

Schubert, Stefan Franz; Broll, Udo

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# Consumption, inflation risk and dynamic hedging

Stefan Franz Schubert<sup>1</sup>, Udo Broll<sup>2</sup>

## ABSTRACT

Our study examines the behavior of a risk-averse investor who faces two sources of uncertainty: a random asset price and inflation risk. Both sources of uncertainty make it difficult to stabilize consumption over time. However, investors can enter risk-sharing markets, such as futures markets, to manage these risks. We develop a dynamic risk management model. Optimal consumption and risk management strategies are derived. It is shown that dynamic hedging increases an investor's welfare in terms of the expected inter-temporal utility of consumption.

## KEY WORDS:

Dynamic hedging, asset price risk, inflation risk, real wealth, consumption

**JEL Classification:** D21, D24

<sup>1</sup>Free University Bozen - Economics, Italy; <sup>2</sup>Technische Universität Dresden - Economics, Germany

## Introduction

The importance of risk management has inspired empirical and theoretical contributions to investment and consumption decision making under uncertainty. Most of the literature on economic risk and risk behavior dealing with investment, consumption and hedging decisions has incorporated the assumption that investors are concerned with random nominal wealth denominated in one currency. However, an investor's wealth may also change when domestic prices change due to unexpected domestic price inflation, for example. An asset holder should not neglect inflation risk because his or her consumption opportunity set also changes. Therefore, economic analysis of dynamic investment and consumption should be imbedded in a framework of inflation risk (Adam-Mueller, 2002; Battermann & Broll, 2001; Bodie, 1976; Broll & Wong, 2013; Eaton, 1980; Elder, 2004; Kawai and Zilcha, 1986; Nocetti and Smith, 2011).

The real value of assets will change with inflation. Inflation causes money to decrease in value regardless of whether the money is invested (Blanchard, Amighini, & Giavazzi, 2013). Real wealth is therefore uncertain for two reasons: The value of accumulated assets is uncertain, and the overall price level is risky. However, the investor can step into futures markets in order to reduce the volatility of real wealth by hedging directly against asset price fluctuations and indirectly against inflation risk due to correlations. Our discussion illustrates a typical use of futures contracts.

In an excellent study, Nocetti and Smith (2011) use a very general framework to analyze how the price uncertainty of commodities affects the (precautionary) saving and welfare of consumers. Nocetti and Smith (2011) develop and solve a model with interesting specific features: recursive preferences, an infinite horizon with many goods, financial assets correlated with the growth rate of prices of multiple commodities. They study the impact of consumer preferences that disentangles risk aversion from preferences towards goods in the absence of risk. Furthermore, they discuss a class

Correspondence concerning this article should be addressed to: **Udo Broll**, Technische Universität Dresden, Dresden 01062, Germany. E-mail: [udo.broll@tu-dresden.de](mailto:udo.broll@tu-dresden.de)

of recursive preferences, appropriately adapted to the case of multiple commodities, that disentangles intertemporal substitution preferences and risk aversion.

In addition to other authors such as Adam-Mueller (2002), Bodie (1976), Eaton (1980), Pelster (2014) and others we focus in this study on the risk-reducing role of futures markets and derive optimal hedge ratios and welfare in cases where an asset price and the price of a composite commodity are uncertain. The existence of risk-sharing markets and risk-reducing institutions in open economies significantly modifies the impact of price and return risk. The problem might be, however, that the economy does not have a complete set of markets, such that existing markets must typically serve several different functions simultaneously. The effect of risk on consumption and welfare will thus depend on whether there is a futures/forward market (for a detailed discussion, see, for example, Froot, Scharfstein, & Stein, 1993; Newbery & Stiglitz, 1981).

In our study, following the standard approach in the literature, we apply time-separable preferences (Briys, Crouhy, & Schlesinger, 1990; Broll, Clark, & Lukas, 2010). We demonstrate that the optimal dynamic hedging strategy can be decomposed in a variance-minimizing component and in a speculative component. In contrast to Nocetti and Smith (2011), in our model, the variance-minimizing component is preference free and depends only on the variance of the hedging instrument and on covariances. The speculative component is influenced by preferences, i.e., it depends on the investor's degree of relative risk aversion.<sup>1</sup> Introducing examples, we prove that the optimal consumption-wealth ratio may be greater or smaller than it is without the hedging instrument, depending on the magnitude of relative risk aversion. However, in any case, we show that the investor's overall welfare is greater with hedging. This analysis can also be applied to aspects of asset returns and the business cycle in the economy as well (Heer & Maußner, 2009).

The rest of this paper is organized into four sections. In the first section, we present the model and show the risk-reducing role of futures markets. Then, we characterize investors' optimal hedging and consumption strategies. Furthermore, we employ a standard iso-elastic utility function to gain further economic insights. The final section concludes the paper. The appendix contains additional calculations.

## The model

Consider a risk-averse individual, called "the investor," who has accumulated real wealth  $W$  defined as

$$W = \frac{SA}{P}, \quad (1)$$

where  $S$  denotes the spot market price of assets,  $A$  is the stock of assets accumulated, and  $P$  represents the overall price level. Both  $S$  and  $P$  are stochastic. Applying Itô's lemma, the change in the investor's real wealth can be written as

$$dW = \left[ \frac{dS}{S} - \frac{dP}{P} - \frac{dS}{S} \frac{dP}{P} + \left( \frac{dP}{P} \right)^2 \right] W + \frac{S}{P} dA. \quad (2)$$

The investor can take a short or long position on the futures market of volume  $H$  (a positive/negative  $H$  denotes a short/long position). To enter the futures market costs nothing, but from thereon, the investor's margin account has to be continuously adjusted due to changes in the futures price  $F$ . Moreover, the investor can consume at the rate  $Cdt$ , where  $C$  denotes the rate of real consumption.  $C$  can be broadly defined as a composite good. Thus, the last term of equation (2), the change in the value of accumulated assets at any instant, given  $S/P$ , is

$$SA = -PCdt - H(dF).$$

Introducing the hedge ratio  $h \equiv \frac{HF}{PW}$ , the change of the real value of accumulated assets can be written as

$$\frac{SdA}{P} = -Cdt - h \frac{dF}{F} W, \quad (3)$$

which says that in addition to price changes, real wealth changes due to real consumption and hedging. Hence, real wealth accumulation (2) becomes

$$dW = \left[ \frac{dS}{S} - \frac{dP}{P} - \frac{dS}{S} \frac{dP}{P} + \left( \frac{dP}{P} \right)^2 \right] W - h \left( \frac{dF}{F} \right) W - Cdt. \quad (2')$$

We assume that the evolution of  $S$ ,  $P$ , and  $F$  can be described by geometric Brownian motions

$$\frac{dS}{S} = \mu_s dt + \sigma_s dz_s \quad (4a)$$

$$\frac{dP}{P} = \mu_p dt + \sigma_p dz_p \quad (4b)$$

$$\frac{dF}{F} = \mu_f dt + \sigma_f dz_f \quad (4c)$$

where  $\mu_i$  represents the expected instantaneous rate of change and  $\sigma_i$  is the volatility parameter. For example,  $\mu_p$  can be interpreted as the trend in the inflation rate, which itself is determined by the growth rate of the money supply.  $\mu_F$  is the expected rate of change of the futures price.  $dz_i$  are standard Wiener processes with mean zero and instantaneous variance  $dt$ . Note that  $\sigma_i dz_i \sigma_j dz_j = \sigma_i \sigma_j \rho_{ij} dt \equiv \sigma_{ij} dt$ , where  $\sigma_{ij} dt$  is the instantaneous covariance between prices  $i$  and  $j$  and  $\rho_{ij}$  is the correlation coefficient between the two standard Wiener processes  $dz_i$  and  $dz_j$ . Furthermore, we have  $\mu_i dt \sigma_j dz_j = o(dt)$ . Inserting (4) into real wealth accumulation (2'), dropping terms of higher order than  $dt$ , and collecting terms, we obtain

$$dW = [(\mu_S - \mu_P - \sigma_{SP} + \sigma_P^2)dt + (\sigma_S dz_S - \sigma_P dz_P) - h(\mu_F dt + \sigma_F dz_F)]W - Cdt. \tag{5}$$

Eq. (5) describes the stochastic real wealth accumulation equation.

### Optimal consumption and risk management

The investor's objective is to maximize the expected present value of utility of consumption over his planning horizon  $T$

$$V(W, t) = \max_{C, h} E_0^T \int_0^T U(C) e^{-\beta t} dt \tag{6}$$

subject to (5). The instantaneous utility function  $U(C)$  with  $U_C > 0, U_{CC} < 0$  is assumed to be time separable. Parameter  $\beta$  denotes the investor's constant rate of time preference. Equation (6) shows that the investor ultimately cares about real consumption; he or she tries to reduce the volatility of wealth as expressed by (5) by choosing an appropriate hedge ratio  $h$ , allowing for a smoother consumption profile over time.

Applying Itô's Lemma to the value function  $V(W, t)$ , as shown in the appendix, the Bellman equation to program (6) subject to (5) is

$$\beta V(W, t) = \max_{C, h} \left\{ U(C) + V_t + V_W [(\mu_S - \mu_P - \sigma_{SP} + \sigma_P^2) - h\mu_F]W - V_W C \right. \\ \left. + \frac{1}{2} V_{WW} [\sigma_S^2 + \sigma_P^2 - 2\sigma_{SP} - 2h(\sigma_{SF} - \sigma_{PF}) + h^2 \sigma_F^2] W^2 \right\} \tag{7}$$

**Proposition 1.** Maximization of expected present value of utility leads to optimal consumption and hedging rules

$$U_C(\hat{C}) = V_W(W, t) \tag{8a}$$

$$\hat{h} = \frac{\sigma_{SF} - \sigma_{PF}}{\sigma_F^2} + \frac{\mu_F V_W(W, t)}{W \sigma_F^2 V_{WW}(W, t)} \tag{8b}$$

where  $\hat{\cdot}$  denotes an optimal value.

Eq. (8a) is a standard optimality condition and equates the marginal utility of consumption  $U_C$  to the marginal utility of wealth,  $V_W$ . Solving this equation for  $C$  yields the investor's rate of real consumption. Eq. (8b) describes the investor's optimal hedge ratio ( $\hat{h} > 0, \hat{h} < 0$  implies a short hedge, long hedge).

The first term on the right-hand side of (8b) represents the preference-free, variance-minimizing hedge ratio and can be understood as follows. On one hand, hedging is used to reduce the variance of the time path of real wealth. On the other hand, hedging introduces an additional risk via the volatility of the futures price. Thus, with higher variance  $\sigma_F^2$ , hedging is less able to reduce fluctuations of wealth, and thus, the lower the optimal hedge ratio. The variance-minimizing hedge ratio depends positively on the covariance between the spot price  $S$  and the futures price  $F$ . In the case of a positive covariance, on average,  $S$  and  $F$  move in the same direction, and thus, via margin account adjustments, a short hedge works against real wealth increases due to increases in the nominal value of assets  $SA$ . The opposite is true for a positive covariance between the futures price and the price level  $P$ . To stabilize real wealth, an increase in the price level and, thus, a reduction of real wealth requires a long hedge because on average, losses of wealth's purchasing power are offset by increases in the margin account. This explains the minus sign of the  $\sigma_{PF}$ -term. In the special case  $\sigma_{SF} = \sigma_{PF}$ , these two hedging components cancel out, and the variance-minimizing hedge becomes zero. Intuitively, on average,  $S$  and  $P$  then move exactly and proportionally in tandem; hence, real wealth is automatically stabilized.

The second term on the right-hand side of (8b) is the speculative component of the hedge ratio. A rational investor will deviate from the variance-minimizing hedge ratio to the extent that hedging pro-

vides an additional source of income. If, for example,  $\mu_F > 0$ , the futures price is expected to increase over time; thus, a long position gives rise to expected profits via the margin account. However, in contrast to the variance-minimizing hedge ratio, the speculative component depends not only on the variance of the futures price but also on the investor's relative risk aversion  $-V_{WW}W/V_W > 0$ . Thus, with higher  $\sigma_F^2$  and greater relative risk aversion, the speculative hedge ratio becomes smaller.

The overall hedge ratio is thus a combination of variance-minimizing and speculative components. Which of these components dominates depends on the volatility parameters  $\sigma_{SF}$ ,  $\sigma_{PF}$ , and  $\sigma_F^2$ , on the expected rate of change  $\mu_F$ , and on the investor's relative risk aversion. In general, relative risk aversion is a function of wealth and time; hence, the optimal hedge ratio is time dependent and will change over time, i.e., the hedge ratio is dynamic.

### Examples

*Example 1* Let us consider the simple case of a deterministic evolution of the price level by setting  $\sigma_P^2 = \sigma_{SF} = \sigma_{PF} = 0$ . The optimal hedge ratio is

$$\hat{h} = \frac{\sigma_{SF}}{\sigma_F^2} + \frac{\mu_F V_W(W, t)}{W \sigma_F^2 V_{WW}(W, t)}$$

Whereas the speculative component remains unchanged, the variance-minimizing hedge rate simplifies, as it depends only on the covariance between the spot asset price and the futures price (and, of course, on its variance). A positive correlation thus implies a short hedge to minimize the variance of wealth. If the asset price and the futures price are also perfectly correlated, i.e., if they obey the same Wiener process ( $\sigma_S dz_S \equiv \sigma_F dz_F$  and, thus,  $\sigma_{SF} = \sigma_F^2$ ), we obtain the well-known result

$$\hat{h} = 1 + \frac{\mu_F V_W(W, t)}{W \sigma_F^2 V_{WW}(W, t)}$$

The variance-minimizing component then equals unity. Because in this case, the spot asset price and the futures price move exactly in tandem, a full short hedge eliminates any wealth fluctuations. This means that the investor incurs a short position in the futures market whose value equals the nominal value of her wealth. However, a rational investor will deviate from

a full hedge to take advantage of the expected change in the futures price to the extent of her relative risk aversion.

Contrarily, if the evolution of the asset's spot price  $S$  but not that of the price level is deterministic,  $\sigma_{SF} = 0$  and the variance-minimizing hedge ratio in equation (8b) simply becomes  $-\sigma_{PF} / \sigma_F^2$ . A positive covariance between the price level and the futures price then requires a long hedge to minimize the variance of wealth accumulation.

The ultimate purpose of hedging is to stabilize the evolution of real wealth, enabling the investor to smooth her consumption profile and to increase her inter-temporal utility  $V(W, t)$ . To observe this more formally, in the next example, we consider a particular utility function.

*Example 2* To obtain more insight into the hedging strategy and its effects, and for analytical convenience, from now on, we assume that the investor's utility function is iso-elastic and that her planning horizon is infinite. The maximization problem becomes

$$V(W) = \max_{C, b} \int_0^\infty \frac{1}{\gamma} C^\gamma e^{-\beta t} dt, \quad -\infty < \gamma < 1 \tag{6'}$$

where  $\gamma - 1$  denotes the investor's degree of relative risk aversion. In case of  $\gamma = 0$ , the utility function is logarithmic. Because the planning horizon is infinite, the utility function is additively separable in time, and the involved stochastic processes (4) do not directly depend on time, the value function  $V(\cdot)$  can be expressed in terms of wealth  $W$  solely.<sup>2</sup> As shown in the appendix, the value function  $V$ , optimal consumption, and the hedge ratio, are

$$V(W) = \frac{1}{\gamma} \left( \frac{\hat{C}}{W} \right)^{\gamma-1} W^\gamma \tag{9a}$$

$$\frac{\hat{C}}{W} = \frac{\beta - \hat{a}\gamma - \hat{b}\gamma(\gamma-1)/2}{1-\gamma}, \tag{9b}$$

$$\hat{h} = \frac{\sigma_{SF} - \sigma_{PF}}{\sigma_F^2} + \frac{\mu_F}{\sigma_F^2(\gamma-1)}, \tag{9c}$$

where

$$\begin{aligned} \hat{a} &\equiv \mu_S - \mu_P - \sigma_{SP} + \sigma_P^2 - \hat{h}\mu_F, \\ \hat{b} &\equiv \sigma_S^2 + \sigma_P^2 - 2\sigma_{SP} - 2\hat{h}(\sigma_{SF} - \sigma_{PF}) + \hat{h}^2\sigma_F^2. \end{aligned} \tag{9d}$$

Term  $\hat{a}$  describes the expected real rate of return on wealth, and  $\hat{b}$  measures the variance of the Wiener process for real wealth accumulation when optimal hedging (9c) is chosen.<sup>3</sup> The following observations can be made.

Eq. (9c) confirms that when the optimal hedge ratio is smaller (in absolute value), the investor's relative risk aversion  $(\gamma - 1)$  is greater, and the variance of the futures price is higher. It is worth noting that in case of an infinite planning horizon combined with an iso-elastic utility function and stochastic processes described by geometric Brownian motions, the optimal hedge ratio is time invariant, i.e., the hedge ratio is static. From (9b) in combination with the time-invariant optimal hedge ratio, entering  $\hat{a}$  and  $\hat{b}$ , it follows that the optimal consumption-wealth ratio is time invariant as well.

Depending on the sign of  $\gamma$ , from equation (9b), we see that an increase in the expected rate of return on real wealth  $\hat{a}$  increases or lowers the consumption-wealth ratio. There are two opposite effects at work. First, according to the income effect, a higher  $\hat{a}$  increases consumption, thus increasing  $C/W$ . Second, due to the substitution effect, a higher expected rate of return encourages wealth accumulation, thus lowering  $C/W$ . Which effect dominates depends on  $\gamma$ . We also observe that the expected real rate of return  $\hat{a}$  depends on the optimal hedging strategy and on the expected growth rate of the futures price, i.e.,  $(-\hat{h}\mu_F)$ . A short hedge ( $\hat{h} > 0$ ) in combination with a positive expected growth rate of the futures price ( $\mu_F > 0$ ) lowers  $\hat{a}$  because of expected losses, as an investor incurring a short position has to pay for marginal account adjustments when the futures price rises. For a long hedge in combination with a negative expected growth rate, the investor has to make payments to his margin account for a declining futures price, thus leading to expected losses. In contrast, if the expected growth rate of the futures price and the hedge ratio have opposite signs, the investor expects additional profits from hedging, hence increasing the expected rate of return on real wealth.

Similar but opposite income and substitution effects are detected for the variance  $\hat{b}$ . A higher variance exercises a negative income effect, thus lowering  $C/W$ , whereas its positive substitution effect stimulates consumption. Which effect dominates depends

again on  $\gamma$ . The optimal hedging strategy itself has two opposite effects on the variance of real wealth: (i) the variance-minimizing component of  $\hat{h}$  lowers the variance by  $-(\sigma_{SF} - \sigma_{PF})^2 / \sigma_F^2$ , whereas (ii) the speculative component unambiguously increases the variance by  $\mu_F^2 / (\sigma_F^2 (\gamma - 1)^2)$ . The overall effect on the variance is therefore unclear.

Inserting (9c) and (9d) into (9b) and simplifying, we obtain for the optimal consumption-wealth ratio

$$\frac{\hat{C}}{W} = \frac{1}{1-\gamma} \left\{ \beta - \gamma(\mu_S - \mu_P - \sigma_{SP} + \sigma_P^2) - (1/2)\gamma(\gamma-1)(\sigma_S^2 + \sigma_P^2 - 2\sigma_{SP}) + (1/2)\gamma(\gamma-1)\sigma_F^2\hat{h}^2 \right\}.$$

Because the first two terms on the right-hand side can be identified as the optimal consumption-wealth ratio without hedging possibilities, i.e.,  $(\hat{C}/W)_{without\ hedging}$ , we can write

$$\frac{\hat{C}}{W} = \left( \frac{\hat{C}}{W} \right)_{without\ hedging} + \frac{1}{1-\gamma} (1/2)\gamma(\gamma-1)\sigma_F^2\hat{h}^2. \quad (9b')$$

Looking at (9b'), we note that the consumption-wealth ratio may be greater or lower than in the case without hedging. Again, this depends on  $\gamma$ . In the case of the logarithmic utility function,  $\gamma = 0$  and the consumption-wealth ratio is unaffected by hedging and is given by  $\hat{C}/W = \beta$ .

*Welfare* It is important to note that it is not the investor's objective to maximize utility by means of choosing the highest possible consumption-wealth ratio. Ultimately, the investor only cares about the utility of real consumption  $C$ . Most importantly, given initial real wealth  $W(0)$  we can infer from (9a) in combination with (9b') that regardless of the specific value of  $\gamma \in (-\infty, 1)$ , at time zero, hedging always leads to an inter-temporal utility that is higher than in the case without hedging.<sup>4</sup> We can thus summarize that in general, the introduction of hedging against inflation risk increases the investor's expected utility, i.e.,

$$V(W(0)) > V(W(0))_{without\ hedging} \quad (10)$$

as long as  $\hat{h} \neq 0$ .

Finally, let us briefly discuss the effects of changes in the trend in the inflation rate  $\mu_P$  and in the inflation volatility  $\sigma_P^2$ , which may be caused by changes in monetary policy, for example. An increase in  $\mu_P$

clearly lowers the expected rate of return on real wealth  $\hat{a}$ , as seen from eq. (9d). An increase in  $\sigma_p^2$  results in a higher variance parameter  $\hat{b}$  of real wealth accumulation and, perhaps a little surprisingly, raises the expected rate of return on real wealth. The reason for this is that due to the convexity of real wealth  $W \equiv AS/P$  in  $P$ , an increase of  $\Delta P$ , say, reduces real wealth by less than a fall of magnitude  $-\Delta P$  increases real wealth; hence, real wealth increases on average. As we have already mentioned, changes in  $\hat{a}$  and  $\hat{b}$  cause income and substitution effects, and their overall effects on the consumption-wealth ratio depend on the elasticity of utility  $\gamma$ . Most interestingly, the optimal hedging strategy  $\hat{h}$  is unaffected as long as only  $\mu_p$  and  $\sigma_p^2$  change, as they do not appear in (9c). This does not mean that the investor will not change his futures position, however, because a constant hedge ratio implies that any change in the nominal value of wealth  $PW$  has to be exactly offset by a proportional change in the futures position  $HF$ . Note that this result holds as long as the investor's relative risk aversion is constant. Otherwise, changes in real wealth change the speculative component of the hedge ratio. Changes in macro monetary policy may also result in changes in the other drift and volatility parameters, exercising further effects on  $\hat{a}$  and  $\hat{b}$  and subsequently on the optimal hedging strategy.

## Conclusions

In a continuous time framework in which an investor faces real wealth risk, we derived the investor's optimal hedging and consumption strategy. We showed that the optimal hedge ratio can be decomposed in a preference-free variance-minimizing component and in a speculative component, depending on the investor's degree of relative risk aversion. We discussed the interaction between optimal dynamic hedging, correlation, the expected growth rate of the futures price, volatility on the futures market, and the sign of the optimal hedge ratio, i.e., a short hedge or long hedge. For the case of an infinite planning horizon, we calculated that the optimal consumption-wealth ratio may be greater or smaller as that in the case without the use of a financial hedging instrument, depending on the magnitude of relative risk aversion. However, in any case, we have shown that the investor's inter-temporal expected utility is greater in the case with futures contracts.

## References

- Adam-Mueller, A. F. A. (2002). Hedging Price Risk when Real Wealth matters. *Journal of International Money and Finance*, 19 (4), 549-560.
- Battermann, H. L., Broll, U. (2001). Inflation Risk, Hedging, and Exports. *Review of Development Economics*, 5 (3), 355-362.
- Blanchard, O., Amighini, A., Giavazzi, F. (2013). *Macroeconomics. A European Perspective* (2<sup>nd</sup> ed.). Harlow: UK, Pearson Education Limited.
- Bodie, Z. (1976). Common Stocks as a Hedge against Inflation. *Journal of Finance*, 31 (2), 459-470.
- Briys, E., Crouhy, M., Schlesinger, H. (1990). Optimal Hedging under Intertemporal Dependent Preferences. *Journal of Finance*, 45 (4), 1315-1324.
- Broll, U., Clark, E., Lukas, E. (2010). Hedging Mean-Reverting Commodities. *IMA Journal of Management Mathematics*, 21 (1), 19-26.
- Broll, U., Wong, K. P. (2013). The Firm under Uncertainty: Real and Financial Decisions. *Decisions in Economics and Finance*, 36 (2), 125-136.
- Dixit, A. K., Pindyck, R. S. (1994). *Investment under Uncertainty*. Princeton, NJ: Princeton University Press.
- Eaton, J. (1980). Price Variability, Utility and Savings. *Review of Economic Studies*, 47 (3), 513-520.
- Elder, J. (2004). Another Perspective on the Effects of Inflation Uncertainty. *Journal of Money, Credit and Banking*, 36 (5), 911-928.
- Froot, K. A., Scharfstein, D. S., Stein, J. C. (1993). Risk Management: Coordinating Corporate Investment and Financing Policies. *Journal of Finance*, 48 (5), 1629-1658.
- Heer, B., Maußner, A. (2009). *Dynamic General Equilibrium Modeling. Computational Methods and Applications* (2<sup>nd</sup> ed.). Berlin: Springer.
- Kawai, M., Zilcha, I. (1986). International Trade with Forward-Futures Markets under Exchange Rate and Price Uncertainty. *Journal of International Economics*, 20 (1-2), 83-98.
- Newbery, D. M. G., Stiglitz, J. E. (1981). *The Theory of Commodity Price Stabilization; A Study in the Economics of Risk*. Oxford, UK: Clarendon Press.
- Nocetti, D., Smith, W. T. (2011). Price Uncertainty, Saving, and Welfare. *Journal of Economic Dynamics & Control*, 35 (7), 1139-1149.



- Pelster, M. (2014). Implications of Financial Transaction Costs on the Real Economy: A Note. *Contemporary Economics*, 8 (1), 113-118.
- Turnovsky, S. J. (2000), *Methods of Macroeconomic Dynamics* (2nd ed.). Cambridge: MA, The MIT Press.

## Endnotes

- <sup>1</sup> *Nocetti and Smith's (2011) Proposition 2 contains a demand for a risky asset, which is used in their model in a similar way as hedging to optimize the agent's welfare.*
- <sup>2</sup> *See, e.g., Dixit and Pindyck (1994) and Turnovsky (2000). The Bellman differential equation is then an ordinary one, allowing it to be solved analytically.*
- <sup>3</sup> *The part of the stochastic wealth accumulation equation not concerning consumption can thus be written as  $dW = \hat{\alpha}Wdt + Wdw$ , where  $dw$  is a Wiener process with zero mean and instantaneous variance  $\hat{\beta}dt$ .*
- <sup>4</sup> *Note that at time zero, wealth  $W(0)$  is the same regardless of hedging possibilities, as signing a futures contract costs nothing.*



## Appendix

### Bellman equation

The Bellman equation of the optimization problem is

$$\beta V(W, t) = \max_{c, h} \left[ U(C) + \frac{E(dV(W, t))}{dt} \right] \quad (A1)$$

where  $dV(W, t)$  is calculated using Itô's lemma:

$$dV(W, t) = V_t dt + V_w dW + \frac{1}{2} V_{ww} (dW)^2 + o(dt)$$

where  $dW$  is defined in equation (5). Inserting  $dV(W, t)$  into (A1), making use of

$$E(dW) = [(\mu_s - \mu_p - \sigma_{sp} + \sigma_p^2)dt - h\mu_f dt]W - Cdt$$

$$E(dW)^2 = [(\sigma_s dz_s - \sigma_p dz_p - h\sigma_f dz_f)]^2 W^2 + o(dt)$$

$$= [\sigma_s^2 + \sigma_p^2 - 2\sigma_{sp} - 2h(\sigma_{sf} - \sigma_{pf}) + h^2 \sigma_f^2] W^2 dt$$

gives

$$\beta V(W, t) = \max_{c, h} \left\{ \begin{aligned} & [U(C) + V_t + V_w [(\mu_s - \mu_p - \sigma_{sp} + \sigma_p^2) - h\mu_f]W - V_w C] \\ & + \frac{1}{2} V_{ww} [\sigma_s^2 + \sigma_p^2 - 2\sigma_{sp} - 2h(\sigma_{sf} - \sigma_{pf}) + h^2 \sigma_f^2] W^2 \end{aligned} \right\} \quad (A2)$$

which is equation (7). Maximization of the right-hand side gives rise to the two first-order conditions

$$U_c(C) - V_w(W, t) = 0, \quad (A3.1)$$

$$-\mu_f V_w W + \frac{1}{2} V_{ww} [(-2)(\sigma_{sf} - \sigma_{pf}) + 2h\sigma_f^2] W^2 = 0, \quad (A3.2)$$

from which (8a) and (8b) immediately follow.

### Iso-elastic utility function

$$U(C) = \frac{1}{\gamma} C^\gamma, \quad -\infty < \gamma < 1$$

Inserting this utility function and the optimized values  $\hat{C}$  and  $\hat{h}$  into the Bellman equation (A2), writing  $V(W)$  instead of  $V(W, t)$ , thus dropping the  $V_t$ -term on the right-hand side, and rearranging yields

$$\left\{ \frac{1}{\gamma} \hat{C}^\gamma + \left[ \hat{a} - \frac{\hat{C}}{W} \right] W V_w(W) + \frac{1}{2} \hat{h} W^2 V_{ww}(W) \right\} - \beta V(W) = 0 \quad (A3)$$

with  $\hat{a}$  and  $\hat{h}$  defined as in the text. This is an ordinary differential equation. The solution of the differential equation (A3) is achieved by trial and error. Looking at (A3), it is quite natural to postulate a solution of the form

$$V(W) = \delta W^\gamma, \quad (A4)$$

where the coefficient  $\delta$  has to be determined (see Turnovsky (2000), chapter 15). Equation (A4) implies

$$V_w = \delta \gamma W^{\gamma-1}, \quad V_{ww} = \delta \gamma (\gamma - 1) W^{\gamma-2}. \quad (A5)$$

Substituting (A4) and (A5) into the first-order conditions (A3.1) and (A3.2), using  $U_c = C^{\gamma-1}$  yields

$$\hat{C}^{\gamma-1} = \delta \gamma W^{\gamma-1}, \quad \text{or} \quad \frac{\hat{C}}{W} = (\delta \gamma)^{\frac{1}{\gamma-1}}, \quad (A6.1)$$

$$-\mu_f \delta \gamma W^\gamma + \delta \gamma (\gamma - 1) W^{\gamma-1} [(-1)(\sigma_{sf} - \sigma_{pf}) + \hat{h} \sigma_f^2] = 0, \quad \text{or, equivalently,}$$

$$\hat{h} = \frac{\sigma_{sf} - \sigma_{pf}}{\sigma_f^2} + \frac{\mu_f}{\sigma_f^2 (\gamma - 1)}, \quad (A6.2)$$

which is equation (9c).

### Solution of the Bellman equation

Next, we solve (A6.1) for  $\hat{C} = (\delta \gamma)^{\frac{1}{\gamma-1}} W$  and insert this together with (A4) and (A5) for  $V(W)$ ,  $V_w$ ,  $V_{ww}$ , respectively, into the Bellman differential equation (A3) to obtain

$$\frac{1}{\gamma} (\delta \gamma)^{\frac{\gamma}{\gamma-1}} + \left[ \hat{a} - (\delta \gamma)^{\frac{1}{\gamma-1}} \right] \delta \gamma + \frac{1}{2} \hat{h} \delta \gamma (\gamma - 1) - \beta \delta = 0.$$

This equation can be solved for

$$(\delta \gamma)^{\frac{1}{\gamma-1}} = \frac{\beta - \hat{a} \gamma - (1/2) \hat{h} \gamma (\gamma - 1)}{1 - \gamma}.$$

Combining this expression with (A6.1), we obtain the consumption-wealth ratio

$$\frac{\hat{C}}{W} = \frac{\beta - \hat{a} \gamma - (1/2) \hat{h} \gamma (\gamma - 1)}{1 - \gamma}. \quad (A7)$$

This is equation (9b). Solving (A6.1) for the coefficient  $\delta$  gives

$$\delta = \frac{1}{\gamma} \left( \frac{\hat{C}}{W} \right)^{\gamma-1}, \quad (A8)$$

where  $\hat{C}/W$  is given by (A7). Thus, the value function becomes

$$V(W) = \frac{1}{\gamma} \left( \frac{\hat{C}}{W} \right)^{\gamma-1} W^\gamma \quad \text{or} \quad V(W) = \frac{W^\gamma}{\gamma \left( \frac{\hat{C}}{W} \right)^{1-\gamma}}, \quad (\text{A9})$$

where the first equation is (9a) in the text. Inserting (A7) into (A9), assuming  $\hat{h} \neq 0$ , and using the initial level of real wealth  $W(0)$ , for the case of  $0 < \gamma < 1$ , we obtain  $(\hat{C}/W(0)) < (\hat{C}/W(0))_{\text{without hedging}}$  and, thus,  $V(W(0)) > V(W(0))_{\text{without hedging}}$ , and for  $-\infty < \gamma < 0$ ,  $(\hat{C}/W(0)) > (\hat{C}/W(0))_{\text{without hedging}}$  and therefore again  $V(W(0)) > V(W(0))_{\text{without hedging}}$  (note that in this case, the denominator of (A9) is negative because of  $\gamma < 0$ ).

In the case of the logarithmic utility function,  $\gamma = 0$  and, thus,  $U(C) = \ln C$ . A solution of the Bellman differential equation (A3), where  $\hat{C}^\gamma/\gamma$  is substituted by  $\ln C$ , is  $V(W) = \lambda + \delta \ln W$ . From (A7), we obtain  $\hat{C}/W = \beta$  both with and without hedging because the optimal hedging strategy enclosed in  $\hat{a}$  and  $\hat{b}$  does not influence the optimal consumption-wealth ratio (A7). However, it can be shown that

$$\lambda = \frac{\beta \ln \beta - \beta + \hat{a} - \hat{b}/2}{\beta^2}, \quad \delta = \frac{1}{\beta} \left( = \left( \frac{\hat{C}}{W} \right)^{-1} \right),$$

and that

$$\hat{a} - \frac{1}{2} \hat{b} = (\mu_S - \mu_P - \sigma_{SP} + \sigma_P^2) - \frac{1}{2} (\sigma_S^2 + \sigma_P^2 - 2\sigma_{SP}) + \frac{1}{2} \sigma_F^2 \hat{h}^2,$$

where the first two terms in parentheses denote  $a - b/2$  in the case without hedging. Hence,  $\lambda > \lambda_{\text{without hedging}}$ , thus proving  $V(W(0)) > V(W(0))_{\text{without hedging}}$ . From these observations, inequality (10) follows.

