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Individual Heterogeneity and Average Welfare*

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Abstract

Individual heterogeneity is an important source of variation in demand. Allowing for general heterogeneity is needed for correct welfare comparisons. We consider general heterogenous demand where preferences and linear budget sets are statistically independent. Only the marginal distribution of demand for each price and income is identified from cross-section data where only one price and income is observed for each individual. Thus, objects that depend on varying price and/or income for an indiviual are not generally identified, including average exact consumer surplus. We use bounds on income effects to derive relatively simple bounds on the average surplus, including for discrete/continous choice. We also sketch an approach to bounding surplus that does not use income effect bounds. We apply the results to gasoline demand. We find tight bounds for average surplus in this application but wider bounds for average deadweight loss.

JEL Classification: C10, C14, C51, C54.
Keywords: Consumer surplus, deadweight loss, identification, quantiles.

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1 Introduction

Unobserved individual heterogeneity is thought to be a large source of variation in empirical demand equations. Often r-squareds are found to be low in cross-section and panel data applications, suggesting that much variation in demand is due to unobserved heterogeneity. The magnitude of heterogeneity in applications makes it important to account correctly for heterogeneity.

Demand functions could vary across individuals in general ways. For example, it seems reasonable to suppose that price and income effects are not confined to a one dimensional curve as they vary across individuals, meaning that heterogeneity is multidimensional. Demand might also arise from combined discrete and continuous choice, where heterogeneity has different effects on discrete and continuous choices. For these reasons it seems important to allow for general heterogeneity in demand analysis. In this paper we do so.

Exact consumer surplus quantifies the welfare effect of price changes, including the deadweight loss of taxes. The average surplus over individuals is a common welfare measure. We show that for continuous demand average surplus is generally not identified from the distribution of demand for a given price and income. Nonidentification motivates a bounds approach. We use bounds on income effects to derive bounds on average surplus. Surplus bounds are constructed from the average of quantity demanded across consumers. In the Appendix we extend these bounds to combined discrete and continuous choice. We also sketch how surplus bounds can be obtained from the distribution of quantity demanded without knowing bounds on income effects.

Empirical application of these bounds is based on independence of preferences and budget sets, possibly conditioned on covariates and control functions. Under independence, average demand is the conditional expectation of quantity, that can be estimated by nonparametric, semiparametric, or parametric methods in cross section data. The distribution of demand can be also estimated in analogous ways.

We apply average surplus bounds to gasoline demand, using data from the 2001 U.S.

National Household Transportation Survey. We find that average surplus bounds based on income effects are quite tight in the application, but that bounds on deadweight loss are wider. We give confidence intervals for an identified set with income effect bounds. We find that bounds are substantially wider when we just impose utility maximization.

In this paper bounds on derivatives lead to useful bounds for objects of interest, as have restrictions like monotonicity and concavity in other settings. The focus here on derivatives is driven by economics, where income effects play a pivotal role in bounding average surplus.

Choice models with general heterogeneity have previously been considered. In their analysis of nonlinear taxes, Burtless and Hausman (1978) allowed the income effect to vary over individuals. McFadden and Richter (1991) and McFadden (2005) allowed for general heterogeneity in a revealed preference framework. Our paper specializes the revealed preference work in imposing single valued, smooth demands to facilitate estimation. Lewbel (2001) provided conditions for the average demand to satisfy the restrictions of utility maximization. Hoderlein and Stoye (2009) showed how to impose the weak axiom of revealed preference. Dette, Hoderlein, and Neumeyer (2011) proposed tests of downward sloping compensated demands. Kitamura and Stoye (2012) gave tests of the revealed preference hypothesis. Blomquist and Newey (2002) derived the form of average demand with nonlinear budget sets; see also Blomquist, Kumar, Liang, and Newey (2014). Recently Bhattacharya (2014) has derived average surplus for discrete demand and general heterogeneity.

Blundell, Horowitz, and Parey (2012), Hoderlein and Vanhems (2010), and Blundell, Kristensen, and Matzkin (2011) have considered identification and estimation of welfare measures when demand depends continuously on a single unobserved variable. Blundell, Kristensen and Matzkin (2011) impose revealed preference restrictions on demand functions in that setting. Recently Lewbel and Pendakur (2013) have considered restricted multivariate heterogeneity. Here we go beyond these specifications and consider general heterogeneity.

The results of this paper build on Hausman and Newey (1995). This paper is about

heterogeneity in demand, which was largely ignored in the previous paper, but we do make use of asymptotic estimation theory from the previous paper. In this paper we find that bounds should be computed from the nonparametric regression of quantity on price and income, and not log quantity or some other function of quantity, because the conditional expectation of quantity averages across individuals in the desired way. The results of Hausman and Newey (1995) can then be applied for inference in large samples.

2 Demand Functions with General Heterogeneity

We consider a demand model where the form of heterogeneity is unrestricted. To describe the model let q denote the quantity of a vector of goods, a the quantity of a numeraire good, p the price vector for q relative to a, and y the individual income level relative to the numeraire price. Also let $x = (p^T, y)^T$, where throughout we adopt the notational convention that vectors are column vectors. The unobserved heterogeneity will be represented by a vector η of unobserved disturbances of unknown dimension. We think of each value of η as corresponding to a consumer but do allow η to be continuously distributed.

For each consumer η the demand function $q(x, \eta)$ will be obtained by maximizing a utility function $U(q, a, \eta)$ that is monotonic increasing in q and a, subject to the budget constraint, with

$$q(x, \eta) = \arg \max_{q \geq 0, a \geq 0} U(q, a, \eta) \text{ s.t. } p^T q + a \leq y. \tag{2.1}$$

Here we assume that demand is single valued and not a correspondence. This assumption is essentially equivalent to strict quasi-concavity of the utility function. We impose no form on the way η enters the utility function U, and hence the form of heterogeneity in $q(x, \eta)$ is also unrestricted. Demand functions are allowed to vary across individuals in general ways.

Utility maximization imposes restrictions on the demand functions as a function of prices and income. For continuously differentiable demands and positive prices and income these restrictions are summarized in the following condition.

ASSUMPTION 1: *For each η the demand function $q(x, \eta)$ is continuously differentiable*

in $x >> 0$ *and* $\partial q(x,\eta)/\partial p + q(x,\eta)[\partial q(x,\eta)/\partial y]^T$ *is symmetric and negative semi-definite for all* $x \in \chi$.

By Hurwicz and Uzawa (1971), this condition is also sufficient for existence of a utility function, with $q(x,\eta)$ maximizing the utility function subject to the budget constraint. In this sense, formulating a model with demand functions satisfying Assumption 1 is equivalent to formulating a model based on utility maximization. In what follows we take as primitive demand functions satisfying Assumption 1. We also need technical conditions in order to make probability statements using these demand functions. For convenience we reserve these technical conditions to Assumption A1 of the Appendix.

Let r denote a possible value of quantity demanded, G the distribution of η, and $F(r|x,q,G)$ the CDF of quantity r when prices and income equal x for all individuals,

$$F(r|x,q,G) = \int 1(q(x,\eta) \leq r) G(d\eta). \tag{2.2}$$

The model we consider is one with a CDF for this form for $q(x,\eta)$ satisfying Assumption 1 and a distribution G of η.

This model is a random utility model (RUM) of the kind considered by McFadden (2005, see also McFadden and Richter, 1991). The model here specializes the RUM to single valued demands that are smooth in prices and income. Single valued, smooth demand specifications are often used in applications. In particular, smoothness has often proven useful in applications of nonparametric models and we expect it will here. Estimation under a RUM with general preference variation is not often done in applications. We do so here for average surplus.

Much of the revealed stochastic preference literature is concerned with deriving restrictions on $F(r|x,q,G)$ as a function of r and x that are necessary and sufficient for a RUM. McFadden (2005) provides a set of inequalities that are necessary and sufficient for the RUM with continuous demands. With two goods and single valued, smooth demand there is a simple, alternative characterization in terms of quantiles that is useful in the identification analysis to follow. The characterization is that each quantile is a demand function. Let $Q(\tau|x) = \inf\{r : F(r|x,q,G) \geq \tau\}$ denote the τ^{th} conditional

quantile corresponding to $F(r|x,q,G)$, where we drop dependence of Q on q and G for notational convenience. The following result holds under technical conditions that are given in Assumption A2 of the Appendix.

THEOREM 1: *Suppose that there are two goods, so that $q(x,\eta)$ and p are scalars. If Assumptions 1 and A2 are satisfied then $Q(\tau|x)$ is a demand function for all $0<\tau<1$ and p, $y>0$. If $Q(\tau|x)$ is a demand function that is continuously differentiable in x for $0<\tau<1$ and p, $y>0$ then there is $\tilde{\eta}$ with $U(0,1)$ and CDF $\tilde{G}$ such that Assumption 1 is satisfied for $\tilde{q}(x,\tilde{\eta})=Q(\tilde{\eta}|x)$ and $F(r|x,q,G)=F(r|x,\tilde{q},\tilde{G})$.*

Dette, Hoderlein, and Neumeyer (2011) showed that the quantile function is a demand function under conditions similar to those of Assumption A2. Theorem 1 also shows that if the quantile is a demand function then a demand model with a one dimensional uniformly distributed η gives the distribution of demand. Together these results show that, under Assumption A2, the CDF of quantity takes the form $F(r|x,q,G)$ for some q satisfying Assumption 1 if and only if each quantile is a demand function. In this sense, for two goods and single valued smooth demands the revealed, stochastic preference conditions are that each quantile is a demand function. This result will be used in the identification analysis to follow and is of interest in its own right.

3 Exact Consumer Surplus

We focus on equivalent variation though a similar analysis could be carried out for compensating variation. Let $e(p,u,\eta)=\min_{q\geq 0,a\geq 0}\{p^Tq+a \text{ s.t. } U(q,a)\geq u\}$ be the expenditure function and $S(\eta)=\bar{y}-e(p^0,u^1,\eta)$ be the equivalent variation for individual η for a price change from p^0 to p^1 with income $\bar{y}$ and u^1 the utility at price p^1. The corresponding deadweight loss is $D(\eta)=S(\eta)-q(p^1,\bar{y},\eta)^T\Delta p$, where $\Delta p=p^1-p^0$.

It is helpful to express surplus and deadweight loss in terms of demand. Let $\{p(t)\}_{t=0}^1$ be a continuously differentiable price path with $p(0)=p^0$ and $p(1)=p^1$. As discussed in Hausman and Newey (1995), Shephard's Lemma implies that for a scalar t the equivalent

variation $S(\eta)$ is the solution $s(0,\eta)$ at $t=0$ to

$$\frac{ds(t,\eta)}{dt} = -q(p(t), \bar{y} - s(t,\eta), \eta)^T \frac{dp(t)}{dt}, \qquad s(1,\eta) = 0, \tag{3.3}$$

where $s(1,\eta) = 0$ pins down the constant of integration in this ordinary differential equation. $S(\eta)$ does not depend on the price path as long as the demand function $q(x,\eta)$ satisfies Assumption 1 and $(p(t)^T, \bar{y} - s(t,\eta))^T$ remains in χ.

A change in the price of a single good, say the first one, is a common example. In that case $p^0 = (p_1^0, \bar{p}_2^T)^T$ and $p^1 = (p_1^1, \bar{p}_2^T)^T$ for some fixed $\bar{p}_2$. A natural choice of price path is $p(t) = tp^1 + (1-t)p^0 = (p_1^0 + t\Delta p_1, \bar{p}_2^T)^T$, where $\Delta p_1 = p_1^1 - p_1^0$. In this case equation (3.3) becomes

$$\frac{ds(t,\eta)}{dt} = -q_1(p_1^0 + t\Delta p_1, \bar{p}_2, \bar{y} - s(t,\eta), \eta)\Delta p_1, \quad s(1,\eta) = 0.$$

Thus with multiple goods the exact consumer surplus for a price change for a single good can be computed from the demand function for that good by varying its price and varying income to keep utility constant, as shown by Hausman (1981).

The objects we will focus on and that are of common interest are the average surplus $\bar{S}$ and deadweight loss $\bar{D}$ across individuals, given by

$$\bar{S} = \int S(\eta) G(d\eta), \bar{D} = \int D(\eta) G(d\eta).$$

As is known from Hicks (1939), when $\bar{S}$ is positive it is possible to redistribute income so that individuals are better off under p^0 than under p^1. Also, $\bar{S}$ is often used as a practical measure of social welfare even though it implicitly evaluates money as having equal weight across individuals. For these reasons we will focus on average surplus and deadweight loss in this paper. Some results could be extended to other interesting objects, like the distribution of surplus. We leave such extensions to future work.

Average surplus depends only on average demand $\bar{q}(p,y) = \int q(x,\eta) G(d\eta)$ when the income effect is constant. Suppose that $\partial q_1(p_1, \bar{p}_2, y, \eta)/\partial y = b$ over $p_1 \in [p_1^0, p_1^1]$, $y \in [\bar{y} - S(\eta), \bar{y}]$, and η. Then $S(\eta)$ is the solution at $t=0$ to

$$\frac{ds(t,\eta)}{dt} = -[q_1(p_1^0 + t\Delta p_1, \bar{p}_2, \bar{y}, \eta) - bs(t,\eta)]\Delta p_1, \qquad s\,(1,\eta) = 0. \tag{3.4}$$

This is a linear differential equation with explicit solution

$$S(\eta) = \Delta p_1 \int_0^1 q_1(p_1^0 + t\Delta p_1, \bar{p}_2, \bar{y}, \eta) \exp(-tb\Delta p_1) dt = \int_{p_1^0}^{p_1^1} q_1(p_1, \bar{p}_2, \bar{y}, \eta) \exp(-b(p_1 - p_1^0)) dp_1.$$

Taking expectations under the integral gives

$$\bar{S} = \int_{p_1^0}^{p_1^1} \bar{q}_1(p_1, \bar{p}_2, \bar{y}) \exp(-b(p_1 - p_1^0)) dp_1.$$

This can also be represented as the solution at $t = 0$ to

$$\frac{d\bar{s}(t)}{dt} = -\bar{q}(p(t), \bar{y} - \bar{s}(t))^T \frac{dp(t)}{dt}, \qquad \bar{s}\,(1) = 0. \tag{3.5}$$

Comparing equation (3.5) with (3.3) we see that, with one price changing and income effect constant for that good, average surplus solves the same differential equation as individual surplus, with average demand replacing individual demand. This result generalizes to multiple price changes where the income effects are constant for all goods with changing prices.

Obtaining average surplus from average demand is consistent with the well known aggregation results of Gorman (1961), who showed that constant income effects are necessary and sufficient for demand aggregation. The preceding discussion is a demonstration of a partial dual result, that when the price of one good is changing and the income effect is constant for that good then exact surplus for average demand is the average of exact surplus. McFadden (2004) derived and used this result in the case where income effects are constant for all goods.

Marshallian surplus solves equation (3.3) while replacing $s(t, \eta)$ on the right-hand side with zero, i.e. while not compensating income to remain on the same indifference curve. Average Marshallian surplus $\bar{S}_M$ is given by

$$\bar{S}_M = \int \{ \int_0^1 q(p(t), \bar{y}, \eta)^T [dp(t)/dt] dt \} G(d\eta) = \int_0^1 \bar{q}(p(t), \bar{y})^T [dp(t)/dt] dt.$$

This surplus measure is a function of average demand in general but it ignores income effects. Ignoring income effects results in a poor approximation to deadweight loss, see Hausman (1981). For this reason we focus on exact surplus in our analysis, though we do find that average Marshallian surplus provides a useful upper bound for average equivalent variation for a normal good, as shown for individual demands by Willig (1976).

4 Identification

We consider identification of objects of interest when we know the CDF $F(r|x,q,G)$ of demand over a set $\bar{\chi}$ of prices and income. This corresponds to knowing the distribution of demand in cross-section data, where we only observe one price and income for each individual. If more than one value of x were observed for each individual, as in panel data, then one could identify some joint distributions of demand at different values of x. We leave consideration of this topic to future research.

We adapt a standard framework to our setting, as in Hsiao (1983), by specifying that a structure is a demand function and heterogeneity distribution pair (q,G), where for notational convenience we suppress the arguments of q and G.

DEFINITION 1: *(q,G) and $(\tilde{q},\tilde{G})$ are observationally equivalent if and only if for all r and $x \in \bar{\chi}$,*

$$F(r|x,q,G) = F(r|x,\tilde{q},\tilde{G}).$$

The set $\bar{\chi}$ will correspond to the set of x that is observed. We allow $\bar{\chi}$ to differ from χ of Assumption 1 in order to allow the Slutzky conditions to be imposed outside the range of the data. We consider identification of an object $\delta(q,G)$ that is a function of the structure (q,G). Here $\delta(q,G)$ is a map from the demand function and the distribution of heterogeneity into some set. The identified set for δ we consider will be the set of values of this function for all structures that are observationally equivalent.

DEFINITION 2: *The identified set for δ corresponding to (q_0,G_0) is $\Lambda(q_0,G_0) = \{\delta(\tilde{q},\tilde{G}) : (q_0,G_0)$ and $(\tilde{q},\tilde{G})$ are observationally equivalent$\}$.*

The (q_0,G_0) in this definition can be thought of as the true values of the demand function and heterogeneity distribution. The identified set $\Lambda(q_0,G_0)$ is the set of δ that is consistent with the distribution of demand $F(r|x,q_0,G_0)$ implied by the true values. The set $\Lambda(q_0,G_0)$ is nonempty since it always includes the true value $\delta(q_0,G_0)$. The set $\Lambda(q_0,G_0)$ is sharp, given only knowledge of $F(r|x,q_0,G_0)$, because it consists exactly of

those δ that correspond to some $(\tilde{q}, \tilde{G})$ that generates the same distribution of demand as the true values. In other words, sharpness of $\Lambda(q_0, G_0)$ holds automatically here because we are explicitly formulating the identified set in terms of all the restrictions on the distribution of demand that are implied by the model, and we are assuming that the distribution of demand is all we know.

Viewing demand as a stochastic process indexed by x helps explain identification. Here $q(x, \eta)$ is a function of x for each η, that varies stochastically with η, i.e. $q(x, \eta)$ is a stochastic process. In this way the structure (q, G) can be thought of as a demand process. In the language of stochastic processes the distribution of $q(x, \eta)$ for fixed x is a marginal distribution, while the distribution of $(q(x^1, \eta), ..., q(x^K, \eta))^T$ for some fixed set $\{x^1, ..., x^K\}$ of prices and income is a joint distribution. In our notation the marginal CDF of this stochastic process is $F(r|x, q, G)$. Thus, two demand processes will be observationally equivalent if and only if they have the same marginal distribution.

Objects $\delta(q, G)$ that depend only on the marginal distribution of the demand process are point identified, because they are the same for all observationally equivalent structures. For example, average demand $\bar{q}(x) = \int q(x, \eta) G(d\eta) = \int r F(dr|x, q, G)$ is identified, as are functionals of it, such as average surplus with constant income effect in equation (3.6).

Joint distributions of the demand process, such as the joint distribution of $(q(\tilde{x}, \eta), q(\bar{x}, \eta))^T$ for two different values of x, will not be identified. We will show this result for certain demand processes below and the intuition is straightforward. Intuitively, joint distributions are not identified from marginal distributions. Because joint distributions are not identified, distributions and averages of objects that depend on varying x for given η will not be identified. As we show rigorously below, such nonidentified objects will include average surplus, which depends on varying both price and income for a given η.

It will generally be impossible to identify demand functions for individuals from the marginal distribution of demand. Again the intuition is straightforward, with individual demands not identified because we only observe one price and income for each individual. More formally, we can think of the ability to identify individual demands as

$q_0(\tilde{x}, \eta)$ being perfectly predictable for each $\tilde{x}$ if we know $q_0(\bar{x}, \eta)$ for some $\bar{x}$, i.e. as $Var(\tilde{q}(\tilde{x}, \tilde{\eta})|\tilde{q}(\bar{x}, \tilde{\eta})) = 0$ for any $(\tilde{q}, \tilde{G})$ that is observationally equivalent to the truth. This is a property of the joint distribution of the demand process, and so is not identified from the marginal distribution of the demand process.

The specific nonidentification results we show are for two goods where Theorem 1 provides the key to the proof. Combining its first and second conclusion implies that $Q(\tau|x)$ is a demand function and that $Q(\tilde{\eta}|x)$ for $\tilde{\eta} \sim U(0, 1)$ gives the same conditional distribution of quantity as true demand. Thus, under the conditions of Theorem 1, the quantile demand is observationally equivalent to true demand. The joint distribution of the quantile process can differ from the true one. For example, the true demand may have $Var(q(\tilde{x}, \eta)|q(\bar{x}, \eta)) > 0$ but $Q(\tilde{\eta}|x)$ will be one-to-one in $\tilde{\eta}$ for each x so $Var(Q(\tilde{\eta}|\tilde{x})|Q(\tilde{\eta}|\bar{x})) = 0$.

For example consider a true demand process that is linear in x with varying coefficients, where

$$q_0(x, \eta) = \eta_1 + \eta_2 p + \eta_3 y.$$

This demand process is a familiar specification. By Theorem 1 quantile demand will be observationally equivalent to the true demand. Thus, there is no way to distinguish nonparametrically a true, linear, varying coefficients process from quantile demand. Also, true average surplus will generally be different than average surplus for quantile demand. Intuitively, true average surplus is the average over η_3 of the surplus for a demand function that is linear in p and y. In contrast, the quantile demand will generally be nonlinear in y because it is the inverse of the CDF of demand. Also, average quantile surplus is obtained by averaging over a scalar uniform distribution. These differences between the true and quantile demand processes will often lead to quantile average surplus being different from true average surplus, except when η_3 is constant.

We prove nonidentification by showing numerically that the true average surplus is different than the average surplus for quantile demand in an example of a linear varying coefficients model.

THEOREM 2: *If* $q_0(x,\eta) = \eta_1 - p + \eta_2 y$, $\eta_1 \sim U(0,1)$, η_2 *is distributed independently of* η_1 *with two point support* $\{1/3, 2/3\}$, *and* $\Pr(\eta_2 = 1/3) = 1/2$, *the average equivalent variation* $\bar{S}$ *is not identified for* $p^0 = .1$, $p^1 = .2$, and $\bar{y} = 3/4$.

It can be seen from equation (3.3) that average surplus should be continuous in the demand function q and distribution G in an appropriate sense. Thus, the nonidentification result of Theorem 2 should hold for demand processes that are close to the example of Theorem 2. We also expect nonidentification of average surplus to be generic in the class of demand processes with varying income effects though it is beyond the scope of this paper to prove this result.

5 Income Effect Bounds

Known bounds on income effects can be used to bound average surplus and deadweight loss using average demand. The idea is to extend Section 2, where constant income effects allow identification of average surplus from average demand, to identify bounds on surplus from average demand. To describe the result, for any constant B let

$$\bar{S}_B = \int_0^1 [\bar{q}(p(t), \bar{y})^T \frac{dp(t)}{dt}] e^{-Bt} dt \tag{5.6}$$

be the solution $\bar{s}_B(t)$ at $t = 0$ to the linear differential equation

$$\frac{d\bar{s}_B(t)}{dt} = -\bar{q}(p(t), \bar{y})^T \frac{dp(t)}{dt} + B\bar{s}_B(t), \qquad \bar{s}_B(1) = 0. \tag{5.7}$$

From Section 2 we see that $\bar{S}_B$ would be the average surplus if just the price of the first good were changing and the demand for the first good had a constant income effect $\partial q_1(p(t), y, \eta)/\partial y = B/\Delta p_1$.

THEOREM 3: *If i)* $q(p(t), \bar{y} - s, \eta)^T dp(t)/dt \geq 0$ for $s \in [0, S(\eta)]$, *ii) there are constants* $\underline{B}$ *and* $\overline{B}$ *such that* $\underline{B} \leq [\partial q(x,\eta)/\partial y]^T dp(t)/dt \leq \overline{B}$ *for all* $x \in \chi$; *iii) all prices in* $p(t)$ *are bounded away from zero then*

$$\bar{S}_{\overline{B}} \leq \bar{S} \leq \bar{S}_{\underline{B}}, \bar{S}_{\overline{B}} - \bar{q}(p^1, \bar{y})^T \Delta p \leq \bar{D} \leq \bar{S}_{\underline{B}} - \bar{q}(p^1, \bar{y})^T \Delta p.$$

Condition i) is a restriction on the price path that is automatically satisfied when only the price of the first good is changing and $p_1^1 > p_1^0$. Also, the bounds in the conclusion are satisfied under weaker conditions than bounded income effects. To conserve space here we give the more general result in the Appendix.

The key ingredient for these average surplus bounds are bounds on the income effect $[\partial q(x,\eta)/\partial y]^T dp(t)/dt$. Economics can deliver such bounds. Consider again, and for the rest of this Section, a price change in the first good, where $\underline{B}$ and $\overline{B}$ are bounds on $\Delta p_1 \partial q_1(x,\eta)/\partial y$ and $\Delta p_1 > 0$. If q_1 is a normal good then the income effect is nonnegative, so we can take $\underline{B} = 0$. Then an upper bound for average equivalent variation and deadweight loss can be obtained from Marshallian surplus for average demand as

$$\bar{S} \leq \bar{S}_M = \int_0^1 \left[\bar{q}(p(t),\bar{y})^T dp(t)/dt\right] dt, \bar{D} \leq \bar{S}_M - \bar{q}(p^1,\bar{y})^T \Delta p.$$

The upper bound on average deadweight loss could be useful for policy purposes, e.g. to proceed with a tax if average public benefits (e.g. environmental benefits) exceed average deadweight loss and the appropriate separability conditions are satisfied.

Economics can also deliver upper bounds on income effects. If no more than a fraction π of additional income is spent on q_1 then

$$\partial q_1(x,\eta)/\partial y \leq \pi/p_1 \leq \pi/p_1^0,$$

so that $\bar{B} = \Delta p_1 \pi/p_1^0 = \pi(p_1^1/p_1^0 - 1)$ is an upper bound on $[\partial q(x,\eta)/\partial y]^T dp(t)/dt$. For example, in the gasoline demand application below we are quite certain that only a small fraction of any increase in income is spent on gasoline, making our choice of $\bar{B}$ very credible. The Slutzky condition also can limit the size of income effects relative to price effects an quantity. In the next Section we consider bounds based on the Slutzky condition.

The quantiles of the demand distribution are informative about income effects. Let $Q_1(\tau|x)$ denote the conditional quantile of the first good, where we continue to suppress dependence on q and G. By Hoderlein and Mammen (2007),

$$\frac{\partial Q_1(\tau|x)}{\partial y} = E[\frac{\partial q_1(x,\eta_i)}{\partial y}|q_1(x,\eta_i) = Q_1(\tau|x)],$$

where η_i is a random variable with distribution G. Note that constancy of the income effect will also imply constancy of $\partial Q_1(\tau|x)/\partial y$ as τ varies. Thus, if $\partial Q_1(\tau|x)/\partial y$ varies with τ the income effect for the first good is heterogenous. Also, a necessary condition for $\underline{B}$ and $\overline{B}$ to bound $\Delta p_1 \partial q_1(x,\eta)/\partial y$ is $\underline{B} \leq \Delta p_1 \partial Q_1(\tau|x)/\partial y \leq \overline{B}$. This result can be used to guide the choice of bounds on income effects. For example, one might choose an upper bound that is much larger than derivatives of many quantiles, as we do in the gasoline application to follow. Note though that this approach does not serve to identify the bounds, because we cannot tell from the quantile derivative how the income effect varies over η with $q_1(x,\eta) = Q_1(\tau|x)$.

The conditional quantile is also informative about the surplus bounds. Let S^τ be the exact surplus obtained by treating $Q_1(\tau|x)$ as if it were a demand function, obtained as the solution $s^\tau(0)$ at $t = 0$ to the differential equation

$$\frac{ds^\tau(t)}{dt} = -Q_1(\tau|p_1^0 + t\Delta p_1, \bar{p}_2, \bar{y} - s^\tau(t))\Delta p_1, \qquad s^\tau(1) = 0.$$

With two goods and scalar heterogeneity, the average surplus would be $\int_0^1 S^\tau d\tau$. It turns out that $\int_0^1 S^\tau d\tau$ is between the surplus bounds in general.

COROLLARY 4: *If the conditions of Theorem 1 are satisfied and only the first price is varying then*

$$\bar{S}_{\overline{B}} \leq \int_0^1 S^\tau d\tau \leq \bar{S}_{\underline{B}}.$$

Surplus bounds are relatively insensitive to income effect bounds when a small proportion of income is spent on the good. This result is related to the Hotelling (1938) result that when expenditure is small approximate consumer surplus is typically close to actual consumer surplus. Differentiate equation (5.6) with respect to B to obtain

$$\begin{aligned} \bar{y}^{-1}\frac{\partial \bar{S}_B}{\partial B} &= -\bar{y}^{-1}\int_0^1 [\bar{q}(p(t),\bar{y})^T dp(t)/dt]te^{-Bt}dt \\ &= -\int_{p_1^0}^{p_1^1} [\bar{q}_1(p_1,\bar{p}_2,\bar{y})p_1/\bar{y}](\frac{1-p_{10}/p_1}{\Delta p_1})\exp(-B\frac{p_1-p_1^0}{\Delta p_1})dp_1. \end{aligned}$$

In this way the bounds are less sensitive to B when share $\bar{q}_1(p_1,\bar{p}_2,\bar{y})p_1/\bar{y}$ of income spent on the first good is smaller.

6 General Bounds with Two Goods

It is possible to drop knowledge of income effects and obtain average surplus bounds based only on utility maximization We do this by finding the supremum and infimum of average surplus over an approximation to the set of demand processes that are consistent with the distribution of demand. We focus here on the two good case. The approximation is based on a series expansion around the quantile demand where the coefficients of the series terms have a discrete distribution. We consider a demand process of the form

$$\tilde{q}(x,\eta) = Q(\tilde{\eta}|x) + \sum_{j=1}^{J} \breve{\eta}_j m_j(x)$$

where $m_j(x)$, $j = 1, ..., J$ are approximating functions, $\tilde{\eta} \sim U(0,1)$, and $\breve{\eta} = (\breve{\eta}_1, ..., \breve{\eta}_J)^T$ has a discrete distribution with L points of support $\{\breve{\eta}^1, ..., \breve{\eta}^L\}$ that is independent of $\tilde{\eta}$. In this approximation we draw the support points $\breve{\eta}^\ell$ at random, keeping only those where $\tilde{q}^\ell(x,\tilde{\eta}) = Q(\tilde{\eta}|x) + \sum_{j=1}^{J} \breve{\eta}_j^\ell m_j(x)$ satisfies the Slutzky condition over a grid of values for $\tilde{\eta}$ and $x \in \chi$. Let $\rho_\ell = \Pr(\breve{\eta} = \breve{\eta}^\ell)$ and $F(r|x) = Q^{-1}(r|x)$ be the CDF corresponding the quantile $Q(\tau|x)$. Integrating over $\tilde{\eta}$ gives

$$F(r|x,\tilde{q},\tilde{G}) = \sum_{\ell=1}^{L} \rho_\ell F(r - \sum_{j=1}^{J} \breve{\eta}_j^\ell m_j(x)|x).$$

The integration over $\tilde{\eta}$ here helps to smooth out the CDF which should provide a better fit in applications.

We allow the mixture probabilities ρ_ℓ to vary and look for the maximum and minimum average surplus subject to restrictions imposed by the data. Let $\tilde{S}^\ell(\tilde{\eta})$ be the surplus for $\tilde{q}^\ell(x,\tilde{\eta})$ and $\bar{S}^\ell = \int_0^1 \tilde{S}^\ell(\tilde{\eta})d\tilde{\eta}$, which can be approximated using a grid of $\tilde{\eta}$ values. We can get an approximate upper bound for surplus by solving the linear program

$$\max_{\rho_1,...,\rho_L} \sum_{\ell=1}^{L} \rho_\ell \bar{S}_\ell \text{ s.t. } F(r_m|x_m) = \sum_{\ell=1}^{L} \rho_\ell F(r_m - \sum_{j=1}^{J} \eta_j^\ell m_j(x_m)|x_m), (r_m,x_m) \in \Gamma, \rho_\ell \geq 0, \sum_{\ell=1}^{L} \rho_\ell = 1,$$

where Γ is a grid where the constraints are imposed. This is a linear program so computation is straightforward. However, as for other estimators of partially identified objects (e.g. Manski and Tamer, 2002), it may be important to include some slackness in the

constraints, by solving instead

$$\max_{\rho_1,...,\rho_L} \sum_{\ell=1}^{L} \rho_\ell \bar{S}_\ell \text{ s.t. } \sum_{(r_m,x_m)\in\Gamma} [F(r_m|x_m) - \sum_{\ell=1}^{L} \rho_\ell F(r_m - \sum_{j=1}^{J} \eta_j^\ell m_j(x_m)|x_m)]^2 \leq \varepsilon, \rho_\ell \geq 0, \sum_{\ell=1}^{L} \rho_\ell = 1.$$

for some $\varepsilon > 0$. This quadratic program can be computed using standard software.

This approach provides approximate bounds for surplus for a series approximation to the set of all demand processes that are consistent with a quantile demand $Q(\tau|x)$. Approximation to the true bounds depends on large L and J. The choice of J and L and the corresponding approximation and inference theory are beyond the scope of this paper. Note though that these bounds are of interest even for some fixed J. As long as $\breve{\eta}^\ell = 0$ for some ℓ the average quantile surplus will be between the bounds, so that the bounds give a measure of how much surplus can vary away from the quantile surplus for other random utility specifications consistent with the data. Further, if they turn out to be wider than bounds based on knowing income effects then we can be assured that using income effects produces narrower bounds, because increasing J will increase the width of the general bounds.

This series approximation approach provides a way of empirically implementing the RUM, i.e. of finding identified sets for objects of interest under revealed stochastic preference conditions. This approach differs from Kitamura and Stoye (2012) where revealed stochastic preference inequalities are imposed. Here we impose the Slutzky conditions on a grid and then interpolate between points using a series approximation. This approach relies on and exploits smoothness in underlying demand functions.

7 Empirical Application

The previous results are based on the average and distribution of demand for fixed price and income. These objects are identified when prices and income in the data are independent of preferences, i.e. when the data are (q_i, x_i), $(i = 1, ..., n)$ with $q_i = q_0(x_i, \eta_i)$ and x_i and η_i are statistically independent. In that case

$$E[q_i|x_i = x] = \bar{q}_0(x), \Pr(q_i \leq r|x_i = x) = F(r|x, q_0, G_0).$$

Here average demand is the conditional expectation of quantity given prices and income in the data, and similarly for the distribution of demand. The conditional expectation of quantity, and not some other function of quantity, such as the log, is special because it equals the average demand which is used in bounds based on income effects. Average demand could also be recovered from the conditional expectation of the share of income spent on q.

The conditional expectation $E[q_i|x_i = x]$ could be estimated by nonparametric regression, as we do in the gasoline demand application below. Alternatively, if there are many goods, so that nonparametric estimation is affected by the curse of dimensionality, a semiparametric or parametric estimate of the conditional expectation of quantity could be used. Those estimators could have functional form misspecification but are useful with high dimensional regressors.

Independence of η_i and x_i encompasses a statistical version of a fundamental hypothesis of consumer demand, that preferences do not vary with prices. It is also based on the individual being small relative to the market of observation, as would hold when different observations come from different markets. The independence of income from preferences has been a concern in some demand specifications where allowance is made for dynamic consumption, but is an important starting point and is commonly imposed in the gasoline demand application we consider.

Independence of η_i and x_i could be relaxed to allow for covariates. Consider an index specification where there are covariates w and it is assumed that there is a vector of functions $v(w, \delta)$ that affect utility such that η_i and $(x_i^T, w_i^T)^T$ are independent. These covariates might be demographic variables that represent observed components of the utility. For example, one could use a single, linear index $v(w, \delta) = w_1 + w_2^T \delta$, with the usual scale and location normalization imposed. The demand function $q_0(x, v(w, \delta_0), \eta)$ would then depend on the index $v(w, \delta_0)$, as would the average demand

$$\bar{q}_0(x, v(w, \delta_0)) = \int q_0(x, v(w, \delta_0), \eta) G(d\eta) = E[q_i|x_i = x, v(w_i, \delta_0) = v]$$

Here average demand is equal to a partial index regression of quantity q_i on x_i and $v(w_i, \delta)$.

Similar approaches to conditioning on covariates are common in demand analysis.

Endogeneity can be accounted for if there is an estimable control variable ξ such that x_i and η_i are independent conditional on ξ_i and the conditional support of ξ_i given x_i equals the marginal support of ξ_i. In that case it follows as in Blundell and Powell (2003) and Imbens and Newey (2009) that

$$\int E[q_i|x_i = x, \xi_i = \xi]F_\xi(d\xi) = \bar{q}_0(x), \int \Pr(q_i \leq r|x_i = x, \xi_i = \xi)F_\xi(d\xi) = F(r|x, q_0, G_0),$$

where $F_\xi(\xi)$ is the CDF of ξ_i. Although conditions for existence of a control variable are quite strong (see Blundell and Matzkin, 2014), this approach does provide a way to allow for some forms of endogeneity.

8 Estimation and Welfare Analysis of Gasoline Demand

In this section we estimate average consumer surplus and deadweight loss from changes in the gasoline tax in the US while allowing for unrestricted multidimensional individual heterogeneity. We use data from the 2001 U.S. National Household Transportation Survey (NHTS). This survey is conducted every 5-8 years by the Federal Highway Administration. The survey is designed to be a nationally representative cross section which captures 24-hour travel behavior of randomly-selected households. Data collected includes detailed trip data and household characteristics such as income, age, and number of drivers. We restrict our estimation sample to households with either one or two gasoline-powered cars, vans, SUVs and pickup trucks. We exclude Alaska and Hawaii. We use daily gasoline consumption, monthly state gasoline prices, and annual household income. The data we use consists of 8,908 observations. Summary statistics are given in Table 1. Note that the mean price of gasoline was \$1.33 per gallon with the mean number of drivers in a household equal to 2.04.

To estimate average gasoline demand we estimate up to a 4th degree polynomial with

interaction and predetermined variables along with price and income for household i:

$$\bar{q}(\widehat{x,w}) = \sum_{j,k,\ell=1}^{4} \hat{\beta}_{j,k,\ell} (\ln p)^j (\ln y)^k (v(w,\hat{\delta}))^\ell \tag{8.8}$$

We estimate equation (8.8) taking the price of gasoline as predetermined assuming a world market for gasoline. We also allow for the gasoline price to be jointly endogenous using state tax rates as instruments and also distance of the state from the Gulf of Mexico, as in Blundell, Horowitz and Parey (2012). Here we take a control function approach where in the first stage we use the instruments z_i, along with household income, and the predetermined variables w_i. We then take the estimated residuals from this first stage $\hat{\xi}_i$ and use them as a control function in equation (8.8), constructing

$$E[q_i|\widehat{x,w},\xi] = \sum_{j,k,\ell,m=1}^{4} \tilde{\beta}_{j,k,\ell} (\ln p)^j (\ln y)^k (w'\delta)^\ell (\hat{\xi})^m, \tag{8.9}$$

where $\tilde{\beta}_{j,k,\ell,m}$ are the coefficients from the regression of q_i on log price, income, the covariates index, and the first stage residual. The average demand is then estimated by averaging over the estimated residuals $\hat{\xi}_i$ holding p, y, and w fixed.

In Figure 1 we plot the OLS average demand estimate for monthly gasoline consumption. Note that it is generally downward sloping except at low prices. In Figure 2 we estimate the demand function using the control function approach and find it to be better behaved. In Table 2 we consider the estimated price elasticities for OLS and IV. We see that the estimated price elasticity has the incorrect sign for the 75th quantile for three out of the four specifications, while the IV estimates all have the correct sign. However, the IV estimates are somewhat large except perhaps for the 3rd and 4th order specification. In Table 3 we see that the estimated income elasticities for both OLS and IV are quite similar and also similar to previous estimates, e.g. Hausman and Newey (1995).

To set bounds on income effects we assume that gasoline is a normal good and so choose the lower bound $\underline{B}$ to be 0.0. To set the upper bound we estimate a local linear quantile regression of log of gasoline demand on log price and log income and evaluate the income derivative of the gasoline quantile at median price and income. We find that

this income effect is increasing in the quantile τ. We take the upper bound on the income effect to be .0197., which is 20 times the quantile derivative at $\tau = .9$. This income effect is very large, corresponding to more than two cents of every additional dollar of income being spent on gasoline. We are confident that no one would have such a large income effect for gasoline. Estimating linear, varying coefficients demand we find a precisely estimated mean income effect of .000726. From the squared residual regression the upper 95% confidence bound on the square root of the coefficient of y^2, which estimates the standard deviation of the income effect, is .00241. Here .0197 is well out in the distribution of income effects, implying the bound is approximately correct; see the Appendix.

In Figure 3 we graph the bounds on the monthly average equivalent variation for a price increase from the stated price on the lower axis to $1.40 per gallon. We use the estimates from the 3rd order power series, with a control function, evaluated at median income. Note that the lower bound and upper bound estimates are almost the same and it is difficult to distinguish between them. This result follows from the small share of gasoline expenditure in overall household expenditure. The results demonstrate that although the welfare function is not point identified, in this type of situation the upper and lower bound estimates are very similar.

In Figure 4 we graph the bounds on deadweight loss for a price increase from the price on the lower axis to $1.40. Again we use the 3rd order power series control function estimates evaluated at the median income. Again, the lower and upper bound estimates are quite similar and difficult to distinguish except for very low gasoline prices. Since deadweight loss is a second order calculation compared to the first order calculation of equivalent variation, e.g. Hausman (1981), the closeness of the bounding estimates allows for policy evaluation, even in the absence of point identification.

We now estimate confidence sets for our estimated bounds. We use the Beresteanu and Molinari (2008) confidence interval for the identified set with .95 coverage probability. Let the estimated set identification regions be given by $\hat{\Theta} = [\hat{\theta}_\ell, \hat{\theta}_u]$ and the joint estimated asymptotic variance matrix $\hat{\theta}_\ell$ and $\hat{\theta}_u$ be $\hat{\Sigma}$. It will follow from Hausman and Newey (1995) that the bounds are joint asymptotically normal. We form $\hat{\Sigma}$ by treating the model as

if it were parametric and applying the delta method, as works for series estimates, as in Newey (1997). Here we use the delta method on the estimated equation (5.6) with the nonparametric series estimator used in place of $\bar{q}$. The results are given in Table 4 for the equivalent variation estimates with the estimate standard errors in parenthesis and the 95% confidence intervals given in brackets. Concentrating on the 3rd order estimates which we plotted before we see that the estimated standard errors are quite small at both the lower bound and the upper bound and the 95th percentile confidence interval goes from \$13.72 to \$16.24 which is small enough for reliable policy analysis. In Table 5 we give the standard errors and bounds and the DWL estimates. Here we find that the standard errors are reasonably small but the estimate confidence intervals are sufficiently large to impact the policy analysis.

We also estimated the general bounds described in Section 6. We used a third order power series in $\ln p$ and $\ln y$ to estimate the quantile of $\ln q$ and for $m_j(x)$, where J corresponded to a cubic in logs specification, analogous to the main empirical specification with income effect bounds given above. We estimated the conditional quantile at many values of τ, imposed the Slutzky condition on a grid, and drew two sets of $L = 1000$ coefficients, with Slutzky imposed at the same values for τ and x as for quantile estimation, with more details given in the Appendix. We calculated the general surplus bounds for a price change from 1.10-1.45 accounting for the distribution constraints at five quantile values for r, including the median, and replacing $F(r|x)$ in the constraints by a smoothed version of $\hat{Q}^{-1}(r|x)$. The results for three values of ε are reported in Table 6. We find that, for the smallest value of ε, the bounds are informative but substantially wider in percentage terms than those we obtained with bounds on income effects. Thus, in this data the surplus bounds based on income effects turn out to be more informative than general bounds, as well as being easier to compute.

We have used our bounds approach to estimate household gasoline demand functions allowing for unrestricted heterogeneity. While the welfare measures are not point identified, we find that the lower and upper bound estimates are close to each other and provide precise information about exact surplus with general heterogeneity.

9 Appendix

In this Appendix we give proofs of the results in the paper along with some supplementary results.

9.1 Proofs of Theorems in the Paper

The following two technical conditions are referred to in the text and used in the proofs.

ASSUMPTION A1: *η belongs to a complete, separable metric space and $q(x,\eta)$ and $\partial q(x,\eta)/\partial x$ are continuous in (x,η).*

ASSUMPTION A2: *$\eta = (u,\varepsilon)$ for scalar ε and Assumption A1 is satisfied for $\eta = (u,\varepsilon)$ for a complete, separable metric space that is the product of a complete separable metric space for u with Euclidean space for ε, $q(x,\eta) = q(x,u,\varepsilon)$ is continuously differentiable in ε, there is $C > 0$ with $\partial q(x,u,\varepsilon)/\partial\varepsilon \geq 1/C$, $\|\partial q(x,\eta)/\partial x\| \leq C$ everywhere, ε_i is continuously distributed conditional on u_i, with conditional pdf $f_\varepsilon(\varepsilon|u)$ that is bounded and continuous in ε.*

Before proving Theorem 1 we give a result on the derivatives of the quantile with respect to x.

LEMMA A1: *If Assumptions 1 and A2 are satisfied then $q(x,\eta_i)$ is continuously distributed for each $x \in \chi$ and $\Pr(q(x,\eta_i) \leq r)$ and $Q(\tau|x)$ are continuously differentiable in r and x and for the pdf $f_q(r)$ of $q(x,\eta_i)$ at r,*

$$\begin{aligned} \frac{\partial \Pr(q(x,\eta_i) \leq r)}{\partial x} &= -f_q(r) E[\frac{\partial q(x,\eta_i)}{\partial x} | q(x,\eta_i) = r], \\ \frac{\partial Q(\tau|x)}{\partial x} &= E[\frac{\partial q(x,\eta_i)}{\partial x} | q(x,\eta_i) = Q(\tau|x)]. \end{aligned}$$

Proof: Let $F_\varepsilon(e|u) = \Pr(\varepsilon \leq e|u) = \int_{-\infty}^{e} f_\varepsilon(r|u)dr$. Then by the fundamental theorem of calculus, $F_\varepsilon(e|\eta)$ is differentiable in e, and the derivative $f_\varepsilon(e|u)$ is continuous in e by hypothesis. Let $q^{-1}(x,u,r)$ denote the inverse function of $q(x,u,\varepsilon)$ as a function of ε. Then

$$\Pr(q(x,\eta_i) \leq r) = E[1(\varepsilon_i \leq q^{-1}(x,u_i,r))] = E[F_\varepsilon(q^{-1}(x,u_i,r)|u_i)].$$

By the inverse function theorem $q^{-1}(x,u,r)$ is continuously differentiable in x and r, with

$$\partial q^{-1}(x,u,r)/\partial r = [\partial q(x,u,q^{-1}(x,u,r))/\partial \varepsilon]^{-1}, \partial q^{-1}(x,u,r)/\partial x = -\frac{\partial q(x,u,q^{-1}(x,u,r))/\partial x}{\partial q(x,u,q^{-1}(x,u,r))/\partial \varepsilon},$$

By Assumption A2 both $\partial q^{-1}(x,u,r)/\partial r$ and $\partial q^{-1}(x,u,r)/\partial x$ are bounded. Then by the chain rule $F(q^{-1}(x,u_i,r)|u_i)$ is differentiable in r and x with bounded continuous derivatives, so that $E[F_\varepsilon(q^{-1}(x,u_i,r)|u_i)]$ is differentiable in r and x with

$$\frac{\partial E[F_\varepsilon(q^{-1}(x,u_i,r)|u_i)]}{\partial r} = E[f_\varepsilon(q^{-1}(x,u_i,r)|u_i)\{\partial q(x,u,q^{-1}(x,u_i,r))/\partial \varepsilon\}^{-1}] = E[f_r(r|u_i)] = f_r(r),$$

where $f_r(r)$ and $f_r(r|u)$ are the marginal and conditional pdf of $q(x,\eta_i)$ respectively and the second equality follows by the change of variables $r = q(x,u_i,\varepsilon_i)$. Similarly,

$$\begin{aligned}
\frac{\partial E[F_\varepsilon(q^{-1}(x,u_i,r)|u_i)]}{\partial x} &= -E[f_\varepsilon(q^{-1}(x,u_i,r)|u_i)\frac{\partial q(x,u_i,q^{-1}(x,u_i,r))/\partial x}{\partial q(x,u_i,q^{-1}(x,u_i,r))/\partial \varepsilon}] \\
&= -\int f_{r,u}(r,u)[\frac{\partial q(x,u,q^{-1}(x,u,r))}{\partial x}]d\mu(u) \\
&= -f_r(r)E[\frac{\partial q(x,\eta_i)}{\partial x}|q(x,\eta_i)=r],
\end{aligned}$$

where $f_{r,u}(r,u)$ is a joint pdf with respect to the product of Lebesgue measure and a dominating measure μ for u_i, and the last equality follows by multiplying and dividing by $f_r(r)$. This results gives the first conclusion. The second conclusion follows by the inverse function theorem. Q.E.D.

Proof of Theorem 1: Since $q(x,\eta)$ satisfies Assumption 1 we have $\partial q(x,\eta_i)/\partial p + q(x,\eta_i)\partial q(x,\eta_i)/\partial p \leq 0$ for all η_i. Therefore, following Dette et. al. (2011), we have by Lemma A1 that for each τ with $0 < \tau < 1$ the quantile $Q(\tau|x)$ is continuously differentiable in x and

$$\begin{aligned}
\frac{\partial Q(\tau|x)}{\partial p} + Q(\tau|x)\frac{\partial Q(\tau|x)}{\partial y} &= E[\frac{\partial q(x,\eta_i)}{\partial p} + Q(\tau|x)\frac{\partial q(x,\eta_i)}{\partial y}|q(x,\eta_i)=Q(\tau|x)] \\
&= E[\frac{\partial q(x,\eta_i)}{\partial p} + q(x,\eta_i)\frac{\partial q(x,\eta_i)}{\partial y}|q(x,\eta_i)=Q(\tau|x)] \leq 0.
\end{aligned}$$

It is well known that with two goods and a continuously differentiable demand function that the Slutzky condition for the non numeraire good suffices for the function to be a demand function for $p > 0, y > 0$, giving the first conclusion.

For the second conclusion note that $\tilde{q}(x,\tilde{\eta})$ satisfies Assumption 1 for $0<\tilde{\eta}<1$ by $Q(\tau|x)$ being a continuously differentiable demand function and therefore the Slutzky condition being satisfied. Also, $Q(\tau|x)\leq r$ if and only $\tau\leq F(r|x,q,G)$ by the definition of $Q(\tau|x)$ and the properties of $F(r|x,q,G)$ as a function of r. To show this suppress the x,q,G arguments in F and Q. Note that by the definition $Q(\tau)=\inf\{\tilde{r}:F(\tilde{r})\geq\tau\}$ we have $Q(\tau)>r$ implies $\tau>F(r)$. Now suppose $\tau>F(r)$. By $F(r)$ continuous from the right there is $\varepsilon>0$ such that $F(\tilde{r})<\tau$ for $\tilde{r}\in[r,r+\varepsilon)$. Also, by $F(r)$ monotonic increasing, $F(\tilde{r})\geq\tau$ implies $\tilde{r}\geq r+\varepsilon$. Therefore $Q(\tau)>r$. It follows that $Q(\tau)>r$ if and only if $\tau>F(r)$. This also implies its contrapositive, $Q(\tau)\leq r$ if and only if $\tau\leq F(r)$. It then follows that

$$\int 1(\tilde{q}(x,\tilde{\eta})\leq r)\tilde{G}(d\tilde{\eta})=\int_0^1 1(Q(\tilde{\eta}|x)\leq r)d\tilde{\eta}=\int_0^1 1(\tilde{\eta}\leq F(r|x,q,G))d\tilde{\eta}=F(r|x,q,G).Q.E.D.$$

Proof of Theorem 2: In this proof we proceed by calculating the true average surplus and the quantile average surplus and finding that they are numerically different for the specification given in the statement of Theorem 2. We first consider the true average surplus for the demand specification

$$q(p,y,\eta)=\eta_1-p+\eta_2 y,\eta_1\sim U(0,1),\Pr(\eta_2=1/3)=\Pr(\eta_3=2/3)=1/2,$$

for a price change with $p^0=.1$, $p^1=.2$, and $\bar{y}=3/4$. Note that for all $\eta_1\in[0,1]$, $\eta_2\in\{1/3,2/3\}$, and $p\in[.1,.2]$ we have

$$\eta_1-p+\eta_2\bar{y}\geq 0,\quad p(\eta_1-p+\eta_2\bar{y})\leq\bar{y},\quad -1+\eta_2(\eta_1-p+\eta_2\bar{y})<0,$$

so that over the range of η and p we consider demand is positive, within the budget constraint, and satisfies the Slutzky condition.

Next, for a linear demand function (which has constant income effect η_3) and two goods we have

$$\begin{aligned}S(\eta) &= \Delta p\int_0^1 q(p^0+t\Delta p,\bar{y},\eta)\exp(-t\eta_3\Delta p)=\int_0^1(A+Bt)\exp(-Ct)dt\\ &= \frac{A}{-C}[\exp(-Ct)]_0^1+\frac{B}{-C}[t\exp(-Ct)]_0^1+\frac{B}{-C^2}[\exp(-Ct)]_0^1\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{C} + \frac{B}{C^2} - e^{-C}(\frac{A}{C} + \frac{B}{C} + \frac{B}{C^2}), \\
A &= \Delta p(\eta_1 + \eta_2 p^0 + \eta_3 \bar{y}), B = (\Delta p)^2 \eta_2, C = \eta_3 \Delta p.
\end{aligned}
$$

Note that A and B are linear in η_1 and η_2. Assuming that (η_1, η_2) is independent of η_3 gives

$$
\begin{aligned}
\int S(\eta) G(d\eta_1, d\eta_2, \eta_3) &= \frac{\bar{A}}{C} + \frac{\bar{B}}{C^2} - e^{-C}(\frac{\bar{A}}{C} + \frac{\bar{B}}{C} + \frac{\bar{B}}{C^2}), \\
\bar{A} &= \Delta p(\bar{\eta}_1 + \bar{\eta}_2 p^0 + \eta_3 \bar{y}), \bar{B} = (\Delta p)^2 \bar{\eta}_2.
\end{aligned}
$$

For $p^0 = .1$, $p^1 = .2$, $\bar{y} = 3/4, \bar{\eta}_1 = 1/2, \bar{\eta}_2 = -1$ and for η_3 equal to $1/3$ or $2/3$ with probability .5 we find that

$$
\bar{S} = 0.070673.
$$

Next we derive the quantile demand and then calculate average surplus. For clarity we do this when $\eta_1 \sim U(0,1)$, η_2 is a constant β, and when $\eta_3 = \underline{b}$ with probability π and $\eta_3 = \bar{b}$ with probability $(1-\pi)$, where $\bar{b} > \underline{b}$ and η_3 is independent of η_1. Define

$$
r_0 = \beta p + \underline{b} y, \quad r_1 = \beta p + \bar{b} y, \quad r_2 = \beta p + 1 + \underline{b} y, \quad r_3 = \beta p + 1 + \bar{b} y.
$$

Assuming that $1 + \underline{b} y > \bar{b} y$ we have that

$$
\begin{aligned}
F(r|x) &= \Pr(q(x,\eta) \le r) = \Pr(\eta_1 + \beta p + \eta_3 y \le r) \\
&= \begin{cases}
0, & r < r_0 \\
\pi(r - r_0), & r_0 \le r < r_1 \\
(r - r_1) + F(r_1|x), & r_1 \le r < r_2 \\
(1-\pi)(r - r_2) + F(r_2|x), & r_2 \le r < r_3 \\
1 & r \ge r_3
\end{cases}.
\end{aligned}
$$

Note that this CDF is a mixture, over two values of η_3, of two CDFs for a $U(0,1)$. It has slope π or $1-\pi$ over the ranges where only one CDF is increasing and slope 1 where both are increasing. Inverting this function as a function of r gives the corresponding quantile function

$$
Q(\tau|p,y) = \begin{cases}
r_0 + \frac{\tau}{\pi}, & 0 < \tau \le \pi(r_1 - r_0) \\
r_1 + \tau - \pi(r_1 - r_0), & \pi(r_1 - r_0) < \tau < r_2 - r_1 + \pi(r_1 - r_0) \\
r_2 + \frac{\tau - [r_2 - r_1 + \pi(r_1 - r_0)]}{1-\pi}, & r_2 - r_1 + \pi(r_1 - r_0) \le \tau < 1.
\end{cases}
$$

In terms of the original parameters the quantile function is given by

$$Q(\tau|p,y) = \begin{cases} \beta p + \underline{b}y + \frac{\tau}{\pi}, & 0 < \tau \leq \pi(\bar{b} - \underline{b})y \\ \beta p + \bar{b}y + \tau - \pi(\bar{b} - \underline{b})y, & \pi(\bar{b} - \underline{b})y < \tau < 1 - (1-\pi)(\bar{b} - \underline{b})y \\ 1 + \beta p + \underline{b}y + \frac{\tau - [1 - (1-\pi)(\bar{b} - \underline{b})y]}{1-\pi}, & 1 - (1-\pi)(\bar{b} - \underline{b})y \leq \tau < 1 \end{cases}$$

Plugging in $\beta = -1$, $\pi = .5$, $\underline{b} = 1/3$, and $\bar{b} = 2/3$ we obtain the quantile demand implied by the true model, equaling

$$Q(\tau|p,y) = \begin{cases} -p + y/3 + 2\tau, & 0 < \tau \leq y/6 \\ -p + y/2 + \tau, & y/6 < \tau < 1 - y/6 \\ -p + 2y/3 + 2\tau - 1 & 1 - y/6 \leq \tau < 1 \end{cases}.$$

We can rewrite this as a function of y for given $\tau = \tilde{\eta}$ as

$$\tilde{q}(p,y,\tilde{\eta}) = \begin{cases} -p + 1(y < 6\tilde{\eta})\,(y/2 + \tilde{\eta}) + 1(y \geq 6\tilde{\eta})(y/3 + 2\tilde{\eta}) & \tilde{\eta} \leq 1/2 \\ -p + 1(y < 6(1-\tilde{\eta}))\,(y/2 + \tilde{\eta}) + 1(y \geq 6(1-\tilde{\eta}))(2y/3 + 2\tilde{\eta} - 1) & \tilde{\eta} > 1/2 \end{cases},$$

where we have used the fact that we only need to evaluate this demand where $y < 3$. This is the quantile demand function, which is observationally equivalent to the true demand by construction. It is nonlinear in y, with an income effect that varies as y crosses over a threshold.

Note that $y \leq 3/4$ is in the income range relevant for our calculation. When $\tilde{\eta} \in [1/8, 7/8]$ the demand function will be linear income over the evaluation range for the consumer surplus calculation, with income effect equal to $1/2$. For smaller $\tilde{\eta}$ or values of $\tilde{\eta}$ closer to 1 the income effect can change with y. The mix of nonlinearities that is evident in the comparison of this complicated demand function with the simple true linear, varying coefficients specification results in the quantile average surplus being different from the true average surplus.

Because the demand function is nonlinear in y for $\tilde{\eta} \notin [1/8, 7/8]$ we compute surplus numerically for each value of $\tilde{\eta}$ and then average. We do this by drawing $50,000$ values of $\tilde{\eta}$ from a $U(0,1)$, computing the equivalent variation from a price change with $p^0 = .1, p^1 = .2, \bar{y} = 3/4, \pi = 1/2, \underline{b} = 1/3, \bar{b} = 2/3$, and averaging across the draws to obtain an average quantile surplus of 0.070774. This value is different than the true average surplus computed above. Therefore, average surplus for the observationally equivalent

quantile demand is different than for true demand and hence average surplus is not identified. *Q.E.D.*

Proof of Theorem 3: By $q(p, y, \eta)$ differentiable in y with derivative that is continuous in p and y we know that the solution $s(t, \eta)$ to the differential equation (3.3) exists and is unique. By condition i) we have $ds(t, \eta)/dt \leq 0$ so that $s(t, \eta) \geq 0$ for all $t \in [0, 1]$ by $s(1, \eta) = 0$. Let

$$s_B(t, \eta) = e^{Bt} \int_t^1 [q(p(s), \bar{y}, \eta)^T \frac{dp(s)}{ds}] e^{-Bs} ds$$

be the solution to

$$\begin{aligned} \frac{ds_B(t, \eta)}{dt} &= a(t, \eta) + B s_B(t, \eta), \qquad s_B(1, \eta) = 0, \\ a(t, \eta) &= -q(p(t), \bar{y}, \eta)^T \frac{dp(t)}{dt}. \end{aligned} \tag{9.10}$$

Then expanding the right-hand side of equation (3.3) around $s = 0$ it follows that

$$\begin{aligned} \frac{ds(t, \eta)}{dt} &= -\left[q(p(t), \bar{y}, \eta) - \frac{\partial q(p, \bar{y} - \dot{s}(t, \eta), \eta)}{\partial y} s(t, \eta) \right]^T dp(t)/dt \\ &\geq -q(p(t), \bar{y}, \eta)^T dp(t)/dt + \underline{B} s(t, \eta) = \frac{ds_{\underline{B}}(t, \eta)}{dt}, \end{aligned} \tag{9.11}$$

where $\dot{s}(t, \eta)$ is a mean value in $[0, s(t, \eta)]$. Since $s(1, \eta) = 0 = s_{\underline{B}}(1, \eta)$ it follows by this inequality that $s_{\underline{B}}(t, \eta) \geq s(t, \eta)$. It follows similarly that $s_{\overline{B}}(t, \eta) \leq s(t, \eta)$, so that evaluating at $t = 0$ we have

$$s_{\overline{B}}(0, \eta) \leq s(0, \eta) = S(\eta) \leq s_{\underline{B}}(0, \eta). \tag{9.12}$$

Evaluating at $t = 0$ we have $s_B(0, \eta) = \int_0^1 q(p(t), \bar{y}, \eta)^T [dp(t)/dt] e^{-Bt} dt$. Also note that $dp(t)/dt$ is bounded by continuity of $dp(t)/dt$ on $[0, 1]$, and that by all the elements of $p(t)$ bounded away from zero and $q(p(t), \bar{y}, \eta)^T p(t) \leq \bar{y}$ the demand vector $q(p(t), \bar{y}, \eta)$ is bounded uniformly in η and t. Therefore by the Fubini theorem,

$$E[s_B(0, \eta_i)] = \int_0^1 \bar{q}(p(t), \bar{y})^T [dp(t)/dt] e^{-Bt} dt = \bar{S}_B.$$

Taking expectations of equation (9.12) then gives

$$\bar{S}_{\overline{B}} = E[s_{\overline{B}}(0,\eta_i)] \leq \bar{S} \leq E[s_{\underline{B}}(0,\eta_i)] = \bar{S}_{\underline{B}}. \quad Q.E.D.$$

Proof of Corollary 4: It follows by Lemma A1 that $Q_1(\tau|x)$ is continuously differentiable in x and

$$\frac{\partial Q_1(\tau|x)}{\partial p} = E[\frac{\partial q_1(x,\eta_i)}{\partial p}|q_1(x,\eta_i) = Q_1(\tau|x)], \frac{\partial Q_1(\tau|x)}{\partial y} = E[\frac{\partial q_1(x,\eta_i)}{\partial y}|q_1(x,\eta_i) = Q_1(\tau|x)].$$

As shown by Dette, Hoderlein, and Neumeyer (2011), it follows that the Slutzky condition for the first price is satisfied by the conditional quantile, i.e.

$$\frac{\partial Q_1(\tau|x)}{\partial p_1} + Q_1(\tau|x)\frac{\partial Q_1(\tau|x)}{\partial y} \leq 0.$$

Therefore, at each $0 < \tau < 1$, $Q_1(\tau|x)$ is a demand function as a function of p_1 and y. Furthermore, by $\underline{B} \leq \Delta p_1 \partial q_1(x,\eta)/\partial y \leq \overline{B}$ we have

$$\underline{B} \leq E[\Delta p_1 \frac{\partial q_1(x,\eta_i)}{\partial y}|q_1(x,\eta_i) = Q_1(\tau|x)] = \Delta p_1 \frac{\partial Q_1(\tau|x)}{\partial y} \leq \overline{B}.$$

Consider the demand process $\tilde{q}_1(x,\tilde{\eta}) = Q_1(\tilde{\eta}|x)$ for $\tilde{\eta} \sim U(0,1)$. Note that $\int_0^1 S^\tau d\tau$ is average surplus for this demand process. Also,

$$\int \tilde{q}_1(x,\tilde{\eta})\tilde{G}(d\tilde{\eta}) = \int_0^1 Q_1(\tilde{\eta}|x)d\tilde{\eta} = \bar{q}_1(x).$$

Therefore, the conclusion follows by the conclusion of Theorem 3. $Q.E.D.$

9.2 The Expenditure Function and Exact Consumer Surplus for Discrete and Continuous Choice

Discrete and continuous choice models are important in applications. For instance, gasoline demand could be modeled as gasoline purchases that are made jointly with the purchase of automobiles. In those models the heterogeneity can influence the discrete choices as well as the demand for a particular commodity, e.g. see Dubin and McFadden (1984) and Hausman (1985). Multiple sources of heterogeneity are an integral part of

these models, with separate disturbances for discrete and continuous choices. The general heterogeneity we consider allows for such multi dimensional heterogeneity. Here we consider discrete and continuous choice with general heterogeneity, focusing on the effect of price changes in the continuous demand. Bhattacharya (2014) has recently considered surplus for changes in the prices of the discrete alternatives with general heterogeneity.

We first consider the individual choice problem and the associated expenditure function. We adopt the framework of Dubin and McFadden (1984) and Hausman (1985), extending previous results to the expenditure function. Suppose that the agent is choosing among J discrete choices in addition to choosing q. The consumer choice problem is

$$\max_{j,q,a} U_j(q,a,\eta) \text{ s.t. } p^T q + a + r_j \leq y \tag{9.13}$$

where r_j is the usage price of choice j relative to the price of the numeraire good a. Here we assume that for each η and j the function $U_j(q,a)$ is strictly quasi-concave (preferences are strictly convex) and satisfies local nonsatiation. Let

$$q_j(p,y,\eta) = \arg\max_q U_j(q,a,\eta) \text{ s.t. } p^T q + a \leq y$$

be the demand function associated with the j^{th} utility function and let

$$V_j(p,y,\eta) = U_j(q_j(p,y,\eta), y - p^T q_j(p,y,\eta), \eta)$$

be the associated indirect utility function. The utility maximizing choice of the discrete good will be $\arg\max_j V_j(p, y - r_j, \eta)$ and the indirect utility function will be $V(p,r,y,\eta) = \max_j V_j(p, y - r_j, \eta)$, where $r = (r_1, ..., r_J)^T$. When there is a unique discrete choice j (depending on p, r, y, and η) that maximizes utility, i.e. where $V_j(p, y - r_j, \eta) > V_k(p, y - r_k, \eta)$ for all $k \neq j$, the demand $q(p,y,r,\eta)$ will be

$$q(p,r,y,\eta) = q_j(p, y - r_j, \eta).$$

When there are multiple values of the discrete choice that maximize utility the demand will generally be a correspondence, containing one point for each value of j that maximizes utility.

In what follows we will assume that $(V_1(p, y-r_1, \eta), ..., V_J(p, y-r_J, \eta))$ is continuously distributed and that the probability of ties is zero. Nevertheless the case with ties is important for us. Surplus is calculated by integrating the demand function as price changes while income is compensated to keep utility constant. As compensated income changes ties may occur and the demand for q may jump. With gasoline demand, compensated income changes could result in a choice of car with different gas mileage, leading to a jump. Such jumps must be accounted for in the bounds analysis.

Turning to welfare analysis, let $e(p, r, u, \eta)$ denote the expenditure function in this discrete/continuous choice setting, defined as

$$e(p, r, u, \eta) = \min\{y \text{ s.t. } \max_{j,q,a}\{U_j(q, a, \eta) \text{ s.t. } p^T q + a + r_j \leq y\} \geq u\}.$$

As usual it is the minimum value of income that allows individual η to attain utility level u. There is a simple, intuitive relationship between this expenditure function, the ones associated with the continuous choice of q for each j, and the indirect utility function $V(p, r, y, \eta) = \max_j V_j(p, y - r_j, \eta)$. Let $e_j(p, u, \eta) = \min_{q,a}\{p^T q + a : U_j(q, a, \eta) \geq u\}$ be the expenditure function for the utility function $U_j(q, a, \eta)$, $(j = 1, ..., J)'$.

LEMMA A2: *If for each j and η the utility $U_j(q, a, \eta)$ is strictly quasi-concave and satisfies local nonsatiation then $e(p, u, \eta) = \min_j\{e_j(p, u, \eta) + r_j\}$, $V(p, r, e(p, r, u, \eta), \eta) = u$, and $e(p, r, V(p, r, y, \eta), \eta) = y$.*

Proof: For notational convenience drop the η argument. Define $\bar{e}(p, r, u) = \min_j\{e_j(p, u) + r_j\}$. By the definition of $\bar{e}(p, r, u)$ it follows that $\bar{e}(p, r, u) = e_{j^*}(p, u) + r_{j^*}$ for some j^* that need not be unique. By the definition of $e_{j^*}(p, u)$ and standard results there is q^* such that $U_{j^*}(q^*, a^*) \geq u$ and $p^T q^* + a^* = e_{j^*}(p, u)$, so $p^T q^* + a^* + r_{j^*} = \bar{e}(p, r, u)$. Since $U_{j^*}(q^*, a^*) \geq u$ and $p^T q^* + a^* + r_{j^*} \leq \bar{e}(p, r, u)$, it follows that

$$\max_{q,j}\{U_j(q, a, j) \text{ s.t. } p^T q + a + r_j \leq \bar{e}(p, r, u)\} \geq U_j(q^*, a^*) \geq u.$$

It follows that $e(p, r, u) \leq \bar{e}(p, r, u)$. Next, consider any $\bar{y} < \bar{e}(p, r, u)$. Then by the

definition of $\bar{e}(p,r,u)$ we have $\bar{y}-r_j < e_j(p,u)$ for all $j \in \{1,\ldots,J\}$. Since $e_j(p,u)$ is the expenditure function it follows that $\max_{q,a}\{U_j(q,a) \text{ s.t. } p^Tq+a \leq \bar{y}-r_j\} < u$ for every j, and so $\max_{q,a,j}\{U_j(q,a) \text{ s.t. } p^Tq+a \leq \bar{y}-r_j\} < u$. It follows that $\bar{y} < e(p,r,u)$. Since this is true for every $\bar{y} < \bar{e}(p,r,u)$ it follows that $\bar{e}(p,r,u) = e(p,r,u)$.

Next, note that by the definition of the expenditure function $e(p,r,u)$ as the minimum income level that will allow an individual to reach utility u we have $V(p,r,e(p,r,u)) \geq u$. Also, $V(p,r,y)$ is monotonically increasing in y by $V_j(p,y-r_j)$ monotonically increasing in y for each j and $V_k(p,e_k(p,u)) = u$ by standard results for indirect utility and expenditure functions. By the definition of $V(p,r,y)$ and monotonicity of $V_j(p,y-r_j)$ in y there is j with

$$V(p,r,e(p,r,u)) = V_j(p,e(p,u)-r_j) \leq V_j(p,e_j(p,u)) = u,$$

where the inequality holds by the first conclusion that implies $e(p,u) \leq e_j(p,u)+r_j$. Therefore we have $V(p,e(p,r,u),r) = u$. Similarly, we have $e(p,r,V(p,r,y)) \leq y$ by the definitions and there is j such that by $e_j(p,u)$ increasing in u,

$$e(p,r,V(p,r,y)) = e_j(p,V(p,r,y))+r_j \geq e_j(p,V_j(p,y-r_j))+r_j = y,$$

so that $e(p,r,V(p,r,y)) = y$. *Q.E.D.*

Turning now to exact surplus for discrete/continuous choice, the equivalent variation for a price change from p^0 to p^1 with income $\bar{y}$ for individual η is $S(\eta) = \bar{y}-e(p^0,r,u^1,\eta)$, where u^1 is the utility at p^1, r, and $\bar{y}$. Consider a price path $p(t)$ as in the body of the paper. Then $s(t,\eta) = \bar{y}-e(p(t),r,u^1,\eta)$ is the equivalent variation for a price change from $p(t)$ to p^1 for income $\bar{y}$, where u^1 is the utility at p^1. The next result gives conditions for $s(t,\eta)$ to satisfy the same differential equation as in the continuous case.

LEMMA A3: *If for each j and η the utility $U_j(q,a,\eta)$ is strictly quasi-concave and satisfies local nonsatiation then at any $p(t)$ and η such that there is j with $V_j(p(t),\bar{y}-s(t,\eta)-r_j,\eta) > V_k(p(t),\bar{y}-s(t,\eta)-r_k,\eta)$ for all $k \neq j$, it follows that $s(t,\eta)$ is differentiable and*

$$\frac{ds(t,\eta)}{dt} = -q(p(t),r,\bar{y}-s(t,\eta),\eta)^T dp(t)/dt.$$

Proof: For notational convenience suppress the η argument and let $p = p(t)$. By definition we have $s(t,\eta) = \bar{y} - e(p(t), r, u^1, \eta)$. Consider j^* such that

$$V(p, r, e(p, r, u^1)) = V_{j^*}(p, e(p, r, u^1) - r_{j^*}).$$

For any $k \neq j^*$ it follows by duality that $V_k(p, e_k(p, u^1)) = u^1$. Therefore, we have

$$V_k(p, e_k(p, u^1)) = u^1 = V(p, r, e(p, r, u^1)) = V_{j^*}(p, e(p, r, u^1) - r_{j^*}) > V_k(p, e(p, r, u^1) - r_k).$$

By $V_k(p, y)$ monotonically increasing in y, it follows that $e_k(p, u^1) > e(p, r, u^1) - r_k$. Since this is true for every $k \neq j^*$ we have

$$e(p, r, u^1) = e_{j^*}(p, u^1) + r_{j^*} < e_k(p, u^1) + r_k, \text{ for all } k \neq j.$$

Also note that by standard duality results, for the Hicksian demand $h_{j^*}(p, u)$

$$\begin{aligned} \frac{\partial e_j(p, u^1)}{\partial p} &= h_{j^*}(p, u) = q_{j^*}(p, e_{j^*}(p, u)) = q_{j^*}(p, e(p, r, u^1) - r_{j^*}) \\ &= q_{j^*}(p, \bar{y} - s(t) - r_{j^*}) = q(p, r, \bar{y} - s(t)), \end{aligned}$$

where the last equality follows by the $q(p, r, y) = q_{j^*}(p, y - r_{j^*})$ when $V_{j^*}(p, y - r_{j^*}) > V_k(p, y - r_k)$ for all $k \neq j^*$. Since each $e_k(p, u^1)$ is continuous in p, the previous inequality continues to hold in a neighborhood of p. Therefore, by $e_{j^*}(p, u^1)$ differentiable, Shephard's lemma, and the chain rule, on that neighborhood $s(t) = y - e(p(t), r, u^1)$ is differentiable and

$$\frac{ds(t)}{dt} = -\frac{de(p(t), r, u^1)}{dt} = -\frac{\partial e_j(p(t), u^1)}{\partial p}^T \frac{dp(t)}{dt} = -q(p(t), r, \bar{y} - s(t))^T \frac{dp(t)}{dt}.$$

Q.E.D.

The discontinuity of individual demand does affect the bounds for average consumer surplus. The previous bounds depend on income effects. With jumps we construct bounds that are based on limits on the size of the jump and on the proportion of individuals whose demand would jump as income is compensated along with the price change. For that purpose we make use of a demand decomposition into continuous and jump components.

ASSUMPTION A3: *There are functions* $\dot{q}(p,r,y,\eta)$, $\check{q}(p,r,y,\eta)$, $A(\eta)$ *and constants* $\underline{B}$, $\overline{B}$ *such that* $\bar{A} = E[A(\eta)]$ *exists and for* $t \in [0,1]$, *and* $0 \leq s \leq s(t,\eta)$,

$$\begin{aligned} q(p(t),r,\bar{y}-s,\eta) &= \dot{q}(p(t),r,\bar{y}-s,\eta) + \check{q}(p(t),r,\bar{y}-s,\eta), \\ \left|\check{q}(p(t),r,\bar{y}-s,\eta)^T dp(t)/dt\right| &\leq A(\eta), \\ \underline{B}s &\leq \left[\dot{q}(p(t),r,\bar{y},\eta) - \dot{q}(p(t),r,\bar{y}-s,\eta)\right]^T dp(t)/dt \leq \overline{B}s. \end{aligned}$$

Here we assume that the demand function can be decomposed into a jump component $\check{q}(p,r,y,\eta)$ and a Lipschitz continuous component $\dot{q}(p,r,y,\eta)$, with lower and upper bounds $\underline{B}$ and $\overline{B}$, respectively, on how much $\dot{q}(p(t),r,\bar{y}-s,\eta)^T dp(t)/dt$ may vary with $s > 0$. The term $A(\eta)$ is an individual specific bound on the jump. It will be zero for individuals whose demand function does not jump as income is compensated up to the surplus amount $S(\eta) = s(0,\eta)$. For example, for gasoline demand it will be zero for individuals who would not change car types over the range of income being compensated.

To describe bounds on average surplus that allow for jumps, let

$$\bar{s}_{a,B}(t) = e^{Bt}\int_t^1 [\bar{q}(p(s),r,\bar{y})^T dp(s)/ds - a]e^{-Bs}ds$$

be the solution to the differential equation

$$\frac{d\bar{s}_{a,B}(t)}{dt} = -\bar{q}(p(t),r,\bar{y})^T\frac{dp(t)}{dt} + a + B\bar{s}_{a,B}(t), \qquad \bar{s}_{a,B}(1) = 0.$$

Letting $\bar{S}_{a,B} = \bar{s}_{a,B}(0)$, we have

$$\bar{S}_{a,B} = \int_0^1 [\bar{q}(p(t),r,\bar{y})^T dp(t)/dt]e^{-Bt} + \frac{a}{B}\left(e^{-B}-1\right).$$

THEOREM A4*: If Assumptions 1, A1, and A3 are satisfied, the elements of* $p(t)$ *are bounded away from zero, and with probability one for all but a finite number of* t *values there is* j *with* $V_j(p(t),\bar{y}-s(t,\eta)-r_j,\eta) > V_k(p(t),\bar{y}-s(t,\eta)-r_k,\eta)$ *for all* $k \neq j$, *it follows that*

$$\bar{S}_{2\bar{A},\overline{B}} \leq \bar{S} \leq \bar{S}_{-2\bar{A},\underline{B}}.$$

Also, if $q(p(t), r, y - s, \eta)^T dp(t)/dt \leq q(p(t), r, y, \eta)^T dp(t)/dt$ *for all* $t \in [0, 1]$ *and* $s \in [0, s(t, \eta)]$ *then* $\bar{S} \leq \bar{S}_0 = \int_0^1 [\bar{q}(p(t), r, y)^T dp(t)/dt] dt$.

Proof: For notational convenience suppress the η argument. Let

$$s_{a,B}(t) = e^{Bt} \int_t^1 [q(p(s), r, \bar{y})^T dp(s)/ds - a] e^{-Bs} ds$$

be the solution to the differential equation

$$\frac{ds_{a,B}(t)}{dt} = -q(p(t), r, \bar{y})^T dp(t)/dt + a + B s_{a,B}(t), \qquad s_{a,B}(1) = 0.$$

By Assumption A3

$$\begin{aligned} & [q(p(t), r, \bar{y}) - q(p(t), r, \bar{y} - s)]^T \frac{dp(t)}{dt} \\ = \; & [\dot{q}(p(t), r, \bar{y}) - \dot{q}(p(t), r, \bar{y} - s) + \check{q}(p(t), r, \bar{y}) - \check{q}(p(t), r, \bar{y} - s)]^T \frac{dp(t)}{dt} \leq \overline{B} s + 2A, \end{aligned}$$

Therefore, by Lemma A3 it follows that at any point where $V_j(p(t), \bar{y} - s(t) - r_j) > V_k(p(t), \bar{y} - s(t) - r_k)$ for all $k \neq j$, $s(t)$ is differentiable and

$$\frac{ds(t)}{dt} = -q(p(t), r, \bar{y} - s(t))^T \frac{dp(t)}{dt} \leq -q(p(t), r, \bar{y})^T \frac{dp(t)}{dt} + \overline{B} s(t) + 2A = \frac{ds_{2A,\overline{B}}(t)}{dt}.$$

Note that $s(t)$ is continuous by continuity of the expenditure function and $p(t)$. Consider the event $\mathcal{E}$ where there are no ties in the values of the indirect utility functions (i.e. where there is j^* depending on t such that $V_{j^*}(p(t), \bar{y} - s(t, \eta) - r_{j^*}) > V_k(p(t), \bar{y} - s(t, \eta) - r_k)$ for all $k \neq j^*$), at all t except a finite number. When $\mathcal{E}$ occurs we have

$$s(t) = -\int_t^1 \frac{ds(u)}{du} du.$$

Similarly we have $s_{2A,\overline{B}}(t) = -\int_t^1 \left[ds_{2A,\overline{B}}(u)/du\right] du$. Then by $ds(t)/dt \leq ds_{2A,\overline{B}}(t)/dt$ it follows that $s(t) \geq s_{2A,\overline{B}}(t)$. Evaluating at $t = 0$ we get $S \geq s_{2A,\overline{B}}(0)$. It follows similarly that $S \leq s_{-2A,\underline{B}}(0)$. Thus, adding back the η notation, when the event $\mathcal{E}$ occurs we have

$$s_{2A,\overline{B}}(0, \eta) \leq S(\eta) \leq s_{-2A,\underline{B}}(0, \eta)$$

Also, it follows similarly to the proof of Theorem 3 that

$$E[s_{a,B}(0, \eta)] = \bar{S}_{a,B}.$$

Since $\Pr(\mathcal{E}) = 1$ taking expectations through the previous inequality gives the first conclusion.

For the second conclusion note that, $q(p(t), r, y - s(t, \eta), \eta)^T dp(t)/dt \leq q(p(t), r, y, \eta)^T dp(t)/dt$, so that

$$\frac{ds(t, \eta)}{dt} = -q(p(t), r, \bar{y} - s(t), \eta)^T \frac{dp(t)}{dt} \geq -q(p(t), r, \bar{y}, \eta)^T \frac{dp(t)}{dt}.$$

The second conclusion then follows similarly to the first one. $Q.E.D.$

These bounds adjust for the possible presence of discontinuity in individual demands by adding $2E[D(\eta)]$ to $-\bar{q}(p, y)$ in the equation for the upper bound and subtracting the same term in the equation for the lower bound. This adjustment will be small when the largest possible jump is small or when the proportion of individuals with a discontinuity is small. One can drop this term for the bound for normal goods.

9.3 Generalized Conditions for Bounds on Exact Consumer Surplus

The purpose of this section is to show that known bounds on income effects are not required for validity of the bounds in Theorem 3. To describe this result, let

$$B_u(\eta) = \max_{t \in [0,1], s \in [0, S(\eta)]} \frac{\partial q(p(t), y - s, \eta)}{\partial y}^T \frac{dp(t)}{dt}.$$

This bound is an individual specific upper bound for income effects. Such bounds always exist for continuous demand functions. This can be thought of as an individual specific version of the income effect bounds. Also let

$$\begin{aligned}
S_u(\eta) &= \int_0^1 [q(p(t), \bar{y}, \eta)^T dp(t)/dt] e^{-B_u(\eta)t} dt, \\
\bar{S}'_u &= \int 1(B_u(\eta) \geq B) S_u(\eta) G(d\eta), \bar{S}''_u = \int 1(B_u(\eta) < B) S_u(\eta) G(d\eta), \\
S_B(\eta) &= \int_0^1 [q(p(t), \bar{y}, \eta)^T dp(t)/dt] e^{-Bt} dt, \\
\bar{S}'_B &= \int 1(B_u(\eta) \geq B) s_B(\eta) G(d\eta), \bar{S}''_B = \int 1(B_u(\eta) < B) s_B(\eta) G(d\eta).
\end{aligned}$$

We have the following result:

THEOREM A5: *Suppose that Assumptions 1 and A1 are satisfied, i)* $q(x,\eta)^T dp(t)/dt \geq 0$; *and ii) all prices in* $p(t)$ *are bounded away from zero. If* $\bar{S}'_B - \bar{S}'_u \leq \bar{S}''_u - \bar{S}''_B$ *then*

$$\bar{S} \geq \bar{S}_B, \bar{D} \geq \bar{S}_B - \bar{q}(p^1, \bar{y})^T \Delta p.$$

Also, if $\bar{S}'_B \leq c$ *then*

$$\bar{S} \geq \bar{S}_B - c, \bar{D} \geq \bar{S}_B - c - \bar{q}(p^1, \bar{y})^T \Delta p.$$

Proof: Let note that $S_u(\eta) = s_u(0,\eta)$ where

$$s_u(t,\eta) = e^{B_u(\eta)t} \int_t^1 [q(p(s), \bar{y}, \eta)^T dp(s)/ds] e^{-B_u(\eta)s} ds$$

is the solution to

$$\begin{aligned} \frac{ds_u(t,\eta)}{dt} &= a(t,\eta) + B_u(\eta) s_u(t,\eta), \qquad s_u(1,\eta) = 0, \\ a(t,\eta) &= -q(p(t), \bar{y}, \eta)^T \frac{dp(t)}{dt}. \end{aligned}$$

It follows exactly in the proof of Theorem 3 that $S(\eta) \geq S_u(\eta)$, so that

$$\bar{S}_u = E[S_u(\eta)] \leq \bar{S}.$$

Also, we have

$$\bar{S}_u = \bar{S}'_u + \bar{S}''_u, \bar{S}_B = \bar{S}'_B + \bar{S}''_B.$$

Therefore, $\bar{S}'_B - \bar{S}'_u \leq \bar{S}''_u - \bar{S}''_B$ if and only if $\bar{S}_B \leq \bar{S}_u$, which implies $\bar{S}_B \leq \bar{S}$.

Now suppose $\bar{S}'_B \leq c$. Note that $\bar{S}''_B \leq \bar{S}''_u$ and $\bar{S}'_u \geq 0$. Then

$$\bar{S}_B - c = \bar{S}''_B + \bar{S}'_B - c \leq \bar{S}''_B \leq \bar{S}''_u \leq \bar{S}_u \leq \bar{S}. \quad Q.E.D.$$

The first conclusion of this result gives a more general condition for validity of the bounds. Although the result is simple the decomposition helps clarify that the surplus bounds hold over a much wider class of conditions than just bounded income effects. When B is well into the tail of the distribution of $B_u(\eta)$ it should be the case that $\bar{S}'_B - \bar{S}'_u$ is small while $\bar{S}''_u - \bar{S}''_B$ is large, leading to the bounds being satisfied.

The second conclusion gives a more general bound that may sometimes be applicable. For example, suppose that only the price of the first good is changing and let $\bar{Q}_B = \sup_{0\leq t\leq 1} E[q_1(p(t), \bar{y}, \eta)|B_u(\eta) \geq B]$. Then by the usual Chebyshev inequality type argument,

$$\begin{aligned}
\bar{S}'_B &= \int 1(B_u(\eta) \geq B)\{\int_0^1 [q(p(t), \bar{y}, \eta)^T dp(t)/dt]e^{-Bt}dt\}G(d\eta) \\
&= \Delta p_1 \int_0^1 \Pr(B_u(\eta) \geq B)E[q_1(p(t), \bar{y}, \eta)|B_u(\eta) \geq B]e^{-Bt}dt \\
&\leq \bar{Q}_B \Delta p_1 \left(\int_0^1 e^{-Bt}dt\right) \Pr(B_u(\eta) \geq B) \leq \frac{\bar{Q}_B \Delta p_1 E[B_u(\eta)^r]}{B^r},
\end{aligned}$$

where the last inequality follows by $e^{-Bt} \leq 1$, and hence $\int_0^1 e^{-Bt}dt \leq 1$, and by the Holder inequality.

9.4 Bounding Surplus Bound Error in Gasoline Application

This reasoning just above applies to the justification of the lower bound for surplus in the gasoline demand example. In a linear varying coefficients model we estimate the bounding term in the above equation for $r = 2$ to be

$$\frac{\bar{Q}_B \Delta p_1 E[B_u(\eta)^2]}{B^2} = \frac{\bar{Q}_B \Delta p_1 [(.000726)^2 + (.00241)^2]}{(.0197)^2} \leq \bar{Q}_B \Delta p_1 (.015),$$

It is reasonable to suppose that average demand for large income effects is not very large relative to overall average demand. If anything, given the essential nature of transportation we might expect that average demand is smaller for those with high income effects. This makes $\bar{Q}_B \Delta p_1 \leq 2\bar{S}_{\bar{B}}$ a very reasonable assumption. Applying the inequality at the end of the last Section we thus find that if the linear random coefficients model were true, $\bar{S}'_B \leq (.03)\bar{S}_B$. Then by the second conclusion of Theorem A5 we have $\bar{S} \geq (.97)\bar{S}_{\bar{B}}$, so that the lower bound given in the empirical application is very close to correct. We note that this calculation of $(.97)\bar{S}_{\bar{B}}$ as a lower bound is very conservative, giving us high confidence in the lower abound used in the empirical application.

9.5 Details for General Bounds Estimation

We used a third order power series in $\ln p$ and $\ln y$ to estimate the quantile of $\ln q$. We also used the same power series for $m_j(x)$, which corresponds to the empirical specification with income effect bounds. We estimated the conditional quantile at 99 evenly spaced values, $\tau \in \{.01, .02, ..., .99\}$. We imposed the Slutzky condition appropriate for the natural log of demand on the quantiles at 81 values of x corresponding to nine price and income values drawn randomly from the range of the data. We drew two sets of $L = 1000$ coefficients, ensuring that each coefficient vector gave a demand at each $\tau \in \{.01, .02, ..., .99\}$ satisfying the Slutzky condition on the same τ and x grid. We evaluated the constraints at five quantile values for r including the median. We calculated the bounds as described in Section 6, using $\hat{F}(r|x) = \sum_{k=1}^{99} \Phi([r - \hat{Q}(.01k|x)]/.01)$ in place of $F(r|x)$ in the constraints, where $\Phi(s)$ is the $N(0,1)$ CDF that is used to smooth out $\hat{Q}^{-1}(r|x)$.

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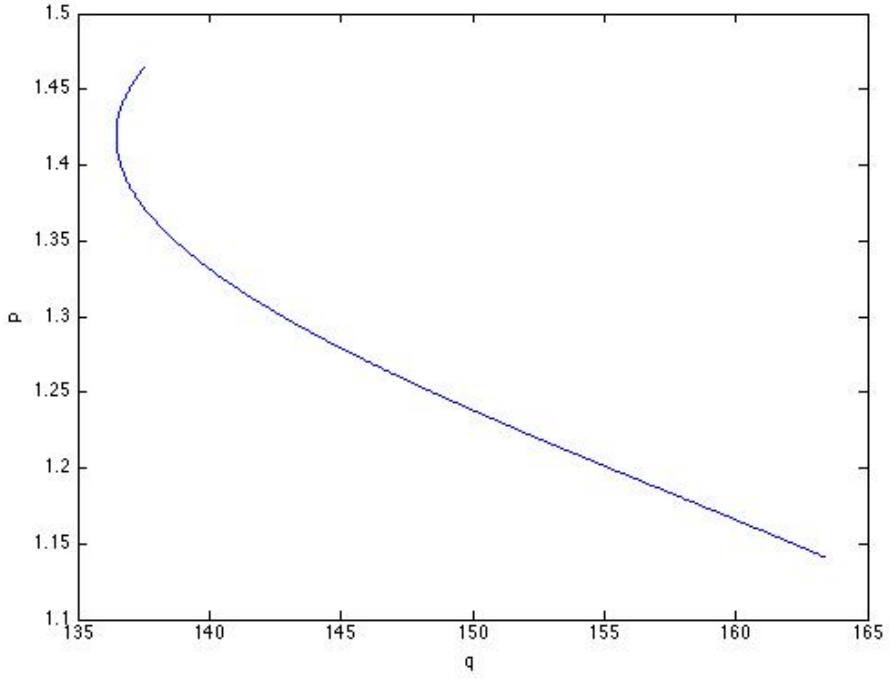
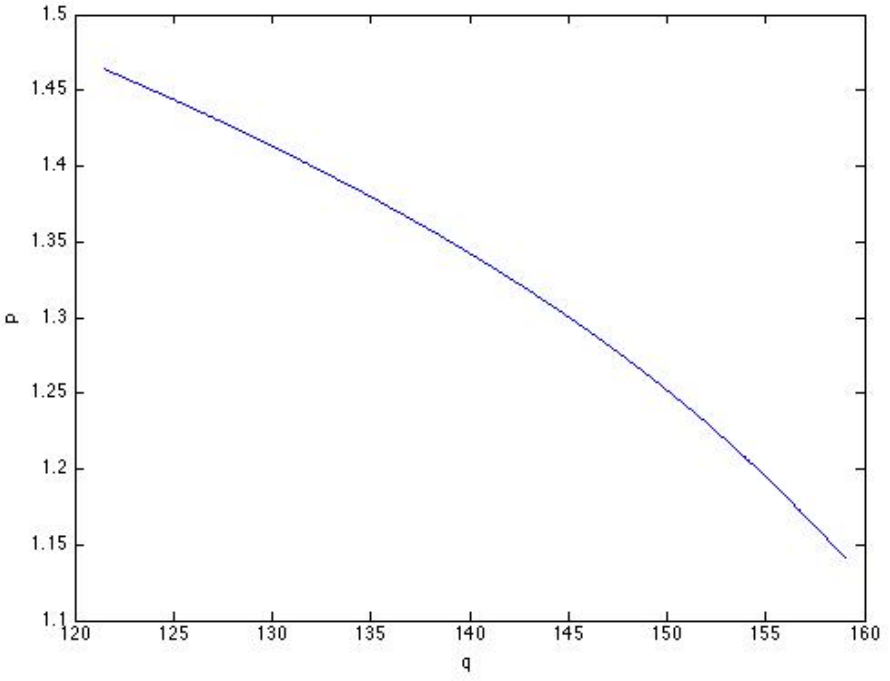
Figure 1. Estimated Demand: OLS



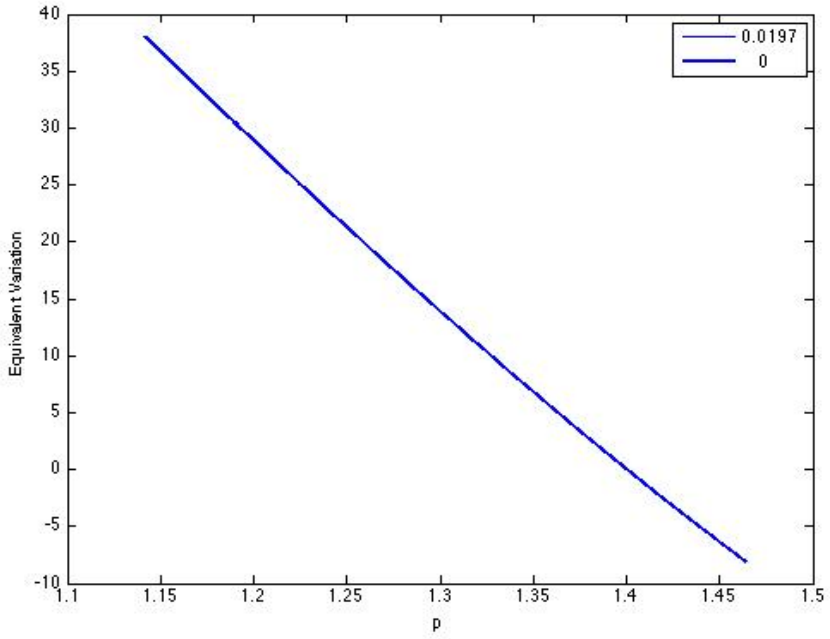
Notes: Demand estimated from 3rd order series regression evaluated at median income.

Figure 2. Estimated Demand: Control Function



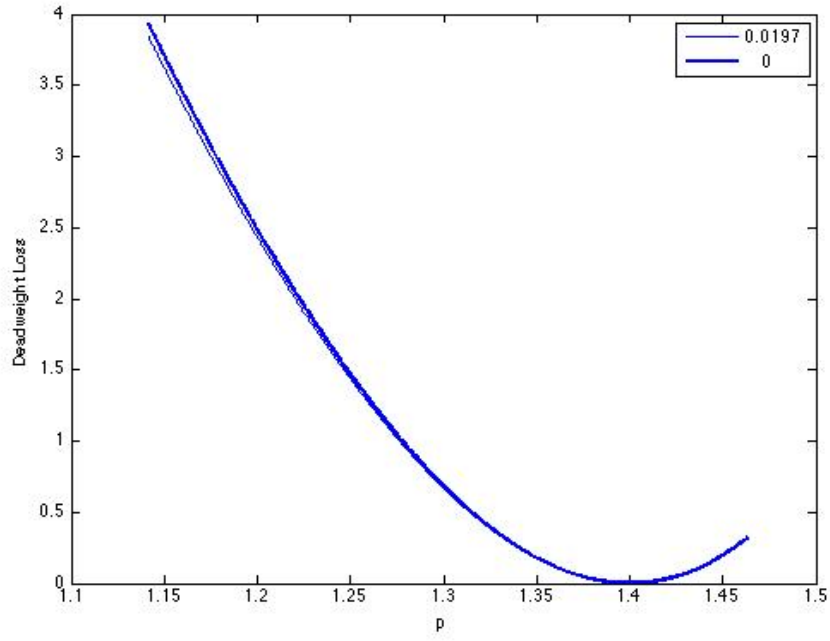
Notes: Demand estimated from 3rd order power series control function regression evaluated at median income.

Figure 3. Equivalent Variation Bounds



Notes: Graph shows change in equivalent variation for a price increase from p to $1.40, evaluated at upper and lower bounds of income derivative and at median income and estimated from 3rd order power series control function estimated demand.

Figure 4. Deadweight Loss Bounds



Notes: Graph shows change in deadweight loss for a price increase from p to $1.40, evaluated at upper and lower bounds of income derivative and at median income and estimated from 3rd order power series control function estimated demand.

Table 1. Summary Statistics

Variable	Mean	Median	Std Dev	Min	Max
price ($)	1.33	1.32	0.08	1.14	1.46
quantity (gallons)	4.90	2.65	7.53	0.01	195.52
income (1,000 $)	62.19	47.5	47.47	2.08	170.72
number of drivers	2.04	2	0.78	1	7
public transit availability	0.24	0	0.42	0	1
Observations	8,908				

Table 2. Estimated Price Elasticities

	OLS Estimates			Control Function Estimates		
Quantiles	0.25	0.5	0.75	0.25	0.5	0.75
Order 1	-0.698	-0.656	-0.631	-1.111	-1.060	-1.043
	(0.254)	(0.244)	(0.243)	(0.282)	(0.280)	(0.291)
Order 2	-1.597	-0.798	0.069	-1.675	-1.350	-1.111
	(0.469)	(0.283)	(0.509)	(0.476)	(0.320)	(0.657)
Order 3	-1.214	-0.798	0.271	-1.037	-1.102	-0.872
	(0.753)	(0.570)	(0.721)	(0.860)	(0.670)	(0.952)
Order 4	-0.713	-0.583	0.140	-0.389	-0.588	-0.853
	(0.877)	(0.623)	(0.801)	(1.158)	(0.707)	(1.032)

Table 3. Estimated Income Elasticities

	OLS Estimates			Control Function Estimates		
Quantiles	0.25	0.5	0.75	0.25	0.5	0.75
Order 1	0.168	0.157	0.151	0.167	0.159	0.157
	(0.022)	(0.019)	(0.017)	(0.022)	(0.019)	(0.018)
Order 2	0.221	0.244	0.261	0.210	0.233	0.262
	(0.032)	(0.025)	(0.031)	(0.032)	(0.025)	(0.034)
Order 3	0.217	0.221	0.236	0.220	0.221	0.256
	(0.057)	(0.040)	(0.039)	(0.059)	(0.041)	(0.043)
Order 4	0.266	0.305	0.275	0.267	0.294	0.277
	(0.067)	(0.054)	(0.047)	(0.074)	(0.058)	(0.052)

Table 4. Bounds on Equivalent Variation Estimates

	From $1.20 to 1.30		From $1.20 to 1.40	
	Lower Bound	Upper Bound	Lower Bound	Upper Bound
Order 1	16.777	16.794	32.281	32.343
	0.349	0.349	0.502	0.501
	[16.104, 17.468]		[31.355, 33.270]	
Order 2	15.147		28.829	28.884
	0.443	0.443	0.753	0.751
	[14.275, 16.005]		[27.405, 30.309]	
Order 3	14.972	14.987	28.845	28.900
	0.647	0.646	0.884	0.882
	[13.715, 16.244]		[27.163, 30.583]	
Order 4	14.625	14.639	28.546	28.601
	0.660	0.659	0.924	0.922
	[13.340, 15.925]		[26.788, 30.360]	

Table 5. Bounds on Deadweight Loss Estimates

	From $1.20 to 1.30		From $1.20 to 1.40	
	Lower Bound	Upper Bound	Lower Bound	Upper Bound
Order 1	0.646	0.663	2.467	2.529
	0.175	0.175	0.669	0.668
	[0.319, 0.990]		[1.219, 3.776]	
Order 2	0.735		2.821	2.876
	0.233	0.233	0.728	0.727
	[0.277, 1.178]		[1.452, 4.245]	
Order 3	0.485	0.500	2.434	2.489
	0.393	0.393	1.04	1.039
	[-0.272, 1.257]		[0.447, 4.475]	
Order 4	0.321	0.335	1.827	1.882
	0.498	0.497	1.137	1.135
	[-0.641, 1.298]		[-0.343, 4.052]	

Table 6: General Average Surplus Bounds

	Data 1			Data 2		
ε	Max CS	Min CS	Dif CS	Max CS	Min CS	Dif CS
.001	56.283	45.771	10.512	56.451	45.774	10.677
.0001	53.073	48.927	4.146	53.070	48.924	4.146
.00001	51.603	50.088	1.515	51.606	50.130	1.476