School of Economics and Finance

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Working Paper No. 766 December 2015 ISSN 1473-0278

Integrated ARCH, FIGARCH and AR models: origins of long memory

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March 30, 2015

Abstract

Although the properties of the ARCH(∞) model are well investigated, the existence of long memory FIGARCH and IARCH solution was not established in the literature. These two popular ARCH type models which are widely used in applied literature, were causing theoretical controversy because of the suspicion that other solutions besides the trivial zero one, do not exist. Since ARCH models with non-zero intercept have a unique stationary solution and exclude long memory, the existence of finite variance FIGARCH and IARCH models and, thus, the possibility of long memory in the ARCH setting was doubtful. The present paper solves this controversy by showing that FIGARCH and IARCH equations have a non-trivial covariance stationary solution, and that such a solution exhibits long memory. The existence and uniqueness of stationary Integrated AR(∞) processes is also discussed, and long memory, as an inherited feature, is established. Summarizing, we show that covariance stationary IARCH, FIEGARCH and IAR(∞) processes exist, their class is wide, and they always have long memory.

Keywords: AR, FIGARCH, IARCH, long memory.
JEL classification: C15; C22

1 Introduction

A non-negative random process $\{\tau_k\} = \{\tau_k,\ k \in \mathbb{Z}\}$ is said to satisfy an ARCH(∞) equation if there exists a sequence of nonnegative i.i.d. random variables $\{\varepsilon_k\}$ with unit mean $E\varepsilon_0 = 1$,

*The second author's research was supported by Research Council of Lithuania grant MIP-063/2013.

a nonnegative number $\omega \geq 0$ and a deterministic sequence $b_j \geq 0$, $j = 1, 2, \cdots$, such that

$$\tau_k \;=\; \varepsilon_k\big(\omega + \sum_{j=1}^{\infty} b_j \tau_{k-j}\big), \qquad k \in \mathbb{Z}. \tag{1.1}$$

In this paper we assume that the process $\{\tau_k\}$ described by equations (1.1) is *causal*, i.e., for any k, τ_k can be represented as a measurable function $f(\varepsilon_k, \varepsilon_{k-1}, \ldots.)$ of the present and past values $\varepsilon_s, s \leq k$. The last property implies that a stationary $\{\tau_k\}$ process is ergodic, and ε_k is independent of $\tau_s, s < k$. Therefore,

$$E[\tau_k | \tau_s, s < k] = h_k, \quad h_k := \omega + \sum_{j=1}^{\infty} b_j \tau_{k-j}.$$

The usual interpretation of τ_k and ε_k in financial econometrics is that of squared returns and squared innovations, viz., $\tau_k = r_k^2$, $\varepsilon_k = \zeta_k^2$, where the return process $\{r_k\}$ satisfies the ARCH equations

$$r_k = \zeta_k h_k^{1/2}, \quad h_k = \omega + \sum_{j=1}^{\infty} b_j r_{k-j}^2, \qquad k \in \mathbb{Z},$$

$\{\zeta_k\}$ is a standardized i.i.d. $(0, 1)$-noise and h_k is volatility. Moreover, since typically random variables in (1.1) are almost surely positive ($\tau_k > 0$), they may represent random durations between transactions. The class of ARCH(∞) processes (1.1) includes the parametric stationary ARCH and GARCH models of Engle (1982) and Bollerslev (1986), and the ACD (Autoregressive Conditional Duration) model of Engle and Russel (1998).

The ARCH(∞) process was introduced by Robinson (1991) and later studied in Kokoszka and Leipus (2000), Giraitis *et al.* (2000a) (see also the review papers by Giraitis *et al.* (2007, 2011), Berkes *et al.* (2004)). In contrast to a standard stationary GARCH(p, q) process whose autocorrelations decay exponentially, the ARCH(∞) process may have autocovariances $\mathrm{cov}(\tau_k, \tau_0)$ decaying to zero at a rate $k^{-\gamma}$ with $\gamma > 1$ arbitrarily close to 1.

In several papers (e.g. Giraitis *et al.* (2000a), Giraitis and Surgailis (2002), Kazakevičius and Leipus (2002)) it is claimed that a finite variance stationary solution to the ARCH equations in (1.1), if exists, has *short memory* or *absolutely summable autocovariance function*, while the existence of such a solution necessarily implies $\sum_{j=1}^{\infty} b_j < 1$. Because of the well known phenomenon of long memory in the squares of financial returns in financial econometrics, the latter finding may be considered as a limitation of ARCH modeling. Subsequently, it initiated and justified the study of other ARCH-type models for which the long memory property can be rigorously established (see Giraitis *et al.* (2007, 2011)).

The above claims are correct if $\omega > 0$. More precisely, Theorem 3.1 and Corollary 3.2 of Giraitis and Surgailis (2002) (below referred to as GS(2002)) say that if a covariance stationary solution $\{\tau_k\}$ of ARCH equation (1.1) with $\omega > 0$ exists, then it is unique and its autocovariance function is nonnegative, and $\sum_{k=0}^{\infty} \mathrm{cov}(\tau_k, \tau_0) < \infty$. Clearly, this implies $\sum_{j=1}^{\infty} b_j < 1$ since $E\tau_k = \omega + (\sum_{j=1}^{\infty} b_j) E\tau_k > (\sum_{j=1}^{\infty} b_j) E\tau_k$.

For $\omega = 0$, however, the situation is different. In this case the existence of a covariance stationary solution of (1.1) implies $\sum_{j=1}^{\infty} b_j = 1$, leading to the integrated ARCH(∞), or IARCH(∞), equation

$$(1.2) \qquad \tau_k \;=\; \varepsilon_k\Big(\sum_{j=1}^{\infty} b_j \tau_{k-j}\Big), \quad k \in \mathbb{Z}, \quad \text{with} \quad \theta := \sum_{j=1}^{\infty} b_j = 1.$$

Obviously, (1.2) has a trivial solution $\{\tau_k \equiv 0\}$. There is not much known in the literature about the existence of a nontrivial covariance stationary solution of (1.2). On the other hand, the existence of a stationary IARCH process of (1.1) with $\omega > 0, \theta = 1$ and infinite mean $E\tau_j = \infty$ was established in Kazakevičius and Leipus (2003) and Douc *et al.* (2006).

A particular case of the IARCH model in (1.2) is the well-known FIGARCH equation with zero intercept $\omega = 0$:

$$(1.3) \qquad \tau_k \;=\; \varepsilon_k\big(1-(1-L)^d\big)\tau_k \;=\; \varepsilon_k\Big(\sum_{j=1}^{\infty} b_j \tau_{k-j}\Big), \quad k \in \mathbb{Z},$$

where $0 < d < 1/2$ is the fractional differencing parameter, L is the backshift operator and the coefficients b_j are determined by the generating function $B(z) = \sum_{j=1}^{\infty} b_j z^j = 1-(1-z)^d$. Note $b_j > 0$, $\sum_{j=1}^{\infty} b_j = 1$ and $b_j = O(j^{-1-d})$ decay hyperbolically with $j \to \infty$. The FIGARCH equation was introduced by Baillie *et al.* (1996) to capture the long memory effect in volatility. They argued that a stationary FIGARCH process, if exist, has long memory. However, the existence of a non-trivial FIGARCH process with finite mean was never shown. See Giraitis *et al.* (2000a), Kazakevičius and Leipus (2003), Mikosch and Stărică (2000, 2003), Davidson (2004) for a discussion of controversies surrounding the FIGARCH. Several papers (Giraitis *et al.* (2000a, 2002), Kazakevičius and Leipus (2003)) claim that the FIGARCH equation has no stationary solution with finite mean $E\tau_k < \infty$ besides the trivial solution $\tau_k \equiv 0$.

The present paper corrects the above claim. We show that for each $\mu > 0$, the FIGARCH equation in (1.3) has a stationary ergodic solution $\{\tau_k\}$ with mean $E\tau_k = \mu$, finite variance (and possibly higher moments), and a nonsummable hyperbolically decaying autocovariance function $\mathrm{cov}(\tau_k, \tau_0) \sim c\, k^{-\gamma}$, $\gamma = 1-2d \in (0,1)$, see Theorem 4.1. Hence, there exist infinitely many stationary solutions of the FIGARCH equation (1.3) with $\omega = 0$ parametrized by the mean value $E\tau_k = \mu \geq 0$. The trivial solution $\{\tau_k \equiv 0\}$ corresponds to $\mu = 0$ and coincides with the stationary solution of (1.1) in the case $\omega > 0$ (Giraitis *et al.* (2000a))

$$(1.4) \qquad \tau_k \;=\; \omega\varepsilon_k\Big(1 + \sum_{m=1}^{\infty} \sum_{s_m < \cdots < s_1 < k} b_{k-s_1} \cdots b_{s_{m-1}-s_m} \varepsilon_{s_1} \cdots \varepsilon_{s_m}\Big)$$

termed also the minimal solution in Kazakevičius and Leipus (2003), as the r.h.s. of (1.4) vanishes if $\omega = 0$. For $\theta = 1$ the series on the r.h.s. of (1.4) does not converges in L_1

and therefore the expansion in (1.4) is not useful for studying the existence of L_1- or L_2-solutions of the FIGARCH and IARCH(∞) equations (1.3) and (1.2). For $\omega > 0$, Douc *et al.* (2006) and Robinson and Zaffaroni (2006) showed the existence of a stationary FIGARCH process $\{\tau_k\}$ in (1.4) with infinite mean $E\tau_k = \infty$, which excludes the desired property of long memory.

A possible explanation of these seemingly confusing statements about stationary solutions of the ARCH(∞) model (1.1) is that many papers on ARCH models explicitly or implicitly assume $\omega > 0$. This assumption, together with covariance stationarity of $\{\tau_k\}$ implies $E\tau_k = \omega + E\tau_k(b_1 + b_2 + ...)$, or

$$E\tau_k = \frac{\omega}{1 - \sum_{j=1}^{\infty} b_j}, \tag{1.5}$$

yielding $E\tau_k < \infty$ if $\theta = \sum_{j=1}^{\infty} b_j < 1$, and $E\tau_k = \infty$ if $\theta = 1$. However, for $\omega = 0$ and $\theta = 1$, which correspond to IARCH model (1.2), the r.h.s. of (1.5) is undefined, and therefore it does not contradict $E\tau_k < \infty$.

The main idea of constructing a stationary L_2-solution of the IARCH equation (1.2) with mean $\mu = E\tau_k > 0$ is the reduction of (1.2) to the linear Integrated AR (IAR) equation for the centered process $Y_k := \tau_k - \mu$:

$$Y_k = \sum_{j=1}^{\infty} b_j Y_{t-j} + z_k, \qquad k \in \mathbb{Z} \tag{1.6}$$

with a conditionally heteroscedastic martingale difference noise $\{z_k\}$ given by

$$z_k := \zeta_k\big(\mu\sigma + \sigma \sum_{j=1}^{\infty} b_j Y_{k-j}\big), \tag{1.7}$$

where $\zeta_k := (\varepsilon_k - 1)/\sigma$, $\sigma^2 := \mathrm{var}(\varepsilon_1) < \infty$. In turn, from (1.6) the process $\{z_k\}$ can be defined as a stationary solution of the LARCH (Linear ARCH) equation (Giraitis *et al.* 2000b, 2004)

$$z_k = \zeta_k\big(\mu\sigma + H(L)z_k\big), \qquad H(L) := \sigma B(L)(1 - B(L))^{-1} = \sum_{j=1}^{\infty} h_j L^j, \tag{1.8}$$

with standardized zero mean i.i.d. innovations $\{\zeta_k\}$, $E\zeta_k = 0, E\zeta_k^2 = 1$. Equation (1.6)-(1.7) is a particular case of the bilinear models studied in Giraitis and Surgailis (2002). The last paper provides a necessary and sufficient condition for the existence of a stationary causal L_2-solution $\{Y_k, z_k\}$ given by convergent orthogonal Volterra series in (3.13). It turns out (see Theorem 4.1 below) that in the case of the FIGARCH equation (1.3), the above necessary and sufficient condition for the existence of $\{Y_k\}$ in (3.13) reduces to the following condition involving $E\varepsilon_0^2$ and the parameter d (see Figure 1):

$$E\varepsilon_0^2 < \frac{\Gamma(1-2d)}{\Gamma(1-2d) - \Gamma^2(1-d)}. \tag{1.9}$$

d	V(d)
0.05	225.794
0.1	52.296
0.15	21.478
0.2	11.133
0.25	6.545
0.3	4.160
0.35	2.786
0.4	1.934
0.45	1.378

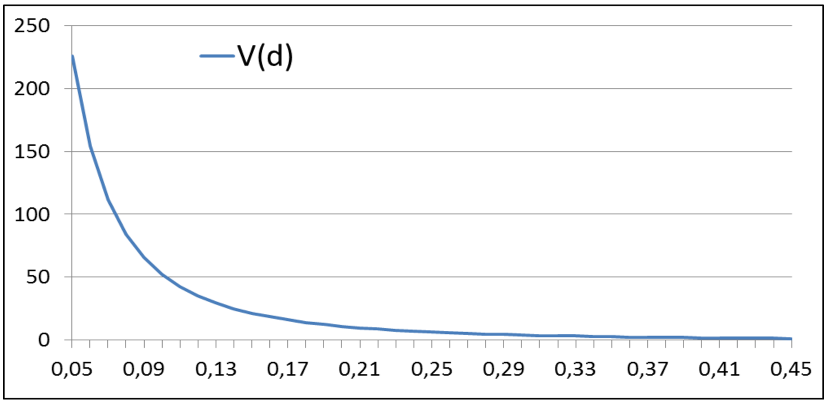

Figure 1: Graph and values of the function $V(d) = \frac{\Gamma(1-2d)}{\Gamma(1-2d)-\Gamma^2(1-d)}$.

We also establish sufficient and necessary conditions for existence of a stationary solution of linear Integrated AR(∞) equation

$$x_k - \sum_{j=1}^{\infty} b_j x_{k-j} = \xi_k, \qquad k \in \mathbb{Z}, \tag{1.10}$$

where b_j's are non-negative, $\sum_{j=1}^{\infty} b_j = 1$, and $\{\xi_k\}$ is stationary short memory process, in particular, a white noise. In Theorems 2.1 and 2.2 we show that stationary solutions of such equations always have long memory, which originates from integration property with infinite number of non-negative b_j's.

For a general non-parametric IAR and IARCH stationary processes we define long memory property of $\{x_k\}$ as the non-sumability of auto-covariances $\sum_{k=0}^{\infty} |\mathrm{cov}(x_k, x_0)| = \infty$, and in spectral domain, by the property that $f(x) \to \infty$, $x \to 0$ of spectral density of $\{x_k\}$.

The paper is structured as follows. Section 2 discusses solvability and second-order properties of IAR(∞) equation (1.10) driven by a general short memory "noise" $\{\xi_k\}$, in particular a general uncorrelated sequence, not necessary having the form in (1.7). This section may have independent interest since we show that long memory is an inherited feature of stationary IAR models whose origins lie in integration rather than in fractional differentiation or filtering. Section 3 discusses stationary L_2-solutions of the ARCH(∞) (1.1) and bilinear (1.6)-(1.7) equations and their mutual relationship. In section 4, the results of the previous sections are used to obtain the existence and uniqueness of stationary L_2-solutions of the FIGARCH, IARCH and ARCH models. We also establish the long/short memory properties of these solutions, including the convergence of normalized partial sums $\sum_{k=1}^{[nt]} \tau_k$ to a fractional/standard Brownian motion. Section 5 contains auxiliary lemmas.

In the sequel, we set $\Pi := [-\pi, \pi]$, and write $a_n \sim b_n$ if $a_n/b_n \to 1$. Moreover, $\to_p$ and $\to_D$ denote the convergence in probability and distribution, respectively. All (in)equalities involving random variables in this paper are supposed to hold almost surely.

2 Stationary Integrated AR(∞) process

In the time series literature, long memory processes are often defined via (or identified with) fractional filtering/differencing operators and ARFIMA(p,d,q) models introduced by Hosking (1981) and Porter-Hudak (1990). Generalizations of fractional filters were discussed in Leipus and Viano (2000). Being a technical tool for generating parametric long memory time series, fractional filtering/differencing cannot fully explain the phenomenon and how long memory is induced, which was often leading to controversies justifying the use of long memory processes and explaining their generating mechanism, see e.g. Lieberman and Phillips (2008) for an illustrative analysis of how long memory may arise in realized volatility.

In this section we consider the Integrated AR(∞) time series model

$$x_k - \sum_{j=1}^{\infty} b_j x_{k-j} = \xi_k, \qquad k \in \mathbb{Z}, \tag{2.1}$$

where the b_j's are non-negative, $\sum_{j=1}^{\infty} b_j = 1$, and $\{\xi_k\}$ is a stationary sequence of uncorrelated noise with $E\xi_k = 0$, $E\xi_k^2 = \sigma_\xi^2$.

Definition 2.1 *We say that a random process $\{x_k\}$ is a L_2-solution of (2.1) if $Ex_k^2 < \infty$ for each $k \in \mathbb{Z}$, the series $\sum_{j=1}^{\infty} b_j x_{k-j}$ converges in mean square and (2.1) holds.*

Similarly, in the IARCH case, it is of interest to find sufficient and necessary conditions for the existence of a stationary finite variance solution to equation (2.1) and investigate its uniqueness and the property of long memory. Contrary to the IARCH model (1.6), the noise $\{\xi_k\}$ does not depend on $\{x_k\}$, so conditions are expected to be less restrictive. As in IARCH case, a stationary solution $\{x_k\}$ of (2.1), if exist, is not unique, since then $\{x_k + \mu\}$ is also a solution of (2.1).

Clearly, b_j's cannot cut off or decay to 0 too fast, viz. a unit root model $x_k - x_{k-1} = \xi_k$ does not have a stationary solution. There are two ways to construct a stationary solution. Firstly, that can be done using power expansion coefficients g_j of the analytic function

$$G(z) = (1 - B(z))^{-1} = \sum_{j=0}^{\infty} g_j z^j, \tag{2.2}$$

where $B(z) = \sum_{j=0}^{\infty} b_j z^j$ is defined on the complex disk $\{|z| < 1\}$. Such g_j's are uniquely determined, and if $\|g\| = (\sum_{j=0}^{\infty} g_j^2)^{1/2} < \infty$, they define a stationary zero mean and finite variance process,

$$\tilde{x}_k = \sum_{j=0}^{\infty} g_j \xi_{k-j}, \qquad k \in \mathbb{Z}. \tag{2.3}$$

Alternatively, if the transfer function $A(x) = (1 - B(\mathrm{e}^{\mathrm{i}x}))^{-1}$, $x \in \Pi$ of the filter (2.1) is L_2-integrable, $\|A\| = (\int_\Pi |A(x)|^2 \mathrm{d}x)^{1/2} < \infty$, then its Fourier coefficients $g'_j = (2\pi)^{-1} \int_\Pi A(x) \mathrm{e}^{-\mathrm{i}xj} \mathrm{d}x$, $j \geq 0$ have property $\|g'\| = (\sum_{j=0}^{\infty} g_j'^2)^{1/2} < \infty$ and define a stationary zero mean and finite variance process,

$$\tilde{x}'_k = \sum_{j=0}^{\infty} g'_j \xi_{k-j}, \qquad k \in \mathbb{Z}. \tag{2.4}$$

Observe that the g_j's in (2.3) are nonnegative and given by

$$g_j = \sum_{m=1}^{j} \sum_{0 < s_{m-1} < \cdots < s_1 < j} b_{j-s_1} b_{s_1 - s_2} \ldots b_{s_{m-2} - s_{m-1}} b_{s_{m-1}}, \quad j \geq 1, \qquad g_0 = 1, \tag{2.5}$$

which follows from the equality $(1 - B(z))^{-1} = \sum_{m=0}^{\infty} B^m(z)$.
The next theorem establishes the equivalence of conditions $\|g\| < \infty$ and $\|A\| < \infty$, the equality of the weights $g_j = g'_j$, and the L_2 equality of the functions $G(\mathrm{e}^{\mathrm{i}x}) = A(x)$, $x \in \Pi$. It obtains conditions for the existence and uniqueness of a stationary L_2-solution of the IAR(∞) equation and establishes its long memory property.

A stationary solution $\{x_k\}$ of (2.1) is said to be *causal* if for each k, x_k can be represented as a measurable function $f(\xi_k, \xi_{k-1}, \ldots)$ of the present and past values ξ_s, $s \leq k$. In addition, we assume that the noise $\{\xi_k\}$ in (2.1) is causal with respect to some i.i.d. noise $\{\eta_j\}$ which implies that $\{\xi_j\}$ is a regular process. The last property is used to prove uniqueness of a stationary IAR solution $\{x_j\}$ that is causal with respect to $\{\eta_j\}$.

Theorem 2.1 *Integrated AR equation (2.1) has the following properties.*

(i) *Assumption $\|g\| < \infty$ is necessary for existence of a stationary L_2-solution of (2.1). It implies $\|A\| < \infty$ and equality $G(\mathrm{e}^{\mathrm{i}x}) = A(x)$ a.e. in Π. Conversely, $\|A\| < \infty$ implies $\|g\| < \infty$.*

(ii) *If* $\|g\| < \infty$*, then with* $\tilde{x}_k$ *as in (2.3) for each real* μ*,*

$$x_k = \mu + \tilde{x}_k, \quad k \in \mathbb{Z}, \tag{2.6}$$

is a unique stationary causal L_2*-solution of (2.1) with* $Ex_k = \mu$*.*

(iii) *Solution (2.6),* x_k*, has non-negative and non-summable covariance function*

$$\mathrm{cov}(x_{0,\mu}, x_{k,\mu}) = \sigma_\xi^2 \sum_{j=0}^{\infty} g_j g_{k+j} \geq 0, \qquad \sum_{k\in\mathbb{Z}} \mathrm{cov}(x_{0,\mu}, x_{k,\mu}) = \infty, \tag{2.7}$$

and unbounded spectral density $f(x) = (\sigma_\xi^2/2\pi)|1 - B(\mathrm{e}^{\mathrm{i}x})|^{-2}$*,* $x \in \Pi$ *such that* $f(x) \to \infty$ *as* $x \to 0$*.*

Proof of Theorem 2.1. (i) the necessity of condition $\|g\| < \infty$ for the existence of a stationary L_2- solution follows from Lemma 5.2(b) while Lemma 5.3 (a) and (b) imply remaining claims of (i).

(ii) It suffices to consider the case $\mu = 0$. In view of (i), $\tilde{x}_k = \tilde{x}'_k$, $k \in \mathbb{Z}$. We will show that $\{\tilde{x}'_k\}$ is zero mean solution of (2.1). Indeed, denote by $\xi_k = \int_\Pi \mathrm{e}^{\mathrm{i}kv} Z_\xi(dv)$ the spectral representation of $\{\xi_k\}$ with the random spectral measure $Z_\xi(\mathrm{d}x)$. Since $\|A\| < \infty$ implies $\sum_{j=0}^\infty g_j'^2 < \infty$, then, by general properties of spectral representation of stationary processes (see, Brockwell and Davis (1989, Thm.4.10.1), see also Giraitis *et al.* (2012, Thm.2.2.1)), the series $x'_k = \sum_{j=0}^\infty g'_j \xi_{k-j}$ converges in mean square and

$$\tilde{x}'_k = \int_\Pi \mathrm{e}^{\mathrm{i}kx} A(x) Z_\xi(\mathrm{d}x), \qquad k \in \mathbb{Z}. \tag{2.8}$$

Hence x'_k is a well-defined stationary process with zero mean, finite variance $E\tilde{x}_k'^2 = (\sigma_\xi^2/2\pi)\int_\Pi |A(x)|^2 dx < \infty$, spectral density $f_{\tilde{x}}(x) = (\sigma_\xi^2/2\pi)|A(x)|^2$ and spectral measure $F_{\tilde{x}}(\mathrm{d}x) = f_{\tilde{x}}(x)\mathrm{d}x$. To show that $\{\tilde{x}'_k\}$ is a L_2-solution of (2.1), observe that the function $1 - B(\mathrm{e}^{\mathrm{i}x})$ is bounded on Π and therefore $L_2(\tilde{F})$ integrable. Then, as above, by the general properties of spectral representation of stationary processes the series $\sum_{j=1}^\infty b_j \tilde{x}'_{k-j}$ converges in mean square and $\tilde{x}'_k - \sum_{j=1}^\infty b_j \tilde{x}'_{k-j} = \int_\Pi \mathrm{e}^{\mathrm{i}kx}(1 - \sum_{j=1}^\infty b_j \mathrm{e}^{\mathrm{i}jx}) A(x) Z_\xi(\mathrm{d}x) = \int_\Pi \mathrm{e}^{\mathrm{i}kx} Z_\xi(\mathrm{d}x) = \xi_k$, proving the above claim.

It remains to show the uniqueness of solution (2.4), $\{\tilde{x}'_k\}$, with the stated properties. Let $\{x'_k\}, \{x''_k\}$ be two causal L_2-solutions of (2.1) with $Ex'_k = Ex''_k$ and $y_k := x'_k - x''_k$. Then $\{y_k\}$ is a causal stationary L_2-solution of the homogeneous equation

$$y_k - \sum_{j=1}^\infty b_j y_{k-j} = 0.$$

Hence using boundedness of $1 - B(\mathrm{e}^{\mathrm{i}x})$ similarly as in the proof of (i) we obtain that

$$\int_\Pi |1 - B(\mathrm{e}^{\mathrm{i}x})|^2 F_y(\mathrm{d}x) = 0, \tag{2.9}$$

where F_y is the spectral measure of $\{y_k\}$. The causality of $\{y_k\}$ implies that $\{y_k\}$ is a regular process since $\{\xi_k\}$ is regular and therefore F_y is absolutely continuous with spectral density f_y, see Ibragimov and Linnik (1971), Thm.17.1.2. Together with Lemma 5.1(i) (2.9) implies that $f_y(x) = 0$ a.e. on Π and hence $F_y = 0$ and $y_k = 0$, proving part (ii).

(iii) The spectral representation (2.8) of the solution implies that it has spectral density $f(x) = (\sigma_\xi^2/2\pi)|1 - B(\mathrm{e}^{\mathrm{i}x})|^{-2}$. The fact that $f(x)$ in unbounded, viz., $\lim_{x\to 0} f(x) = \infty$, follows from continuity of $B(\mathrm{e}^{-\mathrm{i}x})$, $B(1) = 1$, and the fact $|B(x)| < 1$ for $0 < x < x_0$ for some $x_0 > 0$ which holds by Lemma 5.1. The divergence $\sum_{k\in\mathbb{Z}} |\mathrm{cov}(x_0, x_k)| = \infty$ is immediate from the previous fact since the convergence implies boundedness of $f(x)$. Finally, the first claim in (2.7) is a consequence of the moving average representation (2.4) and positivity of g_j's. Theorem 2.1 is proved. □

The requirement of Theorem 2.1 that in the Integrated AR equation (2.1), $\{\xi_k\}$ is an uncorrelated noise is restrictive and can be relaxed. Theorem 2.2 establishes existence and long memory property of stationary Integrated AR process when $\{\xi_k\}$ itself is causal process (with respect to some i.i.d. sequence $\{\eta_j\}$) with short memory, as precised below.

Theorem 2.2 *Let $\{\xi_k\}$ in* IAR(∞) *equation (2.1) be a stationary causal process with mean* 0*, finite variance and the spectral density f_ξ which is bounded away from* 0 *and* ∞*,*

$$c_1 \le f_\xi(x) \le c_2, \quad x \in \Pi, \qquad \exists\, 0 < c_1 < c_2 < \infty. \tag{2.10}$$

Then statements of Theorem 2.1(i), (ii) about stationary solution of IAR equation (2.1) remain valid while (iii) has to be modified as follows:

(iii) Solution (2.6), $\{x_k\}$, has unbounded spectral density $f(x) = |1 - B(\mathrm{e}^{\mathrm{i}x})|^{-2} f_\xi(x)$, $x \in \Pi$ that satisfies $f(x) \to \infty$ as $x \to \infty$, and non-summable autocovariance function

$$\sum_{k\in\mathbb{Z}} |\mathrm{cov}(x_0, x_k)| = \infty. \tag{2.11}$$

Proof of Theorem 2.2. The proof follows using the same arguments as in the proof of Theorem 2.1. □

Remark 2.1 The IAR(∞) model (2.1) does not have a stationary finite variance solution, if the weights b_j decay to zero too fast, i.e. $b_j = O(j^{-\gamma})$ for some $\gamma > 3/2$, e.g. in the unit root model $x_k - x_{k-1} = \xi_k$, which follows from Lemma 5.2(a) and Lemma 5.3(c). However, the IAR(∞) equation has a stationary process, if a singular "unit root" is distributed over infinite number of b_j's that decay not too fast, so that $|1 - B(e^{-ix})|^{-2}$ is integrable.

There exists a large variety of stationary Integrated AR(∞) processes. They always have long memory, their covariances are non-summable and their spectral density is not bounded at zero frequency. However, their covariances may not decay at hyperbolic rate k^{-1+2d}, and spectral densities may not explode at zero frequency at the rate $|x|^{-2d}$. The latter properties are key features of fractionally integrated parametric ARFIMA(p, d, q) models.

2.1 Parametric and semiparametric IAR(∞) long memory models

An **ARFIMA**$(0, d, 0)$ **model** is defined as a stationary solution of the difference equation

$$(1-L)^d x_k = \xi_k, \qquad 0 < d < 1/2,$$

where $\{\xi_k\}$ is a stationary sequence of uncorrelated noise with $E\xi_k = 0$, $E\xi_k^2 = \sigma_\xi^2$. It can be written as an IAR equation

$$x_k - \sum_{j=1}^{\infty} b_j x_{k-j} = \xi_k, \tag{2.12}$$

with b_j's generated by operator $B(L) = 1 - (1-L)^d = \sum_{j=1}^{\infty} b_j L^j$. Here, b_j's are non-negative, and $\sum_{j=1}^{\infty} b_j = 1$. The function $A(x) = (1 - B(\mathrm{e}^{-\mathrm{i}x}))^{-1}$ has the property that $|A(x)|^2 = |1 - \mathrm{e}^{-\mathrm{i}x}|^{-2d} \sim |x|^{-2d}$ as $x \to 0$ and is integrable for $d \in (0, 1/2)$. By Theorem 2.1, the IAR equation (2.12) has a stationary zero mean solution $\{x_k\}$ given by (2.3).

The b_j's and the coefficients of the generating function $G(z) = (1 - B(z))^{-1} = (1-z)^{-d}$ are given by

$$b_j = -\frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}, \quad g_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}, \quad j \geq 1, \quad g_0 = 1. \tag{2.13}$$

They have properties $b_j > 0$, $g_j > 0$ and $\theta = \sum_{j=1}^{\infty} b_j = 1$, and satisfy

$$b_j \sim -j^{-d-1}/\Gamma(-d), \quad g_j \sim j^{d-1}/\Gamma(-d), \qquad j \to \infty, \tag{2.14}$$

so $\|g\| < \infty$. This implies that the covariance $\gamma_k = \mathrm{cov}(x_0, x_k) = \sigma_\xi^2 \sum_{j=0}^{\infty} g_j g_{k+j}$ of the solution $\{x_k\}$ decays hyperbolically:

$$\gamma_k = \sigma_\xi^2 \frac{\Gamma(k+d)}{\Gamma(k-d+1)} \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \sim c_\gamma k^{-1+2d}, \qquad c_\gamma := \frac{\sigma_\xi^2 \Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}, \tag{2.15}$$

see, e.g., Chapter 7 of GKS(2012), and the spectral density is unbounded at the origin:

$$f(x) = (\sigma_\xi^2/2\pi)|1 - \mathrm{e}^{\mathrm{i}x}|^{-2d} \sim c_f |x|^{-2d}, \qquad c_f := \sigma_\xi^2/2\pi. \tag{2.16}$$

Semiparametric IAR model. A wider class of IAR processes generalizing ARFIMA$(0, d, 0)$ is defined by equation (2.12) with uncorrelated noise $\{\xi_k\}$ and b_j's generated by operator

$$B(L) := \big(1 - (1-L)^d\big) P(L) = \sum_{j=1}^{\infty} b_j L^j, \quad 0 < d < 1/2 \tag{2.17}$$

where $P(z) = \sum_{j=0}^{\infty} p_j z^j$ is a generating function with coefficients satisfying

$$p_j \geq 0, \quad p_1 > 0, \quad \textstyle\sum_{j=0}^{\infty} p_j = 1 \quad \text{and} \quad \textstyle\sum_{j=1}^{\infty} j p_j < \infty. \tag{2.18}$$

Then $b_j = \sum_{k=0}^{j-1} p_k b^0_{j-k}$ where b^0_j are coefficients of ARFIMA generating function $1-(1-z)^d = \sum_{j=1}^{\infty} b^0_j z^j$, defined in (2.13), and therefore the b_j's in $B(L) = \sum_{j=1}^{\infty} b_j L^j$ are non-negative and sum up to 1.

Let us show that $|A(x)|^2 = |1-B(\mathrm{e}^{\mathrm{i}x})|^{-2}$ is integrable for $d \in (0, 1/2)$. Then, by Theorem 2.1, the IAR equation (2.12) has a stationary zero mean solution $\{x_k\}$ given by (2.3).

Since $b_1 = p_0 b^0_1 > 0$, by Lemma 5.1(ii) $|A(x)|$ is bounded on $[\epsilon, \pi]$ for any $\epsilon > 0$. Therefore, it suffices to show that $|A(x)|^2$ is integrable at $x = 0$. To this end, rewrite $1 - B(\mathrm{e}^{\mathrm{i}x}) = 1 - \big(1 - (1 - \mathrm{e}^{\mathrm{i}x})^d\big) P(\mathrm{e}^{\mathrm{i}x}) = (1 - \mathrm{e}^{\mathrm{i}x})^d h(x)$, where

$$h(x) := P(\mathrm{e}^{\mathrm{i}x}) - \big(P(\mathrm{e}^{\mathrm{i}x}) - 1\big)(1 - \mathrm{e}^{\mathrm{i}x})^{-d}. \tag{2.19}$$

From (2.18) we have $|P(\mathrm{e}^{\mathrm{i}x}) - 1| = \sum_{j=1}^{\infty} |\mathrm{e}^{\mathrm{i}jx} - 1| p_j \le |x| \sum_{j=1}^{\infty} j p_j = O(|x|) = o(|1 - \mathrm{e}^{\mathrm{i}x}|^d)$ and therefore $h(x) \to h(1) = P(1) = 1$ as $x \to 0$. Hence, $|A(x)|^2 \sim |x|^{-2d}$, $x \to 0$, proving the integrability of $|A(x)|^2$ for $d \in (0, 1/2)$. The corresponding stationary solution $\{x_k\}$ of (2.12) with uncorrelated noise $\{\xi_k\}$ has spectral density

$$f(x) = (\sigma^2_\xi/2\pi)|1 - B(\mathrm{e}^{-\mathrm{i}x})|^{-2} = (\sigma^2_\xi/2\pi)|1 - \mathrm{e}^{-\mathrm{i}x}|^{-2d} |h(x)|^{-2}, \quad x \in \Pi \tag{2.20}$$

with h defined at (2.19), that satisfies $f(x) \sim (\sigma^2_\xi/2\pi)|x|^{-2d}, x \to 0$, and is a continuous bounded function on intervals $[\epsilon, \pi], \epsilon > 0$. Moreover, using (2.20), (2.19), (2.18) and Lemma 2.3.1 in Giraitis *et al.* (2012), one can show that

$$\mathrm{cov}(x_0, x_k) \sim c_\gamma k^{-1+2d}, \tag{2.21}$$

with c_γ as in (2.15) for ARFIMA$(0, d, 0)$ model. Hence, the p_j's are essentially parameters of short run dynamics and do not have an impact on the long-run behavior of $\mathrm{cov}(x_0, x_k)$ and asymptotic of $f(x)$ as $x \to 0$.

IAR(q, d) model. Finally, let us remark that the IAR(∞) model given by equations (2.12), (2.17) with polynomial $P(L)$ defines a parametric class of long memory processes that are different from ARFIMA(p, d, q) models. For example, $B(z) = (1 - (1 - z)^d)(1 + rz)/(1 + r)$ generates a different covariance structure than ARFIMA$(1, d, 0)$ model $(1 - L)^d(1 + rL)x_k = \xi_k$.

It is convenient to parameterize $P(L)$ as

$$P(L) = \frac{1 + r_1 L + ... + r_q L^q}{1 + r_1 + ... r_q}, \quad r_1 \ge 0, \dots, r_q \ge 0. \tag{2.22}$$

Then coefficients $p_i := r_i/(1 + r_1 + ... + r_q)$, $i = 1, ..., q$ and $p_0 := 1/(1 + r_1 + ... r_q)$ are nonnegative and have property $p_0 + p_1 + ... + p_q = 1$, so this model is a separate case of (2.17). For $d \in (0, 1/2)$ it has a stationary long memory solution with the spectral density (2.20) satisfying $f(x) \sim (\sigma^2_\xi/2\pi)|x|^{-2d}$, $x \to 0$ and $\mathrm{cov}(x_0, x_k)$ satisfying (2.21).

Observe that an IAR$(0, d)$ process is an ARFIMA$(0, d, 0)$ process.

Remark 2.2 The well-known ARFIMA(p, d, q) model is obtained from the IAR ARFIMA$(0, d, 0)$ equation (2.12) by replacing $\{\xi_k\}$ by a stationary ARMA(p, q) process $\xi_j = A(L)^{-1}B(L)\eta_j$. Here, $\{\eta_j\}$ is an uncorrelated stationary noise, $E\eta_k = 0$, $E\eta_k^2 = \sigma_\eta^2$ and the polynomials $A(z)$, $B(z)$ do not have zeroes on the complex disk $\{|z| \le 1\}$.

In the IAR(p, d) model, parameterizing weights b_j by $P(L)$ changes the correlations $corr(x_k, x_0)$ at finite lags but the asymptotics of $corr(x_k, x_0)$ remain the same as for ARFIMA$(0, d, 0)$. In ARFIMA(p, d, q), instead of b_j's we are modelling the persistence of the noise $\{\xi_j\}$. The latter has strong impact on $corr(x_k, x_0)$ for large lags, distorting the long memory behaviour generated by b_j and controlled by d.

Example 2.1 Figures 2-5 contain realisations, sample and theoretical ACF's and the spectral densities of

- IAR$(1, d)$ process $x_k = B(L)x_k + \xi_k$, $B(L) = (1 - (1 - z)^d)(1 + rz)/(1 + r)$,
- ARFIMA$(0, d, 0)$ process $(1 - L)^d x_k = \xi_k$,
- ARFIMA$(1, d, 0)$ process $(1 - L)^d(1 - rL)x_k = \xi_k$

where $d = 0.3$, $r = 0.5$ and $\{\xi_k\}$ is standard normal i.i.d. noise. They confirm theoretical findings on long memory properties of IAR$(1, d)$ processes.

It can be seen from Figure 2 that, although realizations of IAR$(1, d)$, ARFIMA$(0, d, 0)$ and ARFIMA(1,d,0) processes are rather similar, persistence slightly increases for ARFIMA$(1, d, 0)$ processes. Different values of parameter r in the IAR$(1, d)$ model lead to different shapes of ACF at very low lags, for example, increasing r would form a peak in ACF at lag 2. Due to the impact of the parameter r on short-range dependence, the ACF is notably lower at lag 1 for IAR$(1, d)$ model compared with ARFIMA$(0, d, 0)$, see Figure 4. This illustrates the important inherent feature of IAR(q, d) models - coefficient r or, more generally, coefficients of polynomial $P(L)$) - are mostly associated with the short-range dependence structure of IAR process, while having a minor impact on long-run behaviour of ACF, in contrast to ARFIMA models where parameters r and d tend to duplicate each other, and therefore it might be hard to separate them in finite samples.

3 ARCH(∞) model and bilinear equation

In this section, we derive important auxiliary results. Our objective is to rewrite the ARCH(∞) process $\{\tau_k\}$ of (1.1) as a bilinear process driven by an i.i.d. zero mean noise and analyse the existence of its stationary solution. In Section 4 we use solutions of bilinear model to find stationary solutions for ARCH(∞) model

$$\tau_k \;=\; \varepsilon_k\big(\omega + \sum_{j=1}^{\infty} b_j\tau_{k-j}\big), \quad k \in \mathbb{Z}. \tag{3.1}$$

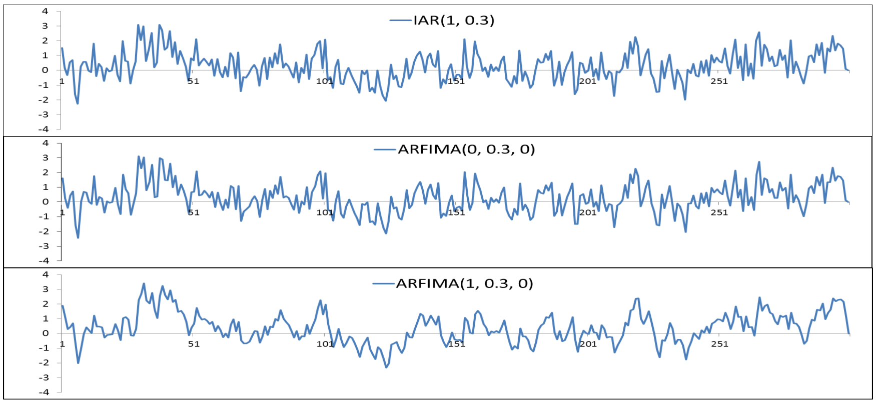

Figure 2: Realizations of $ARFIMA(1, 0.3)$, $ARFIMA(0, 0.3, 0)$, $ARFIMA(1, 0.3, 0)$ with $r = 0.5$

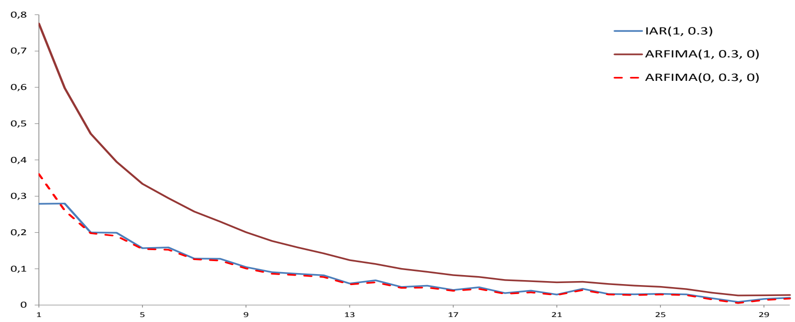

Figure 3: Sample ACF of $ARFIMA(1, 0.3)$, $ARFIMA(0, 0.3, 0)$, $ARFIMA(1, 0.3, 0)$ with $r = 0.5$, sample size $n = 1000$

Specifically, for a stationary ARCH(∞) process in (3.1) with mean $E\tau_k = \mu$, we set

$$\tau_k = (\tau_k - \mu) + \mu = Y_k + \mu, \quad Y_k := \tau_k - \mu.$$

Recall $\theta = \sum_{j=1}^{n} b_j$. We focuss on two cases, a) $\omega > 0$ and $0 < \theta < 1$, and b) $\omega = 0$ and $\theta = 1$. In case a), equation (1.5) implies $\mu = E\tau_k = \omega/(1-\theta)$, while in case b), it does not contradict a free choice of $\mu > 0$. Motivated by these facts, put

$$\mu := \begin{cases} \omega/(1-\theta) & \text{if } \theta < 1, \text{ and } \omega > 0, \\ \text{any positive number } \mu > 0 & \text{if } \theta = 1 \text{ and } \omega = 0. \end{cases}$$

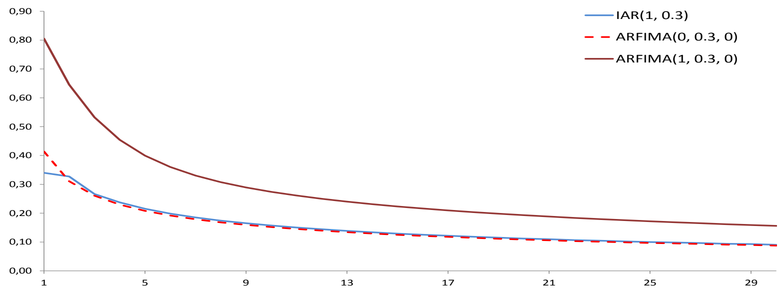

Figure 4: True ACF of $ARFIMA(1, 0.3)$, $ARFIMA(0, 0.3, 0)$, $ARFIMA(1, 0.3, 0)$ with $r = 0.5$

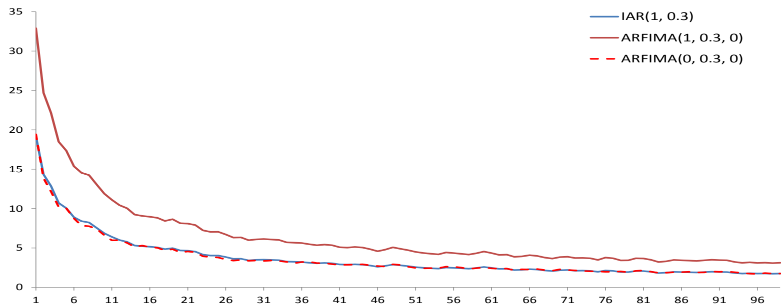

Figure 5: Spectral density of $ARFIMA(1, 0.3)$, $ARFIMA(0, 0.3, 0)$, $ARFIMA(1, 0.3, 0)$ with $r = 0.5$ computed averaging the periodogram at Fourier frequencies u_j, $j = 1, ..., n = 250$.

Assume $\sigma^2 := \mathrm{var}(\varepsilon_1) < \infty$ and let $\{\zeta_k := (\varepsilon_k - 1)/\sigma,\ k \in \mathbb{Z}\}$ be the centered i.i.d. noise. With this notation, the ARCH equation of (3.1) can be written as the bilinear equation

$$Y_k = \sum_{j=1}^{\infty} b_j Y_{k-j} + \zeta_k \Big(\mu\sigma + \sigma \sum_{j=1}^{\infty} b_j Y_{k-j} \Big), \tag{3.2}$$

see also Giraitis and Surgailis (2002). By introducing the generating function

$$H(z) := \frac{\sigma B(z)}{1 - B(z)} = \sum_{j=1}^{\infty} h_j z^j, \qquad |z| < 1, \tag{3.3}$$

and putting $z_k := (1 - B(L))Y_k$, (3.2) can be further rewritten as a system of two equations:

$$(a)\ \ Y_k = \sum_{j=1}^{\infty} b_j Y_{k-j} + z_k, \qquad (b)\ \ z_k = \zeta_k \big(\mu\sigma + \sum_{j=1}^{\infty} h_j z_{k-j} \big). \tag{3.4}$$

Note that the second equation in (3.4) does not contain the Y_k's and coincides with the so-called LARCH model studied in Giraitis et al. (2000b, 2004) and elsewhere. Also note that $\{z_k\}$ is a martingale difference sequence and can be written as

$$z_k = \zeta_k v_k, \qquad v_k = \mu\sigma + \sum_{j=1}^{\infty} h_j z_{k-j}, \tag{3.5}$$

where v_k is interpreted as volatility while the first equation is an IAR equation studied in Section 2. A stationary solution $\{z_k\}$ of the second equation in (3.4) is constructed in terms of causal Volterra series in i.i.d. innovations $\zeta_s, s \le k$, see (3.9), (3.10) below, also Giraitis *et al.* (2000b). Thus, the first equation in (3.4) in the integrated case $\theta = \sum_{j=1}^{\infty} b_j = 1$ represents a particular case of the IAR(∞) model with causal albeit dependent uncorrelated noise $\{z_k\}$ discussed in Theorem 2.1 of the previous section. Accordingly, the stationary solution of the bilinear equation (3.2) is obtained as a solution

$$Y_k = \sum_{j=0}^{\infty} g_j z_{k-j} \tag{3.6}$$

of the IAR equation with martingale difference innovations z_{k-j}'s determined by the second equation in (3.4), or (3.5), and with g_j's as in (2.2) which satisfy $h_j = \sigma g_j$, $j \ge 1$. This seems the most simple way to construct and solve the bilinear equation (3.2) and the ARCH equation (3.1), too.

In the rest of paper, by "causal" we mean a stationary process $\{y_k\}$ written as a measurable function of the present and past values $\zeta_s, s \le k$ or, equivalently, $\varepsilon_s, s \le k$. Similarly as in Definition 2.1, by an L_2-solution of equations (3.1), (3.2), (3.4) we mean a random process with finite second moment such that all series in these equations converge in mean square and the corresponding equations hold for each $k \in \mathbb{Z}$.

The following proposition establishes the relation between solutions of (1.1), (3.2) and (3.4) with ε_k and ζ_k related by $\varepsilon_k = \sigma\zeta_k + 1$, and $\omega = \mu(1-\theta)$. For Y_k in (3.2), we define the "noise" as $z_k := \zeta_k(\mu\sigma + \sigma\sum_{j=1}^{\infty} b_j Y_{k-j})$. For z_k in (3.4), the volatility process v_k is defined in (3.5).

Proposition 3.1 *Let* $0 < \mu < \infty$ *and* $\theta \in (0, 1]$.
(i) *If* $\{\tau_k\}$ *is a causal* L_2-*solution of (1.1) then* $\{Y_k := \tau_k - \mu\}$ *is a causal* L_2-*solution of (3.2) such that* $Y_k \ge -\mu$.
(ii) *If* $\{Y_k\}$ *is a causal* L_2-*solution of (3.2) such that* $Y_k \ge -\mu$, *then* $\{\tau_k := Y_k + \mu\}$ *is a causal* L_2-*solution of equation (1.1).*
(iii) $\{Y_k\}$ *is a causal* L_2-*solution of bilinear equation (3.2) if and only if* $\{Y_k, z_k\}$ *is a causal* L_2-*solution of equation (3.4). Moreover,* $\{Y_k \ge -\mu\}$ *is equivalent to* $\{v_k \ge 0\}$ *with* v_k *as in (3.5).*

Proof. The equivalence of (i) and (ii) is immediate. It remains to prove (iii). Let $\{Y_k\}$ be a causal L_2-solution of (3.2), $z_k := \zeta_k(\mu\sigma + \sigma\sum_{j=1}^\infty b_j Y_{k-j})$ and $v_k := \mu\sigma + \sum_{j=1}^\infty h_j z_{k-j}$. Let us prove that $\{Y_k, z_k\}$ is a causal L_2-solution of (3.4). This follows from (3.2) and equality

$$v_k = \mu\sigma + \sigma \sum_{j=1}^{\infty} b_j Y_{k-j} \tag{3.7}$$

which is verified below. From the definition of z_k and (3.2) it follows that the Y_k's satisfy the IAR equation $Y_k - \sum_{j=1}^\infty b_j Y_{t-j} = z_k$ where $\{z_k\}$ is a causal uncorrelated process with finite variance. Therefore by Theorem 2.1 we have $Y_k = \sum_{j=0}^\infty g_j z_{k-j}$, and $\sigma\sum_{j=1}^\infty b_j Y_{k-j} = \sigma\sum_{j=1}^\infty b_j \sum_{i=0}^\infty g_i z_{k-j-i} = \sum_{j=1}^\infty h_j z_{k-j}$ follows from the definition of h_j in (3.3), proving (3.7) or the fact that $\{Y_k, z_k\}$ is a causal L_2-solution of (3.4). Moreover, $Y_{k-j} \ge -\mu$ and (3.7) imply $v_k \ge \mu\sigma + \sigma(\sum_{j=1}^\infty b_j)(-\mu) = \mu\sigma(1-\theta) \ge 0$.

Conversely, assume that $\{Y_k, z_k\}$ is a causal L_2-solution of (3.4). Then the fact that $\{Y_k\}$ is a causal L_2-solution of (3.2) follows from (3.7) which in turn follows from Theorem 2.1 exactly as above. Finally, from $v_k \ge 0$, (3.7), (3.2) and $\zeta_k \ge -1/\sigma$ we obtain

$$\begin{aligned} Y_k &= \sum_{j=0}^{\infty} b_j Y_{k-j} + \zeta_k v_k \;\ge\; \sum_{j=0}^{\infty} b_j Y_{k-j} - (1/\sigma) v_k \\ &= \sum_{j=0}^{\infty} b_j Y_{k-j} - (1/\sigma)(\mu\sigma + \sigma \sum_{j=0}^{\infty} b_j Y_{k-j}) \;=\; -\mu, \end{aligned}$$

proving part (iii) and the proposition. □

Solution of bilinear equation. The solution $\{Y_k\}$ of equation (3.2) can be obtained by finding a solution, $\{z_k\}$, of the LARCH equation (3.4)(b), and then using it to solve AR equation (3.4)(a) to find $\{Y_k\}$.

The existence of stationary L_2-solution of LARCH equation (3.4)(b) was discussed in Giraitis, Robinson and Surgailis (2000b). As shown in this paper, a necessary and sufficient condition for the existence of such solution is $\|h\|^2 = \sum_{j=1}^\infty h_j^2 < 1$. Using the fact that $h_j = \sigma g_j$, $j \ge 1$, this condition can be written as

$$\|h\|^2 \;=\; \sigma^2 \sum_{j=1}^{\infty} g_j^2 \;<\; 1, \tag{3.8}$$

where the g_j's are obtained using the generating function (2.2), $G(z) = (1 - B(z))^{-1} = \sum_{j=0}^\infty g_j z^j$. Under (3.8), the solution z_k of LARCH equation is written as the sum

$$z_k \;=\; \zeta_k v_k, \qquad v_k := (\mu\sigma) \sum_{m=0}^{\infty} v_k^{(m)}, \tag{3.9}$$

of Volterra series of order m

$$v_k^{(m)} := \sum_{s_m < \cdots < s_1 < s} h_{s-s_1} h_{s_1-s_2} \cdots h_{s_{m-1}-s_m} \zeta_{s_1} \cdots \zeta_{s_m}, \qquad m \ge 1, \qquad v_k^{(0)} := 1 \tag{3.10}$$

which converge in mean square under (3.8), have zero mean $Ev_k^{(m)} = 0$ and variance

$$Ev_k^{(m)\,2} = \sum_{s_m < \cdots < s_1 < s} h^2_{s-s_1} h^2_{s_1-s_2} \cdots h^2_{s_{m-1}-s_m} = \|h\|^{2m} < \infty.$$

By orthogonality of the summands in (3.9), it follows

$$\sigma_z^2 := Ez_k^2 = Ev_k^2 = (\mu\sigma)^2\big(1 + \sum_{m=1}^{\infty} \|h\|^{2m}\big) = (\mu\sigma)^2/(1 - \|h\|^2) < \infty. \tag{3.11}$$

The corresponding solution Y_k of AR equation (3.4)(a) is then written as a moving-average of martingale difference innovations $z_s, s \le k$

$$Y_k = \sum_{j=0}^{\infty} g_j z_{k-j}, \tag{3.12}$$

where $E[z_k|\zeta_s, s < k] = 0$, $E[z_k^2|\zeta_s, s < k] = v_k^2$. With (3.9), (3.10) in mind, Y_k in (3.12) can be rewritten as the Volterra series

$$Y_k = (\mu\sigma) \sum_{m=1}^{\infty} \Big(\sum_{s_m < \cdots < s_1 \le k} g_{k-s_1} h_{s_1-s_2} \cdots h_{s_{m-1}-s_m} \zeta_{s_1} \cdots \zeta_{s_m} \Big). \tag{3.13}$$

The solution of bilinear equation in (3.12) is causal (in particular, ergodic), has zero mean $EY_k = 0$, and covariance-correlation functions

$$\mathrm{cov}(Y_0, Y_k) = \sigma_z^2 \sum_{j=0}^{\infty} g_j g_{k+j}, \qquad \mathrm{corr}(Y_0, Y_k) = \frac{\sum_{j=0}^{\infty} g_j g_{k+j}}{\sum_{j=0}^{\infty} g_j^2}, \quad k \ge 0. \tag{3.14}$$

Assumption (3.8), $\|h\| < 1$, mainly guarantees that solution $\{Y_k\}$ has finite variance. Solution (3.12) has spectral density

$$f(x) = \frac{\sigma_z^2}{2\pi} \Big| \sum_{j=0}^{\infty} g_j \mathrm{e}^{-\mathrm{i}jx} \Big|^2 = \frac{\sigma_z^2}{2\pi} |1 - \sum_{j=1}^{\infty} b_j \mathrm{e}^{-\mathrm{i}jx}|^2, \qquad x \in \Pi,$$

in view of equality $A(x) = (1 - B(\mathrm{e}^{-\mathrm{i}x}))^{-1} = \sum_{j=0}^{\infty} g_j \mathrm{e}^{-\mathrm{i}jx}$ proved in Lemma 5.3(a), and with σ_z^2 defined in (3.11).

The following theorem establishes sufficient and necessary conditions for the existence of a causal L_2-solution of the ARCH(∞) equation (3.1) and bilinear equations (3.2), (3.4). Recall that $\theta = \sum_{k=1}^{\infty} b_k$, $\sigma^2 = \mathrm{var}(\varepsilon_1)$, $\|g\|^2 = \sum_{j=0}^{\infty} g_j^2$ and $\|A\|^2 = \int_{\Pi} |A(x)|^2 \mathrm{d}x$.

Theorem 3.1 (a) *Bilinear equation (3.1) has a nontrivial causal L_2-solution if and only if*

$$\|g\|^2 < (1 + \sigma^2)/\sigma^2. \tag{3.15}$$

Condition (3.15) is equivalent to $\|A\|^2 < 2\pi(1 + \sigma^2)/\sigma^2$.

(b) *Let* (3.15) *be satisfied. Then*

(i) *If* $\omega > 0, 0 < \theta < 1$ *then ARCH equation (3.1) has a unique causal* L_2*-solution* $\{\tau_k\}$ *given by* $\tau_k = \mu + Y_k$*, where* $\{Y_k\}$ *are defined in (3.12) and* $\mu = \omega/(1-B)$*.*

(ii) *If* $\omega = 0, \theta = 1$ *then (3.1) has infinite number of causal* L_2*-solutions. Each such solution* $\{\tau_k\}$ *with* $E\tau_k = \mu > 0$ *is unique and has the form* $\{\tau_k = \mu + Y_k\}$*, where* Y_k *are defined in (3.12) with* $\{z_k\}$ *given in* (3.9)*.*

Proof. (a) Note first that (3.15) is equivalent to (3.8) while the equivalence of (3.15) and $\|A\|^2 < 2\pi(1+\sigma^2)/\sigma^2$ follows from Lemma 5.3(a).

Let us prove the necessity of condition (3.8). Assume that $\{\tau_k\}$ is L_2-solution of ARCH equation (3.1). By Proposition 3.1(i), the last fact implies that $\{Y_k := \tau_k - \mu, z_k := \zeta_k(\mu\sigma + \sigma\sum_{j=1}^{\infty} b_j Y_{k-j})\}$ is an L_2-solution of bilinear equation (3.4). Consequently,

$$\sigma_z^2 = Ez_k^2 = E\big(\mu\sigma + \sum_{j=1}^{\infty} h_j z_{k-j}\big)^2 = (\mu\sigma)^2 + (\sum_{j=1}^{\infty} h_j^2)\sigma_z^2 > (\sum_{j=1}^{\infty} h_j^2)\sigma_z^2$$

yielding (3.8).

Conversely, let us show that (3.8) implies the existence of L_2-solution $\{\tau_k\}$ of (1.1) with $E\tau_k = \mu$ given by $\tau_k = Y_k + \mu$, with Y_k defined in (3.13). As shown above, (3.8) guarantees that the above $\{Y_k\}$ is an L_2-solution of (3.2). Therefore, by Proposition 3.1(ii), it suffices to prove that

$$Y_k \geq -\mu. \tag{3.16}$$

To show (3.16), we approximate Y_k in (3.13) by

$$Y_{k,p} := (\mu\sigma)\sum_{m=1}^{\infty}\Big(\sum_{p<s_m<\cdots<s_1\leq k} g_{k-s_1}h_{s_1-s_2}\cdots h_{s_{m-1}-s_m}\zeta_{s_1}\cdots\zeta_{s_m}\Big),$$

where $p \geq 1$ is a large integer. Observe that for $k > p$ the $Y_{k,p}$'s satisfy equation (3.2), viz.,

$$Y_{k,p} = \zeta_k\Big(\mu\sigma + \sigma\sum_{j=1}^{\infty} b_j Y_{k-j,p}\Big) + \sum_{j=1}^{\infty} b_j Y_{k-j,p}, \qquad \text{for } k > p, \tag{3.17}$$

while $Y_{k,p} = 0$ for $k \leq p$. Moreover, by orthogonality of Volterra series,

$$E(Y_k - Y_{k,p})^2 = (\mu\sigma)^2\sum_{m=1}^{\infty} J_{k,p}^{(m)}, \quad J_{k,p}^{(m)} := \sum_{s_m<\cdots<s_1\leq k,\, s_m\leq p} g_{k-s_1}^2 h_{s_1-s_2}^2\cdots h_{s_{m-1}-s_m}^2.$$

Note that $J_{k,p}^{(m)} \leq \|g\|^2\|h\|^{2(m-1)}$, where $\|h\| < 1$, and hence $\sum_{m=1}^{\infty} J_{k,p}^{(m)}$ is dominated by a converging series. Moreover, for each $m \geq 1$, $J_{k,p}^{(m)} \to 0$ as $p \to -\infty$. Hence, $\lim_{p\to-\infty} E(Y_k -$

$Y_{k,p})^2 = 0$ for any $k \in \mathbb{Z}$ by the dominated convergence theorem. Therefore, (3.16) follows if we show that for any $p \in \mathbb{Z}$,

$$Y_{k,p} \geq -\mu, \qquad k \in \mathbb{Z}. \tag{3.18}$$

To prove (3.18), we use induction on k. Clearly, (3.18) holds for $k \leq p$ because by definition $Y_{k,p} = 0 > -\mu$ for $k \leq p$. Also, (3.18) holds for $k = p+1$ since $Y_{p+1,p} = (\mu\sigma)\zeta_{p+1} \geq -\mu$ because $(\mu\sigma)\zeta_j = (\mu\sigma)(\varepsilon_j - 1)/\sigma \geq -\mu$, for $j \in \mathbb{Z}$. Let $k > p+1$. Assume by induction that (3.18) holds for all $k = k - j$, $j \geq 1$. Then, by (3.17) and the inductive assumption,

$$\begin{aligned} Y_{k,p} &= \zeta_k(\mu\sigma) + (\zeta_k\sigma + 1)\big(\sum_{j=1}^{\infty} b_j Y_{k-j,p}\big) \geq \zeta_k(\mu\sigma) + (\zeta_k\sigma + 1)(\sum_{j=1}^{\infty} b_j)(-\mu) \\ &\geq \zeta_k(\mu\sigma) + (\zeta_k\sigma + 1)(-\mu) \geq -\mu. \end{aligned}$$

This proves the induction step $k-1 \to k$ and (3.18), (3.16), too, thereby proving part (a).

(b) Claim (i) follows from Giraitis and Surgailis (2002), Thm.3.1. Let us prove (ii). By Proposition 3.1, it suffices to show the uniqueness of solution $\{Y_k\}$ in (3.4) with $\{z_k\}$ as in (3.9)-(3.10). Since the above $\{z_k\}$ is regular and $\{Y_k\}$ is causal, the uniqueness follows from Theorem 2.1. Theorem 3.1 is proved. □

4 Stationary solutions of IARCH, FIGARCH and ARCH equations

We now are ready to establish the existence of stationary IARCH and FIGARCH processes with finite variance. Theorem 4.1 and Corollary 4.1 below show that covariance stationary IARCH and FIGARCH processes exist, and that long memory is their inherited feature. The proofs are based on results obtained for a bilinear model (3.2) in Section 3.

The results of the previous section lead to the following conclusions about stationary solutions of the IARCH, FIGARCH and ARCH(∞) equations. The stationary solution $\{\tau_k\}$ with mean $E\tau_k = \mu$ in Theorem 3.1 can be represented as

$$\tau_k = \mu + Y_k, \quad k \in \mathbb{Z}, \tag{4.1}$$

where $\{Y_k\}$ has mean $EY_k = 0$ and is given by

$$\begin{aligned} Y_k &:= \sum_{s=0}^{\infty} g_s z_{k-s}, \qquad z_s := \zeta_s v_s, \\ v_s &:= \mu\sigma^{m+1} \textstyle\sum_{m=1}^{\infty} \big(\sum_{s_m < \cdots < s_1 < s} g_{s-s_1} g_{s_1-s_2} \cdots g_{s_{m-1}-s_m} \zeta_{s_1} \cdots \zeta_{s_m}\big). \end{aligned} \tag{4.2}$$

The sequence $\{z_s = \zeta_s v_s\}$ in (4.2) is a stationary ergodic martingale difference sequence with zero mean and variance $\sigma_z^2 = Ez_s^2 := (\mu\sigma)^2/\big(1 - \sigma^2(\|g\|^2 - 1)\big)$, and assumption $\sigma_z^2 < \infty$ is equivalent to

$$\|g\|^2 \le (1+\sigma^2)/\sigma^2. \tag{4.3}$$

The following theorem shows that under assumption (4.3) there exist an infinite number of IARCH solutions τ_k parameterised by the mean value $\mu > 0$. Part (iii) of the theorem shows that IARCH process always has long memory, viz. its auto-covariance is not summable, and spectral density is unbounded at zero frequency.

Theorem 4.1 *The IARCH equation (1.2) has a non-trivial stationary causal L_2-solution if and only if $\sigma^2 = \mathrm{var}(\varepsilon_1)$ and b_j's satisfy condition (4.3). In the latter case,*
(i) *For each $\mu > 0$, the process $\{\tau_k\}$ in* (4.1) *is a unique causal L_2-solution of (1.2) with mean $E\tau_k = \mu$.*
(ii) *The covariance function of $\{\tau_k\}$ in* (4.1) *is given by*

$$\mathrm{cov}(\tau_0, \tau_k) = \sigma_z^2 \sum_{j=0}^{\infty} g_j g_{k+j}. \tag{4.4}$$

(iii) *The covariance function in* (4.4) *is nonnegative: $\mathrm{cov}(\tau_0, \tau_k) \ge 0$ and nonsummable: $\sum_{k\in\mathbb{Z}} \mathrm{cov}(\tau_0, \tau_k) = \infty$. Moreover, $\{\tau_k\}$ has spectral density $f(x) = (\sigma_z^2/2\pi)|1 - B(\mathrm{e}^{-\mathrm{i}x})|^{-2}$, $x \in \Pi$ that is unbounded at the zero frequency.*

Proof. All claims with the exception of (iii) follow from Theorem 3.1, and claim (iii) follows from Theorem 2.1(iii). □

Remark 4.1 The above corollary together with Lemma 5.2(a) and Lemma 5.3(c) imply that that the IARCH model in (1.2) with $\omega = 0$ does not have a stationary solution with finite variance if the b_j's tend to zero fast enough: exponentially or vanish for j large enough, or decay at rate $b_j = O(j^{-\gamma})$ for some $\gamma > 3/2$. In contrast, the sufficient conditions for the existence of a stationary IARCH process with non-zero intercept $\omega > 0$ and infinite mean $E\tau_k = \infty$ obtained in Kazakevičius and Leipus (2003) and Douc *et al.* (2006) require an exponential decay of b_j's as $j \to \infty$, or the opposite property.

The next corollary details the case of the FIGARCH equation in (1.3). It establishes the existence of stationary long memory FIGARCH processes and shows that their covariance function $\mathrm{cov}(\tau_k, \tau_0)$ decays to zero hyperbolically slowly as in (4.6). Recall that $\sigma^2 = \mathrm{var}(\varepsilon_1)$.

Corollary 4.1 *For the FIGARCH model in (1.3) with $d \in (0, 1/2)$, condition (4.3) is equivalent to*

$$E\varepsilon_0^2 < \frac{\Gamma(1-2d)}{\Gamma(1-2d) - \Gamma^2(1-d)}. \tag{4.5}$$

Under (4.5), the statements of Theorem 3.1 (i) and (ii)(b) hold. Moreover, as $k \to \infty$, the covariance and spectral density of the FIGARCH process $\{\tau_k\}$ with $E\tau_k = \mu$ satisfy

$$\operatorname{cov}(\tau_0, \tau_k) \sim \mu^2 c_\gamma k^{-1+2d} \tag{4.6}$$

where $c_\gamma = \sigma_z^2 \Gamma(1-2d)/\{\Gamma(d)\Gamma(1-d)\}$, $\sigma_z^2 = \sigma^2/\big(1+\sigma^2 - \sigma^2\Gamma(1-2d)/\Gamma^2(1-d)\big)$ *and*

$$f(x) = (\sigma_z^2/2\pi)|1-\mathrm{e}^{\mathrm{i}x}|^{-2d} \sim (\sigma_z^2/2\pi)|x|^{-2d}, \quad x \to 0. \tag{4.7}$$

Proof. From (2.15), we have $\|g\|^2 = \Gamma(1-2d)/\Gamma^2(1-d)$ yielding the equivalence of (4.3) and (4.5). The remaining claims of the corollary follow from Theorem 4.1 and (2.15). □

For comparison, Corollary 4.2 below recovers the results on the existence of a stationary finite variance solution of ARCH(∞) equation with $\theta = \sum_{j=1}^\infty b_j < 1$, obtained in GS(2002). As noted above, the existence of such a solution in this case necessary implies $E\tau_k = \mu = \omega/(1-\theta)$. In sharp contrast to finite variance stationary IARCH processes, which can have only long memory (see Theorem 4.1(iii)), a stationary finite variance process ARCH processes with $\theta < 1$ always has short memory, in the sense that its covariance function is non-negative and absolutely summable.

Corollary 4.2 *An ARCH(∞) equation in (1.1) with $\omega > 0$ and $\theta = \sum_{j=1}^\infty b_j < 1$ has a unique stationary causal L_2-solution if and only if condition (4.3) is satisfied. The above solution is given by $\tau_k = \mu + Y_k$, (4.1), with $\mu = \omega/(1-\theta)$. It has mean $E\tau_k = \mu = \omega/(1-\theta)$ and non-negative covariance function given in (4.4). Moreover,*

$$\sum_{k\in\mathbb{Z}} \operatorname{cov}(\tau_0, \tau_k) < \infty, \qquad \sum_{k\in\mathbb{Z}} g_k < \infty.$$

Proof. All statements with the exception of the last convergence follow from Theorem 3.1. To show it, note that $g_j \ge 0$ in (2.5) satisfy $\sum_{j=0}^\infty g_j \le \sum_{m=0}^\infty \theta^m < \infty$ since $\theta < 1$. □

The following proposition discusses the weak convergence in the Skorohod space $D[0,1]$, denoted by $\to_{D[0,1]}$, of the partial sums process of $\{\tau_k\}$. Below, $\{B(t), t\in[0,1]\}$ denotes a standard Brownian motion with variance $EB^2(t) = t$ and $\{B_{d+1/2}(t), t \in [0,1]\}$ a fractional Brownian motion with variance $EB^2_{d+1/2}(t) = t^{2d+1}$, $0 < d < 1/2$.

Proposition 4.1 *Suppose that (4.3) holds.*

(i) *Let $\omega > 0$, $\theta < 1$ and $\{\tau_k\}$ be the ARCH(∞) process in Corollary 4.2. Then*

$$n^{-1/2}\sum_{k=1}^{[nt]}(\tau_k - E\tau_k) \ \to_{D[0,1]} \ s^2 B(t), \qquad s^2 := \sum_{k\in\mathbb{Z}} \operatorname{cov}(\tau_0,\tau_k). \tag{4.8}$$

(ii) *Let $\{\tau_k\}$ be the FIGARCH process in Corollary 4.1. Then*

$$n^{-1/2+d}\sum_{k=1}^{[nt]}(\tau_k - E\tau_k) \ \to_{D[0,1]} \ s_d B_{d+1/2}(t), \quad s_d^2 := \mu^2 c_\gamma/(d(1+2d)). \tag{4.9}$$

Proof. (i) The convergence in (4.8) is a consequence of the summability $\sum_{k\in\mathbb{Z}} \mathrm{cov}(\tau_0, \tau_k) < \infty$, the associativity property of $\{\tau_k\}$ and the functional CLT by Newman and Wright (1981), see Giraitis *et al.* (2007) for details.
(ii) Using the fact that $\{\tau_k\}$ in (4.2) is a moving average of stationary ergodic martingale differences, (4.9) follows from Theorem 3.1 in Abadir *et al.* (2014) or Theorem 6.2 in Giraitis and Surgailis (2002). □

We end the paper with two examples of integrated ARCH(∞) processes.

Example 4.1 Autoregressive Conditional Duration (ACD) model of Engle and Russell (1998) of order 1 is given by

$$\tau_k \;=\; \varepsilon_k(\omega + b\tau_{k-1}), \tag{4.10}$$

where $\omega > 0$, $0 < b < 1$, $E\varepsilon_1 = 1$ and $\sigma^2 = var(\varepsilon_1) < \infty$. In this case we have that $B(z) = bz$, $G(z) = (1-bz)^{-1} = \sum_{j=0}^{\infty} b^j z^j$, $g_j = b^j$, $h_j = \sigma b^j$, $j \geq 1$ and $\|h\|^2 = \sum_{j=1}^{\infty} h_j^2 = \sigma^2 \sum_{j=1}^{\infty} b^{2j} = \sigma^2 b^2/(1-b^2)$. The corresponding bilinear equation (3.2) for $Y_k = \tau_k - EY_k$ writes as

$$Y_k \;=\; \zeta_k\big(\mu\sigma + \sigma b Y_{k-1}\big) + bY_{k-1}, \quad \mu = \omega/(1-b), \quad \zeta_k = (\varepsilon_k - 1)/\sigma.$$

Condition $\|h\| < 1$ of (3.8) becomes $\sigma^2 b^2/(1-b^2) < 1$, or $b^2 E\varepsilon_1^2 < 1$. By (3.14), the covariance of the ACD process in (4.10) is a multiple of that of AR(1):

$$cov(\tau_0, \tau_k) \;=\; \sigma_z^2 \sum_{j=0}^{\infty} g_j g_{k+j} = \sigma_z^2 \sum_{j=0}^{\infty} b^j b^{k+j} = a\, b^k, \quad k \geq 0,$$

where $a := (\omega\sigma)^2/(1-b^2(1+\sigma^2))$.

Example 4.2 The following extension of the FIGARCH model

$$\tau_k \;=\; \varepsilon_k B(L)\tau_k, \qquad B(L) := (1-(1-L)^d)\frac{1+rL}{1+r} \tag{4.11}$$

where $0 < d < 1/2$ and $r \geq 0$ are parameters, is a particular case of integrated autoregressive operators discussed in example (2.22).

Figures 6-9 contain realisations, sample and theoretical ACF's and the spectral density of the FIGARCH process $\tau_k = FIGARCH(0.2)$ and the long memory ARFIMA process $x_k \sim ARFIMA(0, 0.2, 0) + 1$ with $Ex_k = E\tau_k = 1$ which confirm our theoretical results on long memory properties of FIGARCH processes.

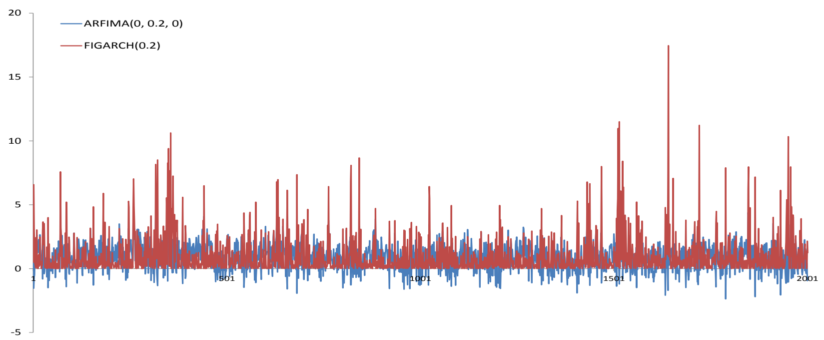

Figure 6: Realizations of $x_k \sim ARFIMA(0, 0.2, 0) + 1$, $\tau_k \sim FIGARCH(0.2)$ processes, $E\tau_k = 1$

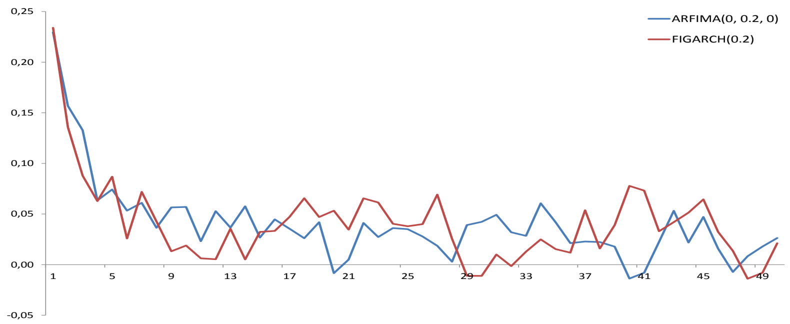

Figure 7: Sample ACF of $x_k \sim ARFIMA(0, 0.2, 0)+1$, $\tau_k \sim FIGARCH(0.2)$, sample size $n = 500$

5 Auxiliary lemmas

This section contains three auxiliary lemmas.

Lemma 5.1 *Let* $\theta = \sum_{j=1}^{\infty} b_j = 1$ *where* b_j*'s are non-negative.*
(i) *The function* $1 - B(\mathrm{e}^{\mathrm{i}x})$, $x \in \Pi$ *has only finite number of zeroes on* Π, *including* $x = 0$.
(ii) *The point* $x = 0$ *is the unique zero of* $1 - B(\mathrm{e}^{\mathrm{i}x})$ *provided either* $b_1 > 0$, *or* $b_k b_{k+1} > 0$ *for some* $k \geq 2$ *hold.*

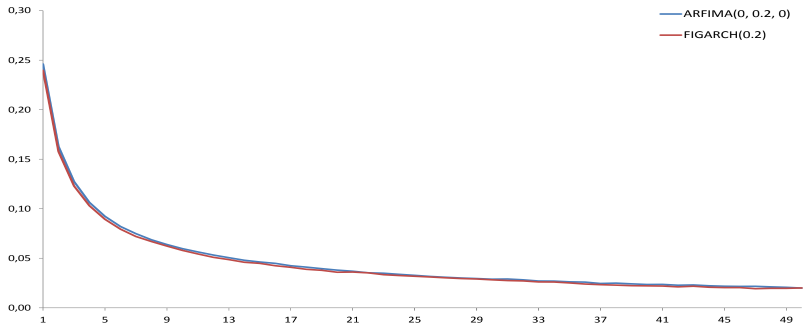

Figure 8: True ACF of $x_k \sim ARFIMA(0, 0.2, 0) + 1$, $\tau_k \sim FIGARCH(0.2)$

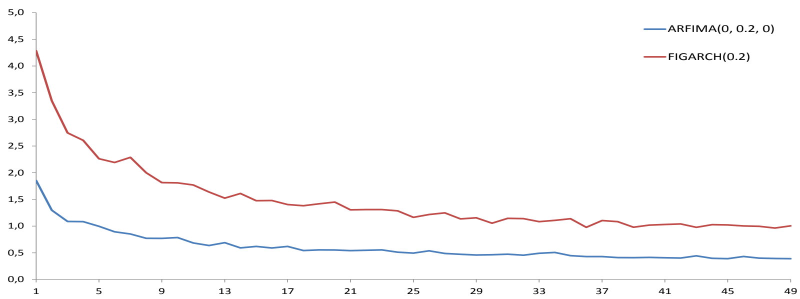

Figure 9: Spectral densities of $x_k \sim ARFIMA(0, 0.2, 0) + 1$, $\tau_k \sim FIGARCH(0.2)$ computed averaging the periodogram at Fourier frequencies u_j, $j = 1, ..., n = 250$.

Proof. (i) Suppose that $B(\mathrm{e}^{\mathrm{i}x_0}) = 1$ and $b_1 = \ldots = b_{p-1} = 0$, $b_p > 0$ for some $x_0 \in \Pi$, $p \geq 1$. We will prove that

$$B(\mathrm{e}^{\mathrm{i}x_0}) = \mathrm{e}^{\mathrm{i}px_0}. \tag{5.1}$$

Then, $1 - B(\mathrm{e}^{\mathrm{i}x_0}) = 1 - \mathrm{e}^{\mathrm{i}px_0} = 0$ yields $px_0 = 0 \pmod{2\pi}$. Hence, $x_0 \in \{0, \pm 2\pi/p, \ldots, \pm 2\pi k/p, 0 \leq k \leq p/2\}$, proving the first statement in (i).

To show (5.1), let $b_{p+1} = \ldots = b_{p'-1} = 0$, $b_{p'} > 0$. (If such $p' > p$ does not exist, then (5.1) holds trivially.) Then $B(\mathrm{e}^{\mathrm{i}x_0}) = \sum_{j=p}^{\infty} b_j \mathrm{e}^{\mathrm{i}jx_0} = \mathrm{e}^{\mathrm{i}px_0} \sum_{j=p}^{\infty} b_j \mathrm{e}^{\mathrm{i}(j-p)x_0} = \mathrm{e}^{\mathrm{i}px_0}(b_p + z + w)$, where $z := b_{p'} \mathrm{e}^{\mathrm{i}(p'-p)x_0}$ and $w := \sum_{j=p'}^{\infty} b_j \mathrm{e}^{\mathrm{i}(j-p)x_0}$. Write $z = b_{p'} \cos((p'-p)x_0) + \mathrm{i} b_{p'} \sin((p'-p)x_0) =: u + \mathrm{i}v$. Then,

$$1 = |B(\mathrm{e}^{\mathrm{i}x_0})| = |b_p + z + w| \leq |b_p + z| + |w| \leq b_p + |z| + |w| \leq \textstyle\sum_{j=1}^{\infty} b_j = 1.$$

This implies $|b_p+z| = b_p+|z|$, or $((b_p+u)^2+v^2)^{1/2} = b_p+(u^2+v^2)^{1/2}$. Taking squares of both sides gives $u = (u^2+v^2)^{1/2}$, or $u \geq 0, v = 0$. Hence, $z = b_{p'}$ and $B(\mathrm{e}^{\mathrm{i}x_0}) = \mathrm{e}^{\mathrm{i}px_0}(b_p + b_{p'} + w)$. Repeating the above argument, we obtain $w = b_{p'+1} + b_{p'+2} + \dots$ and consequently $B(\mathrm{e}^{\mathrm{i}x_0}) = \mathrm{e}^{\mathrm{i}px_0}(b_p + b_{p+1} + \dots) = \mathrm{e}^{\mathrm{i}px_0}(b_1 + b_2 + \dots) = \mathrm{e}^{\mathrm{i}px_0}$, proving (5.1) and part (i).

(ii) Let first $b_1 > 0$. Then (5.1) implies $1 - B(\mathrm{e}^{\mathrm{i}x_0}) = 1 - \mathrm{e}^{\mathrm{i}x_0} = 0$ and hence $x_0 = 0$. Next, if for some $k \geq 2$, $b_k b_{k+1} > 0$, then, with $z' := b_{k+1}\mathrm{e}^{\mathrm{i}x_0}$ and $w' := \sum_{j=1:\, j\neq k,k+1}^{\infty} b_j \mathrm{e}^{\mathrm{i}(j-k)x_0}$,

$$1 = |B(\mathrm{e}^{\mathrm{i}x_0})| = |b_k + z' + w'| \leq |b_k + z'| + |w'| \leq b_k + |z'| + |w'| \leq \textstyle\sum_{j=1}^{\infty} b_j = 1.$$

Hence, $|b_k + z'| = b_k + |z'|$, and the same argument as used in the proof of (5.1) implies that $z' = b_{k+1}$. Therefore $\mathrm{e}^{\mathrm{i}x_0} = 1$, or $x_0 = 0$, proving (ii). Lemma 5.1 is proved. □

Lemma 5.2 *Let $\theta = \sum_{j=1}^{\infty} b_j = 1$, where b_j's are non-negative. Assume that ξ_k is a stationary zero mean process which spectral density f_ξ is bounded away from 0 and ∞.*
Suppose that equation

$$x_k - \textstyle\sum_{j=1}^{\infty} b_j x_{k-j} = \xi_k, \qquad k \in \mathbb{Z}, \tag{5.2}$$

has a stationary zero mean solution x_k with $0 < Ex_k^2 < \infty$. Then,

(a) *$A(x) := (1 - B(\mathrm{e}^{\mathrm{i}x}))^{-1}$ is L_2 -integrable.*

(b) *$\|g\| < \infty$.*

Proof. (a) Let $f(x) = |1 - B(\mathrm{e}^{-\mathrm{i}x})|^{-2} f_\xi(x)$, $x \in \Pi$. Then, by the same argument as in the proof of Theorem 2.1(ii), the corresponding spectral measures F_x and F_ξ of stationary processes $\{x_k\}$ and $\{\xi_k\}$ satisfy

$$|1 - B(\mathrm{e}^{-\mathrm{i}x})|^2 F_x(\mathrm{d}x) = F_\xi(\mathrm{d}x) = f_\xi(x)\mathrm{d}x, \quad x \in \Pi.$$

By Lemma 5.1(i), $1 - B(\mathrm{e}^{-\mathrm{i}x}) = 0$ has finite number of zeros $x_1, ..., x_m$ in Π. Since F_x is non-decreasing, the only difference between F_x and F_ξ is a possible jump at the points $x_1, ..., x_m$, which yields equality $F_x(\mathrm{d}x) = f(x)\mathrm{d}x + \sum_{i=1}^{m} c_i \delta_{x_i}$ where $c_i \geq 0$ are some non-negative constants. Therefore, $\infty > Ex_k = \int_\Pi F_x(\mathrm{d}x) \geq \int_\Pi f(x)\mathrm{d}x \geq c \int_\Pi |1 - B(\mathrm{e}^{-\mathrm{i}x})|^{-2}\mathrm{d}x$, since by assumption $f_\xi(x) \geq c > 0$, $x \in \Pi$, for some $c > 0$, which proves (a).

(b) Next, we show that the existence of a stationary L_2-solution $\{x_k\}$ with the finite second moment $Ex_k^2 < \infty$ contradicts $\|g\| = \infty$. From Lemma 5.3(c), condition $\|A\| < \infty$ implies that $b_j > 0$ for infinite number of $j \geq 1$. For a large $p \geq 1$, denote $b'_j := b_j I(j \leq p)$ and

$$B'(L) := \textstyle\sum_{j=1}^{\infty} b'_j L^j, \qquad G'(L) := (1 - B'(L))^{-1} = \textstyle\sum_{j=0}^{\infty} g'_j L^j.$$

Observe that g'_j satisfy (2.5) with b_j's replaced by b'_j's. Rewrite (5.2) as

$$x_k - \textstyle\sum_{j=1}^{p} b_j x_{k-j} = \xi_k + u_k, \qquad \text{where} \quad u_k := \textstyle\sum_{j=p+1}^{\infty} b_j x_{k-j},$$

or $(1-B'(L))x_k = \xi_k + u_k$. Since $\sum_{j=1}^\infty b'_j < 1$, standard spectral argument implies that

$$x_k = G'(L)(\xi_k + u_k) = \xi'_k + u'_k, \quad \text{where} \quad \xi'_k := G'(L)\xi_k,\ u'_k := G'(L)u_k. \tag{5.3}$$

We claim that under $\|g\| = \infty$, as $p \to \infty$,

$$E(u'_k)^2 \le Ex_k^2 = O(1) \qquad \text{and} \qquad E(\xi'_k)^2 \to \infty. \tag{5.4}$$

On the other hand, since $Ex_k^2 < \infty$, then $E(\xi'_k)^2 = E(x_k - u'_k)^2 \le 2Ex_k^2 + 2E(u'_k)^2 = C < \infty$ which leads to contradiction.

To prove (5.4), notice that from the definition of u'_k as a linear filter and $F_x(\mathrm{d}x)$ the spectral measure of $\{x_k\}$,

$$E(u'_k)^2 = \int_\Pi \Big|\frac{\sum_{j=p+1}^\infty b_j \mathrm{e}^{-\mathrm{i}jx}}{1-\sum_{j=1}^p b_j \mathrm{e}^{-\mathrm{i}jx}}\Big|^2 F_x(\mathrm{d}x) \le \int_\Pi \Big|\frac{\sum_{j=p+1}^\infty b_j}{1-\sum_{j=1}^p b_j}\Big|^2 F_x(\mathrm{d}x) = \int_\Pi F_x(\mathrm{d}x) = Ex_k^2$$

proving the first relation in (5.4). The second relation follows by standard spectral argument using the lower bound $f_\xi(x) \ge c > 0$, $x \in \Pi$,

$$E(\xi'_k)^2 = (\sigma_\xi^2/2\pi)\int_\Pi |G'(\mathrm{e}^{-\mathrm{i}x})|^2 f_\xi(x)\mathrm{d}x \ge \frac{c\,\sigma_\xi^2}{2\pi}\int_\Pi |G'(\mathrm{e}^{-\mathrm{i}x})|^2 \mathrm{d}x = c\sigma_\xi^2 \sum_{j=0}^\infty {g'_j}^2$$

and the fact that the sum $\sum_{j=0}^\infty {g'_j}^2 \to \|g\|^2 = \infty$ tends monotonically to ∞ with $p \to \infty$. This completes the proof of the part (b) of the lemma. $\Box$

The following lemma shows that the analytic function $G(z) = \sum_{j=0}^\infty g_j z^j$, $|z| < 1$ extends to the unit circle $|z| = 1$, and such extension coincides with $A(x) := (1 - B(\mathrm{e}^{\mathrm{i}x}))^{-1}$, $x \in \Pi$ in L_2-norm. Recall notation $\|A\| = (\int_\Pi |A(x)|^2 \mathrm{d}x)^{1/2}$.

Lemma 5.3 *Let $\theta = \sum_{j=1}^\infty b_j = 1$, where b_j's are non-negative.*
(a) *If $\|g\| < \infty$, then $\|A\| < \infty$ and $A(x) = G(\mathrm{e}^{\mathrm{i}x})$ a.e. in Π.*
(b) *If $\|A\| < \infty$, then $\|g\| < \infty$.*
(c) *If $b_k = O(k^{-\gamma})$, $k \to \infty$, for some $\gamma \ge 3/2$, then $|1 - B(\mathrm{e}^{\mathrm{i}x})|^{-2}$ is not integrable on Π.*

Proof. (a) Let us show that $\|g\| < \infty$ implies $\|A\| < \infty$ and $A(x) = G(\mathrm{e}^{\mathrm{i}x})$ *a.e.* in Π. Let $\|g\| < \infty$ and $0 < r < 1$. Then $\int_\Pi |G(r\mathrm{e}^{\mathrm{i}x}) - G(\mathrm{e}^{\mathrm{i}x})|^2 \mathrm{d}x = 2\pi \sum_{j=0}^\infty g_j^2 |r^j - 1|^2 \to 0$ as $r \uparrow 1$ and therefore $\lim_{r\uparrow 1} G(r\mathrm{e}^{\mathrm{i}x}) = G(\mathrm{e}^{\mathrm{i}x})$ a.e. in Π implying $1 = \lim_{r\uparrow 1} G(r\mathrm{e}^{\mathrm{i}x})(1 - B(r\mathrm{e}^{\mathrm{i}x})) = G(\mathrm{e}^{\mathrm{i}x})(1 - B(\mathrm{e}^{\mathrm{i}x}))$ a.e. in Π. Since $1 - B(\mathrm{e}^{\mathrm{i}x}) \ne 0$ a.e. in Π (see Lemma 5.1(i)), this implies $G(\mathrm{e}^{\mathrm{i}x}) = (1 - B(\mathrm{e}^{\mathrm{i}x}))^{-1} \equiv A(x)$ a.e. in Π. Then by Parseval's identity, $(2\pi)^{-1}\|A\|^2 = \|g\|^2 < \infty$ which completes the proof of (a).

Proof of (b). Let $\|A\| < \infty$. We will show that $\|g\| < \infty$. For that we will construct IAR equation

$$y_k - \sum_{j=1}^\infty b_j y_{k-j} = \xi_k, \qquad k \in \mathbb{Z}, \tag{5.5}$$

where ξ_k's are i.i.d. random variables with zero mean and unit variance, and show that this equation has a stationary zero mean and finite variance solution, which by Lemma 5.2(b) implies $\|g\| < \infty$. To construct such solution, denote by $\xi_k = \int_\Pi e^{ikv} Z_\xi(dv)$ the spectral representation of $\{\xi_k\}$, and let

$$\tilde{y}_k = \int_\Pi e^{ikx} A(x) Z_\xi(dx), \quad k \in \mathbb{Z}. \tag{5.6}$$

Since $\|A\| < \infty$, $\{\tilde{y}_k\}$ is a stationary process with the spectral measure $F_{\tilde{y}}(dx) = |A(x)|^2 F_\xi(dx) = (2\pi)^{-1}|A(x)|^2 dx$, $E\tilde{y}_k = 0$ and variance $E\tilde{y}_k^2 = \int_\Pi F_{\tilde{y}}(dx) < \infty$. To show that $\tilde{y}_k$ is a solution of (5.5), observe that the function $1 - \sum_{j=1}^\infty b_j e^{-ijx}$ is bounded and, therefore, $L_2(F_{\tilde{y}})$-integrable. Then, by properties of spectral representation of stationary times series (see, Brockwell and Davis (1989, Thm.4.10.1) $\tilde{y}_k - \sum_{j=1}^\infty b_j \tilde{y}_{k-j} = \int_\Pi e^{ikx}(1 - \sum_{j=1}^\infty b_j e^{-ijx}) A(x) Z_\xi(dx) = \int_\Pi e^{ikx} Z_\xi(dx) = \xi_k$, and therefore $\tilde{y}_k$ is a solution of (5.5). This completes the proof of (b).

Proof of (c). We will prove that $|1 - B(e^{ix})| \le C|x|^{1/2}$ which implies (c): $\int_\Pi |1 - B(e^{ix})|^{-2} dx \ge C^{-1} \int_\Pi |x|^{-1} dx = \infty$. Recall that $\sum_{j=1}^\infty b_j = 1$, $b_j \le Cj^{-\gamma} \le Cj^{-3/2}$ and notice that by the mean value theorem $|1 - e^{ijx}| \le Cj|x|$. Thus, $|1 - B(e^{ix})| = |\sum_{j=1}^\infty b_j (1 - e^{ijx})| \le C \sum_{j=1}^{1/|x|} j^{-1/2}|x| + C \sum_{j \ge 1/|x|} j^{-3/2} \le C|x|^{1/2}$. This proves (c) and completes the proof of the lemma. □

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the School of Economics and Finance at
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