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Research Report

Application of Monte Carlo Methods: Computing Heterogeneous Agent Models Without Aggregate Uncertainty

Application of Monte Carlo Methods: Computing Heterogeneous Agent Models Without Aggregate Uncertainty

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Abstract

In this paper we solve the benchmark hetrogeneous agents model by Aiyagari (1994) using Monte Carlo methods. In addition, the idiosyncratic shocks process is approximated using Tauchen's (1986) method. This we go beyond the 2 by 2 Markov matrix approximation of the AR(1) stochastic process. The code is written in MATLAB. The computation time is much faster than the one written by Heer and Maussner (2008) in FORTRAN. This model also solves Mehra-Prescott's puzzle and generates a risk-free interest rate that is much closer to the one we observe in data.

1 Introduction

In this paper we solve the benchmark hetrogeneous agents model by Aiyagari (1994) using Monte Carlo methods. In addition, the idiosyncratic shocks process is approximated using Tauchen's (1986) method. This we go beyond the 2 by 2 Markov matrix approximation of the AR(1) stochastic process. The code is written in MATLAB. The computation time is much faster than the one written by Heer and Maussner (2008) in FORTRAN.

The heterogeneous agent literature can be viewed as a response to Lucas' asset pricing model and to Mehra and Prescott's "equity premium puzzle" paper. In what follows, we provide a synopsis and the main conclusion from that literature. In Lucas's paper, there is a unit mass of agents. Every agent owns a tree, which is the asset in this economy. In every period, each tree produces fruit, which plays the role of a financial divident. Fruit harvest is uncertain and follows a stochastic process. Agents are allowed to buy other agent's trees, or sell their own tree. Agent's objective is to maximize the discounted present value of utility streams of consumption. In the "equity premium puzzle"' paper, Mehra and Prescott found that one needs an extermely high coefficient of risk aversion to get a reasonable risk-free interest rate. From asset prices data, however, we can put bounds on that coefficient, which results in an implausible risk-free interest rate. (Risk-free interest rate is estimated as the return on equity in the last hundred years in US.)This puzzle led Mehra and Prescott to the conclusion that the precautionary motive on aggregate level is not quantitatively big.

Mehra-Prescott's paper has important limitations. Their model imposes a symmetry in order to solve for the equilibrium: all agents are ex-ante and ex-post identical. That simplification allows them to argue the model can be represented as an economy with a representative Agent. In addition, markets are complete, i.e there are markets for all goods in all states of nature. Those are important assumptions, which will be shown to drive the negative results.

Mark Huggett(1993) and Rao Aiyagari(1994) made important contributions towards the solution of the equity premium puzzle. They claim that heterogeneity of agents is a stylized fact that is an important feature of reality that should be incorporated into the model, if we are to explain the equity premium puzzle. They show that we need to depart from the symmetric RA framework in order to generate a strong precautionary motive. With heterogeneous agents facing individual-specific uncertainty, there will be overaccumulation of assets, which will then bring down the risk-free interest rate.

The necessary ingredients of such a model are agent-specific(idiosyncratic) labor productivity shocks, incomplete markets, and borrowing constraints. The easiest way to model incomplete markets is to assume that markets are exogenously incomplete: some assets do not exist. In particular, agents cannot write contracts contingent of the realization of some shocks. An alternative way is to assume some frictions (asymmetric information, limited enforcement etc.) and derive endogenously which assets are traded in equilibrium. In this paper, we will use the exogenously incomplete markets as a modelling shortcut, but bear in mind it can be obtained by resorting to mechanism design and a lot of messy algebra.

Note that with exogenously incomplete markets, agents become heterogeneous with respect to their history of shocks and their asset position. In this class of economies, we can also study questions related to inequality and distribution. Hugget's (1993) model is partial equilibrium, a heterogeneous version of Lucas' asset pricing model. Rao Aiyagari extended the model to General Equilibrium: interest rate will be determined endogenously as a result of optimal behavior by househols/agents and firms. Both Hugget and Aiyagari's models have no analytical solution. We need to resort to numerical methods and solve the model on the computer. In addition, sequential methods, e.g shooting methods, are of no use. That is due to the fact that asset choice is stochastic and depends on the realization of the idiosyncratic shocks. This stochastic dependence makes it computationally burdensome to guess an infinite sequence. The only viable way to proceed then are recursive methods. An interesting note is that Huggett and Aiyagari did not have the computational power we have today to simulate the model with aggregate shocks as well. Krusell and Smith (1998) provided an "approximate aggregation" algorithm that can tackle this. In this paper, we abstract from aggregate uncertainty, and discuss how to compute an equilibrium in and extension of Aiyagaris model.

2 The Model

The model environment is the following: As in Lucas's model, there is of measure one of agents. Agents are infinitely lived. Their preferences are represented by the utility function

$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$, where $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and $\beta \in (0,1)$ is the discount factor.

Each agent is endowed with some initial assets a_0 and with one unit of time each period. In addition, since leisure is not valued, agents supply their time endowment inelastically.

Each agent is subject to a individual, idiosyncratic shocks e to individual productivity. The variable e follows a finite state, first order Markov process. The transition probabilities are denoted by $\pi_{ee'}$ and the transition matrix is denoted by Π. We can extend the MArkov matrix to as many states as we wish using Tauchen's (1986) procedure for discretizing a continuous process (usually AR(1)). Since labor is inelastically supplied, e equals the efficiency units of labor supplied.

Market Structure is the following: There is a sequence of spot markets for capital, labor, and output. Assets cannot be contingent on e (a possible justiffication is that the realization of e is not observable). In addition, there is a borrowing limit: $a \geq \underline{a}$, where $\underline{a} \leq 0$

On the production side, there is a representative firm which rents capital and labor from agents at competitive prices. This stand-in firm can be regarded as an aggregation of infinitely many identical perfectly competitive firms. The aggregate production function is $Y_t = K_t^{\theta}$, $0 < \theta < 1$ where K is aggregate capital.

To close the model in general equilibrium, we need to specify the market clearing condition. The resource constraint in this economy is

$$C_t + K_{t+1} - (1-\delta)K_t = Y_t = K_t^{\theta},$$

where we have substituted out aggregate investment from the following expression:

$$K_{t+1} = I_t + (1-\delta)K_t.$$

Thus we made sure that output is equal to the sum of its uses in equilibrium.

3 Equilibrium

In order to define a Recursive Competitive Equilibrium (RCE), we need to specify the individual and aggregate state variables. This is important, since in heterogeneous-agent models, the two are not the same. The aggregate state is a distribution (a measure) over the individual state variables. In this model, the state vector is $(e, a, \mu(e, a))$ where $\mu(.,.)$ is the measure over agents type (e, a).

A RCE for this economy is value function $V(e, a, \mu(e, a))$, individual and aggregate policy rules $a'(e, a, \mu(e, a))$, $K'(\mu(e, a))$, and prices $w(e, a, \mu(e, a))$, $r(e, a, \mu(e, a))$ s.t.

(1) Consumer maximization: Taking $K'(\mu(e, a))$,$w(e, a, \mu(e, a))$, $r(e, a, \mu(e, a))$ as given, Consumer solves the following Bellman equation

$V(e, a, \mu(e, a)) = max_{c' \geq 0, a \geq \underline{a}} \{u(c) + \beta \sum_{e' \in E} \pi_{ee'} V(e', a', \mu'(e, a))\}$
s.t $c + a' = we + (1 + r - \delta)a$
$w = F_L(K, L)$
$r = F_K(K, L)$

and $g(e, a, \mu(e, a))$ denotes the optimal decision rule for a'.

(2) Firm maximization: Taking $K'(\mu(e, a))$,$w(e, a, \mu(e, a))$, $r(e, a, \mu(e, a))$ as given,the representative firm solves the static problem

$max_{K,L} F(K, L) - rK - wL$
s.t. $K \geq 0$, $L \geq 0$

(3) Consistency: (the law of motion of the type distribution is consistent with the perceived law of motion by each individual agent)

$\mu'(e', a') = \Phi(\mu)(e', a') = \int_{\{(e,a) \in E \times A, a' = g(e,a)\}} \pi_{ee'} d\mu(e, a)$

(4)Market Clearing

$\int_{E \times A} (c + a') d\mu(e, a) = Y + (1 - \delta)K$ (Output)
$\int_{E \times A} a d\mu(e, a) = K$ (capital)
$\int_{E \times A} e d\mu(e, a) = L$ (labor)

4 Stationary Distribution

However, the recursive competitive equilibrium is almost impossible to solve because Φ is a function from the space of type distributions to itself. To simplify, we focus on the stationary distribution $\mu^* = \Phi(\mu^*)$ Price are then constant over time and the agents problem becomes
$V(e, a) = max_{c' \geq 0, a \geq \underline{a}} \{u(c) + \beta \sum_{e' \in E} \pi_{ee'} V(e', a')\}$
s.t $c + a' = we + (1 + r - \delta)a$

Existence of Stationary Equilibrium: Markov process satisfies the Feller property as required by Stockey and Lucas (1989). Inaddition, Hopenhayn and Prescott (1992) show Feller property is sufficient, but not necessaty. Using lattice methods, they show necessary and sufficient conditions. The key one is the Monotone Mixing Condition (MMC)
- there is a sufficiently large probability that an agent moves from the lower part of the distribution to the upper part and vice versa. In the economic jargon, it is known as the "American Dream and American Nightmare" condition. The reason is that even if you are as rich as Bill Gates, tomorrow you could be hit by such a negative shock, that you can end up begging on the street. Therefore, despite being the richest man on the planet, an agent will still save for precautionary reasons.

5 Solution Algorithm

Stationary Distribution is a Fixed Point in the fuction space: under the assumptions in the model, a unique stationary distribution can be shown to exist (Ljungqvist and Sargent (2006) . Given a path of constant r and w, solve the agents problem for the optimal decision rule. Find the stationary distribution implied by the optimal decision rule. This stationary distribution must yield the constant prices from which you started.

Therefore, the solution algorithm consists of two loops: Outer iteration (guess r and w), and inner iteration - solve the agents problem paying attention to the borrowing constraint $a' \geq \underline{a}$. Increase the value of a' until it holds with equality all larger values must also satisfy the inequality above, so you do not need to check anymore.

There are several options to compute the new distribution μ'. The first possibility is to approximate the measure by discretizing the space of a. Using the same discretization for the outer and the inner iterations, we have $\mu'_{e',a'} = \sum_{\{(e,a):a'=g(e,a)\}} \pi_{ee'} \mu_{ea}$. The second possibility is to simulate the behavior of, say, 100,000 agents using the optimal decision rule, and keep simulating until the type distribution of agents becomes stable, i.e the moments of e and a become stable. In this paper we focus on the second approach.

Monte Carlo Flowchart: Computing the Invariant Distribution Function $F(e, a)$ by Monte Carlo Simulation

Step 1: Choose a sample size N equal to some tens of thousands.

Step 2: Initialize the sample. Each household $i = 1, .., N$ is assigned an initial wealth level a_0^i and productivity realization ϵ_0^i.

Step 3: Compute the next-period wealth level $a'(e^i, a^i)$ for all $i = 1, .., N$.

Step 4: Use a random number generator to obtain ϵ'^i for all $i = 1, .., N$.

Step 5: Compute a set of statistics from this sample, in particular, the mean and the standard deviation of a and e.

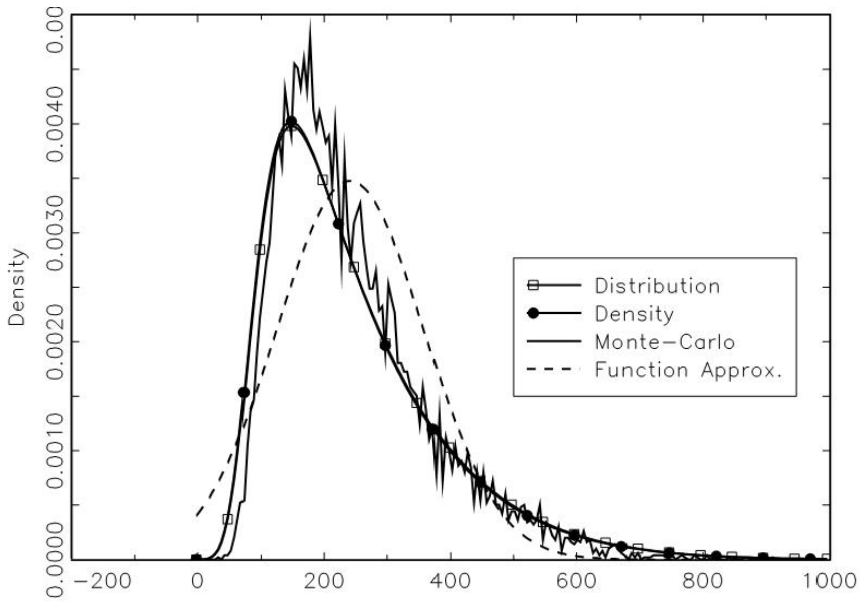
Step 6: Iterate until the distributional statistics converge.

Compute the new aggregate (K', L') and the new prices (r', w'). If the new prices are sufficiently close to the old ones, you are done. Otherwise, set $r = r'$ and $w = w'$ and go back to the first step.

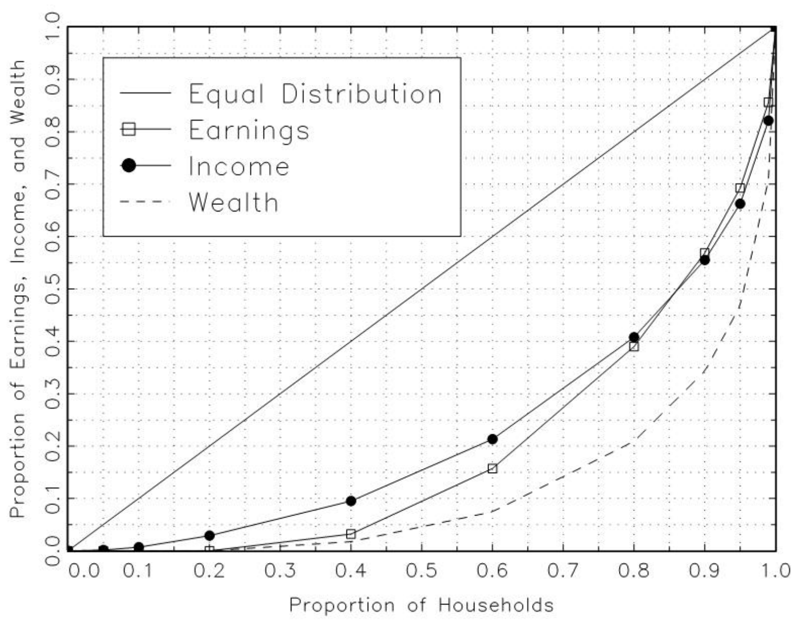
With inelastically labor supply, the invariant distribution of e implies L. In other words, L is exogenous and we just need to iterate on K. The program takes 45 minutes in MATLAB on Intel Pentium Dual-Core processor, which is much faster than the 1 h. 45 min. in FORTRAN that Heer and Maussner (2008) report. Their processor is a bit slower than mine but still FORTRAN is supposed to beat MATLAB in the loop calculation part, since MATLAB was developed for tackling matrix calculations.

In addition, when we solve the model, we obtain the whole distribution for capital income, labor earnings, as well as total wealth (current income plus capital stock). We can compare the moments of the simulated distribution to the moments from US data.

When the RA model is altered only by adding idiosyncratic, uninsurable shock, the resulting stationary wealth distribution is quite unrealistic: there are too few poor agents, and much too little concentration



Density
Distribution
Density
Monte-Carlo
Function Approx.
0.0000
0.0010
0.0020
0.0030
0.0040
0.00
-200
0
200
400
600
800
1000



Equal Distribution
Earnings
Income
Wealth
Proportion of Earnings, Income, and Wealth
Proportion of Households
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Table 1: **Statistics of Inequality**

	Gini	σ/μ	Top 1%	5%	20 %	Btm. 40 %
US Data						
Lab.Income	0.61	2.65	15.3	31.1	60.2	3.8
Tot.Income	0.55	3.57	17.5	32.8	58	9.6
Wealth	0.55	6.53	34.7	57.8	81.7	1
Aiyagari Model						
Labor Income	0.38	0.79	0	11.16	37.07	29.16
Tot.Income	0.46	0.96	7.31	26.71	62.88	5.57
Wealth	0.63	1.40	7.85	28.29	64.88	4.62

of wealth among the very richest. Adding aggregate shocks might help in this dimension.

Table 2: **Results**

	$\sigma = 1.5$	$\sigma = 3$
Credit Limit ($\underline{a}$)	Int. Rate (r)	Int. Rate (r)
-2	-7.1 %	-23 %
-4	2.3 %	-2.6 %
-6	3.4 %	1.8 %
-8	4.0 %	3.7 %

This model also solves Mehra-Prescott's puzzle and generates a risk-free interest rate that is much closer to the one we observe in data.

6 Conclusion

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