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Progressive taxation and (in)stability in an endogenous growth model with human capital accumulation: the case of Bulgaria

Progressive taxation and (in)stability in an endogenous growth model with human capital accumulation. Technical Appendix

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0.1 Log-Linearization of the Stationary Equilibrium System

0.1.1 MU consumption

$$\begin{aligned}
c_t^{-1} &= \lambda_t \\
\ln c_t^{-1} &= \ln \lambda_t \\
-\frac{d}{dt}\ln c_t &= \frac{d}{dt}\ln \lambda_t \\
-\hat{c}_t &= \hat{\lambda}_t
\end{aligned}$$

0.1.2 Progressive taxation

$$\begin{aligned}
\tau_t &= \eta(y_t/y)^{\phi} \\
\ln \tau_t &= \ln \eta + \phi \ln y_t - \phi \ln y \\
\frac{d}{dt}\ln \tau_t &= \frac{d}{dt}\ln \eta + \phi\frac{d}{dt}\ln y_t - \phi\frac{d}{dt}\ln y \\
\hat{\tau}_t &= \phi \hat{y}_t
\end{aligned}$$

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0.1.3 Physical capital Euler

$$
\begin{aligned}
\lambda_t &= \bar{\beta}\lambda_{t+1}\left[(1-\delta^k) + \left(1-(1+\phi)\tau_{t+1}\right)r_{t+1}\right] \\
\ln\lambda_t &= \ln\bar{\beta} + \ln\lambda_{t+1}\left[(1-\delta^k) + \left(1-(1+\phi)\tau_{t+1}\right)r_{t+1}\right] \\
\frac{d}{dt}\ln\lambda_t &= \frac{d}{dt}\ln\bar{\beta} + \frac{d}{dt}\ln\lambda_{t+1}\left[(1-\delta^k) + \left(1-(1+\phi)\tau_{t+1}\right)r_{t+1}\right] \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \frac{d}{dt}\ln\left[(1-\delta^k) + \left(1-(1+\phi)\tau_{t+1}\right)r_{t+1}\right] \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \frac{1}{\left[(1-\delta^k) + \left(1-(1+\phi)\tau\right)r\right]}\frac{d}{dt}\left[(1-\delta^k) + \left(1-(1+\phi)\tau_{t+1}\right)r_{t+1}\right] \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \bar{\beta}\frac{d}{dt}(1-\delta^k) + \beta\frac{d}{dt}\left[\left(1-(1+\phi)\tau_{t+1}\right)r_{t+1}\right] \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \bar{\beta}\frac{d}{dt}\left[\left(1-(1+\phi)\tau_{t+1}\right)r_{t+1}\right] \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \bar{\beta}\left(1-(1+\phi)\tau\right)r\hat{r}_{t+1} - \bar{\beta}(1+\phi)\tau r\hat{\tau}_{t+1}
\end{aligned}
$$

0.1.4 Human capital Euler

Note that $h_{t+1} = 1, \forall t$.

$$
\begin{aligned}
\left(1-(1+\phi)\tau_t\right)r_t &= \left(1-(1+\phi)\tau_t\right)w_t\bar{h} + \delta \\
\ln\left(1-(1+\phi)\tau_t\right)r_t &= \ln\left[\left(1-(1+\phi)\tau_t\right)w_t\bar{h} + \delta\right] \\
\frac{d}{dt}\ln\left(1-(1+\phi)\tau_t\right)r_t &= \frac{d}{dt}\ln\left[\left(1-(1+\phi)\tau_t\right)w_t\bar{h} + \delta\right]
\end{aligned}
$$

$$
\frac{1}{\left(1-(1+\phi)\tau\right)r}\frac{d}{dt}\left(1-(1+\phi)\tau_t\right)r_t = \frac{1}{\left[\left(1-(1+\phi)\tau\right)w\bar{h}+\delta\right]}\frac{d}{dt}\left[\left(1-(1+\phi)\tau\right)w_t\bar{h}+\delta\right]
$$

$$
\begin{aligned}
\frac{d}{dt}\left(1-(1+\phi)\tau_t\right)r_t &= \frac{d}{dt}\left[\left(1-(1+\phi)\tau\right)w_t\bar{h} + \delta\right] \\
\left(1-(1+\phi)\tau\right)r\hat{r}_t - (1+\phi)\tau r\hat{\tau}_t &= \left(1-(1+\phi)\tau\right)w\bar{h}\hat{w}_t - (1+\phi)\tau w\bar{h}\hat{\tau}_t
\end{aligned}
$$

0.1.5 Real interest rate

$$
\begin{aligned}
r_t &= \theta \frac{y_t}{k_t} \\
\ln r_t &= \ln \theta + \ln y_t - \ln k_t \\
\frac{d}{dt} \ln r_t &= \frac{d}{dt} \ln \theta + \frac{d}{dt} \ln y_t - \frac{d}{dt} \ln k_t \\
\hat{r}_t &= \hat{y}_t - \hat{k}_t
\end{aligned}
$$

0.1.6 Wage rate

$$
\begin{aligned}
w_t &= (1-\theta) \frac{y_t}{s_t \bar{h}} \\
\ln w_t &= \ln(1-\theta) + \ln \frac{y_t}{s_t \bar{h}} \\
\frac{d}{dt} \ln w_t &= \frac{d}{dt} \ln(1-\theta) + \frac{d}{dt} \ln \frac{y_t}{s_t \bar{h}} \\
\hat{w}_t &= \hat{y}_t - \hat{s}_t
\end{aligned}
$$

0.1.7 Government Budget Constraint

$$
\begin{aligned}
g_t &= \tau_t [r_t k_t + w_t s_t \bar{h}] \\
g_t &= \tau_t y_t \\
\ln g_t &= \ln \tau_t + \ln y_t \\
\frac{d}{dt} \ln g_t &= \frac{d}{dt} \ln \tau_t + \frac{d}{dt} \ln y_t \\
\hat{g}_t &= \hat{\tau}_t + \hat{y}_t
\end{aligned}
$$

0.1.8 Production function

$$
\begin{aligned}
y_t &= \bar{A} k_t^{\theta} (s_t)^{1-\theta} \\
\ln y_t &= \ln \bar{A} + \theta \ln k_t + (1-\theta) \ln s_t \\
\frac{d}{dt} \ln y_t &= \frac{d}{dt} \ln \bar{A} + \theta \frac{d}{dt} \ln k_t + (1-\theta) \frac{d}{dt} \ln s_t \\
\hat{y}_t &= \theta \hat{k}_t + (1-\theta) \hat{s}_t
\end{aligned}
$$

0.1.9 Market clearing

$$
\begin{aligned}
y_t &= c_t + (1+\gamma)k_{t+1} - (1-\delta^k)k_t + (1+\gamma)s_{t+1} - (1-\delta^s)s_t \\
\ln y_t &= \ln[c_t + (1+\gamma)k_{t+1} - (1-\delta^k)k_t + (1+\gamma)s_{t+1} - (1-\delta^s)s_t] \\
\frac{d}{dt}\ln y_t &= \frac{d}{dt}\ln[c_t + (1+\gamma)k_{t+1} - (1-\delta^k)k_t + (1+\gamma)s_{t+1} - (1-\delta^s)s_t] \\
\hat{y}_t &= \frac{1}{c+\delta^k k + \delta^s s + g}\frac{d}{dt}[c_t + (1+\gamma)k_{t+1} - (1-\delta^k)k_t + (1+\gamma)s_{t+1} - (1-\delta^s)s_t] \\
y\hat{y}_t &= c\hat{c}_t + (1+\gamma)k\hat{k}_{t+1} - (1-\delta^k)k\hat{k}_t + (1+\gamma)s\hat{s}_{t+1} - (1-\delta^s)s\hat{s}_t
\end{aligned}
$$

0.2 Log-linearized System

$$
\begin{aligned}
-\hat{c}_t &= \hat{\lambda}_t \\
\hat{\tau}_t &= \phi\hat{y}_t \\
\Bigg(1-(1+\phi)\tau\Bigg)r\hat{r}_t - (1+\phi)\tau r\hat{\tau}_t &= \Bigg(1-(1+\phi)\tau\Bigg)w\bar{h}\hat{w}_t - (1+\phi)\tau w\bar{h}\hat{\tau}_t \\
\hat{\lambda}_t &= \hat{\lambda}_{t+1} + \bar{\beta}\Bigg(1-(1+\phi)\tau\Bigg)w\hat{w}_{t+1} - \bar{\beta}(1+\phi)\tau w\hat{\tau}_{t+1} \\
\hat{r}_t &= \hat{y}_t - \hat{k}_t \\
\hat{w}_t &= \hat{y}_t - \hat{s}_t \\
\hat{g}_t &= \hat{\tau}_t + \hat{y}_t \\
\hat{y}_t &= \theta\hat{k}_t + (1-\theta)\hat{s}_t \\
y\hat{y}_t &= c\hat{c}_t + (1+\gamma)k\hat{k}_{t+1} - (1-\delta^k)k\hat{k}_t + (1+\gamma)s\hat{s}_{t+1} - (1-\delta^s)s\hat{s}_t
\end{aligned}
$$

0.3 Reduced Log-linearized system

0.3.1 Substituting out the Lagrangean multiplier

$$
\begin{aligned}
\hat{\tau}_t &= \phi \hat{y}_t \\
-\hat{c}_t &= -\hat{c}_{t+1} + \bar{\beta}\Big(1-(1+\phi)\tau\Big) r\hat{r}_{t+1} - \bar{\beta}(1+\phi)\tau r\hat{\tau}_{t+1} \\
\Big(1-(1+\phi)\tau\Big) r\hat{r}_t - (1+\phi)\tau r\hat{\tau}_t &= \Big(1-(1+\phi)\tau\Big) w\bar{h}\hat{w}_t - (1+\phi)\tau w\bar{h}\hat{\tau}_t \\
\hat{r}_t &= \hat{y}_t - \hat{k}_t \\
\hat{w}_t &= \hat{y}_t - \hat{s}_t \\
\hat{g}_t &= \hat{\tau}_t + \hat{y}_t \\
\hat{y}_t &= \theta\hat{k}_t + (1-\theta)\hat{s}_t \\
y\hat{y}_t &= c\hat{c}_t + k\hat{k}_{t+1} - (1-\delta^k)k\hat{k}_t + s\hat{s}_{t+1} - (1-\delta^s)s\hat{s}_t
\end{aligned}
$$

0.3.2 Substitute out taxes

$$
\begin{aligned}
-\hat{c}_t &= -\hat{c}_{t+1} + \bar{\beta}\Big(1-(1+\phi)\tau\Big) r\hat{r}_{t+1} - \bar{\beta}(1+\phi)\tau r\phi\hat{y}_{t+1} \\
\Big(1-(1+\phi)\tau\Big) r\hat{r}_t - (1+\phi)\tau r\phi\hat{y}_t &= \Big(1-(1+\phi)\tau\Big) w\bar{h}\hat{w}_t - (1+\phi)\tau w\bar{h}\phi\hat{y}_t \\
\hat{r}_t &= \hat{y}_t - \hat{k}_t \\
\hat{w}_t &= \hat{y}_t - \hat{s}_t \\
\hat{g}_t &= (1+\phi)\hat{y}_t \\
\hat{y}_t &= \theta\hat{k}_t + (1-\theta)\hat{s}_t \\
y\hat{y}_t &= c\hat{c}_t + k\hat{k}_{t+1} - (1-\delta^k)k\hat{k}_t + s\hat{s}_{t+1} - (1-\delta^s)s\hat{s}_t
\end{aligned}
$$

0.3.3 Substituting out output

$$
-\hat{c}_t = -\hat{c}_{t+1} + \bar{\beta}\Big(1-(1+\phi)\tau\Big) r\hat{r}_{t+1} - \bar{\beta}(1+\phi)\tau r\phi[\theta\hat{k}_{t+1} + (1-\theta)\hat{s}_{t+1}]
$$

$$
\begin{aligned}
\Big(1-(1+\phi)\tau\Big)r\hat{r}_t-(1+\phi)\tau r\phi\hat{y}_t &= \Big(1-(1+\phi)\tau\Big)w\bar{h}\hat{w}_t-(1+\phi)\tau w\bar{h}\phi\hat{y}_t \\
\hat{r}_t &= (\theta-1)\hat{k}_t+(1-\theta)\hat{s}_t \\
\hat{w}_t &= \theta\hat{k}_t-\theta\hat{s}_t \\
y(\theta\hat{k}_t+(1-\theta)\hat{s}_t) &= c\hat{c}_t+(1+\gamma)k\hat{k}_{t+1}-(1-\delta^k)k\hat{k}_t+(1+\gamma)s\hat{s}_{t+1}-(1-\delta^s)s\hat{s}_t
\end{aligned}
$$

0.3.4 Simplify

we will get that the dynamics in prices is proportional, i.e. $\hat{r}_t=\frac{\theta-1}{\theta}\hat{w}_t$, where $\frac{\theta-1}{\theta}<0$.

$$
-\hat{c}_t=-\hat{c}_{t+1}+\bar{\beta}\Big(1-(1+\phi)\tau\Big)r\hat{r}_{t+1}-\bar{\beta}(1+\phi)\tau r\phi[\theta\hat{k}_{t+1}+(1-\theta)\hat{s}_{t+1}]
$$

$$
\begin{aligned}
\Big(1-(1+\phi)\tau\Big)r\hat{r}_t-(1+\phi)\tau r\phi\hat{y}_t &= \Big(1-(1+\phi)\tau\Big)w\bar{h}\hat{w}_t-(1+\phi)\tau w\bar{h}\phi\hat{y}_t \\
\hat{r}_t &= (\theta-1)\hat{k}_t+(1-\theta)\hat{s}_t \\
\hat{w}_t &= \theta\hat{k}_t-\theta\hat{s}_t \\
y(\theta\hat{k}_t+(1-\theta)\hat{s}_t) &= c\hat{c}_t+(1+\gamma)k\hat{k}_{t+1}-(1-\delta^k)k\hat{k}_t+(1+\gamma)s\hat{s}_{t+1}-(1-\delta^s)s\hat{s}_t
\end{aligned}
$$

0.3.5 Substitute out the prices (real interest rate)

$$
\hat{c}_t=\hat{c}_{t+1}-\bar{\beta}\Big(1-(1+\phi)\tau\Big)r(\theta-1)[\hat{k}_{t+1}-s_{t+1}]+\bar{\beta}(1+\phi)\tau r\phi[\theta\hat{k}_{t+1}+(1-\theta)\hat{s}_{t+1}]
$$

$$
\Big(1-(1+\phi)\tau\Big)r(\theta-1)[\hat{k}_t-\hat{s}_t]-(1+\phi)\tau r\phi[\theta\hat{k}_t+(1-\theta)\hat{s}_t]=
$$

$$
\Big(1-(1+\phi)\tau\Big)w\bar{h}\theta(\hat{k}_t-\hat{s}_t)-(1+\phi)\tau w\bar{h}\phi[\theta\hat{k}_t+(1-\theta)\hat{s}_t]
$$

$$
-(1+\gamma)k\hat{k}_{t+1}=c\hat{c}_t-y(\theta\hat{k}_t+(1-\theta)\hat{s}_t)-(1-\delta^k)k\hat{k}_t+(1+\gamma)s\hat{s}_{t+1}-(1-\delta^s)s\hat{s}_t
$$

0.3.6 Simplify Eq3

$$
\hat{c}_t=\hat{c}_{t+1}-\bar{\beta}\Big(1-(1+\phi)\tau\Big)r(\theta-1)[\hat{k}_{t+1}-s_{t+1}]+\bar{\beta}(1+\phi)\tau r\phi[\theta\hat{k}_{t+1}+(1-\theta)\hat{s}_{t+1}]
$$

$$
\Big(1-(1+\phi)\tau\Big)r(\theta-1)[\hat{k}_t-\hat{s}_t]-(1+\phi)\tau r\phi[\theta\hat{k}_t+(1-\theta)\hat{s}_t]=
$$

$$
\Big(1-(1+\phi)\tau\Big)w\bar{h}\theta(\hat{k}_t-\hat{s}_t)-(1+\phi)\tau w\bar{h}\phi[\theta\hat{k}_t+(1-\theta)\hat{s}_t]
$$

$$
(1+\gamma)k\hat{k}_{t+1}+(1+\gamma)s\hat{s}_{t+1}=-c\hat{c}_t+y(\theta\hat{k}_t+(1-\theta)\hat{s}_t)+(1-\delta^k)k\hat{k}_t+(1-\delta^s)s\hat{s}_t
$$

0.3.7 Simplify further Eq3

$$\hat{c}_t = \hat{c}_{t+1} - \bar{\beta}\bigg(1-(1+\phi)\tau\bigg)r(\theta-1)[\hat{k}_{t+1} - s_{t+1}] + \bar{\beta}(1+\phi)\tau r\phi[\theta\hat{k}_{t+1} + (1-\theta)\hat{s}_{t+1}]$$

$$\bigg(1-(1+\phi)\tau\bigg)r(\theta-1)[\hat{k}_t - \hat{s}_t] - (1+\phi)\tau r\phi[\theta\hat{k}_t + (1-\theta)\hat{s}_t] =$$
$$\bigg(1-(1+\phi)\tau\bigg)w\bar{h}\theta(\hat{k}_t - \hat{s}_t) - (1+\phi)\tau w\bar{h}\phi[\theta\hat{k}_t + (1-\theta)\hat{s}_t]$$

$$(1+\gamma)k\hat{k}_{t+1} + (1+\gamma)s\hat{s}_{t+1} = -c\hat{c}_t + [\theta y + (1-\delta^k)k]\hat{k}_t + [(1-\theta)y + (1-\delta^s)s]\hat{s}_t$$

0.3.8 Simplify Eq1

$$\hat{c}_t = \hat{c}_{t+1} + \bar{\beta}r[2(1+\phi)\tau\theta + (1-\theta) - (1+\phi)\tau]\hat{k}_{t+1} + \bar{\beta}r(1-\theta)[\tau\phi^2 - 1 - \tau]\hat{s}_{t+1}$$

$$\bigg(1-(1+\phi)\tau\bigg)r(\theta-1)[\hat{k}_t - \hat{s}_t] - (1+\phi)\tau r\phi[\theta\hat{k}_t + (1-\theta)\hat{s}_t] =$$
$$\bigg(1-(1+\phi)\tau\bigg)w\bar{h}\theta(\hat{k}_t - \hat{s}_t) - (1+\phi)\tau w\bar{h}\phi[\theta\hat{k}_t + (1-\theta)\hat{s}_t]$$

$$(1+\gamma)k\hat{k}_{t+1} + (1+\gamma)s\hat{s}_{t+1} = -c\hat{c}_t + [\theta y + (1-\delta^k)k]\hat{k}_t + [(1-\theta)y + (1-\delta^s)s]\hat{s}_t$$

0.3.9 Simplify Eq2

$$\hat{c}_t = \hat{c}_{t+1} + \bar{\beta}r[2(1+\phi)\tau\theta + (1-\theta) - (1+\phi)\tau]\hat{k}_{t+1} + \bar{\beta}r(1-\theta)[\tau\phi^2 - 1 - \tau]\hat{s}_{t+1}$$

$$\Bigg[\bigg(1-(1+\phi)\tau\bigg)[\theta w\bar{h} - (1-\theta)r] - (1+\phi)\tau\phi\theta[r - w\bar{h}]\Bigg]\hat{k}_t +$$
$$\Bigg[\bigg(1-(1+\phi)\tau\bigg)[\theta w\bar{h} + (1-\theta)r] - (1-\theta)(1+\phi)\tau\phi[r - w\bar{h}]\Bigg]\hat{s}_t = 0$$

$$(1+\gamma)k\hat{k}_{t+1} + (1+\gamma)s\hat{s}_{t+1} = -c\hat{c}_t + [\theta y + (1-\delta^k)k]\hat{k}_t + [(1-\theta)y + (1-\delta^s)s]\hat{s}_t$$

or

$$\hat{c}_t = \hat{c}_{t+1} + \bar{\beta}r[2(1+\phi)\tau\theta + (1-\theta) - (1+\phi)\tau]\hat{k}_{t+1} + \bar{\beta}r(1-\theta)[\tau\phi^2 - 1 - \tau]\hat{s}_{t+1}$$

$$\hat{k}_t = -\frac{\Bigg[\bigg(1-(1+\phi)\tau\bigg)[\theta w\bar{h} + (1-\theta)r] - (1-\theta)(1+\phi)\tau\phi[r - w\bar{h}]\Bigg]}{\Bigg[\bigg(1-(1+\phi)\tau\bigg)[\theta w\bar{h} - (1-\theta)r] - (1+\phi)\tau\phi\theta[r - w\bar{h}]\Bigg]}\hat{s}_t$$

$$(1+\gamma)k\hat{k}_{t+1} + (1+\gamma)s\hat{s}_{t+1} = -c\hat{c}_t + [\theta y + (1-\delta^k)k]\hat{k}_t + [(1-\theta)y + (1-\delta^s)s]\hat{s}_t$$

Let

$$\begin{aligned}
A &= \bar{\beta}r[2(1+\phi)\tau\theta + (1-\theta) - (1+\phi)\tau] \\
B &= \bar{\beta}r(1-\theta)[\tau\phi^2 - 1 - \tau] \\
C &= -\frac{\left[\left(1-(1+\phi)\tau\right)[\theta w\bar{h} + (1-\theta)r] - (1-\theta)(1+\phi)\tau\phi[r - w\bar{h}]\right]}{\left[\left(1-(1+\phi)\tau\right)[\theta w\bar{h} - (1-\theta)r] - (1+\phi)\tau\phi\theta[r - w\bar{h}]\right]} \\
D &= [\theta y + (1-\delta^k)k] \\
E &= [(1-\theta)y + (1-\delta^s)s]
\end{aligned}$$

Then the system becomes

$$\begin{aligned}
\hat{c}_t &= \hat{c}_{t+1} + A\hat{k}_{t+1} + B\hat{s}_{t+1} \\
\hat{k}_t &= C\hat{s}_t \\
(1+\gamma)k\hat{k}_{t+1} + (1+\gamma)s\hat{s}_{t+1} &= -c\hat{c}_t + D\hat{k}_t + E\hat{s}_t
\end{aligned}$$

Substitute out physical capital

$$\begin{aligned}
\hat{c}_t &= \hat{c}_{t+1} + AC\hat{s}_{t+1} + B\hat{s}_{t+1} \\
C(1+\gamma)k\hat{s}_{t+1} + (1+\gamma)s\hat{s}_{t+1} &= -c\hat{c}_t + DC\hat{s}_t + E\hat{s}_t
\end{aligned}$$

Let

$$\begin{aligned}
F &= AC + B \\
G &= C(1+\gamma)k + (1+\gamma)s \\
H &= DC + E
\end{aligned}$$

Thus

$$\begin{aligned}
\hat{c}_t &= \hat{c}_{t+1} + F\hat{s}_{t+1} \\
G\hat{s}_{t+1} &= -c\hat{c}_t + H\hat{s}_t
\end{aligned}$$

0.3.10 The reduced system in matrix form

$$\begin{pmatrix} 1 & F \\ 0 & G \end{pmatrix} \begin{pmatrix} \hat{c}_{t+1} \\ \hat{s}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & H \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{s}_t \end{pmatrix}$$

or (the matrix is invertible as its determinant is $H > 0$)

$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & F \\ 0 & G \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -c & H \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

or

$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{s}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & -F/G \\ 0 & 1/G \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c & H \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{s}_t \end{pmatrix}$$

or

$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{s}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 + Fc/G & -FH/G \\ -c/G & H/G \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{s}_t \end{pmatrix}$$

Let

$$\begin{aligned} I &= 1 + Fc/G \\ J &= -FH/G \\ K &= -c/G \\ L &= H/G \end{aligned}$$

or

$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{s}_{t+1} \end{pmatrix} = \begin{pmatrix} I & J \\ K & L \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{s}_t \end{pmatrix}$$

0.4 Stability

Let

$$Z = \begin{pmatrix} I & J \\ K & L \end{pmatrix}$$

Find the characteristic roots

$$\begin{vmatrix} I-\lambda & J \\ K & L-\lambda \end{vmatrix} = 0$$

The characteristic polynomial is then

$$\lambda^2 - (I+L)\lambda + (IL - JK) = 0$$

$$\lambda_{1,2} = \frac{(I+L) \pm \sqrt{(I+L)^2 - 4(IL-JK)}}{2} =$$

$$= \frac{(I+L) \pm \sqrt{I^2 + 2IL + L^2 - 4IL + JK}}{2} =$$

$$= \frac{(I+L) \pm \sqrt{(I-L)^2 + 4JK}}{2}$$

since $JK > 0$, and $(I-L)^2 + 4JK > 0$, we will have two distinct, real roots.

$$\lambda_1 + \lambda_2 = tr(Z) = I + L$$

$$\lambda_1\lambda_2 = det(Z) = IL - JK = \frac{H}{G} + \frac{FHc}{G^2} - \frac{FHc}{G^2} = \frac{H}{G} = \frac{DC+E}{Ck+s} > 0$$

$$\lambda_1 = \frac{(I+L) + \sqrt{(I-L)^2 + 4JK}}{2}$$

$$\lambda_2 = \frac{(I+L) - \sqrt{(I-L)^2 + 4JK}}{2}$$