

Vasilev, Aleksandar

Preprint

Progressive taxation and (in)stability in an endogenous growth model with human capital accumulation: the case of Bulgaria

Suggested Citation: Vasilev, Aleksandar (2016) : Progressive taxation and (in)stability in an endogenous growth model with human capital accumulation: the case of Bulgaria, ZBW - Deutsche Zentralbibliothek für Wirtschaftswissenschaften, Leibniz-Informationszentrum Wirtschaft, Kiel und Hamburg

This Version is available at:

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Progressive taxation and (in)stability in an endogenous growth model with human capital accumulation.

Technical Appendix

Aleksandar Vasilev*

January 8, 2016

0.1 Log-Linearization of the Stationary Equilibrium System

0.1.1 MU consumption

$$\begin{aligned}c_t^{-1} &= \lambda_t \\ \ln c_t^{-1} &= \ln \lambda_t \\ -\frac{d}{dt} \ln c_t &= \frac{d}{dt} \ln \lambda_t \\ -\hat{c}_t &= \hat{\lambda}_t\end{aligned}$$

0.1.2 Progressive taxation

$$\begin{aligned}\tau_t &= \eta(y_t/y)^\phi \\ \ln \tau_t &= \ln \eta + \phi \ln y_t - \phi \ln y \\ \frac{d}{dt} \ln \tau_t &= \frac{d}{dt} \ln \eta + \phi \frac{d}{dt} \ln y_t - \phi \frac{d}{dt} \ln y \\ \hat{\tau}_t &= \phi \hat{y}_t\end{aligned}$$

*Aleksandar Vasilev is a CERGE-EI Affiliate Fellow and an Asst. Professor at the Department of Economics, American University in Bulgaria, 1 Georgi Izmirliov Sq., Blagoevgrad 2700, Bulgaria. Tel: 00 359 73 888 482. All errors and omissions are mine alone. E-mail for correspondence: avasilev@aubg.edu.

0.1.3 Physical capital Euler

$$\begin{aligned}
\lambda_t &= \bar{\beta}\lambda_{t+1} \left[(1 - \delta^k) + \left(1 - (1 + \phi)\tau_{t+1} \right) r_{t+1} \right] \\
\ln \lambda_t &= \ln \bar{\beta} + \ln \lambda_{t+1} \left[(1 - \delta^k) + \left(1 - (1 + \phi)\tau_{t+1} \right) r_{t+1} \right] \\
\frac{d}{dt} \ln \lambda_t &= \frac{d}{dt} \ln \bar{\beta} + \frac{d}{dt} \ln \lambda_{t+1} \left[(1 - \delta^k) + \left(1 - (1 + \phi)\tau_{t+1} \right) r_{t+1} \right] \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \frac{d}{dt} \ln \left[(1 - \delta^k) + \left(1 - (1 + \phi)\tau_{t+1} \right) r_{t+1} \right] \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \frac{1}{\left[(1 - \delta^k) + \left(1 - (1 + \phi)\tau \right) r \right]} \frac{d}{dt} \left[(1 - \delta^k) + \left(1 - (1 + \phi)\tau_{t+1} \right) r_{t+1} \right] \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \bar{\beta} \frac{d}{dt} (1 - \delta^k) + \beta \frac{d}{dt} \left[\left(1 - (1 + \phi)\tau_{t+1} \right) r_{t+1} \right] \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \bar{\beta} \frac{d}{dt} \left[\left(1 - (1 + \phi)\tau_{t+1} \right) r_{t+1} \right] \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \bar{\beta} \left(1 - (1 + \phi)\tau \right) r \hat{r}_{t+1} - \bar{\beta} (1 + \phi) \tau r \hat{r}_{t+1}
\end{aligned}$$

0.1.4 Human capital Euler

Note that $h_{t+1} = 1, \forall t$.

$$\begin{aligned}
\left(1 - (1 + \phi)\tau_t \right) r_t &= \left(1 - (1 + \phi)\tau_t \right) w_t \bar{h} + \delta \\
\ln \left(1 - (1 + \phi)\tau_t \right) r_t &= \ln \left[\left(1 - (1 + \phi)\tau_t \right) w_t \bar{h} + \delta \right] \\
\frac{d}{dt} \ln \left(1 - (1 + \phi)\tau_t \right) r_t &= \frac{d}{dt} \ln \left[\left(1 - (1 + \phi)\tau_t \right) w_t \bar{h} + \delta \right] \\
\frac{1}{\left(1 - (1 + \phi)\tau \right) r} \frac{d}{dt} \left(1 - (1 + \phi)\tau_t \right) r_t &= \frac{1}{\left[\left(1 - (1 + \phi)\tau \right) w \bar{h} + \delta \right]} \frac{d}{dt} \left[\left(1 - (1 + \phi)\tau \right) w_t \bar{h} + \delta \right] \\
\frac{d}{dt} \left(1 - (1 + \phi)\tau_t \right) r_t &= \frac{d}{dt} \left[\left(1 - (1 + \phi)\tau \right) w_t \bar{h} + \delta \right] \\
\left(1 - (1 + \phi)\tau \right) r \hat{r}_t - (1 + \phi) \tau r \hat{r}_t &= \left(1 - (1 + \phi)\tau \right) w \bar{h} \hat{w}_t - (1 + \phi) \tau w \bar{h} \hat{r}_t
\end{aligned}$$

0.1.5 Real interest rate

$$\begin{aligned}r_t &= \theta \frac{y_t}{k_t} \\ \ln r_t &= \ln \theta + \ln y_t - \ln k_t \\ \frac{d}{dt} \ln r_t &= \frac{d}{dt} \ln \theta + \frac{d}{dt} \ln y_t - \frac{d}{dt} \ln k_t \\ \hat{r}_t &= \hat{y}_t - \hat{k}_t\end{aligned}$$

0.1.6 Wage rate

$$\begin{aligned}w_t &= (1 - \theta) \frac{y_t}{s_t \bar{h}} \\ \ln w_t &= \ln(1 - \theta) + \ln \frac{y_t}{s_t \bar{h}} \\ \frac{d}{dt} \ln w_t &= \frac{d}{dt} \ln(1 - \theta) + \frac{d}{dt} \ln \frac{y_t}{s_t \bar{h}} \\ \hat{w}_t &= \hat{y}_t - \hat{s}_t\end{aligned}$$

0.1.7 Government Budget Constraint

$$\begin{aligned}g_t &= \tau_t [r_t k_t + w_t s_t \bar{h}] \\ g_t &= \tau_t y_t \\ \ln g_t &= \ln \tau_t + \ln y_t \\ \frac{d}{dt} \ln g_t &= \frac{d}{dt} \ln \tau_t + \frac{d}{dt} \ln y_t \\ \hat{g}_t &= \hat{\tau}_t + \hat{y}_t\end{aligned}$$

0.1.8 Production function

$$\begin{aligned}y_t &= \bar{A} k_t^\theta (s_t)^{1-\theta} \\ \ln y_t &= \ln \bar{A} + \theta \ln k_t + (1 - \theta) \ln s_t \\ \frac{d}{dt} \ln y_t &= \frac{d}{dt} \ln \bar{A} + \theta \frac{d}{dt} \ln k_t + (1 - \theta) \frac{d}{dt} \ln s_t \\ \hat{y}_t &= \theta \hat{k}_t + (1 - \theta) \hat{s}_t\end{aligned}$$

0.1.9 Market clearing

$$\begin{aligned}
y_t &= c_t + (1 + \gamma)k_{t+1} - (1 - \delta^k)k_t + (1 + \gamma)s_{t+1} - (1 - \delta^s)s_t \\
\ln y_t &= \ln[c_t + (1 + \gamma)k_{t+1} - (1 - \delta^k)k_t + (1 + \gamma)s_{t+1} - (1 - \delta^s)s_t] \\
\frac{d}{dt} \ln y_t &= \frac{d}{dt} \ln[c_t + (1 + \gamma)k_{t+1} - (1 - \delta^k)k_t + (1 + \gamma)s_{t+1} - (1 - \delta^s)s_t] \\
\hat{y}_t &= \frac{1}{c + \delta^k k + \delta^s s + g} \frac{d}{dt} [c_t + (1 + \gamma)k_{t+1} - (1 - \delta^k)k_t + (1 + \gamma)s_{t+1} - (1 - \delta^s)s_t] \\
y\hat{y}_t &= c\hat{c}_t + (1 + \gamma)k\hat{k}_{t+1} - (1 - \delta^k)k\hat{k}_t + (1 + \gamma)s\hat{s}_{t+1} - (1 - \delta^s)s\hat{s}_t
\end{aligned}$$

0.2 Log-linearized System

$$\begin{aligned}
-\hat{c}_t &= \hat{\lambda}_t \\
\hat{\tau}_t &= \phi \hat{y}_t \\
\left(1 - (1 + \phi)\tau\right)r\hat{r}_t - (1 + \phi)\tau r\hat{\tau}_t &= \left(1 - (1 + \phi)\tau\right)w\bar{h}\hat{w}_t - (1 + \phi)\tau w\bar{h}\hat{\tau}_t \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \bar{\beta} \left(1 - (1 + \phi)\tau\right)w\hat{w}_{t+1} - \bar{\beta}(1 + \phi)\tau w\hat{\tau}_{t+1} \\
\hat{r}_t &= \hat{y}_t - \hat{k}_t \\
\hat{w}_t &= \hat{y}_t - \hat{s}_t \\
\hat{g}_t &= \hat{\tau}_t + \hat{y}_t \\
\hat{y}_t &= \theta \hat{k}_t + (1 - \theta)\hat{s}_t \\
y\hat{y}_t &= c\hat{c}_t + (1 + \gamma)k\hat{k}_{t+1} - (1 - \delta^k)k\hat{k}_t + (1 + \gamma)s\hat{s}_{t+1} - (1 - \delta^s)s\hat{s}_t
\end{aligned}$$

0.3 Reduced Log-linearized system

0.3.1 Substituting out the Lagrangean multiplier

$$\begin{aligned}
\hat{\tau}_t &= \phi \hat{y}_t \\
-\hat{c}_t &= -\hat{c}_{t+1} + \bar{\beta} \left(1 - (1 + \phi)\tau \right) r \hat{r}_{t+1} - \bar{\beta} (1 + \phi) \tau r \hat{r}_{t+1} \\
\left(1 - (1 + \phi)\tau \right) r \hat{r}_t - (1 + \phi) \tau r \hat{r}_t &= \left(1 - (1 + \phi)\tau \right) w \bar{h} \hat{w}_t - (1 + \phi) \tau w \bar{h} \hat{\tau}_t \\
\hat{r}_t &= \hat{y}_t - \hat{k}_t \\
\hat{w}_t &= \hat{y}_t - \hat{s}_t \\
\hat{g}_t &= \hat{\tau}_t + \hat{y}_t \\
\hat{y}_t &= \theta \hat{k}_t + (1 - \theta) \hat{s}_t \\
y \hat{y}_t &= c \hat{c}_t + k \hat{k}_{t+1} - (1 - \delta^k) k \hat{k}_t + s \hat{s}_{t+1} - (1 - \delta^s) s \hat{s}_t
\end{aligned}$$

0.3.2 Substitute out taxes

$$\begin{aligned}
-\hat{c}_t &= -\hat{c}_{t+1} + \bar{\beta} \left(1 - (1 + \phi)\tau \right) r \hat{r}_{t+1} - \bar{\beta} (1 + \phi) \tau r \phi \hat{y}_{t+1} \\
\left(1 - (1 + \phi)\tau \right) r \hat{r}_t - (1 + \phi) \tau r \phi \hat{y}_t &= \left(1 - (1 + \phi)\tau \right) w \bar{h} \hat{w}_t - (1 + \phi) \tau w \bar{h} \phi \hat{y}_t \\
\hat{r}_t &= \hat{y}_t - \hat{k}_t \\
\hat{w}_t &= \hat{y}_t - \hat{s}_t \\
\hat{g}_t &= (1 + \phi) \hat{y}_t \\
\hat{y}_t &= \theta \hat{k}_t + (1 - \theta) \hat{s}_t \\
y \hat{y}_t &= c \hat{c}_t + k \hat{k}_{t+1} - (1 - \delta^k) k \hat{k}_t + s \hat{s}_{t+1} - (1 - \delta^s) s \hat{s}_t
\end{aligned}$$

0.3.3 Substituting out output

$$-\hat{c}_t = -\hat{c}_{t+1} + \bar{\beta} \left(1 - (1 + \phi)\tau \right) r \hat{r}_{t+1} - \bar{\beta} (1 + \phi) \tau r \phi [\theta \hat{k}_{t+1} + (1 - \theta) \hat{s}_{t+1}]$$

$$\begin{aligned}
\left(1 - (1 + \phi)\tau\right)r\hat{r}_t - (1 + \phi)\tau r\phi\hat{y}_t &= \left(1 - (1 + \phi)\tau\right)w\bar{h}\hat{w}_t - (1 + \phi)\tau w\bar{h}\phi\hat{y}_t \\
\hat{r}_t &= (\theta - 1)\hat{k}_t + (1 - \theta)\hat{s}_t \\
\hat{w}_t &= \theta\hat{k}_t - \theta\hat{s}_t \\
y(\theta\hat{k}_t + (1 - \theta)\hat{s}_t) &= c\hat{c}_t + (1 + \gamma)k\hat{k}_{t+1} - (1 - \delta^k)k\hat{k}_t + (1 + \gamma)s\hat{s}_{t+1} - (1 - \delta^s)s\hat{s}_t
\end{aligned}$$

0.3.4 Simplify

we will get that the dynamics in prices is proportional, i.e. $\hat{r}_t = \frac{\theta-1}{\theta}\hat{w}_t$, where $\frac{\theta-1}{\theta} < 0$.

$$\begin{aligned}
-\hat{c}_t &= -\hat{c}_{t+1} + \bar{\beta}\left(1 - (1 + \phi)\tau\right)r\hat{r}_{t+1} - \bar{\beta}(1 + \phi)\tau r\phi[\theta\hat{k}_{t+1} + (1 - \theta)\hat{s}_{t+1}] \\
\left(1 - (1 + \phi)\tau\right)r\hat{r}_t - (1 + \phi)\tau r\phi\hat{y}_t &= \left(1 - (1 + \phi)\tau\right)w\bar{h}\hat{w}_t - (1 + \phi)\tau w\bar{h}\phi\hat{y}_t \\
\hat{r}_t &= (\theta - 1)\hat{k}_t + (1 - \theta)\hat{s}_t \\
\hat{w}_t &= \theta\hat{k}_t - \theta\hat{s}_t \\
y(\theta\hat{k}_t + (1 - \theta)\hat{s}_t) &= c\hat{c}_t + (1 + \gamma)k\hat{k}_{t+1} - (1 - \delta^k)k\hat{k}_t + (1 + \gamma)s\hat{s}_{t+1} - (1 - \delta^s)s\hat{s}_t
\end{aligned}$$

0.3.5 Substitute out the prices (real interest rate)

$$\begin{aligned}
\hat{c}_t &= \hat{c}_{t+1} - \bar{\beta}\left(1 - (1 + \phi)\tau\right)r(\theta - 1)[\hat{k}_{t+1} - s_{t+1}] + \bar{\beta}(1 + \phi)\tau r\phi[\theta\hat{k}_{t+1} + (1 - \theta)\hat{s}_{t+1}] \\
&\quad \left(1 - (1 + \phi)\tau\right)r(\theta - 1)[\hat{k}_t - \hat{s}_t] - (1 + \phi)\tau r\phi[\theta\hat{k}_t + (1 - \theta)\hat{s}_t] = \\
&\quad \left(1 - (1 + \phi)\tau\right)w\bar{h}\theta(\hat{k}_t - \hat{s}_t) - (1 + \phi)\tau w\bar{h}\phi[\theta\hat{k}_t + (1 - \theta)\hat{s}_t] \\
-(1 + \gamma)k\hat{k}_{t+1} &= c\hat{c}_t - y(\theta\hat{k}_t + (1 - \theta)\hat{s}_t) - (1 - \delta^k)k\hat{k}_t + (1 + \gamma)s\hat{s}_{t+1} - (1 - \delta^s)s\hat{s}_t
\end{aligned}$$

0.3.6 Simplify Eq3

$$\begin{aligned}
\hat{c}_t &= \hat{c}_{t+1} - \bar{\beta}\left(1 - (1 + \phi)\tau\right)r(\theta - 1)[\hat{k}_{t+1} - s_{t+1}] + \bar{\beta}(1 + \phi)\tau r\phi[\theta\hat{k}_{t+1} + (1 - \theta)\hat{s}_{t+1}] \\
&\quad \left(1 - (1 + \phi)\tau\right)r(\theta - 1)[\hat{k}_t - \hat{s}_t] - (1 + \phi)\tau r\phi[\theta\hat{k}_t + (1 - \theta)\hat{s}_t] = \\
&\quad \left(1 - (1 + \phi)\tau\right)w\bar{h}\theta(\hat{k}_t - \hat{s}_t) - (1 + \phi)\tau w\bar{h}\phi[\theta\hat{k}_t + (1 - \theta)\hat{s}_t] \\
(1 + \gamma)k\hat{k}_{t+1} + (1 + \gamma)s\hat{s}_{t+1} &= -c\hat{c}_t + y(\theta\hat{k}_t + (1 - \theta)\hat{s}_t) + (1 - \delta^k)k\hat{k}_t + (1 - \delta^s)s\hat{s}_t
\end{aligned}$$

0.3.7 Simplify further Eq3

$$\begin{aligned}\hat{c}_t &= \hat{c}_{t+1} - \bar{\beta} \left(1 - (1 + \phi)\tau \right) r(\theta - 1) [\hat{k}_{t+1} - s_{t+1}] + \bar{\beta}(1 + \phi)\tau r \phi [\theta \hat{k}_{t+1} + (1 - \theta)\hat{s}_{t+1}] \\ &\quad \left(1 - (1 + \phi)\tau \right) r(\theta - 1) [\hat{k}_t - \hat{s}_t] - (1 + \phi)\tau r \phi [\theta \hat{k}_t + (1 - \theta)\hat{s}_t] = \\ &\quad \left(1 - (1 + \phi)\tau \right) w \bar{h} \theta (\hat{k}_t - \hat{s}_t) - (1 + \phi)\tau w \bar{h} \phi [\theta \hat{k}_t + (1 - \theta)\hat{s}_t] \\ (1 + \gamma)k\hat{k}_{t+1} + (1 + \gamma)s\hat{s}_{t+1} &= -c\hat{c}_t + [\theta y + (1 - \delta^k)k]\hat{k}_t + [(1 - \theta)y + (1 - \delta^s)s]\hat{s}_t\end{aligned}$$

0.3.8 Simplify Eq1

$$\begin{aligned}\hat{c}_t &= \hat{c}_{t+1} + \bar{\beta}r[2(1 + \phi)\tau\theta + (1 - \theta) - (1 + \phi)\tau]\hat{k}_{t+1} + \bar{\beta}r(1 - \theta)[\tau\phi^2 - 1 - \tau]\hat{s}_{t+1} \\ &\quad \left(1 - (1 + \phi)\tau \right) r(\theta - 1) [\hat{k}_t - \hat{s}_t] - (1 + \phi)\tau r \phi [\theta \hat{k}_t + (1 - \theta)\hat{s}_t] = \\ &\quad \left(1 - (1 + \phi)\tau \right) w \bar{h} \theta (\hat{k}_t - \hat{s}_t) - (1 + \phi)\tau w \bar{h} \phi [\theta \hat{k}_t + (1 - \theta)\hat{s}_t] \\ (1 + \gamma)k\hat{k}_{t+1} + (1 + \gamma)s\hat{s}_{t+1} &= -c\hat{c}_t + [\theta y + (1 - \delta^k)k]\hat{k}_t + [(1 - \theta)y + (1 - \delta^s)s]\hat{s}_t\end{aligned}$$

0.3.9 Simplify Eq2

$$\begin{aligned}\hat{c}_t &= \hat{c}_{t+1} + \bar{\beta}r[2(1 + \phi)\tau\theta + (1 - \theta) - (1 + \phi)\tau]\hat{k}_{t+1} + \bar{\beta}r(1 - \theta)[\tau\phi^2 - 1 - \tau]\hat{s}_{t+1} \\ &\quad \left[\left(1 - (1 + \phi)\tau \right) [\theta w \bar{h} - (1 - \theta)r] - (1 + \phi)\tau \phi \theta [r - w \bar{h}] \right] \hat{k}_t + \\ &\quad \left[\left(1 - (1 + \phi)\tau \right) [\theta w \bar{h} + (1 - \theta)r] - (1 - \theta)(1 + \phi)\tau \phi [r - w \bar{h}] \right] \hat{s}_t = 0 \\ (1 + \gamma)k\hat{k}_{t+1} + (1 + \gamma)s\hat{s}_{t+1} &= -c\hat{c}_t + [\theta y + (1 - \delta^k)k]\hat{k}_t + [(1 - \theta)y + (1 - \delta^s)s]\hat{s}_t\end{aligned}$$

or

$$\begin{aligned}\hat{c}_t &= \hat{c}_{t+1} + \bar{\beta}r[2(1 + \phi)\tau\theta + (1 - \theta) - (1 + \phi)\tau]\hat{k}_{t+1} + \bar{\beta}r(1 - \theta)[\tau\phi^2 - 1 - \tau]\hat{s}_{t+1} \\ \hat{k}_t &= - \frac{\left[\left(1 - (1 + \phi)\tau \right) [\theta w \bar{h} + (1 - \theta)r] - (1 - \theta)(1 + \phi)\tau \phi [r - w \bar{h}] \right]}{\left[\left(1 - (1 + \phi)\tau \right) [\theta w \bar{h} - (1 - \theta)r] - (1 + \phi)\tau \phi \theta [r - w \bar{h}] \right]} \hat{s}_t\end{aligned}$$

$$(1 + \gamma)k\hat{k}_{t+1} + (1 + \gamma)s\hat{s}_{t+1} = -c\hat{c}_t + [\theta y + (1 - \delta^k)k]\hat{k}_t + [(1 - \theta)y + (1 - \delta^s)s]\hat{s}_t$$

Let

$$\begin{aligned} A &= \bar{\beta}r[2(1 + \phi)\tau\theta + (1 - \theta) - (1 + \phi)\tau] \\ B &= \bar{\beta}r(1 - \theta)[\tau\phi^2 - 1 - \tau] \\ C &= -\frac{\left[\left(1 - (1 + \phi)\tau\right)[\theta w\bar{h} + (1 - \theta)r] - (1 - \theta)(1 + \phi)\tau\phi[r - w\bar{h}]\right]}{\left[\left(1 - (1 + \phi)\tau\right)[\theta w\bar{h} - (1 - \theta)r] - (1 + \phi)\tau\phi\theta[r - w\bar{h}]\right]} \\ D &= [\theta y + (1 - \delta^k)k] \\ E &= [(1 - \theta)y + (1 - \delta^s)s] \end{aligned}$$

Then the system becomes

$$\begin{aligned} \hat{c}_t &= \hat{c}_{t+1} + A\hat{k}_{t+1} + B\hat{s}_{t+1} \\ \hat{k}_t &= C\hat{s}_t \\ (1 + \gamma)k\hat{k}_{t+1} + (1 + \gamma)s\hat{s}_{t+1} &= -c\hat{c}_t + D\hat{k}_t + E\hat{s}_t \end{aligned}$$

Substitute out physical capital

$$\begin{aligned} \hat{c}_t &= \hat{c}_{t+1} + AC\hat{s}_{t+1} + B\hat{s}_{t+1} \\ C(1 + \gamma)k\hat{s}_{t+1} + (1 + \gamma)s\hat{s}_{t+1} &= -c\hat{c}_t + DC\hat{s}_t + E\hat{s}_t \end{aligned}$$

Let

$$\begin{aligned} F &= AC + B \\ G &= C(1 + \gamma)k + (1 + \gamma)s \\ H &= DC + E \end{aligned}$$

Thus

$$\begin{aligned} \hat{c}_t &= \hat{c}_{t+1} + F\hat{s}_{t+1} \\ G\hat{s}_{t+1} &= -c\hat{c}_t + H\hat{s}_t \end{aligned}$$

0.3.10 The reduced system in matrix form

$$\begin{pmatrix} 1 & F \\ 0 & G \end{pmatrix} \begin{pmatrix} \hat{c}_{t+1} \\ \hat{s}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & H \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{s}_t \end{pmatrix}$$

or (the matrix is invertible as its determinant is $H > 0$)

$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & F \\ 0 & G \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -c & H \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

or

$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{s}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & -F/G \\ 0 & 1/G \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c & H \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{s}_t \end{pmatrix}$$

or

$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{s}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 + Fc/G & -FH/G \\ -c/G & H/G \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{s}_t \end{pmatrix}$$

Let

$$I = 1 + Fc/G$$

$$J = -FH/G$$

$$K = -c/G$$

$$L = H/G$$

or

$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{s}_{t+1} \end{pmatrix} = \begin{pmatrix} I & J \\ K & L \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{s}_t \end{pmatrix}$$

0.4 Stability

Let

$$Z = \begin{pmatrix} I & J \\ K & L \end{pmatrix}$$

Find the characteristic roots

$$\begin{vmatrix} I - \lambda & J \\ K & L - \lambda \end{vmatrix} = 0$$

The characteristic polynomial is then

$$\begin{aligned} \lambda^2 - (I + L)\lambda + (IL - JK) &= 0 \\ \lambda_{1,2} &= \frac{(I + L) \pm \sqrt{(I + L)^2 - 4(IL - JK)}}{2} = \\ &= \frac{(I + L) \pm \sqrt{I^2 + 2IL + L^2 - 4IL + 4JK}}{2} = \\ &= \frac{(I + L) \pm \sqrt{(I - L)^2 + 4JK}}{2} \end{aligned}$$

since $JK > 0$, and $(I - L)^2 + 4JK > 0$, we will have two distinct, real roots.

$$\begin{aligned} \lambda_1 + \lambda_2 &= \text{tr}(Z) = I + L \\ \lambda_1 \lambda_2 &= \det(Z) = IL - JK = \frac{H}{G} + \frac{FHc}{G^2} - \frac{FHc}{G^2} = \frac{H}{G} = \frac{DC + E}{Ck + s} > 0 \end{aligned}$$

$$\begin{aligned} \lambda_1 &= \frac{(I + L) + \sqrt{(I - L)^2 + 4JK}}{2} \\ \lambda_2 &= \frac{(I + L) - \sqrt{(I - L)^2 + 4JK}}{2} \end{aligned}$$