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## Fitting a Distribution to Survey Data for the Half-Life of Deviations from PPP

Mark Fisher

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**Abstract:** This note presents a nonparametric Bayesian approach to fitting a distribution to the survey data provided in Kilian and Zha (2002) regarding the prior for the half-life of deviations from purchasing power parity (PPP). A point mass at infinity is included. The unknown density is represented as an average of shape-restricted Bernstein polynomials, each of which has been skewed according to a preliminary parametric fit. A sparsity prior is adopted for regularization.

JEL classification: C11, C14, F31

Key words: nonparametric Bayesian estimation, Bernstein polynomials, simplex regression, importance sampling, PPP half-life deviations

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## 1. INTRODUCTION

Kilian and Zha (2002) present results from a survey of economists asking about prior beliefs for the half-life of deviations from purchasing power parity (PPP) for real exchange rates. The survey data are summarized in Table 1 and displayed in Figure 1. The numbers in the table are averages of the responses from 20 economists to a questionnaire.<sup>1</sup> The data are composed of  $n = 9$  pairs  $(h_i, y_i)$ , where  $y_i = \Pr[h \leq h_i]$  and  $h_i \in \{1, 2, 3, 4, 5, 6, 10, 20, 40\}$  (measured in years). Using the survey data, the authors estimate what they call a “consensus prior,” which they compute through the lens a monthly autoregressive model with 12 lags.

In this note I provide an alternative approach to estimating a smooth distribution from the survey data. I treat the problem as an exercise in Bayesian inference.<sup>2</sup> In particular, I take a Bayesian approach that involves nonparametric regression using Bernstein polynomials subject to shape restrictions.<sup>3</sup> The procedure can be thought of as providing flexible variation around a preliminary parametric fit.

There are two additional novelties regarding the distribution I compute, both of which are related to my own research on PPP.<sup>4</sup> First, I allow for a point mass at infinity. Second, I transform the distribution into a prior for the first-order autoregressive coefficient for annual observations.

## 2. THE MODEL

The model I adopt for the unknown distribution for the half-life  $h$  is a mixture of an atom located at infinity and a density over over the positive real line:

$$p(h|\theta_k, w) = \begin{cases} w & h = \infty \\ (1 - w) f(h|\theta_k) & h \in [0, \infty) \end{cases}, \quad (2.1)$$

where  $\Pr[h = \infty] = w$ . The density component in (2.1) is itself a mixture — a mixture of basis density functions:

$$f(h|\theta_k) := \sum_{j=1}^k \theta_{jk} f_{jk}(h), \quad (2.2)$$

where  $\theta_k = (\theta_{1k}, \dots, \theta_{kk})$  and  $\theta_k \in \Delta^{k-1}$ , the simplex of dimension  $k - 1$ .

The basis density functions are related to Bernstein polynomials. The idea can be found in Quintana et al. (2009), for example. Let  $Q(x)$  denote the cumulative distribution function (CDF) for a continuous random variable defined on the real line. Thus  $q(x) := Q'(x)$  is the probability density function (PDF). (For the half-life,  $Q(x) = 0$  for  $x \leq 0$ .) Define

$$f_{jk}(x) := \text{Beta}(Q(x)|j, k - j + 1) q(x), \quad (2.3)$$

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<sup>1</sup>The paper refers to “a survey of 22 economists.” However, one of the authors confirmed there were only 20 responses.

<sup>2</sup>An approach that is similar in spirit can be found in Gosling et al. (2007).

<sup>3</sup>Fisher (2015) places the approach taken here in the context of what he calls *simplex regression*.

<sup>4</sup>Dwyer and Fisher (2014).

TABLE 1. Survey prior probabilities for half-life.

	$h \leq 1$	$h \leq 2$	$h \leq 3$	$h \leq 4$	$h \leq 5$	$h \leq 6$	$h \leq 10$	$h \leq 20$	$h \leq 40$	$h > 40$
Percent	4.6	14.1	31.4	49.6	64.0	75.8	83.9	91.0	94.1	5.9

*Notes:* [This table replicates of Table I in Kilian and Zha (2002).] Average probabilities based on a survey of [20] economists with a professional interest in the PPP question. The survey was conducted by the authors in July and August 1999.

where  $1 \leq j \leq k \in \mathbb{N}$ . Note

$$\text{Beta}(x|a, b) = \frac{x^{a-1} (1-x)^{b-1}}{B(a, b)}, \quad (2.4)$$

where  $B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx$  is the beta function. Also note  $f_{jk}(x) \geq 0$  for  $x \in (-\infty, \infty)$  and

$$\int_{-\infty}^{\infty} f_{jk}(x) dx = 1. \quad (2.5)$$

Beta densities with integer coefficients can be interpreted as normalized Bernstein polynomial basis functions. With integer coefficients,

$$\text{Beta}(x|j, k-j+1) = \frac{k! x^{j-1} (1-x)^{k-j}}{(k-j)! (j-1)!}, \quad (2.6)$$

which is a polynomial of degree  $k-1$  in  $x$ . Bernstein polynomials have a number of useful properties that have led to their use in nonparametric estimations.<sup>5</sup> For example, the “adding-up” property of Bernstein polynomials amounts to

$$\sum_{j=1}^k \text{Beta}(x|j, k-j+1) = k. \quad (2.7)$$

This property delivers the following result:

$$\sum_{j=1}^k \frac{1}{k} f_{jk}(x) = q(x). \quad (2.8)$$

In particular note  $f_{11}(x) = q(x)$ .

**Cumulative distribution function.** In order to make contact with the survey data, we will need the cumulative distribution function associated with (2.1). To that end define

$$F(x|\theta_k) := \sum_{j=1}^k \theta_{jk} F_{jk}(x), \quad (2.9)$$

<sup>5</sup>See, for example, [http://en.wikipedia.org/wiki/Bernstein\\_polynomial](http://en.wikipedia.org/wiki/Bernstein_polynomial).

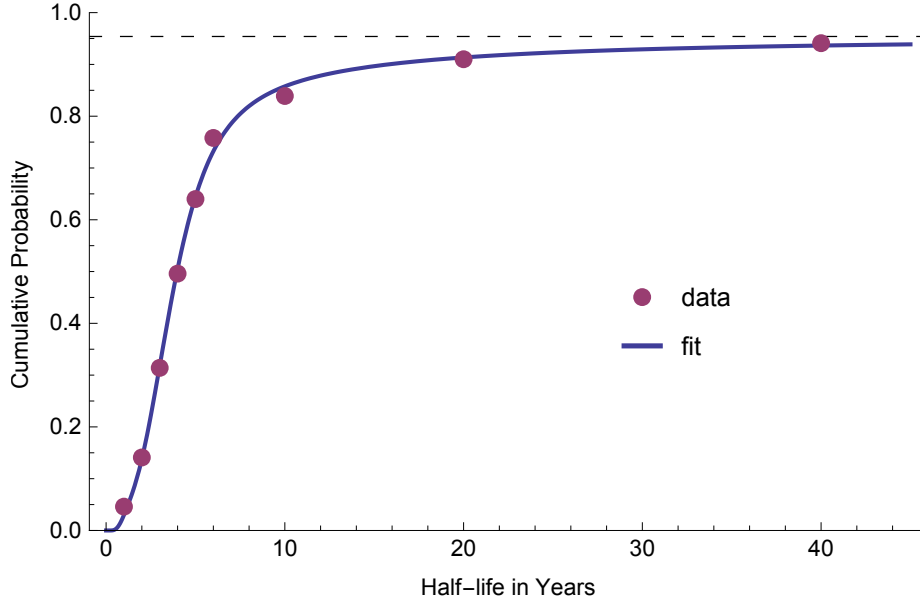


FIGURE 1. The survey data and the survey fit. The fit delivers a 4.6% chance that the half-life is infinite. The dashed line corresponds to the implied asymptote at 0.954.

where

$$\begin{aligned}
 F_{jk}(x) &:= \int_{-\infty}^x f_{jk}(t) dt = \int_{-\infty}^x \text{Beta}(Q(t)|j, k-j+1) q(t) dt \\
 &= \int_0^{Q(x)} \text{Beta}(t|j, k-j+1) dt \\
 &= I_{Q(x)}(j, k-j+1),
 \end{aligned} \tag{2.10}$$

where  $I_x(a, b)$  is the *regularized incomplete beta function*. The adding-up condition (2.8) implies

$$\sum_{j=1}^k \frac{1}{k} F_{jk}(x) = Q(x). \tag{2.11}$$

With (2.8) and (2.11) in mind, I refer to  $Q$  as the *centering function*. The centering function provides location and scale for the fit. Deviation of the weights  $\theta_k$  from uniform (i.e., deviations from  $\theta_{jk} = 1/k$ ) allow for variation around the centering function. Larger values of  $k$  provide greater flexibility.

**Degree elevation.** One of the properties of Bernstein polynomials is that of *degree elevation*, by which lower-degree polynomials can be represented exactly as higher degree polynomials. Degree elevation is useful for combing models with different values of  $k$ .

Applied to mixtures of Beta distributions, degree elevation implies that every mixture of order  $k_0$  can be represented as a mixture of  $k_1 > k_0$ . Define the  $k_1 \times k_0$  matrix

$$A^{k_1, k_0} := A^{k_1, k_1-1} A^{k_1-1, k_1-2} \dots A^{k_0+1, k_0}, \quad (2.12)$$

where the  $(k \times k - 1)$  matrix  $A^{k, k-1}$  is characterized by

$$A_{ij}^{k, k-1} = \begin{cases} 1 - (j/k) & j = i \\ j/k & j = i - 1 \\ 0 & \text{otherwise} \end{cases}. \quad (2.13)$$

In addition, define the row vector

$$f_k(x) := (f_{k1}(x), \dots, f_{kk}(x)). \quad (2.14)$$

One may confirm that

$$f_{k_1}(x) A^{k_1, k_0} \equiv f_{k_0}(x). \quad (2.15)$$

As a consequence (and treating  $\theta_k$  as a column vector),

$$\begin{aligned} f(x|\theta_{k_0}) &= f_{k_0}(x) \theta_{k_0} = (f_{k_1}(x) A^{k_1, k_0}) \theta_{k_0} \\ &= f_{k_1}(x) (A^{k_1, k_0} \theta_{k_0}) = f_{k_1}(x) \theta_{k_1} = f(x|\theta_{k_1}), \end{aligned} \quad (2.16)$$

where  $\theta_{k_1} = A^{k_1, k_0} \theta_{k_0}$ . For example,  $A^{k, 1} \theta_1 = (1/k, \dots, 1/k)^\top$ .

**Reparameterization.** It is convenient to reparameterize the model as follows.

Fix  $K \geq k$  and let

$$\phi = (1 - w) A^{K, k} \theta_k. \quad (2.17)$$

The model [see (2.1)] can be reexpressed as

$$p(h|\phi) = \begin{cases} 1 - \sum_{j=1}^K \phi_j & h = \infty \\ f(h|\phi) & h \in [0, \infty) \end{cases}, \quad (2.18)$$

since

$$1 - \sum_{j=1}^K \phi_j = w \quad \text{and} \quad f(h|\phi) \equiv (1 - w) f(h|\theta_k). \quad (2.19)$$

I will use (2.18) for estimation.

### 3. BAYESIAN APPROACH TO ESTIMATION

The goal is to compute the distribution  $p(h|y)$  for  $h$  conditional on  $y = (y_1, \dots, y_n)$  where the uncertainty regarding the latent variable  $\phi$  has been integrated out. Referring to (2.18), this distribution is given by

$$p(h|y) = \int p(h|\phi) p(\phi|y) d\phi = \begin{cases} 1 - \sum_{j=1}^K \bar{\phi}_j & h = \infty \\ f(h|\bar{\phi}) & h \in [0, \infty) \end{cases}, \quad (3.1)$$

where

$$\bar{\phi} := E[\phi|y]. \quad (3.2)$$

Define

$$\bar{w} := 1 - \sum_{j=1}^K \bar{\phi}_j \quad \text{and} \quad \bar{\theta} := \frac{\bar{\phi}}{1 - \bar{w}}. \quad (3.3)$$

Using (3.3), we can write

$$p(h|y) = \begin{cases} \bar{w} & h = \infty \\ (1 - \bar{w}) f(h|\bar{\theta}) & h \in [0, \infty) \end{cases}. \quad (3.4)$$

Note that  $\bar{\phi}$  is computed from the posterior distribution for  $\phi$ :

$$p(\phi|y) = \frac{p(y|\phi)p(\phi)}{p(y)}, \quad (3.5)$$

where

$$p(y) = \int p(y|\phi)p(\phi) d\phi. \quad (3.6)$$

For future reference let

$$L := p(y). \quad (3.7)$$

We can use  $L$  to compare models with different hyperparameter settings. For example, we can compare the base model to one with no point mass at infinity.

The likelihood  $p(y|\phi)$  and the prior  $p(\phi)$  are described next.

**Likelihood.** I assume the connection between the observations (i.e., the survey data) and the parameters is given by

$$y_i = F(h_i|\phi) + \varepsilon_i, \quad (3.8)$$

where  $\varepsilon_i \stackrel{\text{iid}}{\sim} \mathbf{N}(0, \sigma^2)$ . Note

$$F(h_i|\phi) = \sum_{j=1}^K \phi_j X_{ij}, \quad (3.9)$$

where

$$X_{ij} := F_{jK}(h_i) = I_{Q(h_i)}(j, K - j + 1). \quad (3.10)$$

This setup delivers a linear regression:

$$y = X\phi + \varepsilon, \quad (3.11)$$

where  $X$  is an  $n \times K$  design matrix. For  $K > n$ ,  $X$  cannot have full column rank.

The likelihood including the nuisance parameter  $\sigma^2$  is

$$p(y|\phi, \sigma^2) = \prod_{i=1}^n \mathbf{N}(y_i|F(h_i|\phi), \sigma^2), \quad (3.12)$$

where  $\mathbf{N}(\cdot|\mu, \sigma^2)$  is the PDF of the normal distribution with mean  $\mu$  and variance  $\sigma^2$ . We obtain the marginal likelihood for  $\phi$  by integrating out  $\sigma^2$ , using  $p(\sigma^2) \propto 1/\sigma^2$ :

$$p(y|\phi) = \int p(y|\phi, \sigma^2) p(\sigma^2) d\sigma^2 \propto S(\phi)^{-n/2}, \quad (3.13)$$

where

$$S(\phi) := (y - X\phi)^\top (y - X\phi). \quad (3.14)$$

**Prior.** Recall  $\phi = (1 - w) A^{K,k} \theta_k$ . It is convenient to specify the prior for  $\phi$  via the prior for  $k$ ,  $\theta_k$ , and  $w$ . Let  $p(k, \theta_k, w) = p(\theta_k|k) p(k) p(w)$ , where  $p(w)$  and  $p(k)$  will be specified later. For the time being, we note that we require  $p(k) = 0$  for  $k > K$ .

Let the prior for  $\theta_k$  be given by

$$p(\theta_k|k) = \text{Dirichlet}(\theta_k | (\alpha/k) \iota_k), \quad (3.15)$$

where  $\alpha$  (a fixed hyperparameter) is the concentration parameter and  $\iota_k$  is a vector of  $k$  ones. The PDF of the Dirichlet distribution is given by

$$\text{Dirichlet}(\theta_k | \lambda_k) = \frac{\Gamma(\lambda_{0k})}{\prod_{j=1}^k \Gamma(\lambda_{jk})} \prod_{j=1}^k \theta_{jk}^{\lambda_{jk}-1}, \quad (3.16)$$

where  $\lambda_{jk} > 0$ ,  $\lambda_{0k} := \sum_{j=1}^k \lambda_{jk}$ , and  $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ . Note  $E[\theta_{jk}|k] = \lambda_{jk}/\lambda_{0k}$ . The prior variation around this expectation is inversely related to  $\lambda_{0k}$ , which is called the *concentration parameter*.

For the chosen prior,  $\lambda_{jk} = \alpha/k$  and  $\lambda_{0k} = \alpha$ . Therefore the prior expectation of  $\theta_{jk}$  is  $1/k$  and consequently

$$E[F(x|\theta_k)|k] = \sum_{j=1}^k \frac{1}{k} F_{jk}(x) = Q(x). \quad (3.17)$$

In order to encourage sparsity, I set  $\alpha = 1$ .

**Sampling scheme.** Draws from the posterior are made via importance sampling. Let  $\{\phi^{(r)}\}_{r=1}^R$  represent  $R$  draws of  $\phi$  from its prior. These draws can be made by first drawing  $k$  and  $w$  from their priors, next drawing  $\theta_k$  from its conditional prior (given the draw of  $k$ ), and then setting

$$\phi^{(r)} = A^{K,k^{(r)}} \left( (1 - w^{(r)}) \theta_{k^{(r)}}^{(r)} \right). \quad (3.18)$$

Let

$$\zeta^{(r)} := S(\phi^{(r)})^{-n/2} \quad \text{and} \quad Z := \sum_{r=1}^R \zeta^{(r)}. \quad (3.19)$$

Then

$$\bar{\phi} \approx \hat{\phi} := \frac{1}{Z} \sum_{r=1}^R \zeta^{(r)} \phi^{(r)} \quad \text{and} \quad L \approx \hat{L} := Z/R. \quad (3.20)$$

Approximations to other quantities are  $\bar{w} \approx \hat{w} := 1 - \sum_{j=1}^K \hat{\phi}_j$  and  $\bar{\theta} \approx \hat{\theta} := \hat{\phi}/(1 - \hat{w})$ .



*Computation reduction.* We can reduce the amount of computation by not actually making draws of  $k$  and (more importantly) by delaying the elevation of  $(1 - w)\theta_k$ . [When viewed from the perspective of Bayesian Model Averaging (as applied to a collection of models indexed by  $k$ ), the organization of the computations described in this subsection is natural.]

Let  $R_k \approx p(k)R$  denote the expected number of draws of  $k$  that would be made if  $k$  were drawn from its prior, where  $\sum_{k=1}^K R_k = R$ . For each  $k$ , make  $R_k$  draws of  $\theta_k$  from its conditional prior along with  $R_k$  draws of  $w$  from its prior and set

$$\phi_k^{(r)} = (1 - w^{(r)})\theta_k^{(r)}. \quad (3.21)$$

The relevant draws now consist of  $\{\phi_k^{(r)}\}_{r=1}^{R_k}$  for  $k = 1, \dots, K$ .

Let

$$\zeta_k^{(r)} = S(A^{K,k} \phi_k^{(r)})^{-n/2}. \quad (3.22)$$

A significant reduction in computation comes from

$$S(A^{K,k} \phi_k^{(r)}) \equiv (y - X_k \phi_k^{(r)})^\top (y - X_k \phi_k^{(r)}), \quad (3.23)$$

where  $X_k = XA^{K,k}$ . Since  $X_k$  is computed once,  $X_k \phi_k^{(r)}$  involves fewer operations than  $X(A^{K,k} \phi_k^{(r)})$  as long as  $k < K$ .

Next define

$$Z_k := \sum_{r=1}^{R_k} \zeta_k^{(r)} \quad \text{and} \quad \tilde{\phi}_k := \sum_{r=1}^{R_k} \zeta_k^{(r)} \phi_k^{(r)}. \quad (3.24)$$

Then  $Z = \sum_{k=1}^K Z_k$  and

$$\hat{\phi} = \frac{1}{Z} \sum_{k=1}^K A^{K,k} \tilde{\phi}_k. \quad (3.25)$$

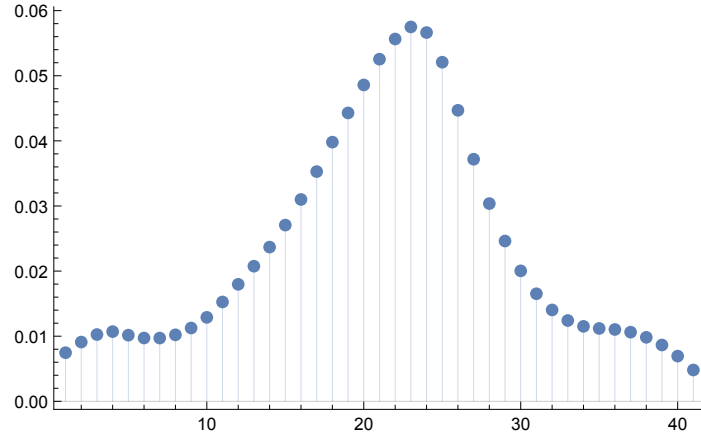
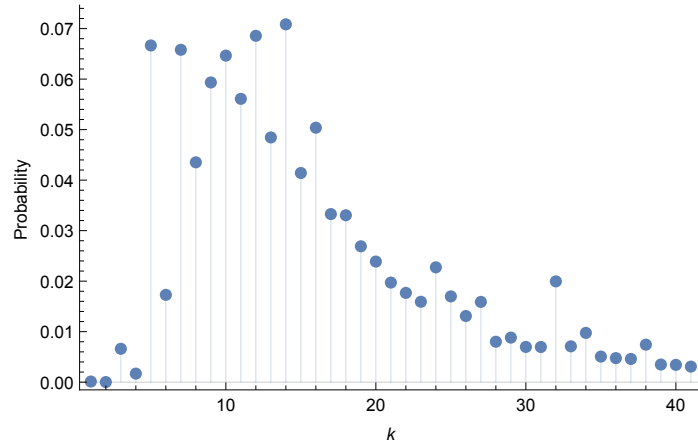
The total number of elevations is reduced from  $R$  to  $K$ .

We can give (3.25) a natural representation:

$$\hat{\phi} = \sum_{k=1}^K \hat{v}_k (A^{K,k} \hat{\phi}_k), \quad (3.26)$$

where  $\hat{v}_k := Z_k/Z$  approximates the posterior probability of  $k$  and  $\hat{\phi}_k := \tilde{\phi}_k/Z_k$  approximates the posterior conditional expectation  $\bar{\phi}_k := E[\phi_k | z_{1:n}, k]$ . Finally, define  $\hat{w}_k := 1 - \sum_{j=1}^k \hat{\phi}_{jk}$  for future reference.

*Adequacy of fit.* The ability of the model to fit a prior depends on both the centering function  $Q$  and the maximum order of the polynomial  $K$ . The more closely the centering function is aligned to the data, the smaller is the required variation around it. In particular, if  $F(h|\hat{\theta})$  fits well, then using it as the centering function should obviate the need for  $k > 1$ . Thus an indication of the adequacy of fit can be obtained by setting  $Q(h) = F(h|\hat{\theta})$ , estimating the model with  $K' \gg 1$ , and checking the posterior probabilities for  $k' = 1, \dots, K'$ .

FIGURE 2.  $\hat{\phi}_{jK}$  for  $j = 1, \dots, K = 41$ .FIGURE 3. Posterior distribution for  $k$ .

#### 4. RESULTS

I chose  $Q(x)$  by fitting a simple parametric distribution to the survey data:  $Q(x) = 2^{-a^*/x}$  where

$$a^* = \operatorname{argmin}_a \sum_{i=1}^n (z_i - (1 - w^*) 2^{-a/h_i})^2. \quad (4.1)$$

In particular,  $a^* = 3.65$  given the chosen value of  $w^* = 0.05$ . Note

$$q(x) = \log(2) a^* 2^{-a^*/x} x^{-2}. \quad (4.2)$$

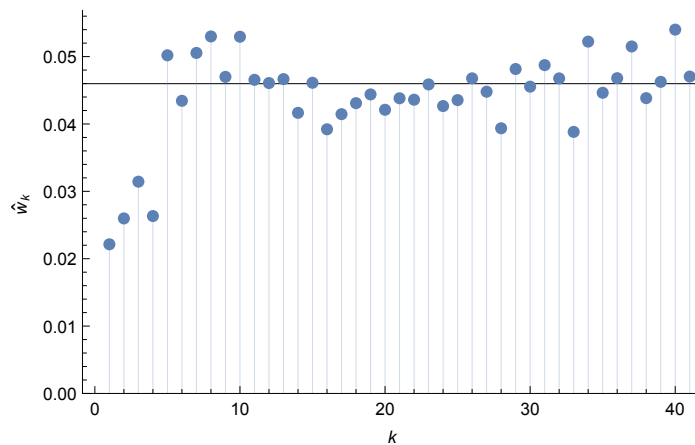


FIGURE 4. Posterior probabilities for the point mass,  $\{\hat{w}_k\}_{k=1}^{41}$  with  $\hat{w} = 0.046$  indicated.

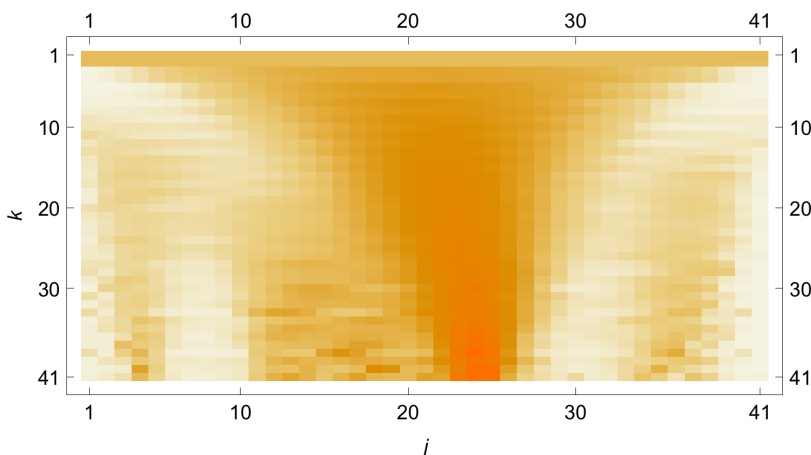


FIGURE 5. Row  $k$  shows  $A^{K,k} \hat{\phi}_k$  for  $k = 1, \dots, K = 41$ .

I let  $p(w) = \text{Beta}(w|1, 19)$ , which has a mean of 0.05. I chose  $K = 41$  and let  $p(k) = 1/K$  for  $k = 1, \dots, K$ . I set  $R = 41 \times 10^7$  for the number of draws from the prior so that  $R_k = 10^7$ .<sup>6</sup>

The central results are  $\hat{w} = 0.046$  and  $\hat{\phi}$  as shown in Figure 2. The posterior distribution for  $k$  is shown in Figure 3. Posterior probabilities  $\hat{w}_k$  for the point mass at infinity are shown in Figure 4 along with the model-averaged  $\hat{w} = 0.046$ . The elevated vectors  $A^{K,k} \hat{\phi}_k$  for each  $k$  are shown row-by-row in Figure 5 and the corresponding weighted vectors  $v_k A^{K,k} \hat{\phi}_k$  are shown in Figure 6. See Figure 1 for a plot of  $F(h|\hat{\phi})$  and Figure 7 for a plot of  $f(h|\hat{\theta})$ .

<sup>6</sup>The calculations were done on my MacBook Pro (circa 2014) using *Mathematica* (with pseudo-compiled code). The entire calculation, which involved generating close to  $10^{10}$  gamma variates, took about 11 minutes using some parallelization.

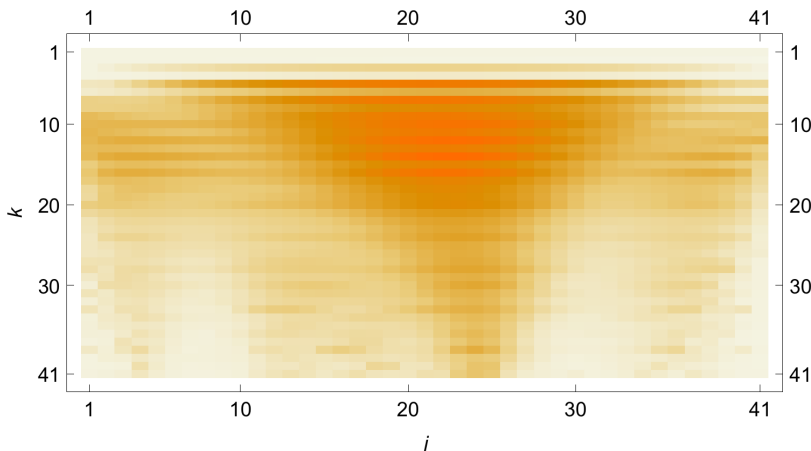


FIGURE 6. Row  $k$  shows  $\hat{v}_k A^{K,k} \hat{\phi}_k$  for  $k = 1, \dots, K = 41$ .

*Adequacy of the fit.* As a check on the adequacy of the fit, I redid the estimation using  $F(h|\hat{\theta})$  as the centering function, constructing the design matrix  $\hat{X}'$  via

$$\hat{X}'_{ij} := I_{F(h_i|\hat{\theta})}(j, K' - j + 1). \quad (4.3)$$

I chose  $K' = 21$  and  $R = 21 \times 10^6$ . The posterior distribution for  $k$  is shown in Figure 8. The first two probabilities account for more than 99%. I found  $F(h|\hat{\phi}')$  to be indistinguishable from  $F(h|\hat{\phi})$ . In summary, this check produced no evidence against the adequacy of the fit.

*Evidence in favor of  $w = 0$ .* I ran the model imposing  $w = 0$ . The centering function was refit under the assumption  $w^* = 0$ , producing  $a^* = 3.96$  [see (4.1)]. The Bayes factor in favor of this restricted model relative to the unrestricted base model is  $\hat{L}'/\hat{L} \approx 0.5$ . In other words, there is very mild evidence in favor of  $w > 0$ .

## 5. FIRST-ORDER AUTOREGRESSIVE COEFFICIENT

The first-order autoregressive model (for the log of the real exchange rate,  $m_t$ ) can be expressed as

$$m_t = \gamma + \beta m_{t-1} + \varepsilon_t, \quad (5.1)$$

where  $\beta$  is the first-order autoregressive coefficient. According to (5.1), the half-life  $h$  is given by  $\beta^h = 1/2$ . This expression can be solved for

$$h(\beta) := \frac{-\log(2)}{\log(\beta)}. \quad (5.2)$$

Note

$$h'(\beta) = \frac{\log(2)}{\beta \log(\beta)^2}. \quad (5.3)$$

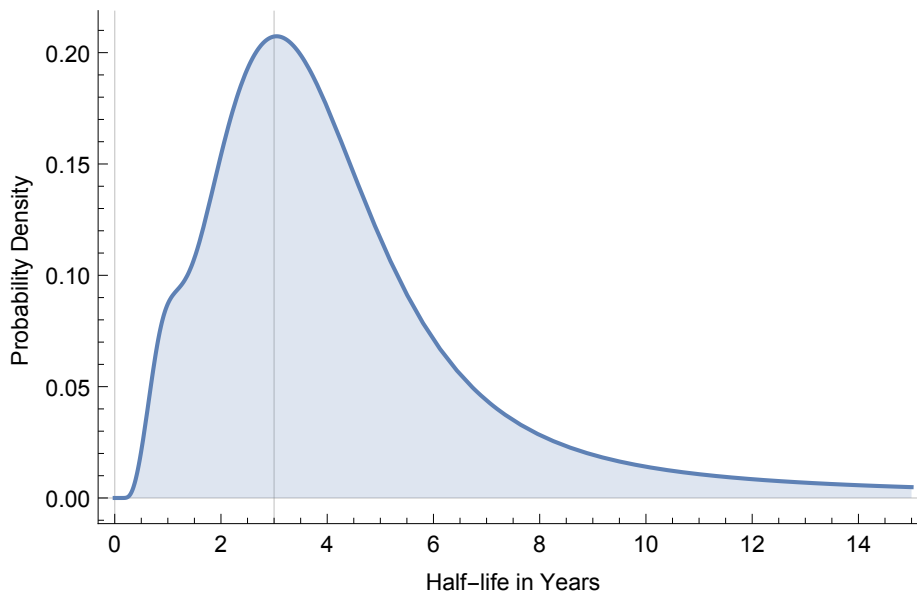


FIGURE 7. PDF for survey fit prior,  $f(h|\hat{\theta})$ . The mode occurs at  $h = 3.0$  years. The fit delivers  $\Pr[h = \infty] = 0.046$ .

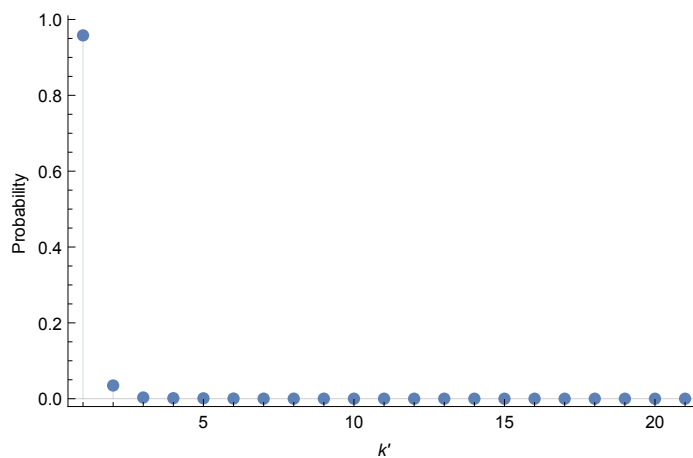


FIGURE 8. Posterior probabilities for  $k = 1, \dots, 21$ , where  $Q(h) = F(h|\hat{\theta})$ .

With these expressions, the model in (2.1) can be written in terms of  $\beta$  as follows:

$$p(\beta|\theta_k, w) = \begin{cases} w & \beta = 1 \\ (1 - w) g(\beta|\theta_k) & \beta \in [0, 1) \end{cases}, \quad (5.4)$$

where

$$g(\beta|\theta_k) := f(h(\beta)|\theta_k) h'(\beta). \quad (5.5)$$

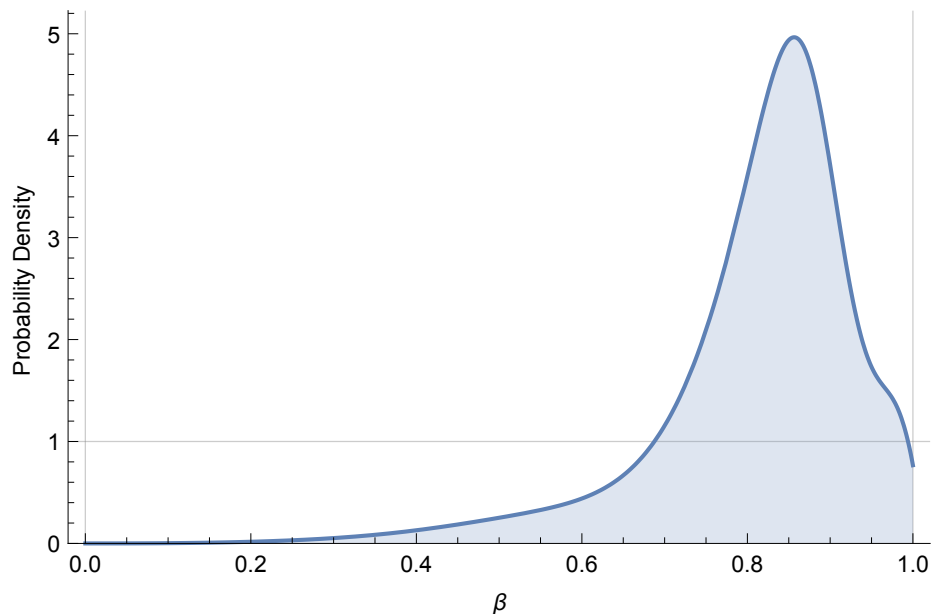


FIGURE 9. PDF for fit survey prior expressed in terms of  $\beta$  (with a uniform distribution for reference). This fit delivers  $\Pr[\beta = 1] = 0.046$ .

Consequently, the posterior probability of a unit root is approximated by  $\hat{w} = 0.046$  and the posterior density over the unit interval is given by  $g(\beta|\hat{\theta})$  as shown in Figure 9.

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