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## Arbitrage with Production, Collateral Constraint and Heterogeneous Belief

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# Arbitrage with Production, Collateral Constraint and Heterogeneous Belief

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## **Abstract**

We construct a dynamic model economy in which households from segmented markets have varying financial asset demand. Intermediaries make profit by exploiting the price difference in segmented financial markets. However, intermediaries have to separately post their physical investment as collateral to trade. We show that the heterogeneous belief will disturb the intermediaries self-recovery process in both financial and real sectors through endogenously determined collateral constraints. The dynamic interaction between belief determined collateral constraint and liquidity supply turns out to be a powerful transmission mechanism by which the effects of shocks persist, amplify and spill over to other sectors.

**Keywords:** collateral constraints, limit of arbitrage, market liquidity, general equilibrium, heterogeneous belief, incomplete market

**JEL Classification:** C68

# 1 Introduction

This paper studies how the heterogeneous belief will disturb the self-recovery process in both financial and real sectors through endogenously determined collateral constraints. In particular, for an economy in which some investors are more sensitive to negative news, we examine how relatively trivial, temporary exogenous shocks in financial markets could lead to dramatic, persistent oscillations in liquidity supply, asset prices and aggregate output. In addition, we investigate whether the effects of sector-specific shocks can spill over to other sectors and get amplified over time.

To address these questions, we consider a dynamic production economy in which collateral constraints arise because of market segmentation. In particular, we construct a dynamic model in which households from two segmented markets have opposite asset demands but can only trade with each other indirectly through competitive intermediaries. The opposite asset demands lead to price discrepancies between identical assets in two markets and create potential arbitrage opportunities for the intermediaries. Whereas the intermediaries can make profit from providing liquidity and exploiting the price differences, they also face separate collateral constraints. In order to trade financial assets, they have to post their physical capital in the production sector as collaterals to prevent them from walking away from their positions in each financial market. As a result, such endogenous financial constraint limits intermediaries positions as a function of their collateral holdings. In our baseline model, we assume that households and intermediaries hold homogeneous belief about the asset demand in each financial market. We find that intermediaries present self-recovery capacity after negative shock in either the production sector or the financial sector. In a nutshell, with homogeneous belief there is no amplification or spillover effects across sectors. Instead, financial profitability and production return both become more favourable after negative shocks, together pushing the intermediaries to revert back to pre-shock state.

However, in our extended model we consider the economy in which intermediaries and households hold heterogeneous belief over the future market demand. Specifically, we assume that the households determine the collateral requirement based on their own beliefs. For example, if the households believe the future demand will increase, they will

require more collateral from intermediaries to prevent default in financial markets. Thus the intermediaries become more collateral constrained independent of their own belief or the true distribution of the future market demand. On the other hand, we also assume that households form their beliefs based on their observation of historical liquidity supply. This renders households more sensitive to negative shocks and easily establish more pessimistic estimations. As a result, the previous liquidity supply affect the tightness of the collateral constraints, which in turn limit the intermediaries current liquidity supply. We show that the dynamic interaction between belief determined collateral constraint and liquidity supply turns out to be a powerful propagation mechanism by which the effects of shocks persist, amplify and spill over to other sectors.

The propagation channel works as follows. Suppose that in some period  $t$  the intermediaries suffer from unexpected large cash outflows in the financial markets due to a temporary demand shock. The sudden loss in total income urges intermediaries to cut down the capital investment, and consequently support less liquidity providing. In the meantime, the abrupt drop in relative liquidity supply triggers households panic. They believe that in the next period the relative demand will continue increasing as the liquidity supply decreases. Thus they expect the price discrepancies will increase and they begin to charge intermediaries with more collateral this period. As a result, the intermediaries feel more constrained in providing liquidities. The further drop of the liquidity causes households more pessimistic estimation of future market demand at  $t + 1$  and tightens the collateral constraints even more. For intermediaries, the unexpected increasing pessimism from the households reduces their collateralization return from capital and to some point they also cut down their capital investment. Intuitively the vicious circle starts. Less liquidity supply triggers more panic. Households growing panic leads to tighter collateral constraints. Finally more restrictive financial constraints force intermediaries to make a deeper cut in capital investment and scale down liquidity supply in financial sectors. Thus the once-off shock has persistent influence and amplifies over time. The spreading panic also impedes capital accumulation and results in less aggregate output. As a consequence, intermediaries self-recovery capacity gets disturbed and households also suffers from insufficient hedge. Also both households and intermediaries receive less income from production sector.

The knock-on effect is shown in Figure 1. As we can see, there are two types of

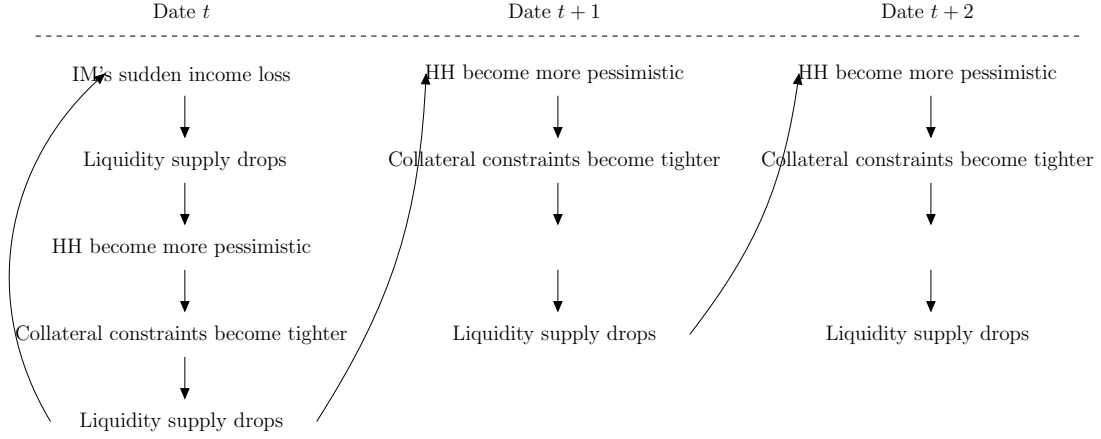


Figure 1: The Multiplier Process.

propagation processes. One is within-period, or static, propagation. Consider the left-hand column of Figure 1, marked “date  $t$ ” (ignore any arrows to the future). The intermediaries income loss forces them to slow down their investment in productive assets; as a result their liquidity supply drops, which triggers households panic. The households pessimistic estimation tightens the intermediaries collateral constraints. Hence it is more restrictive for intermediaries to provide liquidity. The other propagation process is the dynamic, intertemporal one. The reduced liquidity supply gives rise to households more pessimistic estimation of next period market demand. This causes the further tightening of collateral constraints and cutting down in liquidity supply in period  $t+1$  and in subsequent periods. Moreover, the weakened return from collateralization also exacerbates the slowdown of the capital accumulation process.

This paper adds to several strands of the literature. We are the first to set up a theoretical framework to link the shocks from financial markets to the aggregate physical production through the collateral constrained intermediaries. The idea that the persistence and amplification effects of shocks from financial markets manifest in the physical production sectors can be traced back to Kiyotaki and Moore (1997). In their seminal work, they uncover the transmission mechanism by which credit limits and asset prices interact to propagate, amplify and spread out the productivity shocks. Through the impact on the borrowers’ net worth, temporary shocks can have persistent effects on the size of productive assets and collateral. Also, small shocks can get amplified via the feedback spiral as borrowers’ reduced net worth causes the fall of their productive asset

prices which lowers the net worth even more. Brunnermeier and Sanikov (2013) also study the endogenous leverage and its nonlinear amplification effects due to the asset illiquidity. They examine how the exogenous shocks can persist and amplify in the system, causing extreme volatility spikes and higher equilibrium leverage.

Moreover, many previous works have emphasized on the importance of financially constrained intermediaries. For example, Gromb and Vayanos (2002, 2009) formalize the impact of no cross-margining on arbitrageurs' activity and market liquidity. In their model, the financial constraints restrict the intermediaries' positions as a function of their wealth, and thus limited wealth can hinder their attempt to fully eliminate the price wedge between two segmented markets. They also characterize conditions under which intermediaries' liquidity providing activity can play a role of stabilizing or destabilizing in the financial markets.

As far as the computational methodology is concerned, our analysis is closely related to the literature of general equilibrium with incomplete markets. In particular, Kübler and Schmedders (2003) characterize the stationary equilibria by a mapping from the exogenous shock and the current financial wealth distribution to prices and portfolio choices. They develop an algorithm that numerically approximates such a mapping and solves the policy function in recursive equilibria with time iteration methods.

Further, the notion of collateral constraints is more explicitly focused on by Chien and Lustig (2010). They show that in the standard endowment economy with complete markets, the limited-liability technology contributes to the liquidity risk associated with binding collateral constraints. This liquidity risk gives rise to a low risk-free rate, a large equity premium and substantial time variation in risk-premium.

## 2 Baseline Model

We consider an infinite-horizon, discrete-time production economy with one perishable consumption good. In the economy, there are two symmetric yet separated countries (labeled as A and B) populated with identical continuum of citizens or households. The two countries have identical financial markets and agents trade the same risky

financial assets therein. Also there are two types of long-lived homogeneous agents, domestic citizens/households and international intermediaries. These two types of market participants interact with each other through trading risky financial assets within each country.

## 2.1 The Financial Sector

We assume there is only one infinitely lived risky asset in each country's financial market. we label them as asset A and B. They are both in zero net supply. Similar to Gromb and Vayanos (2002), their dividends are identical and equal to a random shock variable  $\theta_t$  revealed in period  $t$ . The shock variables  $\theta_t$  are independent and identically distributed, and the distribution is symmetric around zero in the bounded interval  $[-\bar{\theta}, \bar{\theta}]$ .

The purpose of assuming identical payoffs is two-folded. First it is to simplify the model and the following computation. Second it captures the phenomenon of assets or portfolios that have highly correlated payoffs but are traded at significantly diverging prices in different markets. As illustrated in Gromb and Vayanos (2015), examples of such assets include the Siamese-Twin stocks, traded in different markets but with identical dividend payoffs and identical stocks issued from the same company traded both in Shanghai exchange and Hong Kong exchange. Also bonds with identical coupon rates and time to maturity, like on- and off-the run bonds, can also be generally abstracted and modelled with this feature.

The assumption of zero net supply in both financial markets is to facilitate the later assumption that the intermediaries hold equal amount of but opposite positions so that they bear no aggregate risk from shocks to payoffs. Similarly, the bounded support of the dividend distribution helps quantify the financial constraints (see in Section 2.4).

## 2.2 Households

We refer households in country A and B as A-Households and B-Households. Each side of households form a continuum of measure  $L$  and they are all competitive, infinitely lived, domestic investors. They consume every period and share the same preference. In each period  $t$ , they maximize the following utility:

$$\mathbb{E}_t \sum_{s=t}^{\infty} \beta^s \log(c_s^i), \quad i = A, B.$$

where  $\beta$  is the subjective discount factor and  $c_t^i$  is the consumption at  $t$ .

Households receive income from three potential sources. One stream comes from the natural endowment as in the exchange economy of Gromb and Vayanos (2002). Since we assume this is a production economy, we refer to such endowment as yield from the households self-owned, fixed-size production in our context. Parallel to the settings in Gromb and Vayanos (2002) which assume that households from two countries receive an opposite random endowment shock in each period on top of their natural endowment, we assume that the households' yield in each country suffers from an opposite but identical size of production shock. This induces the households in different countries to have opposite tendency to acquire and hold the risky financial assets. For simplicity, we model each  $i$ -Household ( $i=A, B$ ) receives an output equal to

$$y_{H,t} = bK_H + u_{t-1}^i \theta_t, \quad i \in \{A, B\}, \quad t \in \{1, 2, \dots\}.$$

where  $K_H$  is assumed to be a fixed amount of capital that does not bear any depreciation and cannot be incremented nor converted to consumption.  $b$  is a constant positive coefficient of the capital  $K_H$  for the linear production function.  $\{\theta_t\}_{t=1}^{\infty}$  is the same sequence of independent and identically distributed (i.i.d.) random variables as the payoffs from the risky asset in the financial markets mentioned in Section 2.1. Specifically, the shock  $\theta_t$  is revealed in period  $t$  and follows a symmetric distribution around zero on the bounded support  $\mathcal{S} = [-\bar{\theta}, \bar{\theta}]$ , where  $\bar{\theta} \in \mathbb{R}_+$ .  $u_{t-1}^i$  is revealed at  $t-1$ , which is the “supply shock” in Gromb and Vayanos (2002) to indicate the negative effects on asset demands. We refer to  $u_t^i$  as the production shock intensity (henceforth shock intensity) in the context of production economy. In particular,  $u_t^i$  are of the opposite sign in the two markets and identical in magnitude, that is  $u_{A,t} = -u_{B,t} =: u_t$ . They follow a sequence  $\{u_t\}_{t=0}^{\infty}$  of i.i.d. random variables independent of the production shocks  $\{\theta_t\}_{t=1}^{\infty}$  and each  $u_t$  has bounded support  $\mathcal{U} = [\underline{u}, \bar{u}]$ , where  $-\infty < \underline{u} < \bar{u} < \infty$ .

The reason for assuming opposite signs of shock intensity is to create price differences between two segmented financial markets. Diverse shock intensity gives rise to different demand for assets. The contrasting signs of the shock intensity simplify the later analysis



and lead to opposite households propensity to acquire the financial assets. For example, when  $u_t > 0$ , A-Households' self-production income becomes positively correlated with the random shock and the financial asset payoff  $\theta_t$ . Thus, holding a positive number of financial assets will further increase A-Households' exposure to risk. As a result, A-Households tend to have an aggregate demand to obtain some negative positions of financial assets, whereas B-Households are willing to buy risky assets to offset the negative total amount of shocks to their income.

The demand divergence is a realistic assumption in many contexts. For instance, as illustrated in Gromb and Vayanos (2015), the demand of the domestic-equity mutual funds, which can only invest in their own domestic markets, are influenced by inflows and outflows of the investing capital. Also the bonds demand from different pension funds could be driven by their diverse needs for asset-liability management.

In addition, households can also incur income changes from trading financial assets within their own markets. Through trading the risky assets not only can households offset the exposure from the production shocks to their income, but also can they receive the capital gains from price fluctuations between periods.

Finally, households can earn the labor income by working in the production sectors operated by the intermediaries. We will cover more details later.

### **2.2.1 Market Segmentation**

The financial markets for households are segmented. Households in each country can only trade the financial assets within their own domestic market without access to other markets. Or we can think of this as households face prohibitive high transaction costs when trading in foreign markets due to the regulation restrictions or information disadvantages.

Market segmentation, together with the households' diverging demand for financial assets arising from opposite production shocks, contribute to the price discrepancies between two markets.

## 2.3 Intermediaries

### 2.3.1 Expertise

In our settings, the intermediaries are special third-party investors who can trade simultaneously in both markets. Also they are uniquely capable of transforming the consumption goods into production capital and vice versa. In particular, we assume that the intermediaries organize a production sector in which they invest capital and hire households from both countries as labor input. Such assumptions capture the sophisticated investors in reality such as hedge funds with wide investment spectrum, since they are less constrained by regulatory restrictions or informational asymmetry that lead to segmentation. For instance, some international hedge funds diversify their investment across sectors ranging from high-tech start-ups to stock markets across borders.

We assume that intermediaries are competitive, constitute a continuum of measure 1 and consume in each period with initial physical capital  $K_0$ . They maximize their logarithmic utility by choosing the optimal positions in both financial markets, the physical investment level and the consumption in each period. In particular, their utility writes as:

$$\mathbb{E}_t \sum_{s=t}^{\infty} \rho^s \log(c_s^{\text{IM}}),$$

where  $\rho$  is the subjective discount factor.

The financial market segmentation and the opposite income shocks create a role for the intermediaries to exploit price discrepancies between two financial markets and thus lock in a profit. The intermediaries can make advantage of the price wedge between two markets and make a profit by closing out the gap. As explained in Gromb and Vayanos (2002), suppose the A-Households encounter a positive intensity shock from their fixed-production and meanwhile the B-Households suffer a negative shock of the same magnitude. Then the A-Households are more willing to enter negative positions of the risky assets in aggregate to reduce their exposure whereas B-Households are inclined to have long positions of the assets. As a consequence, the price in market A is lower than that in market B. Thus the intermediaries would buy the risky assets in market A with a lower price and sell the assets to the B-households at a higher rate. Through such trading activities the intermediaries

provide liquidity to both markets and also make profits by exploiting the price differences. Subsequently, the resulting price disparity declines.

Moreover, for simplicity we assume that the intermediaries can only hold zero net amount of total financial assets to avoid aggregate risk from the dividends. Or, they will incur infinitely high cost when they bear unbalanced positions across markets. The assumptions of the zero net supply of financial assets together with the opposite signs of shock intensity also facilitate the intermediaries' balanced holdings. As a consequence, the intermediaries will only trade identical size but opposite positions in the two markets.

Finally, we model that the return of the households' production is either unattractive to the intermediaries or it is prohibitive by law for them to invest. Therefore, intermediaries would not be interested in holding any capital  $K_H$  or investing in this production process in either country.

### 2.3.2 Production Sector P

Production sector P is an independent technological process from other sectors. We assume that its output at period  $t+1$  follows the Cobb-Douglas production function as below:

$$y_{P,t+1} = aK_{P,t}^\alpha L_t^\gamma + (1 - \delta)K_{P,t}, \text{ for } t = 1, 2, \dots$$

with the capital input  $K_{P,t}$  and labor supply  $L_t$ . Here, the first term on the right-hand side of the equation is the production output and the second is the leftover capital after depreciation. In particular,  $a$  is the total productivity factor, and  $\alpha \in (0, 1)$  and  $\gamma \in (0, 1)$  are the output elasticity for the capital and labor input respectively.  $\delta \in [0, 1]$  is the depreciation rate. In addition, we assume that the technology exhibits constant returns to scale, that is  $\alpha + \gamma = 1$ .

Since this is a one-good economy, the only good can serve as both consumption and capital input in the production sector. However, the capacity of managing the production process differs between the intermediaries and ordinary households. As mentioned before, we assume that only the intermediaries are able to transform the consumption goods into capital input. We can think of it as if the intermediaries collectively run the production sector and choose the size of capital investment in each period. Households from both

countries, on the other hand, can only provide their labor force and receive a wage of the prevailing competitive rate  $w_t$  as return. Since later we will assume that the households gain no utility from leisure, the households will invest their entire labor endowment in each period. Thus the resulting competitive wage rate at time  $t$  is

$$w_t = a\gamma K_{P,t-1}^\alpha L_t^{\gamma-1}, \text{ for } t = 1, 2, \dots$$

At the beginning of each period, the production activity yields output and incurs certain depreciation to intermediaries physical investment invested in previous period. After paying out the labor income, the intermediaries receive the rest as total capital return. At the end of each period, the intermediaries decide the capital investment for the next period production.

Apart from being the capital input, the intermediaries physical investment in this sector can also serve as collateral for gaining access to trading certain positions of risky assets in both financial markets. We will elaborate this extra merit of the physical investment later in Section 2.4.

For simplicity, we will henceforth refer production sector  $P$  to production sector and reduce the expression of  $K_{P,t}$  to  $K_t$ .

## 2.4 The Collateral Constraint

To avoid modelling default, we assume that intermediaries are subject to collateral constraints. As they are foreign investors for both markets, they have to collateralize their asset positions to prevent them from walking away without delivering the due payoffs. For instance, if an intermediary wants to buy an asset and make a loan to finance the purchase, she has to post sufficient amount of collateral to guarantee that the loan will be paid back. Also, if another intermediary needs to short assets, he would first borrow the assets from others and meanwhile he must put enough collateral to ensure that such borrowing will be safely repaid. In Gromb and Vayanos (2002), they refer to such restriction as financial constraints. we will use them interchangeably henceforth.

Without such collateral requirement, the intermediaries can absorb all demand from both markets and immediately close out any price differences by taking whatever arbitrage

positions they need. Thus the prices for the same asset in both markets will be the identical. Even if there is any price wedge, the intermediaries can buy the cheaper ones from one market and sell the same number of expensive ones in the other. As long as the position size is equal, they can use the asset payoffs from the long positions to offset the liability from short positions. Through this, intermediaries can keep earning arbitrage profits until the price discrepancy disappears.

Similar to the settings of Gromb and Vayanos (2015), we model the collateral constraint such that not only the intermediaries must fully collateralize their asset positions, but also they are obliged to put collateral separately in each market. Due to the market segmentation, the position they take in one market is not recognized by the custodians in the other country and hence cannot serve as collateral to back up the opposite positions taken in other market. Instead, in this model we require the intermediaries to post claims to the ownership of their current period physical investment in the production sector or the capital for the next period production as collateral. Specifically, the intermediaries are required to at least collateralize a certain minimal amount of their current period physical investment to make sure the total value of the asset positions and the residual value of collaterals are non-negative in the next period at each individual market. Also because of the segmentation, they have to obey this requirement market by market. That is, the part of physical investment which is used as collateral in one market cannot be simultaneously posted in the other market.

Consequently, the positions of intermediaries in one markets margin account are composed of a position in the financial asset and a legally binding claim to the ownership of the required amount of capital investment in the production sector. This combined position is such that it will not incur any liability in the next period. In particular, the liability is calculated individually at each market instead of summing up from the two countries. Note that while the part of the capitals ownership has been transferred to the clearinghouse or directly to the households in both markets as collateral, we assume that the intermediaries can still run the production process and produce output with their entire capital investment. In addition, in case of default in the next period, only intermediaries residual capital after depreciation can be seized to offset the liability. In other words, the collaterals value in the next period is equal to the amount of the depreciated capital. The intermediaries next period capital rent income from the production output, on the other

hand, is protected and cannot be confiscated to fulfil any obligations. This assumption reflects the limited liability practice in reality and is common in literature, e.g., Chien and Lustig (2010). That is, when bankruptcy happens the lender is only allowed to collect certain assets from the debtor, but not her labor income.

To illustrate the financial constraint in this model, denote  $x_t^i$  as intermediaries position in asset  $i$  and  $K_t^i$  as her collateral of physical investment posted in this market. The value of the intermediaries combined position in  $i$  in period  $t+1$  is

$$\sum_{i \in \{A, B\}} \min\left\{\min_{p_{t+1}^i, \theta_t} \{x_t^i (p_{t+1}^i + \theta_t)\}, 0\right\} + \sum_{i \in \{A, B\}} (1 - \delta)K_t^i \geq 0.$$

To prevent the intermediaries from walking away from any liability, the combined position value has to be non-negative for all the asset prices and dividend shocks. The bounded-support settings of the shocks  $\theta_t$  and shock intensity  $u_{i,t}$  now make calculating the maximum liability in the next period possible. Aggregating across two markets, we can have the following collateral constraint:

$$\sum_{i \in \{A, B\}} \min\left\{\min_{p_{t+1}^i, \theta_t} \{x_t^i (p_{t+1}^i + \theta_t)\}, 0\right\} + (1 - \delta)K_t \geq 0.$$

In sum, the intermediaries must have enough capital investment for the residuals to cover the maximum possible liability in each single market.

With 2.4, we can imply that the collateral constraints become more restrictive when the dividend become more volatile, i.e.,  $\bar{\theta}$  is higher. This is due to the maximum liability per unit of position incurred is larger. Also if the price gap at  $t + 1$  is larger, the more severe is the collateral constraints, as this is the “cash outflow resulted from each unit of the previous position.

Different from Gromb and Vayanos (2015), in this model we do not model a riskless asset with exogenous return as collateral. Instead, we generalize the riskless asset to the capital investment of some technology without uncertainty. The production return of the collateral is therefore endogenous. This allows for investigating the interactions between financial markets and real production sectors under the influence of the intermediaries financial constraints. Also, rather than making use of the total return of the collateral to offset the maximum loss in each market, here we only allow the depreciated capital to

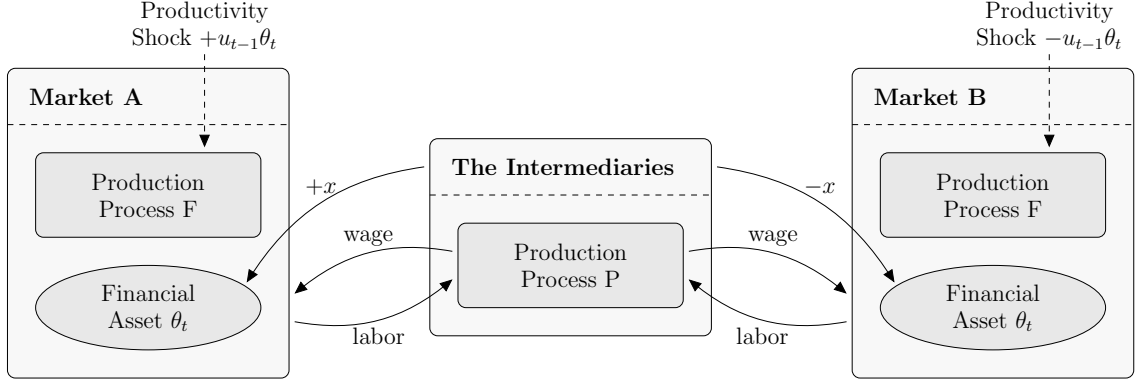


Figure 2: The structure of the economic system.

cover for potential liability. This constraint is of course much more restrictive and can be relaxed for other setting options.

The collateral constraint limits the intermediaries positions and their ability to provide liquidity as a function of their capital investment. As a result, the capital investment derives value not only from production but also from the ability to serve as collateral in the financial markets.

### 3 Agents' Optimization Problems

#### 3.1 Intermediaries

The intermediaries optimization problem is to maximize their utility by choosing in each period the optimal level of consumption  $c_t^{\text{IM}}$ , physical investment  $K_t$  in the production sector and positions  $x_t^i$  ( $i \in \{A, B\}$ ) in both financial markets:

$$\max_{c_s^{\text{IM}}, x_s^i, K_s} \mathbb{E} \left[ \sum_{s=t}^{\infty} \rho^s \log(c_s^{\text{IM}}) \right], \quad i \in \{A, B\}.$$

Denote the price of the financial asset in period  $t$   $p_t^i$  and the capital level at the beginning of the period  $K_{t-1}$ . The intermediaries are subject to the following budget constraints.

$$c_t^{\text{IM}} = \underbrace{\sum_{i \in \{A, B\}} x_{t-1}^i p_t^i}_{\text{value of previous period's investment in financial asset } i} - \underbrace{\sum_{i \in \{A, B\}} x_t^i p_t^i}_{\text{current cost of taking new positions or immediate arbitrage profit}} + \underbrace{a(1 - \gamma)K_{t-1}^\alpha L_t^\gamma + (1 - \delta)K_{t-1} - K_t}_{\text{expert income: net production output minus wage and investment}}, \quad i \in \{A, B\}.$$

Since I assume that the intermediaries incur prohibitively high cost if their net position across markets is non-zero, I implicitly impose  $x_{A,t} = -x_{B,t} = x_t$ . Therefore, we can write the intermediaries budget constraint as below.

$$c_t^{\text{IM}} = (x_{t-1} - x_t)(p_t^A - p_t^B) + a(1 - \gamma)K_{t-1}^\alpha L_t^\gamma + (1 - \delta)K_{t-1} - K_t, \quad i \in \{A, B\}.$$

Meanwhile, the intermediaries also have to satisfy the following collateral constraint.

$$\sum_{i \in \{A, B\}} \min\left\{ \min_{p_{t+1}^i, \theta_t} \{x_t^i (p_{t+1}^i + \theta_t)\}, 0 \right\} + (1 - \delta)K_t \geq 0.$$

The corresponding transversality condition is

$$\lim_{t \rightarrow \infty} \rho^t \mathbb{E} \left[ \frac{1}{c_t^{\text{IM}}} \left( a(1 - \alpha)K_t^\alpha L_t^\gamma + \sum_{i \in \{A, B\}} (x_{t-1}^i - x_t^i) p_t^i \right) \right] = 0.$$

### 3.2 Households

Compare to the intermediaries, households are not restricted by the collateral constraint. Therefore, their optimization problem amounts to maximizing their utility only subject to budget constraints. They do so by choosing the optimal consumption, the positions in the domestic financial markets and labor input invested in the production sector. Since their utility does not include leisure, they will put their entire labor endowment  $L_t$  to earn a competitive wage rate  $w_t = a\gamma K_{t-1}^\alpha L_t^{\gamma-1}$ .

As mentioned earlier, in each period households receive income from three sources: (1) the endowment or income from fixed self-production; (2) the gain from trading financial assets; (3) and their labor income from participating in the production sector. Denote



their consumption in period  $t$  as  $c_t^i$  for  $i \in \{A, B\}$  and their positions in the financial market  $y_t^i$ . The optimization problem can write as

$$\max_{c_s^i, y_s^i} \mathbb{E} \left[ \sum_{s=t}^{\infty} \beta^s \log(c_s^i) \right]$$

subject to the dynamic budget constraints

$$c_t^i = bK_H + (y_{t-1}^i - y_t^i)p_t^i + a\gamma K_{t-1}^\alpha L_t^{\gamma-1} + (y_t^i + u_t^i)\theta_t, \quad \text{for } i = A, B.$$

Ideally, given enough liquidity supply, the households would like to take position  $y_t^i = -u_t^i$  to eliminate all the income risk arising from  $\theta_t$ .

The budget constraint also indicates that the intermediaries investment in the production sector not only affects their own income level and limits their ability to provide liquidity, but also determines the households wage level and hence their labor income.

## 4 Equilibrium

### 4.1 Sequential Equilibrium

In this section, we derive a competitive equilibrium.

In this model with financial frictions, equilibrium is characterized by a map from productivity shock histories to prices, asset allocation and capital accumulation that maximize agents' utility and clear the markets.

For any initial capital endowment, an equilibrium is described by the price process  $p_{i,t}$ , capital  $P$  accumulation  $K_{P,t}$ , financial asset positions  $y_{i,t}$  and  $x_{i,t}$ , and consumption choices  $c_t^{\text{IM}}$  and  $c_{i,t}^{\text{HH}}$  for  $i \in \{A, B\}$  such that

- all agents solve their optimization problems given prices and
- markets clear for financial assets, that is  $y_{i,t} + x_{i,t} = 0$ .

## 4.2 Certainty Case

To shed light on the market liquidity, price volatility and capital accumulation in equilibrium, I will characterize the equilibrium under deterministic shock intensity and derive analytical form of agents value function in this section. Later, I will present the existence of the equilibrium, show the steady state with this degree of certainty and finally discuss the contagion effect with the extended setting of arbitrage opportunities from multiple pairs of markets (This part has not been written).

Since the shock intensity  $u_t^i$  is deterministic, the only source of uncertainty comes from the production shock  $\theta_t$ . The intermediaries position in the financial markets, the capital investment decision are unaffected  $\theta_t$  and thus are deterministic consequently. Similarly the price difference between two markets also stays certain.

### 4.2.1 Intermediaries

As a result, the intermediaries optimization problem reduces to choose the optimal amount of capital input in the production sector and positions in both financial markets. Especially, when the technology is linear function of capital, one can solve this simplified optimization through the conjecture that the value function in period  $t$  is

$$V_t^{IM}(K_{t-1}, x_{t-1}) = C \log(a(1 - \gamma)K_{t-1}^\alpha L^\gamma + (1 - \delta)K_{t-1} + x_{t-1}(P_t^A - P_t^B)) + D_t \quad (1)$$

where  $C$  is a constant and  $D_t$  is a deterministic function of  $t$ .

**Proposition 1.** *When the production output is a linear function of capital investment, e.g.,  $\alpha = 1$ , the value function of an intermediary in period  $t$  is given by (1), where  $C = \frac{\rho}{1-\rho}$ . The intermediaries optimal consumption is*

$$c_t^{IM} = \frac{\rho}{1-\rho} [a(1 - \gamma)K_{t-1}^\alpha L^\gamma + (1 - \delta)K_{t-1} + x_{t-1}(P_t^A - P_t^B)].$$

If the price difference is zero, then the intermediaries are indifferent of any possible positions in both financial markets.

### 4.2.2 Households

Unlike the intermediaries, the households still face the uncertainty from the shock  $\theta_t$  even if  $u_t$  is constant. From their budget constraint, we can see that the households exposure to  $\theta_{t+1}$  is the sum of their asset position  $y_t^i$  and the shock intensity  $u_t$ .

We conjecture that under the certainty case, the households value function in period  $t$  is

$$V_{i,t}(y_t^i) = \mathbb{E}[A \log(y_t^i + N_t^i) + B_t^i] \quad (2)$$

where  $A$  is a constant,  $B_t^i$  and  $N_t^i$  a deterministic function of the next period shock  $\theta_{t+1}$  and price level  $P_{t+1}^i$ .

**Proposition 2.** *The value function of an  $i$ -household in period  $t$  is given by (2), where  $A = \frac{\beta}{1-\beta}$ . In particular, if the  $\theta_t$  is deterministic, the households optimal position in asset  $i$  in period  $t$  takes the following form by the first-order condition:*

$$y_t^i = \frac{A y_{t-1}(\theta_t + P_t^i) + A a \gamma K_{t-1}^\alpha L^{\gamma-1} + A u_{t-1}^i \theta_t - P_t^i N_t^i}{(1+A)P_t^i}. \quad (3)$$

where  $A = \frac{\beta}{1-\beta}$  and  $N_t^i$  is deterministic function of  $i$  and  $t$ .

### 4.2.3 Steady State

We now consider a stationary version of the riskless arbitrage case. Stationarity lays foundation for characterizing the equilibrium and analyzing the dynamics of intermediaries' capital accumulation and market liquidity.

**Proposition 3.** • *If the intensity shocks  $u_t$  are deterministic and  $[\frac{1-\rho(1-\delta)}{a\rho\alpha(1-\gamma)L^\gamma}]^{\frac{1}{\alpha-1}} \geq \frac{2\bar{u}\bar{\theta}}{1-\delta}$ , then the intermediaries' physical capital  $K_t$  converges over time to the unique steady-state level  $K^* = [\frac{1-\rho(1-\delta)}{a\rho\alpha(1-\gamma)L^\gamma}]^{\frac{1}{\alpha-1}}$  and the price discrepancies  $\psi_t$  decreases to  $\psi^*=0$ . That is, the intermediaries eventually close out the price difference and the collateral constraint becomes slack. If  $K_t > K^*$ , then  $K_t$  decreases towards  $K^*$  and the price difference stays at zero. If  $K_t < K^*$ , then  $K_t$  increases towards  $K^*$  and the price difference converges to zero if it is non-zero at the beginning.*

• *Otherwise, if the intensity shocks  $u_t$  stay at a constant level and  $[\frac{1-\rho(1-\delta)}{a\rho\alpha(1-\gamma)L^\gamma}]^{\frac{1}{\alpha-1}} <$*

$\frac{2\bar{u}\bar{\theta}}{1-\delta}$ , then given the distribution of  $\theta_t$  there exists a unique constant price difference level  $\psi^* < 0$ , such that  $K_t$  monotonically converges to a unique steady state  $K^* = \left[ \frac{2\bar{\theta} - \delta\psi^* - 2(1-\delta)\rho\bar{\theta}}{(2\theta + \psi^*)\rho\alpha(1-\gamma)\alpha L^\gamma} \right]^{\frac{1}{\alpha-1}} < \frac{2\bar{u}\bar{\theta}}{1-\delta}$ . In this case, the collateral constraint binds at such steady state and the intermediaries can only provide partial liquidity to both markets. If  $K_t > K^*$ , then  $K_t$  decreases towards  $K^*$  and the absolute value of price difference decreases to  $|\psi^*|$ . If  $K_t < K^*$ , then  $K_t$  increases towards  $K^*$  and the absolute size of price gap  $\psi_t$  increases to  $|\psi^*|$ .

When the intensity shock is deterministic and the intermediaries are patient enough to accumulate capital until the collateral constraint eventually becomes slack all the time, i.e.,  $K^* > \frac{2\bar{u}\bar{\theta}}{1-\delta}$ , the economy can reach a steady state in which the intermediaries provide full liquidity to households in both markets. As a result, the price disparege between markets is completely eliminated and both prices converge to and stay at zero. In this steady state, the intermediaries receive fixed income only from the production sector as they can gain no arbitrage profit through exploiting the price gap. In turn, they invest the same amount of their income in each period as production capital, which can fulfill the margin requirement of their positions in the financial markets to satisfy all the equity demands, i.e.,  $x_t = u_t$ . Households from both countries, on the other hand, realize full risk sharing through trading risky asset with the intermediaries and enjoy smooth consumption over time. Consequently, there is no risk premium for the financial assets, as prices stay at zero in both markets.

If the intensity shock is constant and the intermediaries are not patient enough to accumulate sufficient capital to keep the collateral constraint slack, then the intermediaries can also manage to maintain a steady state with constant capital investment in the production sector and liquidity providing in both financial markets. In this steady state, however, as the collateral constraint is binding all the time, the intermediaries can only provide partial liquidity to both markets and the price difference stays at a constant level  $\psi^* < 0$ . Due to the constant price gap and fixed liquidity providing over time, the intermediaries cannot profit from arbitrage neither. Therefore, they can only receive a steady state income from the production sector and invest the same amount in every period as capital. Consequently, in this case the capital return is higher than the pure production return since the physical investment also has positive marginal returns from serving as collateral. However, the need for immediate consumption prevents the intermediaries

from further accumulating capital despite of higher return through production and closing out the price disparege. As the intermediaries cannot fully satisfy the asset demand, households from both markets are exposed to a fixed degree of income risk. Their consumption varies over time with the realization of  $\theta_t$ .

#### 4.2.4 Equilibrium

**Proposition 4.** *A competitive equilibrium exists in which the price difference  $\psi_t$ , intermediaries' capital investment  $K_t$  and the positions of the financial assets  $x_t^i$  are deterministic.*

#### 4.2.5 Shock Reaction

In this section, we aim to investigate the economy's intermediate and long run reaction following shocks to physical capital and financial losses. Specifically, we ask the following questions: 1) how the financial markets respond over time after a sudden shock on intermediaries' physical capital? 2) How the intermediaries have to adjust their real sector production plan following an expected loss from the financial markets? We focus our shock analysis relative to the steady state we discussed in Section 4.2.3, especially on the version in which there is a constant non-zero price disparege over time.

Corollary 1. Suppose that the intermediaries' capital investment encounters an unexpected loss at the end of the period  $t$  after their positions in the financial markets are already realized.

- Depending on the size of the unexpected loss to the intermediaries' capital, if the resulting total income of intermediaries drops to zero or below, then the intermediaries are forced to liquidate all the assets and go bankrupt. The market liquidity dries out completely and there will be zero capital investment in the production sector, resulting in no aggregate output or labor income for the next period.

$$K_t = 0, \quad x_t = 0, \quad |\psi_t| > |\psi^*|, \quad |y_t^i| = 0.$$

- The markets' immediate reaction is that at  $t+1$  the price gap increases, liquidity

and collateral decrease, and the intermediaries reduce their positions in the financial markets and their investment in the production sector:

$$|\psi_{t+1}| > |\psi^*|, \quad |y_{t+1}^i| < |y^*|, \quad x_{t+1} < x^*, \quad K_{t+1} < K^*.$$

And the marginal return of capital investment both as production input and as collateral increases.

- In the long term, the price gap, market liquidity, collateral amount, intermediaries' positions in the financial markets and their physical investment in the production gradually recover and monotonically converge to their steady state level:

$$\begin{aligned} \psi_t > \psi_{t+1} > \cdots > \psi^*, \quad |y_t^i| < |y_{t+1}^i| < \cdots < |y^*|, \quad x_t < x_{t+1} < \cdots < x^*, \\ K_t < K_{t+1} < \cdots < K^*. \end{aligned}$$

And the marginal return of capital investment also gradually reduces to the steady state level.

Suppose the shock to the capital happens at the end of time  $t$  right before the intermediaries transform it into production output, then the direct effect at  $t + 1$  is that the intermediaries capital income suddenly drops. To smooth the consumption after the shock, the intermediaries inevitably have to reduce their physical investment at  $t + 1$ , which tightens further the collateral constraint, which is already binding at the pre-shock level in the steady state. Thus, the size of positions, which intermediaries can afford in both financial markets, also decreases because of the diminishing collateral. Subsequently, the price discrepancy at  $t + 1$  enlarges as a result of less liquidity supply in the markets.

Meanwhile, such market reaction also gives rise to the intermediaries loss in the financial markets, which reduces their total income even further. This is because the previous steady state positions  $x^*$  has now incurred a larger “cash outflow” than the arbitrage profit from taking the current position  $x_{t+1}$ . Thus the marginal return of the capital investment at  $t + 1$  becomes higher, as it now not only contributes higher than the steady state production return but also indirectly helps lessen the immediate financial loss for the intermediaries.

Households, on the other hands, suffer from sudden less liquidity in the equity markets and obtain lower labor income at  $t + 1$ . However, they also receive positive capital gain from the unexpected price changes.

In the long run, as the capital below the steady state level can result in both higher production return and collateralization value, the intermediaries have incentive to invest more in the production sector over time. As a result, the growing capital not only contributes to more production income for the intermediaries and higher wage level for the households, but also enables more liquidity supply in both financial markets, bringing the price differences down over time. During this process, the intermediaries also yield positive arbitrage cash flows as the price gaps decrease while their position size increases over time. Thus, the economy recovers and eventually converges back to its pre-shock steady state.

Nevertheless, in case there is an extreme blow to the production capital, the intermediaries might become so constrained and the price gap might explode to a degree that they cannot afford to cover the losses in the financial markets. In this disastrous case, the intermediaries will have to liquidate and go bankrupt immediately, without being able to absorb any price differences. Therefore, the liquidity is totally and permanently dried out and further external aid is needed to restore the economy.

Corollary 2. Suppose that there is a sudden and permanent significant increase in the size of the intensity shock at  $t$ , which raises the market demand of liquidity and causes a deficit to the intermediaries' income from financial markets.

- Depending on the size of the unexpected deficit to the intermediaries' income, if it exceeds their total income from production, then the intermediaries are forced to liquidate all the assets and go bankrupt. The market liquidity dries out completely and there will be zero capital investment in the production sector, resulting in no aggregate output or labor income for the next period.

$$K_t = 0, \quad x_t = 0, \quad |\psi_t| > |\psi^*|, \quad |y_t^i| = 0.$$

- If the unexpected deficit is less than the total production income for the intermediaries:

- Then the intermediaries' immediate response is to liquidate part of the production return and reduce their current consumption to cover for the loss, and the resulting less income might further force them to reduce their physical investment for the next period. Accordingly, the market liquidity drops and the price discrepancies enlarges due to the lack of collateral. However, the profitability from providing liquidity in the financial markets increases. In the meantime, the market immediate reaction is that the price gap jump above  $|\psi_t| > |\psi^*|$  and the liquidity supply drops below  $x_t < x^*$ .

$$x_t < x^*, \quad |\psi_t| > |\psi^*|, \quad |y_t^i| < |y^*|.$$

- However, in the long run, the price gap, market liquidity, intermediaries' positions in the financial markets and their physical investment in the production gradually recover to their new steady state level:

$$\begin{aligned} \psi_t > \psi_{t+1} > \dots > \psi_n^*, \quad |y_t^i| < |y_{t+1}^i| < \dots < |y_n^*|, \quad x_t < x_{t+1} < \dots < x_n^*, \\ K_t < K_{t+1} < \dots < K_n^*. \end{aligned}$$

When the shock intensity at  $t$   $u_t$  suddenly jumps to a significantly higher level and is expected to stay at this level in the future, then the relative demand from both markets increases permanently. Thus the price disparage enlarges subsequently as the liquidity supply in the market is not sufficient to match the increased demand. For one thing, the financial constraint implies that the future enlarged price gap will require more collateral for each position taken by the intermediaries. Hence the intermediaries become more severely constrained to provide liquidity. For another, the intermediaries' decision to invest physical capital is affected both by the income effect and the substitution effect. On one hand, the intermediaries' budget constraint shrinks because of the loss in the financial markets, which induces the intermediaries to cut down both consumptions and the physical investment. On the other hand, the increased price difference gives rise to higher profitability of arbitrage in the financial market, thus rendering the collateral return more appealing to encourage the intermediaries to invest more in the production sector.

However, in the long run the profit from arbitrage in the financial markets allows



the intermediaries to increase investment in the capital and provide more liquidity, thus bringing the price difference down over time.

## 5 Risky Arbitrage

In this section, we explore the situations of the risky arbitrage, in which intermediaries now face uncertainty from the production shock intensity  $u_t$  compared to the previous deterministic case. We aim to investigate the effects on the market liquidity providing, price volatility, aggregate production and social welfare through documenting the changes of intermediaries investment policy on both physical capital and financial assets. Furthermore, we also look at the intermediaries capital evolution process after unexpected shocks and discuss the policy implication on the welfare and aid solution to potential liquidity dry out in the short term. At the moment, we only consider random intensity shock with non-trivial known distribution.

In particular, we conduct a concise three-period model to numerically analyse the dynamics of capital accumulation, market liquidity and asset prices in equilibrium. We also compute for the infinite-horizon case and find the same qualitative results hold. So we provide the detailed recursive formation and computation algorithm only in the appendix.

### 5.1 Three Period Model

The three period model inherits the same setup from the previous general model, except both the assets' and the agents' lifespan, as the name suggested, extend only three time intervals i.e.,  $t = 1, t = 2$  and  $t = 3$ . In particular, at  $t = 3$  there will be no trading of the financial assets in neither market, since after this period all agents die out and no households demand the assets to hedge their risk anymore. As a result, the prices converge to zero in both markets.

Thus, the intermediaries' optimization problem in the three period model with initial

physical capital  $K_0$  and asset position  $x_0$  can be shown as the following:

$$\max_{\substack{C_1^{\text{IM}}, C_2^{\text{IM}}, \\ C_3^{\text{IM}}, K_1, \\ K_2, x_1, \\ x_2}} \log C_1^{\text{IM}} + \rho \mathbb{E} \log C_2^{\text{IM}} + \rho^2 \mathbb{E} \log C_3^{\text{IM}}$$

subject to the budget constraints at  $t = 1, 2, 3$

$$C_1^{\text{IM}} = (P_1^{\text{A}} - P_1^{\text{B}})(x_0 - x_1) + a(1 - \gamma)K_0^\alpha L^\gamma + (1 - \delta)K_0 - K_1$$

$$C_2^{\text{IM}} = (P_2^{\text{A}} - P_2^{\text{B}})(x_1 - x_2) + a(1 - \gamma)K_1^\alpha L^\gamma + (1 - \delta)K_1 - K_2$$

$$C_3^{\text{IM}} = a(1 - \gamma)K_2^\alpha L^\gamma + (1 - \delta)K_2$$

and the financial constraints at  $t = 1, 2$ :

$$\min_{P_2^{\text{A}}, \theta_2} \{ \min_{P_2^{\text{A}}, \theta_2} (P_2^{\text{A}} + \theta_2)x_1, 0 \} + \min_{P_2^{\text{B}}, \theta_2} \{ \min_{P_2^{\text{B}}, \theta_2} -(P_2^{\text{B}} + \theta_2)x_1, 0 \} + (1 - \delta)K_1 \geq 0$$

$$\min_{\theta_3} \{ \min_{\theta_3} \theta_3 x_2, 0 \} + \min_{\theta_3} \{ \min_{\theta_3} -\theta_3 x_2, 0 \} + (1 - \delta)K_2 \geq 0$$

Note that for simplicity here and henceforth we assume  $u_t = u_t^{\text{A}} = -u_t^{\text{B}} > 0$  and therefore in equilibrium  $x_t = x_t^{\text{A}} = -x_t^{\text{B}} > 0$  for  $t = 1, 2, 3$ . Intermediaries constitute a continuum of population of measure 1. Accordingly we assume that in the three period model both markets have a continuum of households of measure  $L/2$ . Thus, households' optimization problem reduces as below.

For Household A

$$\max_{\substack{C_1^{\text{A}}, C_2^{\text{A}}, \\ C_3^{\text{A}}, y_1^{\text{A}}, \\ y_2^{\text{A}}}} \log C_1^{\text{A}} + \beta \mathbb{E} \log C_2^{\text{A}} + \beta^2 \mathbb{E} \log C_3^{\text{A}}$$

subject to

$$C_1^{\text{A}} = P_1^{\text{A}}(y_0^{\text{A}} - y_1^{\text{A}}) + a\gamma K_0^\alpha L^{\gamma-1} + bK_F + (u_0 + y_0^{\text{A}})\theta_1$$

$$C_2^{\text{A}} = P_2^{\text{A}}(y_1^{\text{A}} - y_2^{\text{A}}) + a\gamma K_1^\alpha L^{\gamma-1} + bK_F + (u_1 + y_1^{\text{A}})\theta_2$$

$$C_3^{\text{A}} = a\gamma K_2^\alpha L^{\gamma-1} + bK_F + (u_2 + y_2^{\text{A}})\theta_3.$$

For Household B,

$$\max_{\substack{C_1^B, C_2^B, \\ C_3^B, y_1^B, \\ y_2^B}} \log C_1^B + \beta \mathbb{E} \log C_2^B + \beta^2 \mathbb{E} \log C_3^B$$

subject to

$$C_1^B = P_1^B(y_0^B - y_1^B) + a\gamma K_0^\alpha L^{\gamma-1} + bK_F + (-u_0 + y_0^B)\theta_1$$

$$C_2^B = P_2^B(y_1^B - y_{2ln}^B) + a\gamma K_1^\alpha L^{\gamma-1} + bK_F + (-u_1 + y_1^B)\theta_2$$

$$C_3^B = a\gamma K_2^\alpha L^{\gamma-1} + bK_F + (-u_2 + y_2^B)\theta_3$$

### 5.1.1 Equilibrium

In the three period model setup, a competitive equilibrium consists of price  $P_t^A$ ,  $P_t^B$ , intermediaries' capital investment  $K_t$  and positions in the financial assets  $x_t$  for the intermediaries and  $y_t^A$ ,  $y_t^B$  for the households in market A and B, for  $t = 1, 2$ , such that capital investment and positions are optimal given prices and the markets for financial assets clear:

$$x_t + \frac{L}{2}y_t^A = 0 \quad \text{and} \quad -x_t + \frac{L}{2}y_t^B = 0, \quad \text{for } t = 1, 2.$$

### 5.1.2 General Numerical Set-ups

For simplicity, we assume both the volatility of the asset  $\theta_t$  for  $t = 2, 3$  and the shock intensity at  $t = 2$ ,  $u_2$  follow binomial distributions. Specifically, we assume that  $\theta_t$  take values of either  $-\bar{\theta}$  or  $\bar{\theta}$  with probability equal to  $p_n$  and  $1 - p_n$  each. Similarly, we set  $u_2$  to be a random variable with probability  $p_l$  to be  $u_{\max}$  or  $1 - p_l$  to be  $u_{\min}$ . The following Table 1 shows the general parameter values as benchmark for later computation.

$\alpha$	$\beta$	$\gamma$	$\rho$	$\delta$	a	b	$K_0$	$K_H$	$x_0$	$u_0$	$u_1$	$p_l$	$p_n$	$\theta_1$	$u_{\max}$	L	$u_{\min}$
0.5	0.9	0.5	0.9	0.5	4	2	10	7	0	1	1	0.5	0.5	2	7	2	1

Table 1: The Set of the Parameter Values

### 5.1.3 General Findings under Uncertainty

In this section, we conduct a simple numerical thought experiment and aim to examine agents' reaction when facing uncertainty about the intensity shock compared to the deterministic counterpart.

In the framework of three period model, we control for all the other parameter values except for the shock intensity setting. In particular, we set the distribution of  $u_t$  in the uncertain case to be binomial, with equal chance of  $u_t$  equal to  $u_{\max}$  or  $u_{\min} = 1$  each, while setting the intensity value to be the average of  $u_{\max}$  and  $u_{\min} = 1$  for the certainty case. By varying the upper bound  $u_{\max}$ , we can observe the changes of the difference in liquidity providing, capital investment and price discrepancies under the influence of different spread size of shock intensity values. Thus we can have a straightforward comparison of [one period] difference in equilibrium between the certainty case and the uncertain one. Table 2 shows the key comparison with varying spread size between two extreme bounds of  $u_t$  for the uncertain case.

We can see that at  $t = 1$ , the liquidity supply  $x_1$  under certainty case is much greater than that of the uncertain counterpart. The capital investment  $K_1$  under certainty is also greater for not-so-extreme intensity values. For the extremely high spread of  $u_{\max}$  the capital input  $K_1$  under uncertainty exceeds the certainty counterpart. The price gap sizes  $|\psi_1|$  for the certainty case is uniformly lower than those under uncertainty.

Due to the uncertainty the intermediaries have to post extra collateral to cover for the next period extreme realization of prices instead of covering the actual prices in the certainty case. That's why the positions in the uncertainty case are in general more 'expensive' in terms of collateral demand. Also compared to the certainty case, the intermediaries have to consider the potential large cash outflow in the next period resulting from  $x_1$  with the extreme realization of demand intensity. As a result, the intermediaries are more cautious in providing liquidity at  $t=1$  compared with the certainty case. Consequently, the price discrepancies widen up.

u_max	5	6	7	8	9	10	11	12	13	14
u_c	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5
x_1.c	0.743379	0.713549	0.681009	0.64696	0.612821	0.580257	0.551205	0.527871	0.512734	0.508568
x_1.u	0.556861	0.486255	0.426191	0.382883	0.356555	0.344081	0.343235	0.35417	0.379443	0.423349
x_2.c	0.600624	0.632826	0.66998	0.713824	0.766598	0.831117	0.910728	1.009107	1.12988	1.276194
x_2.h	0.697447	0.815524	0.99931	1.264602	1.606528	2.007873	2.452965	2.932136	3.440927	3.978075
x_2.l	0.549979	0.55873	0.567246	0.577027	0.58968	0.606868	0.630818	0.664928	0.714104	0.784133
K_1.c	8.105979	8.374472	8.636953	8.898694	9.167717	9.456206	9.782648	10.17495	10.6747	11.34205
K_1.u	7.901905	8.070072	8.236823	8.434913	8.699071	9.065845	9.586342	10.34184	11.45602	13.08774
PG_1.c	1.966048	2.434913	2.944904	3.509514	4.14457	4.868652	5.703473	6.674121	7.808512	9.133429
PG_1.u	2.242601	2.917839	3.708459	4.610777	5.635395	6.831328	8.275605	10.05002	12.19974	14.66128
PG_2.c	1.161352	1.428253	1.714475	2.022037	2.351392	2.700278	3.06242	3.426667	3.777491	4.097396
PG_2.h	2.433528	3.220739	4.057395	4.822142	5.425768	5.862811	6.167881	6.375251	6.508521	6.583255
PG_2.l	0.20291	0.198768	0.194767	0.190224	0.184407	0.176557	0.165666	0.150213	0.128013	0.096522

Table 2: Comparison of Equilibrium between certainty and uncertainty

## 5.2 Static Analysis

In this section, we analyze the impact of asset volatility, shock intensity spread, intermediaries' technology in the production sector on the market liquidity, capital investment and asset prices through solving the three period model.

### 5.2.1 Asset Volatility

To study the effect of asset volatility on intermediaries' investment policy, liquidity supply and aggregate production, we conduct a test in which all else is equal except with a series of different volatility magnitude. For the choice of parameter value, we inherit the set in Table 1 except with changing  $\bar{\theta}$  and setting the initial realization of  $\theta_1$  to be equal to  $\bar{\theta}$ .

Table 3 exhibits the results of equilibrium with varying size of asset volatility. From the table, we find that intermediaries increase their physical investment at both  $t=1$  and  $t=2$  to adjust for the higher asset volatility. Meanwhile, they scaled down their positions in the financial markets in both periods. As a result, the price discrepancy widens and the market liquidity keeps drying with increasing asset volatility. However, the utility increases for the intermediaries whereas it decreases in general for the households in both countries.

The underlying reason for the above is quite straightforward. As the assets become more volatile, the intermediaries' are more constrained by the collateral requirement. Put differently, the larger volatility entails the intermediaries to post more capital investment as collateral per share of asset they want to trade. Thus in order to exploit the arbitrage

Table 3: Equilibrium with varying  $\bar{\theta}$ 

$\bar{\theta}$	0.5	1	1.5	2	2.5	3
K_1	6.762634	6.963507	7.264324	7.620384	7.973951	8.418317
K_2_low	3.881615	3.961824	4.118505	4.300608	4.489009	4.71662
K_2_high	4.135514	4.612966	5.540046	7.675339	11.82015	17.51266
X_1	0.94271	0.804705	0.590939	0.433436	0.319597	0.255688
X_2_low	1	0.990456	0.686417	0.537576	0.448901	0.393052
X_2_high	2.067757	1.153241	0.923341	0.959417	1.182015	1.459388
Price Gap 1	0.067606	0.359995	1.058263	2.33929	4.27212	6.970416
Price Gap 2_low	0	0.001077	0.079529	0.208401	0.38819	0.616367
Price Gap 2_high	0.140146	0.723523	1.980895	4.174387	6.829409	9.313355
Utility_IM	2.32616	2.340667	2.365819	2.408052	2.466206	2.527165
Utility_A	3.390426	3.403229	3.406998	3.399519	3.384145	3.355648
Utility_B	3.36691	3.356161	3.33631	3.305106	3.265865	3.213323
$\bar{\theta}$	3.5	4	4.5	5	5.5	6
K_1	9.155935	10.60143	13.75348	19.74006	28.27147	38.43909
K_2_low	5.058153	5.663683	6.866801	8.963627	11.73475	14.84461
K_2_high	24.08004	31.25926	39.10309	47.71742	56.90512	66.42369
X_1	0.225689	0.221652	0.252967	0.329723	0.437474	0.557012
X_2_low	0.361297	0.35398	0.381489	0.448181	0.533398	0.618526
X_2_high	1.720003	1.953704	2.172394	2.385871	2.586596	2.767654
Price Gap 1	10.99254	17.45504	27.39025	38.79093	48.45191	56.11539
Price Gap 2_low	0.884254	1.168947	1.412325	1.543401	1.563222	1.505753
Price Gap 2_high	11.56913	13.63312	15.48026	17.10763	18.63323	20.15219
Utility_IM	2.590565	2.667577	2.776663	2.917226	3.058032	3.182217
Utility_A	3.307897	3.232916	3.118838	2.946612	2.65965	2.000363
Utility_B	3.141308	3.041804	2.9029	2.705495	2.392955	1.707632

profit by providing liquidity to both markets, the intermediaries in equilibrium have to invest more in physical capital to support the positions in financial assets with higher volatility. Intuitively, the equilibrium positions they manage to take also decrease with the wider spread of asset payoffs. Consequently, the price differences in equilibrium enlarges due to the drying liquidity supply. As a side effect of the increased physical investment, the aggregate production in period  $t=2,3$  increases, bringing more production income to the intermediaries and higher wage level to households.

Thus with increasing profitability from providing liquidity in the financial markets and higher production income from physical investment, the intermediaries gain more utility from the growing asset volatility. On the contrast, the households suffer from less liquidity and hence have to confront more exposure to risk in both relative and absolute measure, leading to a decreasing utility level with increasing asset volatility.

### 5.2.2 Riskiness of Risky Arbitrage

In order to further investigate the effects of varying degrees of arbitrage risk for the intermediaries on their decisions of capital accumulation and arbitrage behavior in the financial markets, we conduct static analysis with respect to the upper bound of the magnitude of the shock intensity. Such intensity also determines the size of liquidity demand in both financial markets, and therefore its absolute size will have direct influence on asset price elasticity.

As before we also adopt the parameter value set from Table 1 except that we will alter the size of  $u_{\max}$ . In detail, we set the upper bound of the shock intensity varying from 5.0 to 14.0 while keeping the rest of parameter value unchanged.

We present the results in Table 4. From the table, we observe that with expanding spread of the shock intensity, the intermediaries increase their equilibrium physical investment level at both  $t=1$  and  $t=2$ . Also the intermediaries first scale down their positions at time  $t=1$  in the financial markets with the growing riskiness of arbitrage and then increases for the extreme values, whereas they strictly increase their positions at  $t=2$ ,  $x_2$ . The resulting price gap  $\psi_1$  keep widening while the price gap for the next period  $\psi_2$  strictly diminishes with the enlarging spread of the shock intensity.

max u2	5	6	7	8	9	10	11	12	13	14
K.1	7.412342	7.512617	7.620384	7.795735	7.967607	8.18628	8.477446	8.883838	9.480717	10.40219
K.2_low	4.20919	4.254165	4.300608	4.369626	4.437222	4.521347	4.631587	4.783304	5.002729	5.334932
K.2_h	5.330276	6.231574	7.675339	9.786489	12.53241	15.74421	19.28436	23.06693	27.04944	31.225
X.1	0.561388	0.490104	0.433436	0.418374	0.398737	0.3909	0.392366	0.402778	0.424295	0.46247
X.2_low	0.526149	0.531771	0.537576	0.546203	0.554653	0.565168	0.578948	0.597913	0.625341	0.666867
X.2_h	0.666285	0.778947	0.959417	1.223311	1.566551	1.968027	2.410545	2.883366	3.38118	3.903124
Price Gap 1	1.415956	1.843832	2.33929	2.859451	3.411006	4.006225	4.69195	5.530058	6.59697	7.973049
Price Gap 2_low	0.21391	0.211179	0.208401	0.204407	0.200495	0.195662	0.18936	0.180714	0.168246	0.14943
Price Gap 2_h	2.479931	3.298893	4.174387	4.993106	5.615713	6.061871	6.375337	6.595696	6.748942	6.849614
Utility_IM	2.365228	2.38342	2.408052	2.438345	2.471134	2.504328	2.537734	2.572525	2.610941	2.656349
Utility_A	3.398884	3.383889	3.365099	3.342919	3.316678	3.285774	3.249042	3.204971	3.151595	3.086503
Utility_B	3.304471	3.289476	3.270686	3.248506	3.222265	3.191361	3.154629	3.110558	3.057181	2.99209

Table 4: Equilibrium Results with Varying Riskiness of Risky Arbitrage

The intermediaries scale up their physical investment with respond to the increasing upper bound of the shock because of the collateral constraints. Previously in the certainty case, the financial constraint only enforces the intermediaries to collateralize for the actual cash outflow together with the payoff volatility caused by previous position. In the risky arbitrage situation, however, the intermediaries have to post collateral to cover for the

maximum possible cash outflow incurred only when the extreme intensity realizes. Thus as the extreme value of intensity increases, the intermediaries have to increase their capital collateral to fulfill the more severe financial constraints. Accordingly, in equilibrium the physical investment in both period  $t=1$  and  $t=2$  will increase with expanding  $u_{\max}$ . Again as a side effect the production output multiplies, benefiting the intermediaries with more capital rent and the households with more labor income.

With the more extreme bound of asset demand, in equilibrium the intermediaries will decide their optimal asset positions with a trade-off between the immediate arbitrage profit and potentially more costly cash outflow caused by the previous position. The more extreme upper bound of the demand will lead to higher price gap in the next period, giving rise to greater outflow resulting from the previous transaction.

In period  $t=2$  however, by construction of three period model, intermediaries do not have to worry about future consequence of current position. Therefore, they would take as much position as they can to exploit the arbitrage profit. As the  $u_{\max}$  increases, the households become more eager to hedge their risk and less price sensitive towards liquidity, the price gap thus enlarges for the extreme realization of  $u_2$ . The opposite happens when the demand realize to be the lower bound because of the intermediaries increased positions at  $t=2$ .

The utility of the intermediaries increases slowly with the demand upper-bound due to the increment of physical production and better investment opportunity in the financial markets despite the higher risks. However, the utility of households drops with more extreme possible value of  $u_{\max}$  since they become more eager for liquidity whereas the liquidity supply decreases relative to their demand.

### 5.3 Spillover From Production Sector to Financial Markets

To study the potential spillover effects from production sector to the financial markets, for example, the response to a sudden change in physical capital to the market liquidity and asset prices, we conduct a static analysis of the initial physical capital in the three period model framework. In particular, by setting different initial capital  $K_0$  while controlling for other parameter values in Table 1, we aim to simulate the intermediaries' reaction



K_0	5	6	7	8	9	10	11	12	13	14	15
K_1	4.779384	5.39665	5.988264	6.559397	7.113641	7.620384	8.084474	8.541122	8.991304	9.435804	9.875262
X_1	0.266311	0.305224	0.343162	0.380349	0.416936	0.433436	0.430442	0.427589	0.42489	0.422353	0.419981
K_2_low	3.150685	3.412543	3.656221	3.885673	4.103602	4.300608	4.47956	4.653077	4.821828	4.986355	5.147099
K_2_high	6.375252	6.671727	6.944411	7.198575	7.437803	7.675339	7.913143	8.142067	8.363237	8.57756	8.785782
Lagr_multi_1	0.009458	0.006193	0.003814	0.002009	0.000595	-0.1526	-0.44961	-0.73793	-1.01851	-1.29218	-1.5596
Price Gap 1	2.62257	2.552019	2.489057	2.431796	2.378995	2.33929	2.311372	2.285212	2.260532	2.237112	2.214781
Price Gap 2 low	0.274855	0.259598	0.245479	0.232244	0.21972	0.208401	0.19812	0.188183	0.178546	0.169175	0.1600404
Price Gap 2 high	4.580101	4.480301	4.39449	4.319124	4.251871	4.174387	4.08795	4.008668	3.935462	3.86748	3.8040361
Financial Income_1	0.698419	0.778938	0.854151	0.92493	0.991888	1.013933	0.994912	0.977131	0.960477	0.944851	0.930166

Table 5: Equilibrium with Varying Initial Capital  $K_0$

after changes of their capital and show the immediate effects to the rest. We exhibit the equilibrium results in Table 5.

As the initial physical investment  $K_0$  diminishes, the equilibrium physical investment at  $t=1,2$ ,  $K_1$  and  $K_2$  decrease monotonically. Moreover the intermediaries' optimal position size  $x_1$  is hump-shaped, decreasing when their collateral constraint becomes binding and increasing when it is slack. Moreover, the price discrepancies at both periods increase uniformly. Intermediaries' income from financial market at  $t=1$  keeps decreasing while the financial constraints are still binding.

It makes sense for the equilibrium position size  $x_1$  with varying  $K_0$  to behave differently before and after the collateral becomes loose. That's because that when  $K_1$  is sufficient to support a non-binding financial constraint, the optimality condition for choosing the position size is

$$\frac{1}{C_1^{\text{IM}}}(P_1^{\text{B}} - P_1^{\text{A}}) = \rho \mathbb{E} \frac{P_2^{\text{B}} - P_2^{\text{A}}}{C_2^{\text{IM}}}$$

which indicates that the marginal utility from consumption at  $t=1$  multiplying the marginal profit from taking one unit of  $x_1$  is equal to the expected marginal loss of utility multiplying the marginal cash outflow at  $t=2$  caused by  $x_1$ . When the collateral constraint is binding, however, the first-order condition with respect to  $x_1$  becomes

$$(P_1^{\text{B}} - P_1^{\text{A}}) \frac{1}{C_1^{\text{IM}}} = \rho \mathbb{E} \frac{P_2^{\text{B}} - P_2^{\text{A}}}{C_2^{\text{IM}}} + \lambda_1 \left[ \mathbb{1}_{(P_{2h}^{\text{B}} + \theta_2)x_1 < 0} (P_{2h}^{\text{B}} - \text{sgn}(x_1)\bar{\theta}) - \mathbb{1}_{-[P_{2h}^{\text{A}} + \bar{\theta}\text{sgn}(x_1)]x_1 < 0} (P_{2h}^{\text{A}} + \text{sgn}(x_1)\bar{\theta}) \right]$$

The last item on the RHS is the marginal loss associated with losing collateral caused by taking an extra unit of position  $x_1$ . Hence when collateral constraint is binding, the marginal loss from financial market at  $t=2$  is less than the marginal gain at  $t=1$ . In this situation when  $K_0$  decreases and so does the collateral capital  $K_1$ , one expects the optimal position  $x_1$  also shrinks.

Suppose following a sudden drop of the initial physical capital  $K_0$ , e.g. from 11 to 5, we find that the intermediaries now face a tighter budget constraint, consequently they reduce the physical investment  $K_1$ . In the meantime, the intermediaries also become more financial constrained and have to scaled down their positions  $x_1$ , which leads to a reduction of their income from financial market. However, intermediaries' profitability in the financial markets increases, as reflected by the widened price gap at  $t=1$ . Higher returns from trading positions together with higher production returns due to the technology function help the intermediaries accumulate more physical capital for the next period and eventually revert to the "normal" level over time. As indicated from the results,  $\Delta K_2 \leq \Delta K_1$ . Therefore within certain parameter range, the financial market helps the production sectors to recover.

## 6 Heterogeneous Belief

Previously we focus on the implications derived from the assumption that both intermediaries and households hold the same belief over the distribution of future shock intensities. However to show how the heterogeneous belief will disturb the self-recovery process in both financial and real sectors through endogenously determined collateral constraints, now we extend the model by allowing them to hold different beliefs.

For simplicity, we assume for now both sides of households hold the same belief over the future shock intensity all the time. Also we assume that each period households only form their estimation for the shock intensity one period ahead and agree with the intermediaries over further periods distributions.

### 6.1 Constant Heterogeneous Belief

We start illustrating the basic effects that heterogeneous beliefs plays in agents' activities by the simplest model. In particular, we assume the following

- Constant intensity shocks.
- Constant but heterogeneous beliefs

- HH determines the IMs collateral constraint based on their belief of the next period shock intensity or prices.
- IM knows the exact future shock intensity while HH differs with the belief
- Case 1: HH holds constant but more optimistic belief
- Case 2: HH holds constant but more pessimistic belief

**Proposition 5.** *The steady states If IM manage to close the price gap in the benchmark case then  $K_p^* \geq K^* \geq K_o^*$ ,  $x_p^* \leq x^* = x_o^* = u$ .*

*If the FC are all binding at the steady state for all three cases, then  $x_p^* < x^* < x_o^*$ ,  $|\psi_p| > |\psi^*| > |\psi_o|$ , the capital level relationship is unambiguous.*

## 6.2 Changing Heterogeneous Belief

Assume the following:

- Suppose each agent form their estimation of the next period intensive shock at each period. They use unconditional expectation to form those of even further periods.
- HH from both sides form the same estimation of the next period intensity shock each period
- HH determines the IMs collateral constraint based on their belief of the next period shock intensity or prices
- Assume IMs belief over the next period intensity shock is always accurate.
- IM do not know HHs estimation in advance.
- However, HHs belief is changing and deviating more from the realization of the shock intensity
  - Case 1: HH becomes more and more optimistic over time
  - Case 2: HH becomes more and more pessimistic over time

**Proposition 6.** *For case 1:*

*IMs FC becomes looser and looser, thus providing more and more liquidity on the markets. Eventually FC disappears and IM manage to close out all the price gaps in the steady state, realizing full risk sharing.*

*For case 2:*

*IMs FC becomes more and more restricting. IM have to decrease consumptions and can provide less and less liquidity given the capital they invest. As a result, the price gap increases and IM incur financial losses as they tend to over take positions in the financial markets. IM will first increase their capital but eventually the physical investment level will decrease and liquidity supply will dry out.*

### **6.3 HHs belief interact with the liquidity providing**

Basic Assumptions:

- Suppose each agent form their estimation of the next period intensive shock at each period. They use unconditional expectation to form those of even further periods.
- HH from both sides form the same estimation of the next period intensity shock each period. (can be relaxed)
- HH determines the IMs collateral constraint based on their belief of the next period shock intensity or prices
- Assume IMs knows the true distribution of the next period intensity shock. (For simplicity, but can be relaxed)
- IM do not know HHs estimation in advance but they assume (naively) that although HH might currently differ with them in the opinion for the next period intensity estimation  $e_t$ , but HH will agree with them for the one-period-ahead estimation of the intensity in all future periods. Same for HH? (Also can be relaxed)
- HHs estimation is more pessimistic than IM and are more sensitive to shocks;
- HH form their estimation for the next period intensity shock based on the difference from the last period liquidity supply and their expected level from the steady state as well as historical maximal shock. e.g. suppose the unconditional expectation of

shock intensity is  $u$ . If at  $t$  the liquidity supply is  $x_t$ , then at time  $t+1$  HH form their estimation of the shock intensity for the next period is  $e_{t+1} = f(x_t) = u + \lambda_1(x_t - x^*) + \lambda_2 * (u + \Delta u)$ .

Suppose the shock intensity is constant and the economy is at its steady state. Initially HH and IM share the same precise belief of all the future intensity distribution, i.e.  $e_t = u$ . However compared to IM, HH are more sensitive to shocks and are more emotional in that any exogenous shock or decrease in liquidity will cause them to permanently change their estimation over the next period shock intensity. If suddenly at  $t$ , there is a once-off exogenous shock to the intensity such that it experiences an abrupt jump, i.e.,  $u_t = u + \Delta u$ . The immediate reaction are:

- The current liquidity demand subsequently increases;
- Sensitive HH adjust their expectation over the next period shock intensity to be  $e_t = u + \mu \Delta u$ , with  $\mu > 1$ . As a result, IMs FC becomes more restrictive due to HHs pessimistic expectation;
- The liquidity supply drops as IM cannot support the same volume of asset trading even though they might increase their physical investment;
- Thus the price gap increases and IM suffers from financial loss.

At  $t+1$ , though the realized intensity  $u_{t+1}$  revert to normal, the liquidity drop at time  $t$  triggers HHs panic. Their estimation for the next period intensity  $u_{t+2}$  is  $e_{t+1}^{HH} = u + \mu \Delta u + \lambda(x^* - x_t) = e_t^{HH} + \lambda(x^* - x_t)$ , with  $\lambda > 1$ . Thus IMs FC become even more restrictive, causing the trading more expensive in terms of collateral requirement. On the other hand, the  $t+1$  asset demand reverts to the normal level, making the return of trading relatively less. As a result, IM might end up providing less liquidity and suffers from unexpected financial loss. Also IM might also scale down their physical investment. The dropping liquidity leads to HHs greater panic and pessimism in the next period belief.

As a consequence, the liquidity supply keeps drying out over time and the price gap will start to diverge more. IM suffers from tighter and tighter FC and keeps incurring financial loss. At some point of time, IM start to reduce their physical investment, resulting in less income from production. Eventually IM will go bankrupt and liquidity completely dry

out. HH becomes more and more panic and form more pessimistic estimation of the future market prices. In the meantime, they suffer from less and less risk sharing and less labour income from the production industry. The physical industry also suffers from less capital input. The total production decreases and the consumption of agents become less and more unstable. The economy goes into recession.

## 7 Conclusion

This paper studies how the heterogeneous belief will disturb the self-recovery process in both financial and real sectors through endogenously determined collateral constraints. In particular, for an economy in which some investors are more sensitive to negative news, we examine how relatively trivial, temporary exogenous shocks in financial markets could lead to dramatic, persistent oscillations in liquidity supply, asset prices and aggregate output. In addition, we investigate whether the effects of sector-specific shocks can spill over to other sectors and get amplified over time.

We find that in our baseline model intermediaries present self-recovery capacities after negative shock in either the production sector or the financial sector. In a nutshell, with homogeneous belief there is no amplification or lasting spillover effects across sectors. Instead, financial profitability and production return both become more favourable after negative shocks, together pushing the intermediaries to revert to pre-shock state.

However, We show that with heterogeneous belief the dynamic interaction between belief determined collateral constraint and liquidity supply turns out to be a powerful propagation mechanism by which the effects of shocks persist, amplify and spill over to other sectors.

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## Appendices

### A Proof of Proposition 1

*Proof.* The intermediaries Bellman equation is

$$V_t(W_t) = \max_{c_{t+1}^{\text{IM}}, x_{t+1}, K_{t+1}} \{\rho \log(c_{t+1}^{\text{IM}}) + \rho V_{t+1}(W_{t+1})\} \quad (4)$$

where

$$W_t = W(K_{t-1}, x_{t-1}) = F(K_{t-1}) + \psi_t x_{t-1} = a(1 - \gamma)K_{t-1}^\alpha L^\gamma + (1 - \delta)K_{t-1} + x_{t-1}(P_t^A - P_t^B)$$

$$W_{t+1} = F(K_t) + \psi_{t+1} x_t = a(1 - \gamma)K_t^\alpha L^\gamma + (1 - \delta)K_t + x_t(P_{t+1}^A - P_{t+1}^B)$$

As my conjecture is

$$V_t(W_t) = C \log(W_t) + D_t$$

Substituting it to (4), we have

$$C \log(W_t) + D_t = \max_{c_{t+1}^{\text{IM}}, x_{t+1}, K_{t+1}} \{\rho \log(c_{t+1}^{\text{IM}}) + \rho C \log[F(W_t - x_{t+1}\psi_{t+1} - c_{t+1}^{\text{IM}}) + x_{t+1}\psi_{t+2}] + \rho D_{t+1}\} \quad (5)$$

The first-order condition with respect to consumption is

$$\frac{1}{c_{t+1}^{\text{IM}}} - \frac{CF'((W_t - x_{t+1}\psi_{t+1} - c_{t+1}^{\text{IM}}))}{F(W_t - x_{t+1}\psi_{t+1} - c_{t+1}^{\text{IM}}) + x_{t+1}\psi_{t+2}} = 0 \quad (6)$$

When production function  $F$  is linear with capital input, one can find a unique positive solution of  $c_{t+1}^{\text{IM}}$ . For other parameters of the Cobb-Douglas function, there might be multiple solutions. In general, we should choose the highest positive value for  $c_{t+1}^{\text{IM}}$ .

When  $\alpha = 1$ , the solution for  $c_{t+1}^{\text{IM}}$  in terms of  $W_t$  and  $x_{t+1}$  is

$$c_{t+1}^{\text{IM}} = \frac{(a + 1 - \delta)W_t + x_{t+1}[\psi_{t+2} - \psi_{t+1}(a + 1 - \delta)]}{(C + 1)(a + 1 - \delta)} = \frac{W_{t+1}}{C(a + 1 - \delta)}$$



With this, we can write the right-hand side of (4) as

$$\begin{aligned} \max_{x_{t+1}} \{ & \rho(C+1) \log[(a+1-\delta)W_t + x_{t+1}(\psi_{t+2} - (a+1-\delta)\psi_{t+1})] - \rho \log(a+1-\delta) \\ & + \rho C \log C - \rho(C+1) \log(C+1) + \rho D_{t+1} \} \end{aligned} \quad (7)$$

The maximization in (7) is subject to the financial constraint (2.4).

(a) When the collateral constraint is not binding, applying first order condition with respect to  $x_{t+1}$  yields

$$\psi_{t+2} - (a+1-\delta)\psi_{t+1} = 0$$

That means, when the financial constraint is not binding, the intermediaries are indifferent between any position value of  $x_{t+1}$ . The above maximization becomes

$$C \log(W_t) + D_t = \rho(C+1) \log(W_t) + \rho C \log(a+1-\delta) + \rho C \log C - \rho(C+1) \log(C+1) + \rho D_{t+1}$$

Equating the coefficients in front of  $\log(W_t)$ , we have:

$$\begin{aligned} C &= \frac{\rho}{1-\rho} \\ D_t &= \frac{\rho^2}{1-\rho} \log(a+1-\delta) + \rho \log(1-\rho) + \rho D_{t+1} \end{aligned}$$

And the transversality condition is  $\lim_{s \rightarrow \infty} \rho^s D_{t+s} = 0$  determines  $D_t$ .

(b) Otherwise, from the collateral constraint (2.4) we get

$$x_{t+1} = \frac{(1-\delta)K_{t+1}}{2\bar{\theta} - \psi_{t+2}}$$

$$\text{Thus } c_{t+1}^{\text{IM}} = \frac{(a\psi_{t+2} - 2\bar{\theta}(a+1-\delta))W_t}{a\psi_{t+2} - 2\bar{\theta}(a+1-\delta) + C(a+1-\delta)(-\psi_{t+1}(1-\delta) + \psi_{t+2} - 2\bar{\theta})} \quad \text{and} \quad W_{t+1} = \frac{C(a+1-\delta)(a\psi_{t+2} - 2\bar{\theta}(a+1-\delta))W_t}{a\psi_{t+2} - 2\bar{\theta}(a+1-\delta) + C(a+1-\delta)(-\psi_{t+1}(1-\delta) + \psi_{t+2} - 2\bar{\theta})}.$$

Substitute these to (7), the maximization is now

$$\begin{aligned} C \log(W_t) + D_t &= \rho(1+C) \log(W_t) \\ &\quad - \rho(1+C) \log(a\psi_{t+2} - 2\bar{\theta}(a+1-\delta) + C(a+1-\delta)(-\psi_{t+1}(1-\delta) + \psi_{t+2} - 2\bar{\theta})) \\ &\quad + \rho(1+C) \log((a\psi_{t+2} - 2\bar{\theta}(a+1-\delta)) + \rho C \log C + \rho C \log(1+a-\delta) + \rho D_{t+1} \end{aligned}$$

Again, equating the coefficients in front of  $\log(W_t)$  and the constant terms, we have

$$C = \frac{\rho}{1 - \rho}$$

$$D_t = -\rho(1 + C) \log(a\psi_{t+2} - 2\bar{\theta}(a + 1 - \delta) + C(a + 1 - \delta)(-\psi_{t+1}(1 - \delta) + \psi_{t+2} - 2\bar{\theta}))$$

$$+ \rho(1 + C) \log((a\psi_{t+2} - 2\bar{\theta}(a + 1 - \delta)) + \rho C \log C + \rho C \log(1 + a - \delta) + \rho D_{t+1}$$

The transversality condition is  $\lim_{s \rightarrow \infty} \rho^s D_{t+s} = 0$  determines  $D_t$ .

For other parameters of  $\alpha$  than 1, it can be shown that  $C_{t+1}^{\text{IM}}$  also has at least one positive solution, though maybe not a closed form one, from which we can back out intermediaries' optimal investment  $K_{t+1}$  and financial position  $x_{t+1}$  given  $K_t$  and  $x_t$ .

□

## B Proof of Proposition 2

*Proof.* The households' Bellman equation is

$$V_{i,t}(y_t^i) = \max_{c_{t+1}^i, y_{i,t+1}} \mathbb{E}_t \{ \beta \log(c_{t+1}^i) + \beta V_{i,t+1}(y_{t+1}^i) \}. \quad (8)$$

substitute the budget constraint (1) in period  $t+1$  and the conjecture (2) and apply iterated conditional expectation, we get

$$\begin{aligned} & \mathbb{E}[A \log(y_t^i + N_t^i) + B_t^i] \\ &= \max_{c_{t+1}^i, y_{i,t+1}} \mathbb{E}_t \{ \beta \log(y_t^i(\theta_{t+1} + P_{t+1}^i) + a\gamma K_t^\alpha L^{\gamma-1} + u_t^i \theta_{t+1} - P_{t+1}^i y_{t+1}^i) \\ &+ \beta A \log(y_{t+1}^i + N_{t+1}^i) + \beta B_{t+1}^i \} \end{aligned} \quad (9)$$

Apply the first-order condition with respect to the financial position  $y_{t+1}$  and we have

$$y_{t+1}^i = \frac{A y_t(\theta_{t+1} + P_{t+1}^i) + A a \gamma K_t^\alpha L^{\gamma-1} + A u_t^i \theta_{t+1} - P_{t+1}^i N_{t+1}^i}{(1 + A) P_{t+1}^i}. \quad (10)$$

Therefore, we can write the right-hand side of (9) as

$$\begin{aligned}
& \beta \mathbb{E}_t \left\{ \log[y_t^i(\theta_{t+1} + P_{t+1}^i) + a\gamma K_t^\alpha L^{\gamma-1} + u_t^i \theta_{t+1} - \frac{Ay_t(\theta_{t+1} + P_{t+1}^i) + Aa\gamma K_t^\alpha L^{\gamma-1} + Au_t^i \theta_{t+1}}{(1+A)} \right. \\
& \quad \left. + \frac{P_{t+1}^i N_{t+1}^i}{1+A}] + A \log\left[\frac{Ay_t(\theta_{t+1} + P_{t+1}^i) + Aa\gamma K_t^\alpha L^{\gamma-1} + Au_t^i \theta_{t+1} - P_{t+1}^i N_{t+1}^i}{(1+A)P_{t+1}^i} + N_{t+1}^i\right] + B_{t+1}^i \right\} \\
& = \beta \mathbb{E}_t \left\{ (1+A) \log(y_t^i(\theta_{t+1} + P_{t+1}^i) + a\gamma K_t^\alpha L^{\gamma-1} + u_t^i \theta_{t+1} + N_{t+1}^i P_{t+1}^i) - (A+1) \log(A+1) \right. \\
& \quad \left. + A \log A - A \log(P_{t+1}^i) + B_{t+1}^i \right\} \\
& = \beta \mathbb{E}_t \left\{ (1+A) [\log(y_t^i + N_t^i) + \log(\theta_{t+1} + P_{t+1}^i) + \log(1+A)] + A \log A - A \log(P_{t+1}^i) + B_{t+1}^i \right\}
\end{aligned}$$

where I let  $N_t^i = \frac{a\gamma K_t^\alpha L^{\gamma-1} + u_t^i \theta_{t+1} + N_{t+1}^i P_{t+1}^i}{\theta_{t+1} + P_{t+1}^i}$ .

Equating the coefficients in front of  $\log(y_t^i + N_t^i)$ ,

$$\beta(1+A) = A$$

thus we get  $A = \frac{\beta}{1-\beta}$ .

Equating others makes the Bellman equation hold for all values of  $y_t^i$  if

$$\begin{aligned}
A &= \frac{\beta}{1-\beta} \\
N_t^i &= \frac{a\gamma K_t^\alpha L^{\gamma-1} + u_t^i \theta_{t+1} + N_{t+1}^i P_{t+1}^i}{\theta_{t+1} + P_{t+1}^i} \\
B_t^i &= \beta(1+A) [\log(\theta_{t+1} + P_{t+1}^i) + \log(1+A)] + \beta A \log A - \beta A \log(P_{t+1}^i) + \beta B_{t+1}^i.
\end{aligned}$$

The transversality condition  $\lim_{s \rightarrow \infty} \beta^s B_{t+s}^i = 0$  determines  $B_t^i$ . Similarly if we define  $R_t^i \equiv \frac{P_{t+1}^i + \theta_{t+1}}{P_t^i}$ , then the transversality condition  $\lim_{s \rightarrow \infty} \frac{N_{t+s}^i}{\prod_{m=t+1}^s R_m^i} = 0$  also decides  $N_t^i$ .  $\square$

## C Proof of Proposition 3

*Proof.* First we illustrate the case when the steady state capital stays above the minimal level of keeping the collateral constraint slack all the time, that is,  $K^* \geq \frac{2\bar{u}\bar{\theta}}{1-\delta}$ . The first-order condition with respect to capital investment  $K_t$  with unbinding collateral constraint

is

$$-\frac{1}{c_t^{\text{IM}}} + \frac{F'(K_t)\rho}{c_{t+1}^{\text{IM}}} = 0.$$

Since there is no uncertainty for the intermediaries, they will consume and invest the same amount in the steady state for every period. Thus equating  $c_{t+1}^{\text{IM}}$  and  $c_t^{\text{IM}}$  to the steady state consumption, we get  $\rho F'(K^*) = 1$  and the resulting steady state capital level is  $[\frac{1-\rho(1-\delta)}{a\rho\alpha(1-\gamma)L^\gamma}]^{\frac{1}{\alpha-1}}$ . If this expression turns out to have multiple values, then we take  $K^*$  to be the one such that it maximizes the steady state consumption level  $F(K^*) - K^*$ .

In this case, as long as the initial capital stays above the threshold that keeps the collateral constraint slack, it is an ordinary dynamic programming problem and it is obvious that the above steady state has global convergence. When the initial capital is below the level that keeps the financial constraint slack, that is, when the collateral constraint is binding initially, we can show that the capital will keep growing until the physical investment is sufficient to support a non-binding collateral constraint.

From the first-order conditions with respect to the capital investment at time  $t$

$$-\frac{1}{c_t^{\text{IM}}} + \frac{F'(K_t)\rho}{c_{t+1}^{\text{IM}}} + \mu_t(1 - \delta) = 0$$

we can see that compared with the first-order condition without binding collateral constraint, the capital investment not only have higher production return  $F'(K_t)$  but also have positive return from serving as collateral. Therefore, the optimality condition determines that the intermediaries will tend to invest relatively more capital with binding collateral constraint. Thus the intermediaries will eventually accumulate sufficient capital to support a non-binding collateral constraint. Since there exist global convergence for the non-binding case, the same will apply for those with low level of initial capital.

Second, let us look at the possible case when the steady state capital level is not enough to make the collateral constraint slack all the time, e.g.,  $K^* < \frac{2\bar{u}\bar{\theta}}{1-\delta}$ . Then the intermediaries cannot close all the price gaps between two markets. We claim there exists no steady state if the shock intensity is deterministic but not constant over time. Suppose there do exist a steady state level for the capital investment from the intermediaries and a constant price gap  $\psi^*$  can also be sustained. Then different intensity shock will cause the intermediaries to provide different liquidity to both markets to maintain the same price gap  $\psi^*$ . However, since the financial collateral is binding, the intermediaries can only

provide a fixed amount of liquidity to the market, which leads to a contradiction. Thus it is only possible for the intermediaries to reach a steady state when the intensity shock is constant. In this case, the intermediaries provide the same liquidity to households in each period and the price difference stays the same over time. As a result, the intermediaries will not profit from arbitraging in the financial markets. The first-order condition with respect to the capital investment is

$$-\frac{1}{c_t^{\text{IM}}} + \frac{F'(K_t)\rho}{c_{t+1}^{\text{IM}}} + (1 - \delta)\mu_t = 0,$$

where  $\mu_t > 0$  is the Lagrange multiplier associated with the financial constraint at  $t$ . In the steady state, this indicates

$$-\frac{1}{c^*} + \frac{F'(K^*)\rho}{c^*} + (1 - \delta)\mu^* = 0,$$

where  $\mu^*$  is the steady state multiplier. On the other hand, the first-order condition with respect to the position  $x_t$  in the steady state also implicate

$$-\psi^* + \rho\psi^* - \mu^*(2\bar{\theta} + \psi^*)(F(K^*) - K^*) = 0 \quad (11)$$

Together with the binding collateral constraint

$$(1 - \delta)K^* - (2\bar{\theta} + \psi^*)x^* = 0$$

we can deduce that  $F'(K^*) = \frac{2\bar{\theta} + \rho\psi^*(1 - \delta) - \delta\psi^*}{\rho(2\bar{\theta} + \psi^*)}$ . The unique  $K^*$  can be backed out to be  $K^* = [\frac{2\bar{\theta} - \delta\psi^* - 2(1 - \delta)\rho\bar{\theta}}{(2\bar{\theta} + \psi^*)\rho\alpha(1 - \gamma)\alpha L^\gamma}]^{\frac{1}{\alpha - 1}}$  and in case of multiple solutions we choose the one such that it maximizes  $F(K^*) - K^*$ .

We can find out the steady state price difference  $\psi^*$  by looking at households' optimization problem. The first-order condition of both sides households with respect to their position in the financial asset at  $t$  is

$$\frac{P_t^A}{(u - x^*)\theta_t + a\gamma K^{*\alpha} L^{\gamma - 1} + bK_H} = \mathbb{E} \frac{\beta(P_{t+1}^A + \theta_{t+1})}{(u - x^*)\theta_{t+1} + a\gamma K^{*\alpha} L^{\gamma - 1} + bK_H},$$

$$\frac{P_t^B}{(-u + x^*)\theta_t + a\gamma K^{*\alpha} L^{\gamma - 1} + bK_H} = \mathbb{E} \frac{\beta(P_{t+1}^B + \theta_{t+1})}{(-u + x^*)\theta_{t+1} + a\gamma K^{*\alpha} L^{\gamma - 1} + bK_H}.$$

With the specification of distribution of  $\theta_t$  and in steady state by definition  $\phi^* = P_t^A - P_t^B$ ,

$\phi^* = P_{t+1}^A - P_{t+1}^B$ , we can solve out a unique value for  $\phi^*$  and thus the steady state capital investment  $K^*$ . Of course,  $K^*$  has to be less than the minimal requirement for keeping the collateral constraint slack, which is  $\frac{2\bar{u}\bar{\theta}}{1-\delta}$ .

Also global convergence can be proved through the formation of a recursive optimization of the intermediaries and ruling out other possible steady state with properties of production function and logarithmic utility.

□

## D Proof of Proposition 4

*Proof.* Suppose that in equilibrium (1) the positions in market A and B are opposite with the same absolute size; (2) given the price gap, the intermediaries and the households' optimization problem is solved in Section 4.2.1 and Section 4.2.2; (3) the price gap  $\psi_t$  are given either by solving the agents' optimization problem backwards with the terminal condition  $\psi_T=0$  for the slack case or  $\psi_T = \psi^*$  for the tight case in Section 4.2.3 through market clearing conditions. Thus the equilibrium exist.

□

## E Proof of Corollary 1

*Proof.* It is obvious for the case that when there is an extreme blow to the capital, which leads to little production return to the intermediaries, the intermediaries will end up having a total income of zero value or below. Let's suppose that the intermediaries do not have sufficient collateral initially to take arbitrage positions in the financial markets, therefore the price gap enlarges and thus cause further loss to the intermediaries. If such loss is greater than their production income, then they have to liquidate all their capital to cover for the loss, thus are forced out of the market.

For the non-bankrupt case, when the capital suddenly drops, so do the output and intermediaries' production income. As the income effect dominates the substitution effect,

the intermediaries' immediate reaction is to cut down their investment in the capital. As a result, they can support less positions in the financial markets, leading to an increased price gaps. Also as the position  $x_{t+1}$  is less than the previous steady state one,  $x^*$ , the intermediaries also suffers from cash outflow in the financial market, which further reduced the total income. In the long run, according to Proposition 3, the capital investment, the liquidity supply and the asset price will revert eventually to their steady state level.

□

## F Proof of Corollary 2

The budget constraint of the intermediaries at time  $t$  is

$$c_t^{\text{IM}} = (x_{t-1} - x_t)\psi_t + a(1 - \gamma)K_{t-1}^\alpha L^\gamma + (1 - \delta)K_{t-1} - K_t.$$

If  $u_t$  suddenly increases and incurs intermediaries a loss from the financial markets, then it indicates that the liquidity supply decreases at  $t$ , or the size of  $x_t$  is smaller than  $x_{t-1}$ , i.e.,  $x_t < x^*$ . Thus with enlarged demand and less liquidity supply, the price difference will increase  $|\psi_t| > |\psi^*|$ . In turn,  $|y_t^i| < |y^*|$ .

In the long run, according to Proposition 3, the liquidity supply, price difference and capital investment will converge to a new steady state.

## G The Infinite Horizon Model with Risky Arbitrage

### A Recursive Formulation

For the general case, it is remarkably difficult to analytically solve for the prices and capital accumulation in the sequential trading economy, especially we do not know the mappings from the historical path of productivity shocks and shock intensity. Therefore, we try to avoid the difficulty by restating the equilibrium through recursive formulation. Based

on the recursive structure, we consider applying numerical algorithms to find equilibrium prices and asset allocations.

### A.1 Intermediaries

Denote the intermediaries' current position in financial market A as  $x = x_{A,t}$ . Therefore, their position in market B is  $-x$ . Similarly, the A-households' position in equilibrium will be  $y_A = -x$  and the B-households' position is  $y_B = x$ . Also in the following formulation, we denote the intermediaries' capital input and position in market A for the previous period by  $K_{P,-}$  and  $x_-$ . Similarly, the shock intensity of the previous and current periods in market A are  $u_-$  and  $u$  respectively.

To form a recursive expression for the intermediaries, we use the last period's and current period's shock intensities  $u_-$  and  $u$ , the current period's productivity shock  $\theta$ , the intermediaries' physical capital P holding at the beginning of the period  $K_{P,-}$  within each market, and the previous period's position in the financial asset in market A  $x_- = x_{A,-}$  as state variables.

Also we denote the next period production shock and asset prices as  $\theta_+$  and  $P_{i,+}$  for  $i \in \{A, B\}$ .

The recursive formulation for the intermediaries can be expressed as

$$V^{\text{IM}}(K_{P,-}, x_-, u_-, \theta) = \max_{K_P, x, c^{\text{IM}}} \{ \log(c^{\text{IM}}) + \rho \mathbb{E}[V^{\text{IM}}(K_P, x, u, \theta_+)] \}$$

subject to the budget constraints

$$c^{\text{IM}} = (P_A - P_B)(x_- - x) + a(1 - \gamma)K_{P,-}^\alpha L^\gamma + (1 - \delta)K_{P,-} - K_P, \quad (12)$$

$$c_+^{\text{IM}} = (P_{A,+} - P_{B,+})(x - x_+) + a(1 - \gamma)K_P^\alpha L^\gamma + (1 - \delta)K_P - K_{P,+}. \quad (13)$$

and subject to the collateral constraints

$$\begin{aligned} \text{(type I)} \quad 0 &\leq \min\{\min_{P_{A,+}}\{x(P_{A,+} - P_A)\}, 0\} + \min\{\min_{P_{B,+}}\{(-x)(P_{B,+} - P_B)\}, 0\} + a(1 - \gamma)K_P^\alpha L^\gamma \\ &\quad + (1 - \delta)K_P, \end{aligned}$$

$$\text{(type II)} \quad 0 \leq \min\{\min_{P_{A,+}}\{x(P_{A,+} - P_A)\}, 0\} + \min\{\min_{P_{B,+}}\{(-x)(P_{B,+} - P_B)\}, 0\} + K_P.$$



## A.2 Households

For households, the recursive form is easier as they are not subject to any collateral constraints. Their maximization problem can also be expressed in a recursive form with state variables  $K_{P,-}$ ,  $x_-$ ,  $u_-$ ,  $u$ , and  $\theta$ . This is because of the market clearing condition and the symmetry between two markets. The A-households' position in the financial market is  $y = -x$ , and the B-households' position is therefore equal to  $x$  in equilibrium. Thus,  $x$  can serve as state variable in the households' recursive formulation as well.

The A-households' optimization problem in a recursive form is given by

$$V^A(K_{P,-}, x_-, u_-, \theta) = \max_{y_A, c^A} \{ \log(c^A) + \beta \mathbb{E}[V^A(K_P, x, u, \theta_+)] \}$$

$$c^A = -P_A(x_- + y_A) + \frac{1}{2}a\gamma K_{P,-}^\alpha L^\gamma + bK_F + (u_- - x_-)\theta, \quad (14)$$

$$c_+^A = -P_{A,+}(-y_A + y_{A,+}) + \frac{1}{2}a\gamma K_P^\alpha L^\gamma + bK_F + (u + y_A)\theta_+. \quad (15)$$

The corresponding B-households' optimization problem is given by

$$V^B(K_{P,-}, x_-, u_-, \theta) = \max_{y_B, c^B} \{ \log(c^B) + \beta \mathbb{E}[V^B(K_P, x, u, \theta_+)] \}$$

subject to the budget constraints

$$c^B = P_B(x_- - y_B) + \frac{1}{2}a\gamma K_{P,-}^\alpha L^\gamma + bK_F - (u_- - x_-)\theta, \quad (16)$$

$$c_+^B = P_{B,+}(y_B - y_{B,+}) + \frac{1}{2}a\gamma K_P^\alpha L^\gamma + bK_F - (u - y_B)\theta_+. \quad (17)$$

The market clearing condition implies that

$$y_A = -x, \quad (18)$$

$$y_B = x, \quad (19)$$

$$y_{A,+} = -x_+, \quad (20)$$

$$y_{B,+} = x_+. \quad (21)$$

## B Garcia and Zangwill (1981) Trick

In order to transform the inequality into an equality, we apply a change of variables following Garcia and Zangwill (1981). Then the transformed equality for the intermediaries' optimality problem is given by

$$V^{\text{IM}}(K_{\text{P},-}, x_-, u_-, \theta) = \max_{K_{\text{P}}, x, c^{\text{IM}}} \{ \log(c^{\text{IM}}) + \rho \mathbb{E}[V^{\text{IM}}(K_{\text{P}}, x, u, \theta_+)] \}$$

subject to

$$c^{\text{IM}} = (P_{\text{A}} - P_{\text{B}})(x_- - x) + a(1 - \gamma)K_{\text{P},-}^{\alpha}L^{\gamma} + (1 - \delta)K_{\text{P},-} - K_{\text{P}}$$

and

$$\begin{aligned} \text{(type I)} \quad 0 &= \lambda_{1,P_+}(-) - \{ \min[x(P_{\text{A},+} - P_{\text{A}})] + \min[(-x)(P_{\text{B},+} - P_{\text{B}})] + a(1 - \gamma)K_{\text{P}}^{\alpha}L^{\gamma} \\ &\quad + (1 - \delta)K_{\text{P}} \}, \end{aligned} \tag{22a}$$

$$\begin{aligned} \text{(type II)} \quad 0 &= \lambda_{2,P_+}(-) - \{ \min[x(P_{\text{A},+} - P_{\text{A}})] + \min[(-x)(P_{\text{B},+} - P_{\text{B}})] + K_{\text{P}} \} \end{aligned} \tag{22b}$$

for all  $u_+ \in \mathcal{U}$ . Here,

$$\begin{aligned} \lambda_{1,P_+}(-) &= \left( \max \{ -\lambda_{1,P_+}, 0 \} \right)^2, \\ \lambda_{2,P_+}(-) &= \left( \max \{ -\lambda_{2,P_+}, 0 \} \right)^2 \end{aligned}$$

are the inequality multipliers in Garcia and Zangwill (1981).

## C First Order Conditions

The first order conditions for the intermediaries with two different friction types are given by

$$\begin{aligned} \frac{1}{c_{\text{IM}}} &= \rho \mathbb{E} \left[ \frac{1}{c_{+}^{\text{IM}}} (a\alpha(1-\gamma)K_{\text{P}}^{\alpha-1}L^{\gamma} + (1-\delta)) \right] \\ &\quad + \lambda_{1,P_{+}}(+)(a\alpha(1-\gamma)K_{\text{P}}^{\alpha-1}L^{\gamma} + (1-\delta)), \end{aligned} \quad (23a)$$

$$\begin{aligned} \frac{1}{c_{\text{IM}}} (P_{\text{A}} - P_{\text{B}}) &= \rho \mathbb{E} \left[ \frac{1}{c_{+}^{\text{IM}}} (P_{\text{A},+} - P_{\text{B},+}) \right] \\ &\quad + \lambda_{1,P_{+}}(+)\{\mathbb{1}_x \min(P_{\text{A},+}) + (1 - \mathbb{1}_x) \max(P_{\text{A},+}) - P_{\text{A}} \\ &\quad - [\mathbb{1}_x \max(P_{\text{B},+}) + (1 - \mathbb{1}_x) \min(P_{\text{B},+})] + P_{\text{B}}\}. \end{aligned} \quad (24a)$$

in case of type I frictions and

$$\frac{1}{c_{\text{IM}}} = \rho \mathbb{E} \left[ \frac{1}{c_{+}^{\text{IM}}} (a\alpha(1-\gamma)K_{\text{P}}^{\alpha-1}L^{\gamma} + (1-\delta)) \right] + \lambda_{2,P_{+}}(+), \quad (23b)$$

$$\begin{aligned} \frac{1}{c_{\text{IM}}} (P_{\text{A}} - P_{\text{B}}) &= \rho \mathbb{E} \left[ \frac{1}{c_{+}^{\text{IM}}} (P_{\text{A},+} - P_{\text{B},+}) \right] \\ &\quad + \lambda_{2,P_{+}}(+)\{\mathbb{1}_x \min(P_{\text{A},+}) + (1 - \mathbb{1}_x) \max(P_{\text{A},+}) - P_{\text{A}} \\ &\quad - [\mathbb{1}_x \max(P_{\text{B},+}) + (1 - \mathbb{1}_x) \min(P_{\text{B},+})] + P_{\text{B}}\} \end{aligned} \quad (24b)$$

in case of type II frictions, where  $\mathbb{1}_x$  is the sign indicator of  $x$ . Here,

$$\begin{aligned} \lambda_{1,P_{+}}(+) &= (\max\{\lambda_{1,P_{+}}, 0\})^2, \\ \lambda_{2,P_{+}}(+) &= (\max\{\lambda_{2,P_{+}}, 0\})^2 \end{aligned}$$

are the inequality multipliers in Garcia and Zangwill (1981).

The corresponding first order conditions for the households read

$$\frac{P_{\text{A}}}{c^{\text{A}}} = \beta \mathbb{E} \left[ \frac{P_{\text{A},+} + \theta_{+}}{c_{+}^{\text{A}}} \right], \quad (25)$$

$$\frac{P_{\text{B}}}{c^{\text{B}}} = \beta \mathbb{E} \left[ \frac{P_{\text{B},+} + \theta_{+}}{c_{+}^{\text{B}}} \right]. \quad (26)$$

## H Numerical Implementation

To derive a quantitative characterization of the equilibrium, we apply a similar approximation algorithm to approximate the equilibrium asset allocations and prices as in Kübler and Schmedders (2003).

The six budget constraint equations (12) through (17), the four market clearing conditions (18) through (21), together with one collateral constraint (22a)/(22b) and four first order condition equations (23a), (23b), (24a) and (24b) constitute a non-linear system of equations in 19 unknowns. It consists of

- four price variables  $P_A, P_{A,+}, P_B, P_{B,+}$ ,
- six consumption variables  $c^{\text{IM}}, c_+^{\text{IM}}, c^A, c_+^A, c^B, c_+^B$ ,
- asset position variables  $x, x_+, y_A, y_{A,+}, y_B, y_{B,+}$ ,
- two capital accumulation levels  $K_P, K_{P,+}$ ,
- and one multiplier  $\lambda_{1,P_+}/\lambda_{2,P_+}$ .

Denote the above equations as  $\mathbf{F}(x_-, K_{P,-}, u, u_-, \theta) = \mathbf{0}_{15}$ , where  $\mathbf{0}_n$  is an  $n$ -dimensional column vector of zeros.

We compute an approximate policy function via an iterative algorithm. In particular, we assume that the values of the prices, the endogenous intermediaries' financial asset allocation and capital accumulation in the next period,  $P_{A,+}, P_{B,+}, x_+$  and  $K_{P,+}$ , are functions of the intermediaries' current capital accumulation  $K_{P,-}$  and financial asset holdings  $x_-$ , which we denote as

$$\zeta : \underbrace{[-u, u]}_{x_-} \times \underbrace{\mathbb{R}_+}_{K_{P,-}} \times \mathcal{U} \times \mathcal{U} \times \mathcal{S} \rightarrow \underbrace{[-u, u]}_{x_+} \times \underbrace{\mathbb{R}}_{P_{A,+}} \times \underbrace{\mathbb{R}}_{P_{B,+}} \times \underbrace{\mathbb{R}_+}_{K_{P,+}}.$$

By approximating those variables as functions of current state, we manage to transform  $\mathbf{F}(x_-, K_{P,-}, u, u_-, \theta) = 0$  into a well-defined system of equations. As a starting point, we choose a continuous function  $\zeta^0$  to serve as an initial guess to approximate the next period's prices, asset allocations and capital accumulation levels. During each iteration of

the algorithm, given the approximated next period prices and asset holdings  $\zeta^n$ , we solve the well-defined system to obtain the equilibrium prices and holdings in the current period. We then move one period back, and update the approximation of  $\zeta^{n+1}$  by mapping the current period values to the state variables. We define the convergence of the iteration by a predetermined criterion. For some predetermined  $\epsilon > 0$ ,

$$\sup_{\substack{x_-, K_{P,-}, \\ u, u_-, \theta}} \|\zeta^{n+1} - \zeta^n\| \leq \epsilon.$$

The algorithm terminates once  $\zeta^{n+1}$  reaches convergence and we accept  $\zeta^* = \zeta^{n+1}$  as approximated price and policy functions for the next period.

At the end, we will compute the maximum relative errors in Euler equations after substituting the approximated value into the first order conditions to examine the quality of the approximation. For errors that exceeds our preset criterion, we will restart the above iteration with a lower  $\epsilon$  as convergence threshold.

## A Implementation Procedure

Specifically, we construct a piecewise linear spline with coefficient  $\xi^0$  to obtain approximation  $\zeta^0$  as an initial set up. In each iteration given  $\zeta^n$ , we solve the above system of nonlinear equations (12) through (26) and thus obtain the value of current period prices and control variables. Then we interpolate them against the state variables and get an updated  $\zeta^{n+1}$  by updating the coefficient vectors  $\xi^{n+1}$  and. Repeat this procedure until the convergence of  $\xi^{n+1}$ .

## B Algorithm

We applied the following time iteration linear collocation algorithm similar to Kübler and Schmedders (2003) and Judd (1998).

- Step 0: select an error tolerance  $\epsilon$  for the stopping criterion  $\sup \|\zeta^{n+1} - \zeta^n\| \leq \epsilon$ , a finite grid composed of  $K_{P,-} \in [0, K]$  and  $x_- \in [-u, u]$  for each combination of  $(u_-, u, \theta)$  and the piecewise linear coefficients  $\xi^0$  for a starting point  $\zeta^0$ .

- Step 1: Given the piecewise linear coefficients  $\xi^n$ , or the approximation  $\zeta^n$ , solve the system of nonlinear equations (12) through (26), for the finite grids composed of  $x \in [-u, u]$  and  $K_{P,-} \in [0, K]$ , finding a solution  $(P_A, P_B, x, K_P)$  in terms of  $(K_{P,-}, x_-)$ .
- Step 2: Compute the new approximations  $\zeta^{n+1}$ , that is, the new coefficient vectors  $\xi^{n+1}$  by interpolating  $(P_A, P_B, x, K_P)$  on  $(K_{P,-}, x_-)$ .
- Step 3: Check stopping criterion. If  $\sup \|\zeta^{n+1} - \zeta^n\| \leq \delta$  then go to Step 4. Otherwise increase  $n$  by 1 and go to Step 1.
- Step 4: The algorithm terminates. Set  $\zeta^* = \zeta^{n+1}$ .