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Institute for Fiscal Studies

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IFS Working Paper W15/20

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Mutually Consistent Revealed Preference Bounds

By Abi Adams*

Revealed preference restrictions are increasingly used to bound demand responses and as shape restrictions in nonparametric estimation exercises. However, the restrictions imposed are not sufficient for rationality when predictions are made at more than a single price regime. We highlight the nonlinearities in revealed preference restrictions and the nonconvexities in the set of predictions that arise when making multiple predictions. We develop a mixed integer programming characterisation of the problem that can be used to impose rationality on multiple predictions. The approach is applied to the UK Family Expenditure Survey to recover jointly rational nonparametric estimates of income expansion paths.

JEL: C14, C60, D11, D12

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The revealed preference approach to demand prediction uses the behavioural hypothesis of utility maximisation in conjunction with a finite set of observations on a consumer's past behaviour to set identify rational demand responses at new budgets of interest. The benefits of such an approach are well understood: bounds can be placed on behavioural responses and welfare effects without the need for restrictive assumptions on consumer preferences. As Blundell (2005) argues, it is possible "to accomplish all that is required from parametric models of consumer behaviour using only nonparametric regression and revealed preference theory"; crucially, however, without placing strong restrictions on income and

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price responses.

Since Varian (1982), revealed preference arguments have been successfully applied to predict consumer behaviour in a number of different scenarios including bounding demand responses for gasoline (Blundell, Kristensen and Matzkin, 2014), food (Blundell, Browning and Crawford, 2008), broadband (Varian, 2012) and leisure (Manski (2014), Klein and Tartari (2014)) among others. Recent methodological advances have, furthermore, the potential to extend the informativeness and scope of the approach. Nonparametric regression and rationality restrictions are increasingly combined to refine the set of revealed preference demand predictions (Blundell, Browning and Crawford (2003, 2008), Blundell, Kristensen and Matzkin (2014), Blundell et al. (2015)) or to recover theory-consistent, well-disciplined point estimates (Blundell, Horowitz and Parey (2012)). The toolkit has also been developed to account for (nonadditive) heterogeneity in consumer preferences, moving the literature beyond deterministic choice models to allow for bounds to be placed on expected demands, and on other features of the distribution of demand (Blundell, Kristensen and Matzkin (2014), Kitamura and Stoye (2013), Hoderlein and Stoye (2014a, 2014b)).

This paper is concerned with whether the revealed preference restrictions imposed in the literature are, in fact, sufficient for the set of predictions to be rational. It will be shown that, in general, when predictions are made for a set of budgets, where the set has a cardinality strictly greater than one, they are not. In this context, application of the present restrictions (as set out in Varian (1982)) does not guarantee that the resulting set of predictions are jointly rational.

This is of interest because applied problems often require demand behaviour to be forecast at multiple budgets. For example, researchers may be interested in comparing behavioural responses across a number of different policy reforms simultaneously. Furthermore, when revealed preference inequalities are used as shape restrictions to constrain nonparametric regression models, demand predictions are typically made over a grid of budget parameters. While theory con-

sistency constraints may be imposed for a number of reasons beyond ensuring rationality of predictions (for example, decreasing the variance of nonparametric estimates and improving out-of-sample extrapolation properties — see Matzkin (1994) for a review), rationality is an important property for demand predictions if they are to be used for welfare analysis.[1]

In this paper, the revealed preference methodology is extended to address the prediction of rational demands over a set of new budgets. Revealed preference bounds do not extend without modification to scenarios in which multiple demand predictions are to be made because predictions across different budgets must be consistent with one another for them jointly to satisfy rationality restrictions—that is, rationality restrictions must hold between predictions. This requirement of 'mutual consistency' of predictions generates nonlinearities in the standard revealed preference-type inequalities and results in a non-convex set of demand predictions. These complications have not yet been recognised explicitly in the literature.

It is shown that the revealed preference restrictions associated with this problem can be characterised as a mixed integer programme (MIP), which can be implemented with reasonable computational resources. Approaches for enhancing the efficiency of the programming problem are put forward and connections are made to the recent methodological advances of Hoderlein and Stoye (2014a, 2014b) and Kitamura and Stoye (2013). Conditional on a hypothesised preference ordering over new budgets (or a particular choice path), mutual consistency of predictions weakly tightens the bounds on demand forecasts and is shown to discipline nonparametric estimates of consumer Engel curves. The practical use of the methodology is demonstrated via an illustrative application to consumer microdata from the UK Family Expenditure Survey, in which rational income expansion paths and mutually consistent revealed preference bounds are recovered.

[1] If, for example, RP-constrained nonparametric demand predictions jointly fail rationality, methods applied to these predictions to estimate consumer surplus and there welfare metrics will suffer from path dependency.

The ability to recover theory-consistent income expansion paths with relative ease is of use to those looking to recover e-bounds, as defined by Blundell, Browning and Crawford (2008).

I. Revealed Preference Support Sets

We are interested in a consumer's demand behaviour at a finite number of budgets $\mathscr{B} = \{1, ..., N\}$, which are parameterised by $\{\mathbf{p}_b\}_{b \in \mathscr{B}}$ with $\mathbf{p}_b \in \mathbb{R}^K_{++}$.[2] The associated set of budget planes, $\{B_i\}_{i=1,...,N}$, are defined as:[3]

$$B_i = B(\mathbf{p}_i) := \{\mathbf{q} : \mathbf{q} \geq \mathbf{0} \quad \& \quad \mathbf{p}_i'\mathbf{q} = 1\} \tag{1}$$

As is standard in the revealed preference (RP) literature, it is assumed that a consumer's demand is observed at a subset of these budgets: $\mathscr{D} \subset \mathscr{B}$, where $|\mathscr{D}| = T$. Thus, one works with the panel $\{\mathbf{p}_t, \mathbf{q}_t\}_{t \in \mathscr{D}}$ in order to predict behaviour over the remaining subset of budgets, $\mathscr{Q} = \mathscr{B}/\mathscr{D}$.

Throughout this paper, we assume that observed choice behaviour satisfies the axioms of revealed preference and can, therefore, be considered consistent with rationality.[4] Rationality is equated with choice behaviour that satisfies the Generalised Axiom of Revealed Preference (GARP), defined below. GARP is a consistency condition. Intuitively, if one 'reveals a preference' for some bundle $\mathbf{q}_t$ over an other bundle $\mathbf{q}_s$ by selecting $\mathbf{q}_t$ when $\mathbf{q}_s$ is available, then that individual cannot chose $\mathbf{q}_s$ over $\mathbf{q}_t$ in an alternative choice scenario in which $\mathbf{q}_t$ is available.

Direct revealed preference. If $\mathbf{p}_t^{'}\mathbf{q}_s \leq 1$, then $\mathbf{q}_t$ is directly revealed preferred to $\mathbf{q}_s$, or $\mathbf{q}_t \mathbb{R}^0 \mathbf{q}_s$.

[2] We assume an absence of money illusion on the part of the consumer, implying that the implied total expenditure normalisation is irrelevant for our analysis. Afriat (1967) refers to budget normalised prices as the 'balance vector'.

[3] Our notation is similar to that employed by McFadden and Richter (1991) and Kitamura and Stoye (2013).

[4] See Apestuglia and Baluster (2015), Dean and Martin (2015), Echenique, Lee and Shum (2011) and Varian (1990) for approaches to measure violations of rationality.

Revealed preference. If $\mathbf{q}_t \mathbb{R}^0 \mathbf{q}_u, \mathbf{q}_u \mathbb{R}^0 \mathbf{q}_v, ..., \mathbf{q}_w \mathbb{R}^0 \mathbf{q}_s$, then $\mathbf{q}_t$ is revealed preferred to $\mathbf{q}_s$, $\mathbf{q}_t \mathbb{R} \mathbf{q}_s$.

Generalised Axiom of Revealed Preference (GARP). If $\mathbf{q}_t \mathbb{R} \mathbf{q}_s$, then $\mathbf{p}_s^{'} \mathbf{q}_t \geq 1$.

A. Single Predictions

The axioms of revealed preference can be applied to set identify rational demands without any functional assumptions beyond nonsatiation of the utility function (see Varian, 1982). Not all bundles that are affordable at a budget $b \in \mathscr{Q}$ will be consistent (i.e. jointly rational) with observed demand behaviour at budgets $t \in \mathscr{D}$. The RP approach relies on this consistency requirement to bound behavioural responses. Each element of the set of predictions identified by RP arguments can be jointly rationalised with past observations by a monotonic, concave, and non-degenerate utility function.[5]

This approach to demand prediction is well developed when interest lies in predicting behaviour at a single new budget of interest, i.e. $|\mathscr{Q}| = 1$. (We will here work with $\mathscr{Q} = \{1\}$ for notional convenience.) Following Varian (1982), the set of rational demands that are recovered by RP arguments is referred to as the 'support set', $S^V(B_1) \subseteq B_1$. Any element of $S^V(B_1)$ satisfies GARP in union with observed past consumption choices, whilst any demand in the set's complement violates GARP.

Support Set. Given the set of past observations $\{\mathbf{p}_t, \mathbf{q}_t\}_{t \in \mathscr{D}}$ and a new budget,

[5]Monotonicity of the utility function is a maintained assumption throughout this paper and ensures that recovered demands lie on budget hyperplanes.

B_1, the support set is defined as:

$$S^V(B_1) = \begin{cases} & \mathbf{q}_1 \geq \mathbf{0} \\ \mathbf{q}_1 : & \mathbf{p}_1' \mathbf{q}_1 = 1 \\ & \{\{\mathbf{p}_t, \mathbf{q}_t\}_{t \in \mathscr{D}}; \mathbf{p}_1, \mathbf{q}_1\} \text{ satisfy GARP} \end{cases} \tag{2}$$

$S^V(B_1)$ is the set of demands that satisfy the following linear programme:[6]

1) $\mathbf{p}_1' \mathbf{q}_1 = 1$

2) $\mathbf{p}_s' \mathbf{q}_1 \geq 1$ for all $s \in \mathscr{D}$ for which $\mathbf{p}_1 \mathbb{R} \mathbf{p}_s$

3) $\mathbf{p}_s' \mathbf{q}_1 > 1$ for all $s \in \mathscr{D}$ for which $\mathbf{p}_1 \mathbb{P} \mathbf{p}_s$.

The first condition imposes budget exhaustion, as required by the assumption of monotonicity. The second condition imposes that if the new budget is revealed preferred to an observed budget, then predicted bundles must be revealed preferred to the observed demands at these dominated budgets. The final condition imposes the strict extension of the second condition (see Varian (1982) for further discussion).

The support set is closed and convex, endowing it with a number of convenient properties for applied work, and is easily recovered as the convex hull of the set vertices $\{\underline{q}_1^{V,k}, \overline{q}_1^{V,k}\}^{k=1,\ldots,K}$, where:

$$\underline{q}_1^{V,k} = \min_{\mathbf{q}} \sum_{j=1}^{K} \mathbb{1}\left(j == k\right) q^k \qquad \text{s.t.} \qquad \mathbf{q} \in S^V(B_1) \tag{3}$$

$$\overline{q}_1^{V,k} = \max_{\mathbf{q}} \sum_{j=1}^{K} \mathbb{1}\left(j == k\right) q^k \qquad \text{s.t.} \qquad \mathbf{q} \in S^V(B_1) \tag{4}$$

[6] $\mathbb{R}$ and $\mathbb{P}$ extend the concept of revealed preference to price vectors. A price vector $\mathbf{p}_t$ is directly revealed preferred to the budget $\mathbf{p}_s$, $\mathbf{p}_t \mathbf{R}^0 \mathbf{p}_s$, if $\mathbf{p}_t' \mathbf{q}_s \leq 1$. A price vector $\mathbf{p}_t$ is directly revealed strictly preferred to the budget $\mathbf{p}_s$, $\mathbf{p}_t \mathbf{P}^0 \mathbf{p}_s$, if $\mathbf{p}_t' \mathbf{q}_s < 1$. The definitions of revealed preference and strict revealed preference follow straightforwardly.

B. Multiple predictions

Researchers are often interested in comparing consumer behaviour at a number of alternative new budgets as, for example, when looking to compare the impact of alternative tax regimes on consumer welfare and the public finances. Further, nonparametric demand estimation subject to economic theory restrictions is essentially a recovery exercise over a finite set of new budgets because application of such techniques typically proceeds pointwise over a grid of new budget parameters.[7]

When the aim is to recover demands at a set of new budgets (where $|\mathscr{Q}| > 1$), theory consistency of the set of predictions requires that forecasts be mutually consistent with one another, such that they are jointly rationalisable by a single, non-degenerate utility function. A violation of rationality may create problems for welfare analysis and for further recovery exercises — for example, unrestricted income expansion paths in Blundell, Browning and Crawford (2008) and Blundell, Kristensen and Matzkin (2014) jointly violated RP and thus could not be used to bound demand responses without modification.

In what follows, let the support set when demands are recovered at multiple budgets, and at which mutual consistency of forecasts is imposed, be referred to as the 'sufficient support set', S^S. S^S is the set of demand responses at each of the budgets $\{B_b\}_{b\in\mathscr{Q}}$ that simultaneously satisfy GARP when conjoined with observed demand behaviour.

Sufficient Support Set. Given the set of budgets $\mathscr{B}$, for which demand behaviour is observed for the subset $\mathscr{D} \subset \mathscr{B}$ and is to be predicted over the subset

[7]For example, Blundell, Kristensen and Matzkin (2014) check whether RP inequalities hold on a discrete grid of values when estimating income expansion paths — we return to this application in Section V

$\mathscr{Q} = \mathscr{B}/\mathscr{D}$, the sufficient support set is defined as:

$$S^S = \begin{cases} & \mathbf{q}_b \geq \mathbf{0} & \text{for all } b \in \mathscr{Q} \quad [S1] \\ \{\mathbf{q}_b\}_{b\in\mathscr{Q}} : & \mathbf{p}_b'\mathbf{q}_b = 1 & \text{for all } b \in \mathscr{Q} \quad [S2] \\ & \{\mathbf{p}_i, \mathbf{q}_i\}_{i\in\mathscr{B}} \text{ satisfy GARP} & [S3] \end{cases}$$

S^S is a subset of the Cartesian product of the support sets associated with each budget: $S^S \subseteq \times_{b\in\mathscr{Q}} S^V(B_b)$. If the Varian support sets of any new budgets intersect at an interior point, this subset is strict and nonconvex — see Proposition 1, which is proven formally (along with all further Propositions) in the Appendix. In this instance, not all combinations of demands drawn from each $S^V(B_b)$ for $b \in \mathscr{Q}$ will mutually satisfy rationality. Failure to recognise this point can result in, for example, RP-constrained nonparametric regression estimates jointly failing rationality.

Proposition 1. — S^S is not convex if there exist budgets $a, b \in \mathscr{Q}$ at which $\exists\tilde{\mathbf{q}}$ such that $\tilde{\mathbf{q}} \in \text{int}(S^V(B_a))$ and $\tilde{\mathbf{q}} \in \text{int}(S^V(B_b))$.

To illustrate, consider the problem that is depicted in Figure 1 of recovering demands at the budgets defined by $\mathbf{p}_1$ and $\mathbf{p}_2$ given observations on demand at the budgets $\mathbf{p}_3$ and $\mathbf{p}_4$. As previously, past observations and utility maximisation together constrain predictions at each budget to lie within the respective support sets, $S^V(B_1)$ and $S^V(B_2)$.

$$\begin{aligned} \mathbf{q}_1 &\in S^V(B_1) \\ \mathbf{q}_2 &\in S^V(B_2) \end{aligned} \tag{5}$$

In panel (a) of Figure 1, membership of $S^V(B_1)$ constrains demand responses

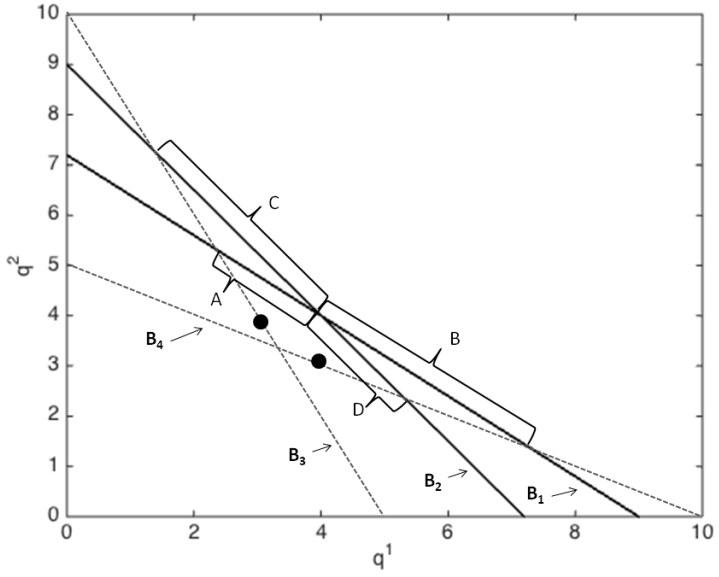
at B_1 to lie in AB and membership of $S^V(B_2)$ constrains demand responses at B_2 to lie in the partition CD. As we are working with a 2-good example, every feasible budget share specification at the new budgets, $\{\boldsymbol{\omega}_b\}_{b=1,2}$, where $\boldsymbol{\omega}_b = (\omega_b^1, \omega_b^2) = (\omega_b^1, 1-\omega_b^1)$, can be represented in a two-dimensional diagram as in panel (b) of Figure 1. Demand predictions that are consistent with RP, and thus are elements of the sufficient support set, are given in dark grey.

Not all combinations of demands drawn from $S^V(B_1)$ and $S^V(B_2)$ are mutually consistent, i.e. some combinations of demands drawn from these support sets cannot be rationalised by the same utility function. Thus, the sufficient support set is a strict subset of the set formed by the Cartesian product of $S^V(B_1)$ and $S^V(B_2)$. Mutual consistency of demand predictions requires that if $\mathbf{q}_1 \in A$, then $\mathbf{q}_2 \notin D$, and if $\mathbf{q}_2 \in D$, then $\mathbf{q}_1 \notin A$.

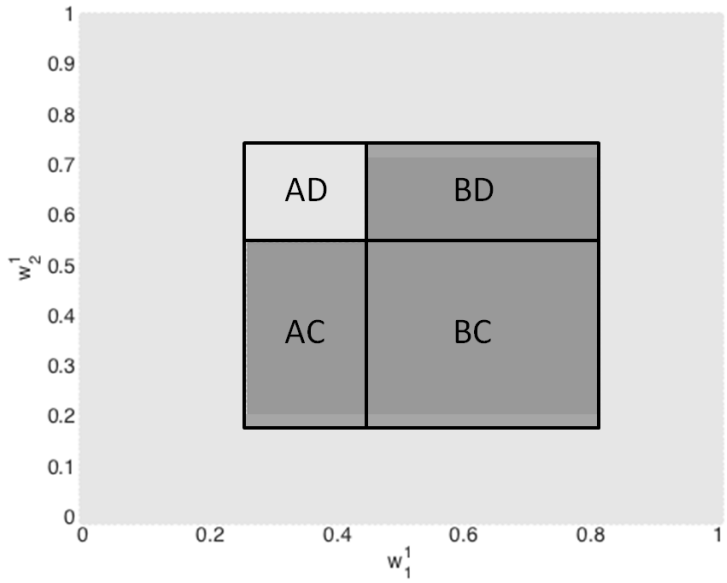
The characterisation of the problem in budget share space gives the impact of the mutual consistency requirement upon the support set. Considering panel (b) of Figure 1, we have that $\{\omega_b^i\}_{b=1,2} \in (AC \cup BC \cup BD)$ but, unlike previously, $\{\omega_b^i\}_{b=3,4} \notin AD$. Without the requirement of mutual consistency, AD would also be permissible. This characterisation of the sufficient support set as a set of 'rational choice types', $\{AC, BC, BD\}$ serves to build a connection to recent work by Hoderlein and Stoye (2014a) and Kitamura and Stoye (2013) — please see Section III for more details.

As is made clear by Figure 1, the requirement that if $\mathbf{q}_1 \in A$, then $\mathbf{q}_2 \notin D$, and if $\mathbf{q}_2 \in D$, then $\mathbf{q}_1 \notin A$, introduces a non-convexity to the sufficient support set. This precludes the construction of the support set by linear programming methods, increasing the complexity of the characterisation of the support set and complicating optimisation over its elements.[8] The aim of the rest of this paper is to develop a practical characterisation of the sufficient support set for empirical work, which can be applied to bound demand responses across a set of budgets

[8] Any linear program is a convex optimisation problem (see, for example, Boyd and Vandenberghe (2004)).



(a) Observed Demands and Budget Lines



(b) Support Set

FIGURE 1. SUFFICIENT SUPPORT SET

Note: Imposing the mutual consistency of demand predictions at B_1 and B_2 introduces a non-convexity to the support set. The light grey region of panel (b) represents bundles ruled out by revealed preference arguments. Letters correspond to the regions highlighted in panel (a).

of interest and to impose a sufficient condition for rationality of predictions on nonparametric regression estimates.

II. Mixed Integer Programme Representation

When $|\mathscr{Q}| > 1$, the sufficient support set cannot necessarily be characterised by a linear programme given the interdependence of rational demands across new budgets. However, a mixer integer programming (MIP) representation of the necessary and sufficient constraints that define S^S can be formulated.[9]

PROPOSITION 2: MIP REPRESENTATION OF S^S. — Given the set of budgets $\mathscr{B}$, for which demand behaviour is observed for the subset $\mathscr{D} \subset \mathscr{B}$ and is to be predicted over the subset $\mathscr{Q} = \mathscr{B}/\mathscr{D}$, the MIP representation of the sufficient support set is defined as:

$$
(6) \qquad S^S(\mathscr{B}) = \left\{ \{\mathbf{q}_b\}_{b\in\mathscr{Q}} : \begin{array}{lcll}
\mathbf{q}_b & \geq & 0 & \text{for all } b \in \mathscr{Q} \quad [1] \\
\mathbf{p}_b'\mathbf{q}_b & = & 1 & \text{for all } b \in \mathscr{Q} \quad [2] \\
\mathbf{p}_b^{'}\mathbf{q}_a & > & 1 - R_{ba} & \text{for all } a, b \in \mathscr{B} \quad [3] \\
R_{ab} + R_{bc} & \leq & 1 + R_{ac} & \text{for all } a, b, c \in \mathscr{B} \quad [4] \\
\mathbf{p}_b^{'}\mathbf{q}_a & \geq & R_{ab} & \text{for all } a, b \in \mathscr{B} \quad [5] \\
R_{ab} & = & \{0, 1\} & \text{for all } a, b \in \mathscr{B} \quad [6]
\end{array} \right.
$$

Constraints [1] through [5] are linear in unknowns and provide an operational methodology with which to practically characterise the sufficient support set. Constraint [6] links the integer variable R_{ab} to the revealed preference relation. $\mathbf{q}_a \mathbb{R} \mathbf{q}_b$ is computationally represented by $R_{ab} = 1$.

Constraints [1] and [2] impose that predictions respect the standard nonnegativity and adding up requirements. Constraint [3] imposes the requirement that

[9] Cherchye, De Rock and Vermeulen (2008) characterise a necessary condition for collective rationality using a MIP representation.

$\mathbf{q}_a \mathbb{R} \mathbf{q}_b$ (i.e. $R_{ab} = 1$) if $1 \geq \mathbf{p}'_a \mathbf{q}_b$, thereby defining the direct revealed preference relation. Constraint [4] imposes transitivity upon the revealed preferred relation: for any budget a that is preferred to budget b, that is revealed preferred to budget c, $R_{bc} = 1$, it must be the case that a is preferred to c, $R_{ac} = 1$. Constraint [5] imposes GARP: for any bundle $\mathbf{q}_a$ that is preferred to a bundle $\mathbf{q}_b$ (and thus $R_{ab} = 1$), it must be the case that $\mathbf{q}_a$ is more expensive than $\mathbf{p}'_b \mathbf{q}_b = 1$.

To illustrate how demands in the region AD in Figure 1 fail these conditions, imagine that at B_1 we draw some $\mathbf{q}_a \in A$ and at B_2 we draw some $\mathbf{q}_d \in D$. We have that $\mathbf{p}'_1 \mathbf{q}_d < 1$ and $\mathbf{p}'_2 \mathbf{q}_a < 1$. Thus, for Constraint [3] to be satisfied $R_{12} = 1$ and $R_{21} = 1$. However, this leads to a contradiction at Constraint [5], which requires $R_{12} = 0$ and $R_{21} = 0$. Yet, for demands drawn from the subsets BC, AC, and BD, a specification for the binary variables can be found such that the MILP constraints are satisfied.

A. Efficiency Enhancements

The computational complexity of the MIP characterisation of the sufficient support set is growing in the size of the data set that one is predicting over. There are $N(N-1)$ integer variables in the problem above and thus, for moderately sized data sets and/or prediction problems, the procedure becomes computationally demanding. However, some budget comparisons are irrelevant for mutual consistency of the set of predictions, potentially allowing the size of the prediction problem to be greatly reduced.

Mutual consistency of predictions does not need to be imposed across sets of non-intersecting new budgets. Let the undirected intersection relation between budgets $a, b \in \mathscr{D}$ be given as $I_{ab} = 1$ if there is an intersection path between the budgets, and $I_{ab} = 0$ otherwise. This can be constructed as follows. First, compute the intersection relation I^0, where:

$$I^0_{ab} = I^0_{ba} = 1 \quad \text{if} \quad \exists \mathbf{q} > \mathbf{0} \text{ such that } \mathbf{p}'_a \mathbf{q} = \mathbf{p}'_b \mathbf{q}. \tag{7}$$

$I^0_{ab} = I^0_{ba} = 0$ otherwise.[10] Warshall's Algorithm (Warshall, 1962) can then be applied to compute the transitive closure of this relation to give I_{ab}.

Subsets of non-intersecting budgets $\mathscr{Q}_m$ are defined such that within a subset there is a path between intersecting budgets ($\forall a, b \in \mathscr{Q}_m, I_{ab} = 1$) but between subsets there is no path of intersecting budgets ($\forall a \in \mathscr{Q}_m$, $b \in \mathscr{Q}_n$ with $m \neq n$, $I_{ab} = 0$). Mutual consistency of predictions can be independently imposed on each subset of new budgets $\mathscr{Q}_m$ for joint rationality of the full set of predictions. That is, mutual consistency restrictions will never be binding for demands predicted at budgets in different subsets of non-intersecting budgets.

PROPOSITION 3. — Define subsets $\mathscr{Q}_m$, such that $\mathscr{Q} = \cup_{m=1}^{M} \mathscr{Q}_m$ and $\forall a, b \in \mathscr{Q}_m, I_{ab} = 1$ and $\forall c \in \mathscr{Q}_m$, $d \in \mathscr{Q}_n$ with $m \neq n$, $I_{cd} = 0$. Then, the sufficient support set is defined as:

(8)

$$S^S(\mathscr{B}) = \left\{ \{\mathbf{q}_b\}_{b\in\mathscr{Q}} : \begin{array}{llll} \mathbf{q}_b & \in & S^V(B_b) & \text{for all } b \in \mathscr{Q} \quad [1] \\ \text{for } m = 1, ..., M & & & \\ \mathbf{p}_b^{'}\mathbf{q}_a & > & 1 - R_{ba} & \text{for all } a, b \in \mathscr{Q}_m \quad [2] \\ \mathbf{p}_b^{'}\mathbf{q}_a & \geq & R_{ab} & \text{for all } a, b \in \mathscr{Q}_m \quad [3] \\ R_{ab} + R_{bc} & \leq & 1 + R_{ac} & \text{for all } a, b, c \in \{\mathscr{Q}_m \cup \mathscr{D}\} \quad [4] \end{array} \right.$$

where $R_{ab} = \{0, 1\}$.

Proposition 3 is particularly useful if, for example, one is interested in predicting demands along a price path where only the relative price of good k is varied. Define this price path as $\{\mathbf{p}_b\}_{b=1,....,B}$ at which $\mathbf{p}^{\neg k}_{b+1}/\mathbf{p}^j_{b+1} = \mathbf{p}^{\neg k}_b/\mathbf{p}^j_{b+1}$ for $b = 1, ..., B-1$ and $\mathbf{p}^k_{b+1}/\mathbf{p}^j_{b+1} = \delta \mathbf{p}^k_b/\mathbf{p}^j_{b+1}$, where $j \neq k$ and $\delta > 1$. No budget in this set intersects with another at strictly positive quantities. Thus, each non-intersecting subset is a singleton containing a single budget, $|\mathscr{Q}_m| = 1$ for all

[10] This is easily constructed with standard linear programming methods.

$m = 1, ..., B$. This prediction problem is then easily solved as B independent linear programmes each with K continuous variables. In the empirical exercise of Section V, we recover demands at 36 budgets, although these can be split into two subsets of non-intersecting budgets, reducing the number of integer variables from 1296 to 720.

Further, if $K = 2$ (i.e. there are only two goods in the modelled demand system), then Constraint [4] of Proposition 3 is trivially satisfied. Constraint [4] imposes transitivity of the preference relation over new budgets. However, as first proven by Rose (1958), transitivity has no empirical content when K=2. In this special case then, that set of constraints may be dropped from the programming problem.[11]

If $K > 2$, transitivity must be imposed on the revealed preference relation for rationality of the set of predictions. Following Cherchye et al. (2013, 2015), the number of constraints associated with Proposition 2 [4] can be reduced using insights from graph theory. Cherchye et al. (2013) Proposition 3 gives the following alternative characterisation of GARP:

PROPOSITION 3, CHERCHYE ET AL. (2013). — The set $\{\mathbf{p}_b, \mathbf{q}_b\}_{b \in \mathscr{B}}$ satisfies GARP if there exist $u_b \in \mathbb{R}$ for all $b \in \mathscr{B}$ such that (i) if $\mathbf{p}_b'\mathbf{q}_b \geq \mathbf{p}_b'\mathbf{q}_a$ then $u_b \geq u_a$, and (ii) if $\mathbf{p}_b'\mathbf{q}_b > \mathbf{p}_b'\mathbf{q}_a$ then $u_b > u_a$.

This representation allows for Constraints [2] and [4] of Proposition 3 to be replaced by:

$$u_a - u_b < R_{ab} \tag{9}$$

$$(R_{ab} - 1) \leq u_a - u_b \tag{10}$$

Cherchye et al. (2013) give evidence for the efficiency savings associated with this

[11] See Blundell et al. (2015) for a further discussion of the complications caused by transitivity for testing and imposing SARP at a single budget.

alternative characterisation.

III. Rational Choice Path Representation

An alternative, less computationally amenable, yet more intuitive, characterisation of the sufficient support set comes from viewing the prediction problem as analogous to a discrete choice problem. Hoderlein and Stoye (2014a) and Kitamura and Stoye (2013) use this insight to construct tests for the Weak Axiom of Revealed Stochastic Preference (ARSP) and the Axiom of Revealed Stochastic Preference (ARSP) respectively. This section serves to build a connection to this work.

The sufficient support set can instead be defined as a finite list of rational 'choice types' over the set of new budgets. For example, in the example of Figure 1, the rational choice types can be described as: $\{AC, BC, BD\}$. Thus, rather than characterise the sufficient support set as the solution to a MIP, one can identify the set of rational choice types.

Rational choice types are defined by a combination of 'patches' across budgets of interest. Using Kitamura and Stoye's (2013) definition, a 'patch' of demand on budget B_b, $x_{r|b}$ is an element of the set $\{x_{1|b}, ..., x_{R_b|b}\}$, where $B_b = \cup_{r=1}^{R_b} x_{r|b}$, that forms the coarsest partition of B_b such that no budget plane other than B_b intersects the interior of any one element of the partition. For example, looking to Figure 1, there are four patches of demand at B_1: the partition between the intersection of B_1 and the y-axis and the intersection of B_1 and B_3; A; B; and the partition between the intersection of B_1 and B_4 and the intersection of B_1 and the x-axis. However, only two of these patches, A and B, are consistent with past demand behaviour.

All patches at new budgets of demand are collected in the vector $\Omega = [x_{1|1}, x_{2|1}, ..., x_{R_B|B}]$.[12] A pattern of predicted individual behaviour over new budgets $\mathscr{Q}$ is represented by the vector $\phi^j = [\phi^j_{1|1}, \phi^j_{2|1}, ..., \phi^j_{R_B|B}]$, where $\phi^j_{r|b} = 1$

[12] We here assume that with $\mathscr{Q} = 1, ..., B$.

if, on choice path j, an element of $x_{r|b}$ is chosen and $\phi^j_{r|b} = 0$ otherwise.

Computation of the patches and rational choice paths of the budgets (given by different specifications of Ω) is covered in Kitamura and Stoye (2013). The only modification required from their approach is for patches that are not consistent with past observations to be excluded from all possible choice paths. This is done by imposing $\phi^j_{r|b} = 0$, $\forall j$ if $\exists t \in \mathscr{D}$ such that:

$$\mathbf{p}_t' \mathbf{q}_{r|b} < 1 \tag{11}$$

$$\mathbf{p}_b' \mathbf{q}_t \leq 1 \tag{12}$$

$$\mathbf{q}_{r|b} \in x_{r|b} \tag{13}$$

In the Appendix, this methodology is used to determine the number of mutually consistent rational choice paths in a simple empirical example.

IV. Support Set Cardinality

It may appear that one is able to refine the support set, thereby tightening the bounds on demand responses, simply by predicting demands at multiple budgets. This impression is false. We here reiterate that the restrictions in this paper serve to restrict *combinations* of demand predictions at new budgets of interest. Indeed, there will exist paths of rational demands over each new budget of interest such that each element of every Varian support set features in S^S.

Proposition 4. — For each $\tilde{\mathbf{q}}_a \in S^V(B_a)$, there will exist a set of demand predictions $\{\tilde{\mathbf{q}}_b\}_{b \in \mathscr{Q}/a}$ such that $\tilde{\mathbf{q}}_a \cup \{\tilde{\mathbf{q}}_c\}_{c \in \mathscr{Q}} \in S^S(\mathscr{B})$.

Proposition 4 implies that it is not possible for *every* combination of demand from the Varian support sets to be jointly irrational. Only a proper subset of the Cartesian product of Varian support sets is irrational.[13] It further leads to

[13] Where $\varnothing$ is considered a proper subset.

a distinction between unconditional and conditional bounds on demand at any particular new budget of interest. Let the unconditional bounds on demand responses for good k at budget b be defined as:

$$\underline{q}_b^{U,k} = \min_{\mathbf{q}} \sum_{b \in \mathscr{Q}} \sum_{j=1}^{K} \mathbb{1}\left(j == k\right) q_b^k \qquad \text{s.t.} \qquad \mathbf{q} \in S^S(\mathscr{B}) \tag{14}$$

$$\overline{q}_b^{U,k} = \max_{\mathbf{q}} \sum_{b \in \mathscr{Q}} \sum_{j=1}^{K} \mathbb{1}\left(j == k\right) q_b^k \qquad \text{s.t.} \qquad \mathbf{q} \in S^S(\mathscr{B}) \tag{15}$$

From Proposition 4, $\{\underline{q}_b^{U,k}, \overline{q}_b^{U,k}\} = \{\underline{q}_b^{V,k}, \overline{q}_b^{V,k}\}$.

However, conditional on a rational preference ordering (or 'rational choice type') over a set of new budgets, the bounds on demand responses will be weakly tightened — strictly if Varian support sets intersect at the interior. For example, in Figure 1, imposing that $R_{12} = 1$ and $R_{21} = 0$, narrows the bounds on demand responses for good-1 at B_1 from $[2.3, 7.3]$ to $[4, 7.3]$ and at B_2 from $[1.4, 5.3]$ to $[4, 5.3]$

V. Empirical Illustration

To illustrate the application of these methods, two empirical exercises are performed with data from the UK Family Expenditure Survey (FES). First, nonparametric estimates of income expansion curves are constrained to jointly satisfy revealed preference restrictions across different price regimes. This application serves to demonstrate an alternative strategy to Blundell, Browning and Crawford (2008) and Blundell, Kristensen and Matzkin (2014) for recovering jointly rational 'intersection demands'. Second, in the Appendix, rational choice paths over a set of new budgets are recovered using the data of Blundell, Browning and Crawford (2008).

We note that these exercises serve as illustrations of the techniques described in the paper; bringing these methods to data in a more comprehensive way requires

dealing with empirical issues such as price and income endogeneity, measurement error, and unobserved preference heterogeneity. Such issues are abstracted from here for reasons of compactness but the reader is referred to Blundell, Browning and Crawford (2003, 2008) and Blundell, Kristensen and Matzkin (2014) for methodological extensions to overcome these challenges, which are directly applicable to the framework introduced here.

A. Mutually consistent regression predictions

A number of recent papers in the revealed preference tradition make use of the Blundell, Browning and Crawford (2008) concept of 'e-bounds', or 'expansion path based bounds'. Rather than use observed (or average) demands to bound choice at a new budget $\{\mathbf{p}_0, x_0\}$, e-bounds are defined using 'intersection demands'. Intersection demands, $\mathbf{q}_t(\tilde{x}_t)$, are defined such that:

$$\mathbf{p}_0' \mathbf{q}_t(\tilde{x}_t) = x_0, \tag{16}$$

where $\mathbf{q}_t(x)$ gives the income expansion path in period t. Intuitively, by 'controlling for' income variation, the use of intersection demands allows for much tighter bounds to be placed on behavioural responses (see also Blundell et al. (2015)).

Recovering e-bounds requires the estimation of nonparametric Engel curves. However, due to sampling variation (, and other measurement and specification error), estimated Engel curves may themselves violate revealed preference restrictions, resulting in the recovered set of intersection demands failing GARP. Without modification, such intersection demands cannot then be used to bound choices at $\{\mathbf{p}_0, x_0\}$. Different strategies are used to correct for this problem. For example, Blundell, Browning and Crawford (2008) perturb unconstrained intersection demands to satisfy GARP. However, without using the integer constraints set out in this paper, the constraints for this problem are nonlinear, and thus not easy to implement. See also an alternative approach in Blundell, Kristensen and

Matzkin (2014).

Here we estimate nonparametric income expansion paths subject to joint rationality using mixed integer programming methods. The method of Blundell, Horowitz and Parey (2012) is adapted to impose mutual consistency on nonparametric estimates of income expansion paths at T different price regimes.[14] In each period t, we observe the consumption and total expenditure of a set of households n_t, $\{\mathbf{q}_i, x_i\}_{i \in n_t}$, at a price regime $\mathbf{p}_t$. Let the full set of observations across price regimes be given as $n = \cup_{t=1}^{T} n_t$, and $|n| = N$.

We assume that, for each good k at price regime $\mathbf{p}_t$, demands and expenditure are related by the stochastic Engel curve:

$$q_{it}^k = g_t^k(x_i) + \varepsilon_i^k, \tag{17}$$

where, for each household i, the error term satisfies $E(\varepsilon_i^k|x) = 0$ and $var(\varepsilon_i^k|x) = \sigma^2(x)$ for $k = 1, ..., K$.

Formally, our rationality-constrained estimator, $\hat{g}_t^k(x)$, is:

$$\hat{g}_t^k(x) = \frac{1}{h\hat{f}(x)} \sum_{i \in n_t} \omega_{it} q_i^k \mathbb{K}\left(\frac{x - x_i}{h}\right), \tag{18}$$

for $k = 1, ..., K-1$, where

$$\hat{f}(x) = \frac{1}{n_t h} \sum_{i \in n_t} \mathbb{K}\left(\frac{x - x_i}{h}\right) \tag{19}$$

and $\mathbb{K}$ is a kernel function, i.e. a bounded, differentiable probability density function that is symmetric about 0, and ω_{it} are nonnegative weights satisfying $\sum_{i \in n_t} \omega_{it} = 1$. The demand for good-$K$ is obtained by adding-up.[15]

[14] In Blundell, Horowitz and Parey (2012), the Slutsky condition is imposed on a nonparametric estimate of the demand function to yield well-behaved estimates.

[15] Provided that the same bandwidth and kernel are used to estimate each $g^k(x)$, adding up will be automatically satisfied and there is no efficiency gain from combining equations. See Deaton (1983) and

The weights required for joint-rationality of demand predictions along different income expansion paths are obtained by solving the following optimisation problem:

$$\min_{\boldsymbol{\omega}} \sum_{t=1}^{T} \sum_{i \in n_t} (\omega_{it} - \frac{1}{n_t})^2 \tag{20}$$

subject to:

$$\{\hat{\mathbf{g}}_t(x_b)\}_{b \in \mathscr{Q}}^{t=1,\ldots,T} \in S^S(\mathscr{Q}) \tag{21}$$

$$\sum_{i \in n_t} \omega_{it} = 1 \quad , \forall t \tag{22}$$

$$\omega_{it} \geq 0 \quad , \forall i, t \tag{23}$$

for a set of new total expenditures, $\mathscr{Q}$. For the unconstrained estimator, $\omega_{it} = 1/n_t$ for all $i \in n_t$.

To solve this programme, the mixed integer constraints of Section II are imposed to ensure that constraint (21) is satisfied.[16] This is done using `intlinprog` in Matlab, iteratively solving a sequence of mixed integer linear problems that locally approximate the mixed integer quadratic programming problem, i.e. a cutting plane method (Kelley, 1960).[17] A Gaussian kernel and Silverman plug-in bandwidth are employed for the kernel weighting function.

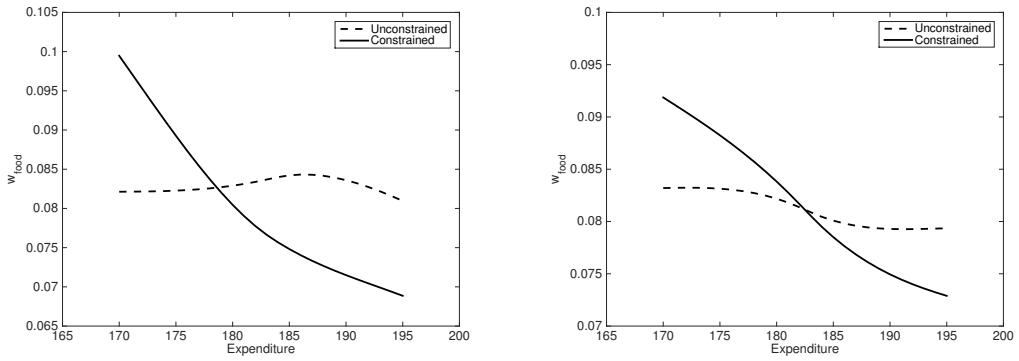
B. Data and Results

Consumer microdata drawn from the UK Family Expenditure Survey (FES) is used to implement the approach. The FES is a repeated cross-section survey consisting of approximately 7,000 households per year. We restrict attention to

Blundell, Browning and Crawford (2003).

[16] Note that we do not impose consistency with observed demands as we here consider the demands of a representative consumer.

[17] The optimisation can be done in a single step using commercially available solvers (e.g. `cvx` Professional). However, the procedure was carried out using standard software to demonstrate the feasibility of implementation.



(a) 1991 (b) 1993

FIGURE 2. INCOME EXPANSION PATHS FOR FOOD

the period 1991-1996 for the sub-sample of couples without children, in which the head of the household left school at 16 years old and is aged between 20 and 60 years old. This leaves us with between 759 and 900 observations per year, and a total sample size of 4909 households over the six different price regimes. We model demand for food, services and other nondurables,[18] and define total expenditure as spending on these commodities.

Income expansion paths are recovered at a set of points between 25^{th} and 75^{th} percentiles of the expenditure distribution at each observed price regime. The set of unconstrained estimates fail GARP. The magnitude of irrationality in recovered predictions is small; the Afriat Efficiency Index for the set of demands is 0.9990, suggesting that only 0.1% of the budget is wasted through inefficiency. Yet, despite the small extent of irrationality, these demands cannot be used to bound demand responses at a new budget as they themselves are inconsistent.

We recover rational intersection demands by applying the mixed integer programming procedure. To increase the efficiency of the procedure, we search for subsets of non-intersecting budgets to apply the insights of Section II.A. Given the set of new total expenditure levels and the observed price vectors, the set of

[18]The underlying commodity list is given in the Appendix.

36 budgets considered could be split into two subsets of non-intersecting budgets — one subset of 12 budgets, the other of 24 budgets. This reduced the number of integer variables associated with the programming problem from 1296 to 720. The programming problem was then feasible to implement on a standard desktop.

Figure 2 gives the unconstrained and rationality-constrained estimates of the income expansion paths in 1991 and 1993. Imposition of rationality generates more steeply downward sloping income expansion paths, and greater consistency in the slope of curves across price regimes. The resulting demand predictions can also, crucially, be used to recover e-bounds as the estimated income expansion paths now jointly satisfy GARP at the set of income levels considered.

VI. Conclusion

The revealed preference restrictions that are commonly imposed on demand predictions are not sufficient for rationality when predicting behaviour at a set of new budgets. When predicting over a set of intersecting budgets, not all combinations of demands from the Varian support sets will satisfy revealed preference restrictions. To ensure rationality of the combined set of predictions, mutual consistency must be imposed across predictions. The requirement of mutual consistency generates non-linearities in the typical revealed preference inequalities and can introduce non-convexities to the revealed preference support set. This prevents standard linear programming methods from being employed to recover the support set.

This paper has provided a Mixed Integer Programming representation of the prediction problem, which can be applied with reasonable computational resources. Routes to enhance the efficiency of the procedure have been noted, giving possibilities for reducing the computational burden of the method. An empirical illustration using data from the UK Family Expenditure Survey served to demonstrate the implementation of the method to impose mutual consistency on nonparametric estimates of income expansion paths. This served to demon-

strate the utility of the method for returning a rational set of 'intersection demands' from estimated income expansion paths, of the type required by Blundell, Browning and Crawford (2008) and Blundell, Kristensen and Matzkin (2014).

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Appendix

Proof of Proposition 1. — Without loss of generality, let $\mathscr{Q} = \{a, b\}$. Define,

$$H_-(\mathbf{p}) = \{\mathbf{q} : \mathbf{p}'\mathbf{q} \leq 1\} \tag{A1}$$

$$H_+(\mathbf{p}) = \{\mathbf{q} : \mathbf{p}'\mathbf{q} \geq 1\} \tag{A2}$$

Since $S^V(B_a)$ and $S^V(B_b)$ intersect in the interior, $\exists \mathbf{q}_a \in H_-(\mathbf{p}_b) \cap S^V(B_a)$ and $\exists \mathbf{q}_b \in H_-(\mathbf{p}_a) \cap S^V(B_b)$, with $\mathbf{q}_a \neq \tilde{\mathbf{q}}$ and $\mathbf{q}_b \neq \tilde{\mathbf{q}}$. Define:

$$\bar{\mathbf{q}}_a \in \left\{\mathbf{q} : \mathbf{q} \in S^V(B_a) \cap H_+(\mathbf{p}_b), ||\mathbf{q} - \tilde{\mathbf{q}}|| < \varepsilon\right\} \tag{A3}$$

$$\bar{\mathbf{q}}_b \in \left\{\mathbf{q} : \mathbf{q} \in S^V(B_b) \cap H_+(\mathbf{p}_a), ||\mathbf{q} - \tilde{\mathbf{q}}|| < \varepsilon\right\}. \tag{A4}$$

Then, taking the data set $D = \{\mathbf{p}_a, \mathbf{p}_b, \{\mathbf{p}_t\}_{t\in\mathscr{D}}; \mathbf{q}_a, \bar{\mathbf{q}}_b, \{\mathbf{q}_t\}_{t\in\mathscr{D}}\}$, observe that:

- D satisfies GARP.

  From Kitamura and Stoye (2013), if there is a choice cycle of any finite

length, then there is a cycle of length 2 or 3 — where a cycle of length 2 is a WARP violation.

There are no cycles of length two — it is assumed that $\{\mathbf{p}_t, \mathbf{q}_t\}_{t\in\mathscr{D}}$ satisfies WARP, that $\{\mathbf{p}_a, \{\mathbf{p}_t\}_{t\in\mathscr{D}}; \mathbf{q}_a, \{\mathbf{q}_t\}_{t\in\mathscr{D}}\}$ satisfies WARP by virtue of $\mathbf{q}_a \in S^V(B_a)$, and that $\{\mathbf{p}_b, \{\mathbf{p}_t\}_{t\in\mathscr{D}}; \bar{\mathbf{q}}_b, \{\mathbf{q}_t\}_{t\in\mathscr{D}}\}$ satisfies WARP by virtue of $\mathbf{q}_b \in S^V(B_b)$. $\{\mathbf{p}_a, \mathbf{p}_b; \mathbf{q}_a, \bar{\mathbf{q}}_b\}$ do not induce a WARP violation given that by construction:

$$\text{(A5)} \qquad \mathbf{p}_b'\mathbf{q}_a \;<\; 1$$

$$\text{(A6)} \qquad \mathbf{p}_a'\bar{\mathbf{q}}_b \;>\; 1$$

and $\neg\left(\mathbf{q}_a \mathbb{R} \bar{\mathbf{q}}_b\right)$.

There are no cycles of length three — Take $\mathbf{q}_t \in H_-(\mathbf{p}_a, 1) \cap H_+(\mathbf{p}_b, 1)$. We have: $\bar{\mathbf{q}}_b \mathbb{R}^0 \mathbf{q}_a \mathbb{R}^0 \mathbf{q}_t$. Thus, $\bar{\mathbf{q}}_b \mathbb{R} \mathbf{q}_t$. This would induce an indirect revealed preference violation if $\mathbf{q}_t \mathbb{R} \bar{\mathbf{q}}_b$. However, for arbitrarily small ϵ, $\bar{\mathbf{q}}_b \approx \tilde{\mathbf{q}}$ and $\tilde{\mathbf{q}} \mathbb{R} \mathbf{q}_t$.

- The indices a and b can be switched such that these arguments apply to $D' = \{\mathbf{p}_a, \mathbf{p}_b, \{\mathbf{p}_t\}_{t\in\mathscr{D}}; \bar{\mathbf{q}}_a, \mathbf{q}_b, \{\mathbf{q}_t\}_{t\in\mathscr{D}}\}$

Let an element of S^S, i.e. a single prediction, be given as the stacked column vector: $Q = [\mathbf{q}_a, \mathbf{q}_b]$. Let $Q^a = [\mathbf{q}_a, \bar{\mathbf{q}}_b]$ and $Q^b = [\bar{\mathbf{q}}_a, \mathbf{q}_b]$. From above, $Q^a, Q^b \in S^S$.Yet,

$$\text{(A7)} \qquad \lambda Q^a + (1-\lambda) Q^b \;\notin\; S^S$$

for some $\lambda \in (0, 1)$. To see this note,

$$\text{(A8)} \qquad \lambda Q^a + (1-\lambda) Q^b \;=\; \begin{bmatrix} \lambda \mathbf{q}_a + (1-\lambda)\bar{\mathbf{q}}_a \\ \lambda \bar{\mathbf{q}}_b + (1-\lambda)\mathbf{q}_b \end{bmatrix},$$

and that the choices $\{\lambda \mathbf{q}_a + (1-\lambda)\bar{\mathbf{q}}_a, \lambda \bar{\mathbf{q}}_b + (1-\lambda)\mathbf{q}_b\}$ violate WARP. It is the case that

$$\mathbf{p}'_a \mathbf{q}_b < 1 \tag{A9}$$

$$\mathbf{p}'_a \bar{\mathbf{q}}_b \approx 1 \tag{A10}$$

Thus, for ε sufficiently small, there exists a λ such that

$$\mathbf{p}'_a \left(\lambda \bar{\mathbf{q}}_b + (1-\lambda)\mathbf{q}_b\right) < 1. \tag{A11}$$

Yet, by the same reasoning:

$$\mathbf{p}'_b \left(\lambda \bar{\mathbf{q}}_a + (1-\lambda)\mathbf{q}_a\right) < 1, \tag{A12}$$

leading to a revealed preference violation.

Proof of Proposition 2. — Constraints [1] and [2] flow directly from [S1] and [S2]. Constraints [3], [4], [5] impose GARP, and thus [S3] on $\mathscr{B}$.

Constraint [3] summarises the direct revealed preference relation. First, let $\mathbf{p}'_a \mathbf{q}_b \leq 1$. We require that $\mathbf{q}_a \mathbb{R}^0 \mathbf{q}_b$. For Constraint [3] to be satisfied, $R_{ab} = 1$. If $\mathbf{p}'_a \mathbf{q}_b > 1$, $R_{ab} = 1$ or $R_{ab} = 1 = 0$ are permissible.

Constraint [4] imposes transitivity of the revealed preference relation. If $\mathbf{q}_a \mathbb{R}^0 \mathbf{q}_b$ and $\mathbf{q}_b \mathbb{R}^0 \mathbf{q}_c$, then we require $\mathbf{q}_a \mathbb{R} \mathbf{q}_c$. If $R_{ab} = 1$ and $R_{bc} = 1$, Constraint [4] is violated unless $R_{ac} = 1$. The integer variables R_{ij} are thus equivalent to the revealed preference relation.

Constraint [5] imposes GARP. If $R_{ab} = 1$, then $\mathbf{p}'_b \mathbf{q}_a \geq 1$, or the constraint is violated. If $R_{ab} = 0$, $\mathbf{p}'_b \mathbf{q}_a \lesseqgtr 1$.

PROOF OF PROPOSITION 3. — For $a \in \mathscr{Q}_m$ and $b \in \mathscr{Q}_n$ with $m \neq n$, either $p_m^k \leq p_n^k$ or $p_n^k \leq p_m^k$ for all $k = 1, ..., K$. Let the former hold.

By construction, $\mathbf{q}_b \in S^V(B_b)$ for all $b \in \mathscr{D}$. Choices at B_i, $i \in \mathscr{Q}_m$, and at B_j, $j \in \mathscr{Q}_n$, cannot generate a length two cycle because $\mathbf{p}_m'\mathbf{q}_n \leq 1$ for all $\mathbf{q}_n \in S^V(B_n)$ and $\mathbf{p}_n'\mathbf{q}_m \geq 1$ for all $\mathbf{q}_m \in S^V(B_m)$, with equality only possible if at a corner.

No length three cycles can be generated. By construction, $\mathbf{q}_m \mathbb{R}^0 \mathbf{q}_n$ for all $\mathbf{q}_m \in S^V(B_m)$ and $\mathbf{q}_n \in S^V(B_n)$. For any, $\mathbf{q}_t$ such that $\mathbf{q}_n \mathbb{R}^0 \mathbf{q}_t$, $\mathbf{q}_m \mathbb{R}^0 \mathbf{q}_t$ as $p_m^k \leq p_n^k$. For any $\mathbf{q}_t$ such that $\mathbf{q}_t \mathbb{R}^0 \mathbf{q}_m$, $\mathbf{q}_t \mathbb{R}^0 \mathbf{q}_n$ as $p_m^k \leq p_n^k$.

PROOF OF PROPOSITION 4. — Take some $\tilde{\mathbf{q}}_b \in S^V(B_b)$. By definition, $\{\tilde{\mathbf{q}}_b, \{\mathbf{q}_t\}_{t \in \mathscr{D}}\}$ satisfies GARP. The following utility function exists to rationalise these choices (Varian, 1982):

$$u(\mathbf{q}) = \min_{t \in \mathscr{T} \cup b} \left\{ u_t + \lambda_t \mathbf{p}_t'(\mathbf{q} - \mathbf{q}_t) \right\} \tag{A13}$$

Let choices at $\forall a \in \mathscr{D}/b$ be given as:

$$\tilde{\mathbf{q}}_a = \text{argmax}_{\mathbf{q}} \{ \min_{t \in \mathscr{T} \cup b} \{ u_t + \lambda_t \mathbf{p}_t'(\mathbf{q} - \mathbf{q}_t) \} \} \tag{A14}$$

subject to $\mathbf{p}_a'\mathbf{q} = 1$. By definition, $\{\{\tilde{\mathbf{q}}_a\}_{a \in \mathscr{D}/b}, \{\mathbf{q}_t\}_{t \in \mathscr{D}}\}$ satisfies GARP $\forall a \in \mathscr{D}/b$. Thus, $\{\tilde{\mathbf{q}}_a\}_{a \in \mathscr{Q}} \in S^S$.

A1. Good construction

In Section V, a second empirical example serves to show how the method can be applied to constrain nonparametric estimates. The commodity definitions for this exercise are as follows:

- *Food*: {bread, cereals, biscuits & cakes, beef, lamb, pork, bacon, poultry, other meats & fish, butter, oil & fats, cheese, eggs, fresh milk, milk products,

tea, coffee, soft drinks, sugar & preserves, sweets & chocolate, potatoes, other vegetables, fruit, other foods, canteen meals, other restaurant meals & snacks}.

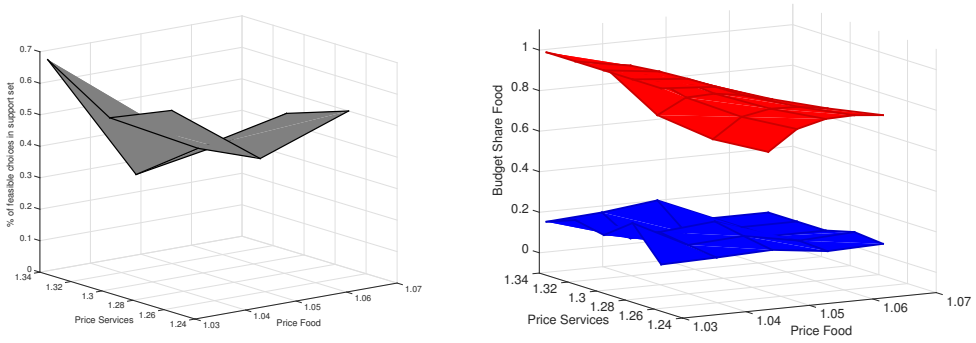
- *Services*: {coal, electric, gas, petrol & oil, postage, telephone, domestic services, fess & subscriptions, personal services, maintenance of motor vehicles, vehicle tax & insurance, travel fares}
- *Nondurables*: {household consumables, pet care, chemist goods, audio visual goods, records & toys, book & newspapers, entertainment}

A2. Mutually consistent bounds

As an additional empirical exercise to demonstrate the recovery of mutually consistent revealed preference bounds and rational choice paths, the mean budget shares, relative price, and total expenditure data from Blundell, Browning and Crawford (2008) Appendix Table A.I are used to bound mean demand responses at a set of new budgets of interest. The data covers food, service, and nondurable consumption from 1973-1999, drawn from the FES.[19] The price of food and of services is varied on a grid in the convex hull of the observed price space, with total expenditure kept constant at its 75^{th} percentile.

Figure A1 (a) shows the percentage of potential choices that are consistent with past observations at each element of the budget grid (i.e. that are members of the Varian support set). A non-negligible, if not a high, proportion of feasible choices are ruled out by rationality at each budget — between 32% and 73% of choices are ruled out at the nine budgets we consider (i.e. between 68% and 27% of potential choices at each budget are in the support set). As, in some cases, a rather large proportion of potential choices are consistent with past observations, the bounds on rational budget shares are rather wide. For example, Figure A1 (b) gives the revealed preference bounds on the budget share of food over this grid of new

[19]The FES is a repeated cross-section survey consisting of around 7,000 households in each year. The sub-sample of couples with children who own a car is used for their analysis.



(a) % Consistent Choices (b) Bounds on w_{food}

FIGURE A1. REVEALED PREFERENCE BOUNDS WITHOUT IMPOSING MUTUAL CONSISTENCY

budgets. The cardinality of the support sets and the width of the bounds could be reduced if we employed the Sequential Maximum Power path methodology but we do not so for reasons of expositional simplicity — see Blundell, Browning and Crawford (2008) and Blundell et al. (2015) for details.

While all demands within the upper and lower bounds shown in Figure A1 (b) are consistent with observed demands, they are not consistent with one another. Employing the adjusted Kitamura and Stoye-type methodology of Section III, the set of rational choice paths over the grid of budgets is recovered.[20] Table A1 gives the number of feasible choice types, and the number of rational choice types for the simple problem considered. A significant number of feasible choice types are excluded by the requirement of mutual consistency — 84.3% in this example.

For any rational choice type, the mutually consistent bounds on demand re-

[20]Their 'decision tree crawling algorithm' reduces the computation time of this process significantly and was feasible to implement on a standard desktop.

TABLE A1—RATIONAL CHOICE PATHS

Budgets	Feasible Choice Paths	Rational Choice Paths	% Irrational Choice Paths
9	6480	1019	84.3

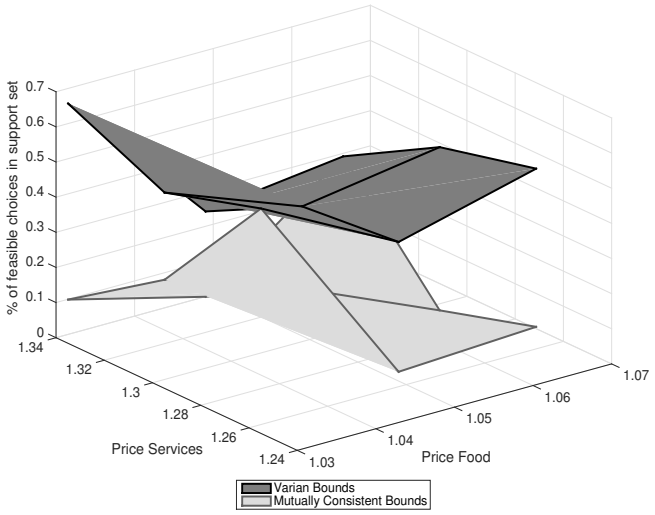


Figure A2. Cardinality of Varian Support Set and a Mutually Consistent Choice Path

sponses across the set of budgets are weakly tighter than those without mutual consistency imposed. To illustrate, Figure A2 compares the cardinality of the support set associated with one mutually consistent rational choice path over the set of budgets to that of the Varian support sets. As expected, a much smaller area is permissible at many budget sets — only where a new budget does not intersect any other new budget is the percentage of choices consistent with rationality the same at the Varian support set and the sufficient support set.