for all  $(w_m, w_s) \in (sr)$ . Hence any contract in region (sr) is dominated by  $w = (y^L, y^L)$ , which in turn either coincides with  $w^{ss}$  or is dominated itself by  $w^{ss}$ . If the solution candidate of region (rr) does not exist, it can be verified in exactly the same way that the contract  $(\omega, \omega)$  dominates any contract of region (rr). This contract in turn either coincides or is dominated by the best contract of region (sr) provided that the respective subproblem has a solution. Otherwise  $(\omega, \omega)$  is inferior to  $w^{ss}$  as argued above. Finally, if the candidate of region (rs) does not exist, then all contracts in this region are dominated by  $(\omega_m, y^L)$  and therefore by  $w^{ss}$ .  $\square$

*Proof of Proposition 4.3.* The proof is a generalization of the proof of Proposition 4.2. We express the objective function as  $V_g(w) := p(\theta_g(w))\mathbb{E}G_m(w, g(w))$  where  $\theta_g(w)$  is the free entry curve (B.1). In a first step, we prove that if the candidate in (ss) is interior, it dominates the candidate in region (sr) if both co-exist. Denote with  $w^{ss}$  the wage level of the safe and

with  $w^{sr}$  the wage level of the risky candidate. According to Proposition B.3, the latter satisfies  $\Psi_{\text{id}}(w^{sr}) \geq 0$ . Whereas we know from Proposition B.1 that  $\Psi_{\text{id}}(w^{ss}) = 0$  because  $w^{ss}$  is interior. Define the auxiliary function

$$f_g(w) := \frac{c\varepsilon(\theta_g(w))}{1 - \varepsilon(\theta_g(w))} \theta_g(w) u'(w)$$

where  $g : \mathbb{R} \rightarrow \mathbb{R}$ . Substituting the free entry curve (B.1) into the definition of  $\Psi_{\text{id}}$  reveals  $V_{\text{id}}(w^{ss}) = f_{\text{id}}(w^{ss})$  as well as  $V_{\text{id}}(w^{sr}) \leq f_{\text{id}}(w^{sr})$ . By Lemma B.2 and Assumption 4.1,  $f_{\text{id}}$  is strictly decreasing and thus  $V_{\text{id}}(w^{ss}) > V_{\text{id}}(w^{sr})$ . In the same way we can verify that  $w^{ss}$  also dominates the solution candidate of region (rr) if both co-exist. This follows from  $V_{\text{id}}(w^{ss}) = f_{\text{id}}(w^{ss}) > f_{\text{id}}(w^{rr}) = V_{\text{id}}(w^{rr})$ , taking into account Proposition B.4. The same holds for the solution candidate of region (sr). Assuming existence, Proposition B.2 implies that  $V_g(w^{rs}) = f_g(w^{rs})$ , where  $g = \min\{\text{id}, y^L\}$ . Lemma B.2 then reveals  $V_g(w^{ss}) = f_g(w^{ss}) > f_g(w^{rs}) = V_g(w^{rs})$ . Putting these results together, an interior  $w^{ss}$  dominates any other solution candidate. Since by Proposition B.5 the equilibrium must be among the set of solution candidates,  $w^{ss}$  is the unique optimum.

The second part of the statement is to verify that the candidate  $w^{sr}$  exists if  $w^{ss}$  is not interior. Since  $w \rightarrow \mathbb{E}J_m(w, w)$  and  $\theta_{\text{id}}(w)$  are continuous at  $w = y^L$  while  $w \rightarrow \mathbb{E}G_m(w, w)$  jumps downwards, we observe  $\Psi_{\text{id}}(y^L) < \Psi_{\text{id}}^+(y^L)$ . Therefore,  $\Psi_{\text{id}}(y^L) \geq 0$  implies  $\Psi_{\text{id}}^+(y^L) > 0$ . From this, the result follows immediately by Propositions B.1 and B.3.

The last point is to show that an interior  $w^{sr}$  dominates any contract in regions (sr) and (rr). If no solution candidate exists in these regions, this immediately follows from Proposition B.5. Assume the solution candidate in region (rr) exists. Then in the same manner as above,  $V_{\text{id}}(w^{sr}) = f_{\text{id}}(w^{sr}) > f_{\text{id}}(w^{rr}) = V_{\text{id}}(w^{rr})$ . Hence  $w^{rr}$  is dominated by  $w^{sr}$ . Assume the solution candidate in region (rs) exists. Then monotonicity implies  $V_{y^L}(w^{rs}) = f_{y^L}(w^{rs}) < f_{y^L}(\omega_m)$  as well as  $V_{\text{id}}(w^{sr}) = f_{\text{id}}(w^{sr}) \geq f_{\text{id}}(\omega)$ . It thus suffices to show  $f_{y^L}(\omega_m) < f_{\text{id}}(\omega)$ . First, observe that  $\mathbb{E}J_m(\omega_m, y^L) = \mathbb{E}J_m(\omega, \omega)$  and therefore  $\theta_{y^L}^L(\omega_m) = \theta_{\text{id}}(\omega)$ . On the other hand,  $\omega_m > \omega$  and therefore  $u'(\omega_m) < u'(\omega)$ . Considering the definition of  $f_g$  reveals  $f_{y^L}(\omega_m) < f_{\text{id}}(\omega)$ , which completes the proof.  $\square$

*Proof of Theorem 4.3.* By Lemma B.3(ii),  $\Psi_{\text{id}}(y^L; 1) \leq 0$  implies that  $\Psi_{\text{id}}(y^L; \phi_s) < 0$  for any  $\phi_s \in (0, 1)$ . This in turn means that the candidate in region (ss) is interior and therefore optimal by Proposition B.1 and Proposition 4.3.

Proving the statement for  $\Psi_{\text{id}}(y^L; 1) < 0$  is more involved. First, we have to check that  $w^{sr}$  indeed exists for sufficiently high  $\phi_s$ . By Proposition 4.3(i), a sufficient condition for this is that  $w^{ss}$  is at the boundary. If  $\Psi_{\text{id}}(y^L; 0) < 0$ , Lemma B.3(ii) ensures a unique  $\tilde{\phi}_s \in (0, 1)$  such that  $\Psi_{\text{id}}(y^L; \tilde{\phi}_s) = 0$ . If  $\Psi_{\text{id}}(y^L; 0) \geq 0$ , on the other hand,  $w^{ss}$  is at the boundary for all  $\phi \in (0, 1)$ . In this case, we define  $\tilde{\phi}_s := 0$ . Altogether,  $w^{ss}$  is at the boundary for  $\phi_s \geq \tilde{\phi}_s$  and existence of  $w^{sr}$  if granted by Proposition 4.3(i).

Next, we show existence of a threshold  $\phi'_s \in (\tilde{\phi}_s, 1)$  as postulated by the proposition. To this end, we write the objective function as  $V(w_m, w_s; \phi_s) = p(\theta(w_m, w_s; \phi_s)) \mathbb{E}G_m(w_m, w_s; \phi_s)$

where  $\theta(w_m, w_s; \phi_s)$  is the free entry curve. Let us first show that  $w^{sr}$  dominates  $w^{ss} = (w_m^{ss}, y^L)$  for  $\phi_s = 1$ . Therefore, consider  $h(w) = w_m^{ss} - \delta(w - y^L)$  and notice that  $w^{ss} = (h(y^L), y^L)$  as well as  $(h(w), w) \in (sr)$  for all  $w \in (y^L, y^H]$ . For  $\phi_s = 1$ , expected worker surplus and hence the objective function  $V$  is continuous at  $w_s = y^L$ . This implies  $V(w_m^{ss}, y^L; 1) = \lim_{w \rightarrow y^L+} V(h(w), w; 1) < V(w^{sr}, w^{sr}; 1)$  because  $w^{sr} > y^L$  is the unique optimum in region  $(sr)$ .<sup>38</sup> This reveals that  $w^{sr}$  dominates  $w^{ss}$  for  $\phi_s$ . To show existence of a threshold, define the difference in values

$$\Delta(\phi_s) := V(w^{sr}(\phi_s), w^{sr}(\phi_s); \phi_s) - V(w_m^{ss}(\phi_s), y^L; \phi_s). \quad (\text{B.10})$$

Note that also the wage levels depend on  $\phi_s$ .  $\Delta$  is well-defined for  $\phi_s \in [\tilde{\phi}_s, 1]$  and we have just verified that  $\Delta(1) > 0$ . If  $\phi'_s > 0$ , then, by definition,  $w_m^{ss}(\tilde{\phi}) = y^L$  and thus  $\Delta(\tilde{\phi}_s) = V(w^{sr}, w^{sr}; \tilde{\phi}_s) - V(y^L, y^L; \tilde{\phi}_s) \leq f(w_m^{sr}) - f(y^L) < 0$  in the notation of the proof of Proposition 4.3. If  $\tilde{\phi}_s = 0$ , on the other hand, observe that for any  $(w_m, w_s) \in (sr)$  the free entry curve satisfies  $\theta(w_m, w_s; 0) = \theta(w_m, y^L; 0)$ , whereas  $\mathbb{E}G_m(w_m, w_s; 0) < \mathbb{E}G_m(w_m, y^L; 0)$ . This implies that  $\Delta(\tilde{\phi}_s) < 0$  also in this case. By continuity,  $\Delta$  has at least one root in  $(\tilde{\phi}_s, 1)$ . To verify that the root is unique, we demonstrate that  $\Delta'(\phi_s) > 0$  holds at any point satisfying  $\Delta(\phi_s) = 0$ . This restricts  $\Delta$  to the positive domain after reaching zero.

To this end, we need the following Lemma:

**Lemma B.4.** *Let  $\Delta(\phi_s) = 0$ . Then, with the notation from before,  $\mathbb{E}J_m(w_m^{ss}, y^L) \geq \mathbb{E}J_m(w^{sr}, w^{sr})$ .*

*Proof.* We distinguish two cases, depending on the location of  $w^{sr}$ . First, assume that the point is at the boundary,  $w^{sr} = \omega$ . In this case,  $\mathbb{E}J_m(w^{sr}, w^{sr}) = \mathbb{E}J(\omega, \omega) = \mathbb{E}J_m(\omega_m, y^L) \leq \mathbb{E}J_m(w_m^{ss}, y^L)$  since  $\omega_m \geq w_m^{ss}$ . If  $w^{sr}$  is interior, it is characterized by  $\Psi_{\text{id}}(w^{sr}) = 0$ . The safe contract, on the other hand, satisfies  $\Psi_{y^L}(w_m^{ss}) \geq 0$ . Define

$$f(w_m, w_s) := \frac{c\varepsilon(\theta_m(w_m, w_s))}{1 - \varepsilon(\theta_m(w_m, w_s))} \theta_m(w_m, w_s) u'(w_m)$$

and notice that the above conditions are equivalent to  $V(w^{sr}, w^{sr}) = f(w^{sr}, w^{sr})$  and  $V(w_m^{ss}, y^L) \leq f(w_m^{ss}, y^L)$ . From here, we complete the proof by contradiction. Assume that  $\mathbb{E}J_m(w_m^{ss}, y^L) < \mathbb{E}J_m(w^{sr}, w^{sr})$ . The free entry condition then implies  $\theta_m(w_m^{ss}, y^L) < \theta_m(w^{sr}, w^{sr})$ . Furthermore, by using the particular expressions for  $\mathbb{E}J_m$  from Table B.1, we find  $w_m^{ss} > w^{sr}$ . Together with Assumption 4.1, this implies  $f(w_m^{ss}, y^L) < f(w^{sr}, w^{sr})$  and thus  $\Delta \geq f(w^{sr}, w^{sr}) - f(w_m^{ss}, y^L) > 0$  which is a contradiction to  $\Delta = 0$ .  $\square$

Now, to determine the derivative  $\Delta'(\phi_s)|_{\Delta=0}$ , we use the Envelope Theorem. This gives

$$\begin{aligned} \frac{dV(w^{sr}, w^{sr}; \phi_s)}{d\phi_s} &= p(\theta_m^{sr}) \delta(u(w^{sr}) - u(b)) + \lambda^{sr} q(\theta_m^{sr}) \delta(y^H - w^{sr}) \\ &= \frac{1 - \varepsilon(\theta_m^{sr})}{\varepsilon(\theta_m^{sr})} \delta(y^H - y^L) \frac{V(w^{sr}, w^{sr})}{\mathbb{E}J_m(w^{sr}, w^{sr})} + p(\theta_m^{sr}) \delta[u(w^{sr}) - u(b) + u'(w^{sr})(y^L - w^{sr})] \end{aligned}$$

<sup>38</sup>Choosing another function  $h$  to approach  $w^{ss}$ , such as  $h(w) \equiv w_m^{ss}$ , would lead out of region  $(sr)$  if  $w^{ss}$  sits at the corner.



and

$$\frac{dV(w^{ss}, y^L; \phi_s)}{d\phi_s} = \lambda^{ss} q(\theta_m^{ss}) \delta(y^H - y^L) = \frac{1 - \varepsilon(\theta_m^{ss})}{\varepsilon(\theta_m^{ss})} \delta(y^H - y^L) \frac{V(w^{ss}, y^L)}{\mathbb{E}J_m(w^{ss}, y^L)}$$

by substituting the Lagrange multiplier  $\lambda^i = \frac{1 - \varepsilon(\theta_m^i)}{\varepsilon(\theta_m^i)} \theta_m^i \frac{\mathbb{E}G_m^i}{\mathbb{E}J_m^i}$ ,  $i \in \{ss, sr\}$ . Evaluated in  $\Delta(\phi_s) = 0$ , we have  $V(w^{sr}, w^{sr}) = V(w^{ss}, y^L)$  and, according to the above lemma,  $\mathbb{E}J_m(w^{sr}, w^{sr}) \leq \mathbb{E}J_m(w^{ss}, y^L)$ , which in turn implies  $\theta_m(w^{sr}, w^{sr}) \leq \theta_m(w^{ss}, y^L)$ . Therefore,

$$\Delta'(\phi_s)|_{\Delta=0} \geq p(\theta_m^{sr}) \delta[u(w^{sr}) - u(b) + u'(w^{sr})(y^L - w^{sr})] \geq p(\theta_m^{sr}) \delta(u(y^L) - u(b)) > 0,$$

where the second inequality follows from the concavity of the utility function. Above, we have already noted that  $\Delta$  has at least one root. Since  $\Delta$  is upwards sloping at every root  $\Delta(\phi_s) = 0$  and  $\Delta$  is continuous, there exists a unique root  $\phi'_s \in (\tilde{\phi}_s, 1)$ .  $\square$

*Proof of Theorem 4.4.* Let  $\phi \geq \phi'_s$  and consider  $\omega$  defined in (B.2) as a function of  $\phi$ . The monotonicity property stated in Lemma B.3(iii) ensures  $\Psi_{\text{id}}(\omega(\phi); \phi) < 0$ . By Proposition B.1, this means that  $w^{sr}$  is interior for any  $\phi > \phi'_s$  and, by Proposition 4.3, dominates any contract in regions (rs) and (rr).  $\square$

*Proof of Theorem 4.5.* In the proof of Theorem 4.3 the threshold  $\phi'_s$  was defined as the unique root of  $\Delta(\phi_s) = 0$ . Using total differentiation, the marginal effect of  $b$  can be calculated as

$$\frac{d\phi'_s}{db} = - \frac{d\Delta/db}{d\Delta/d\phi_s}$$

where the derivatives on the right-hand side are evaluated at  $\phi_s = \phi'_s$ . Above, it has already been shown that  $d\Delta/d\phi_s > 0$ . Furthermore, the Envelope Theorem can be used to obtain

$$\frac{d\Delta}{db} = [p(\theta^{ss}) - p(\theta^{sr})]u'(b)[1 + \delta(1 - \phi_o^* p(\theta_o))] + p(\theta^{sr})u'(b)\delta(1 - \phi_s) > 0,$$

where  $\phi_o^* = 1$  if  $w_o \leq y^L$  and  $\phi_o^* = \phi_o$  otherwise. By Lemma B.4 and the free entry condition, the first term is nonnegative and thus  $d\phi'_s/db < 0$ .  $\square$

*Proof of Theorem 7.1.* Let  $\phi_s \in (0, 1)$  and  $(w_m, w_s) \in (sr)$ . We show existence of a  $(w'_m, w'_s) \in (ss)$  that strictly dominates  $(w_m, w_s)$  in utility terms. Define  $w'_m := w_m + \delta\phi_s(w_s - y^L)$  and  $w'_s := y^L$ . It is straightforward to verify that this contract is indeed in region (ss) and satisfies  $\mathbb{E}J_m(w_m, w_s) = \mathbb{E}J_m(w'_m, w'_s)$ . Therefore, both contracts give rise to the same labor market tightness, and the safe contract  $(w'_m, w'_s)$  dominates since  $\mathbb{E}G_m(w'_m, w'_s) - \mathbb{E}G_m(w_m, w_s) = \delta(1 - \phi_s)(y^L - b) > 0$ . In the same way it can be shown that any contract in (rr) is dominated by a contract in (rs).  $\square$

*Proof of Proposition 7.1.* Define  $\mathbb{E}G_q(w_q; w_s) := \mathbb{E}G_o(w_q) - f(w_s)$ . Then the incumbent worker seeks to maximize  $p(\theta_q)\mathbb{E}G_q(w_q; w_s)$  with respect to  $(w_q, \theta_q)$  subject to the free entry curve  $q(\theta_q)\mathbb{E}J(w_q) = c$ . The structure of the problem is the same as that of problem (O). Hence any

interior solution candidate satisfies  $\Phi(w_q) = 0$  with  $\mathbb{E}G_q(w_q; w_s)$  substituted for  $\mathbb{E}G_o(w_q)$ . Since  $\mathbb{E}G_q(w_q; w_s)$  is the only term that depends on  $w_s$ , it is easy to see that  $\Phi$  strictly increases in  $w_s$ . By Lemma B.1(ii), the wage  $w_q$  of any interior solution candidate must be strictly increasing in  $w_s$  on the two regions under consideration. Alternatively, assume that the safe candidate is located at the corner  $w_q = y^L$  for some level of  $w_s$ . Then it satisfies  $\Phi(y^L) \geq 0$  and since  $\Phi$  is strictly increasing in  $w_s$ , the wage will stay in the corner for any higher levels of  $w_s$ . Putting things together, every solution candidate is (at least) weakly increasing in  $w_s$ , and therefore the optimal wage level  $w_q(w_s)$  is weakly increasing as well. Since the free entry curve is negatively sloped, a higher optimal wage is accompanied by a lower job finding probability  $p(\theta_q(w_s))$ .  $\square$

*Proof of Proposition 7.2.* With linear utility, an interior optimum satisfies the familiar Nash sharing rule, as evident from (B.8), and the wage profile  $(w_m, w_s)$  must maximize expected joint surplus,  $\mathbb{E}S_m := \mathbb{E}G_m + \mathbb{E}J_m$ . In region (ss) we have  $\mathbb{E}S_m = \bar{y}_m - b + \delta[\bar{y}_s - b + \max\{V_q(w_s), 0\} - V_o - p(\theta_q(w_s))(\bar{y}_s - w_s)]$  and hence the optimal senior wage maximizes

$$\max\{V_q(w_s), 0\} - p(\theta_q(w_s))(\bar{y}_s - w_s) = \begin{cases} p(\theta_q(w_s))[\mathbb{E}G_o(w_q(w_s)) - (\bar{y}_s - b)] & \text{if } V_q(w_s) > 0 \\ -p(\theta_q(w_s))(\bar{y}_s - w_s) & \text{if } V_q(w_s) \leq 0 \end{cases}. \quad (\text{B.11})$$

It can be shown that this must also hold for any boundary solution of region (ss). If  $V_q(w_s) \geq 0$  it is clear that it is optimal not to leave the firm. In case  $V_q(w_s) > 0$ , the term in square brackets is negative since  $\mathbb{E}G_o(w_q) = w_q - b \leq y^L - b < \bar{y}_s - b$  if  $w_q \leq y^L$  and  $\mathbb{E}G_o(w_q) = \phi_o(w_q - b) \leq \phi_s(y^H - b) < \bar{y}_s - b$  if  $w_q > y^L$ . Therefore maximizing (B.11) is equivalent to minimizing  $p(\theta_q(w_s))$ . It remains to verify that the optimal solution to (B.11) is feasible, i.e.  $\mathbb{E}G_m(w_m, w_s) > 0$ . For  $w_m$  interior this is equivalent to  $\mathbb{E}S_m > 0$  which is granted by  $\mathbb{E}S_m > \bar{y}_m - b + \delta[\mathbb{E}G_o(w_q(w_s)) - V_o] > 0$  since  $\mathbb{E}G_o(w_q(w_s)) \geq \mathbb{E}G_o(w_q(b)) > V_o$ . For  $w_m$  at the upper boundary this follows from Assumption 4.3(ii).

Deriving optimality conditions in region (rs) leads to the same insights, and, provided  $\phi_s \geq \frac{y^L - b}{y^H - b}$ , this is also the case in the two remaining regions. As a result, if in equilibrium  $p(\theta_q(w_s)) > 0$ , the worker is worse off compared to a situation without on-the-job search (equivalent to  $p(\theta_q(w_s)) = 0$ ).  $\square$

*Proof of Theorem 7.2.* Since  $\underline{\phi} \geq \max\{\phi_o, \frac{y^L - b}{y^H - b}\}$  we know from Proposition 7.2, workers seek to minimize  $p(\theta_q(w_s))$ . This, by Proposition 7.1, is a weakly decreasing function in  $w_s$  on the two half-panes separated by  $w_s = y^L$ . Hence it is sufficient to focus on contracts with  $w_s = y^L$  and  $w_s = y^H$ , since any feasible contract is (weakly) dominated by one with this property. Similar to the situation studied in Theorem 4.4, the remaining two assumptions on  $\underline{\phi}$  guarantee that the equilibrium contract is located in region (ss) or (sr) for  $\phi_s \geq \underline{\phi}$ , and that the optimal  $w_m$  is strictly below the upper boundary on  $w_m$ . Denote with  $(w_m^{ss}, y^L)$  and  $(w_m^{sr}, y^H)$  the (unique) solution candidate with  $w_s = y^L$  and  $w_s = y^H$ , respectively.<sup>39</sup> The globally optimal contract

<sup>39</sup>The corresponding period 1 wage,  $w_m^{ss}$  and  $w_m^{sr}$ , is uniquely pinned down by the Nash sharing rule (B.7) and the free entry condition (B.9).

is the one that maximizes expected joint surplus,  $\mathbb{E}S_m(w_m, w_s)$ . Since no firm is willing to pay more than  $y^H$  in the second period,  $p(\theta_q(y^H)) = 0$ , and thus

$$\begin{aligned}\Delta(\phi_s) &:= \mathbb{E}S(w_m^{ss}, y^L; \phi_s) - \mathbb{E}S(w_m^{sr}, y^H; \phi_s) \\ &= \delta(1 - \phi_s)(y^L - b) + \begin{cases} \delta p(\theta_q(y^L))[\mathbb{E}G_o(w_q(y^L)) - (\bar{y}_s - b)] & \text{if } V_q(y^L) > 0 \\ 0 & \text{if } V_q(y^L) \leq 0 \end{cases}.\end{aligned}$$

If  $w_q(y^L) \leq y^L$ , then  $V_q(y^L) = p(\theta_q(y^L))[w_q(y^L) - y^L] \leq 0$  and thus  $\Delta(\phi_s) > 0$  for all  $\phi_s \in (0, 1)$ . If  $w_q(y^L) > y^L$ , on the other hand,  $V_q(y^L) = \max_{w_q} p(\theta_o(w_q))[\mathbb{E}G_o(w_q) - (y^L - b)] > p(\theta_o(y^L))[\mathbb{E}G_o(y^L) - (y^L - b)] = 0$  since  $(y^L, \theta_o(y^L))$  with  $\theta_o(y^L)$  determined by (5) is a feasible, but not an optimal pair for (9). Straightforward algebra then yields that  $\Delta(\phi_s) < 0$  if and only if  $\phi_s > \phi'_s$  where  $\phi'_s := [y^L - b + V_q(y^L)]/[y^L - b + p(\theta_q(y^L))(y^H - y^L)]$ . Since  $V_q(y^L) < p(\theta_q(y^L))(y^H - y^L)$ , the threshold  $\phi'_s$  is indeed less than unity.  $\square$

*Proof of Proposition 7.3.* Assume such a PBE exists. Since the firm plays  $S$  in case  $Y_i = y^H$ , the expected payoff from  $S$  at least equals that of  $R$ , i.e.  $\mathcal{J}_i(w_i|y^H) \geq P[Y]\mathcal{J}_i(\bar{w}_i|y^H) + P[N]\mathcal{J}_i(w_i|y^H)$ . This is equivalent to  $P[Y]\{\mathcal{J}_i(w_i|y^H) - \mathcal{J}_i(\bar{w}_i|y^H)\} \geq 0$ . Since  $\mathcal{J}_i(w_i|y^H) < \mathcal{J}_i(\bar{w}_i|y^H)$  this requires that workers always refuse to renegotiate,  $P[Y] = 0$ . This is consistent with a rationally behaving worker only if  $\mathcal{G}_i(\bar{w}_i) \leq P[y^H|R]\mathcal{G}_i(w_i)$ . But this inequality cannot hold because Bayes' rule implies  $P[y^H|R] = \frac{P[R|y^H]\phi_i}{P[R|y^H]\phi_i + P[R|y^L](1-\phi_i)} = 0$ , as  $P[R|y^H] = 0$  and  $P[R|y^L] > 0$  by assumption.  $\square$

*Proof of Theorem 7.3.* For layoffs to occur with positive probability, two properties must hold: (i) the firm must propose renegotiation in case of low productivity with positive probability,  $\gamma = P[R|y^L] > 0$ , and (ii) there is a chance that the worker accepts,  $P[Y] > 0$ . The latter implies that the firm always plays  $R$  in the high state,  $P[R|y^H] = 1$ , since  $P[Y]\{\mathcal{J}_i(w_i|y^H) - \mathcal{J}_i(\bar{w}_i|y^H)\} > 0$ . Bayes' rule then implies that  $P[y^H|R] = \frac{\phi_i}{\phi_i + \gamma(1-\phi_i)}$ . Thus for  $P[Y] > 0$  to be rational, the inequality  $\mathcal{G}_i(\bar{w}_i) \geq P[y^H|R]\mathcal{G}_i(w_i)$  must hold. Since  $P[y^H|R] \geq \phi_i$  this is possible if and only if  $\mathcal{G}_i(\bar{w}_i) \geq \phi_i\mathcal{G}_i(w_i)$ . Clearly, the only PBE without layoffs is implemented by  $\gamma = 1$  and  $P[Y] = 1$ .  $\square$

*Proof of Theorem 7.4.* If  $P[Y] = 0$  the statements about the payoffs and layoff probabilities can directly be seen from Figure 11. To verify existence, it suffices to show that  $\{(R, S), N\}$  always constitutes a PBE. Because  $P[Y] = 0$ , firms are in both productivity states indifferent between proposing renegotiation or not. The pure strategy  $(S, R)$  is therefore consistent with the worker's strategy. On the other hand,  $P[R|y^L] = 0$  and  $P[R|y^H] = 1$  imply  $P[y^H|R] = 1$ , and workers will indeed refrain from renegotiation since  $\mathcal{G}_i(\bar{w}_i) < P[y^H|R]\mathcal{G}_i(w_i) = \mathcal{G}_i(w_i)$ .  $\square$

*Proof of Theorem 7.5.* First, note that a contract that promises  $\bar{w}_i$  in the first place implies  $\mathbb{E}\mathcal{J}_i = \phi_i\mathcal{J}_i(\bar{w}_i|y^H)$  and  $\mathbb{E}\mathcal{G}_i = \mathcal{G}_i(\bar{w}_i)$ . For a PBE without renegotiation, the expected payoffs are given by Theorem 7.3, and we note that  $\phi_i\mathcal{J}_i(\bar{w}_i|y^H) > \phi_i\mathcal{J}_i(w_i|y^H)$  as well as  $\mathcal{G}_i(\bar{w}_i) \geq \phi_i\mathcal{G}_i(w_i)$  by assumption. Now consider a PBE with renegotiation, using  $\gamma = P[R|y^L] > 0$ ,

$P[R|y^H] = 1$ , and  $\eta = P[Y] > 0$ . The expected payoffs are  $\mathbb{E}\mathcal{J}_i = \phi_i[\eta\mathcal{J}_i(\bar{w}_i|y^H) + (1 - \eta)\mathcal{J}_i(w_i|y^H)]$  and  $\mathbb{E}\mathcal{G}_i = \phi_i[\eta\mathcal{G}_i(\bar{w}_i) + (1 - \eta)\mathcal{G}_i(w_i)] + (1 - \phi_i)\gamma\eta\mathcal{G}_i(\bar{w}_i)$ . This implies  $\mathbb{E}\mathcal{J}_i \leq \phi_i\mathcal{J}_i(\bar{w}_i|y^H)$  as well as  $\mathbb{E}\mathcal{G}_i \leq \eta\mathcal{G}_i(\bar{w}_i) + (1 - \eta)\phi_i\mathcal{G}_i(w_i) \leq \mathcal{G}(\bar{w}_i)$ , where the assumption of the Theorem was used for the last inequality.  $\square$

*Proof of Theorem 7.7.* As in the proof of Theorem 7.6 we verify that any solution to (P) is renegotiation-proof,  $(w_m^*, w_s^*) \in \Omega_m$ . Suppose the contract violates renegotiation-proofness at the senior stage, i.e.  $w_s^* \in (y^L, y^H]$  and  $\phi_s G_s^*(w_s^*) \leq G_s^*(y^L)$ . In the same way as above, it can be shown that the alternative contract  $(w_m^*, y^L)$  satisfies  $\mathbb{E}J_s(w_s^*) < \mathbb{E}J_s(y^L)$  and  $\mathbb{E}G_s(w_s^*) \leq \mathbb{E}G_s(y^L)$ . Because the expected surplus functions (7)–(8) are strictly increasing in the continuation value, this translates into  $\mathbb{E}J_m(w_m, w_s^*) < \mathbb{E}J_m(w_m, y^L)$  and  $\mathbb{E}G_m(w_m, w_s^*) \leq \mathbb{E}G_m(w_m, y^L)$ . The alternative contract  $(w_m, w_s^*)$  therefore implements a strictly higher value of the objective function. This reveals that an optimal contract must be renegotiation-proof in the last period. On the other hand, renegotiation may occur in the first period if  $w_m^* \in (y^L + \delta\mathbb{E}J_s(w_s), y^H + \delta\mathbb{E}J_s(w_s))$  and  $\phi_m \mathbb{E}G_m^*(w_m^*, w_s) \leq \mathbb{E}G_m^*(y^L + \delta\mathbb{E}J_s(w_s), w_s)$ , using the notation of page 15. In this case it can be shown that a contract  $(w_m, w_s) = (y^L + \delta\mathbb{E}J_s(w_s), w_s)$  would be strictly better than  $(w_m^*, w_s^*)$ .  $\square$