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# Efficient estimation with many weak instruments using regularization techniques 

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# Efficient estimation with many weak instruments 

## using regularization techniques*

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#### Abstract

The problem of weak instruments is due to a very small concentration parameter. To boost the concentration parameter, we propose to increase the number of instruments to a large number or even up to a continuum. However, in finite samples, the inclusion of an excessive number of moments may be harmful. To address this issue, we use regularization techniques as in Carrasco (2012) and Carrasco and Tchuente (2013). We show that normalized regularized 2SLS and LIML are consistent and asymptotically normally distributed. Moreover, their asymptotic variances reach the semiparametric efficiency bound unlike most competing estimators. Our simulations show that the leading regularized estimators (LF and T of LIML) work very well (are nearly median unbiased) even in the case of relatively weak instruments. An application to the effect of institutions on output growth completes the paper.


Key Words: Many weak instruments, LIML, 2SLS, regularization methods. JEL: C13

[^0]
## Non-technical summary

The identification of coefficients on endogenous variables in linear structural equations is achieved using instrumental variables that are assumed to be correlated with the right hand side endogenous variables (strong) and uncorrelated with the structural error (valid). However, when instruments are weak, when the correlation between the instrument and the endogenous variable is low, conventional estimation procedure can be misleading.

The problem of weak instruments is due to a very small concentration parameter. The concentration parameter is a measure of the strength of the instruments. To boost the concentration parameter, we propose to increase the number of instruments to a large number. An excessive number of moments may induce a bias. The effect of the number of instruments is mitigated by using regularization techniques.

In the many weak instruments framework, we show that normalized regularized 2SLS and limited information maximum likelihood (LIML) are consistent and asymptotically normally distributed. Moreover, their asymptotic variances are smaller than most competing estimators. A Monte Carlo experiment shows that the leading regularized estimators (Tikhonov and Landweber-Fridman of LIML) work very well (are nearly median unbiased) even in the case of relatively weak instruments.

An application to the effect of institutions on output growth completes the paper. Hall and Jones (1999) argue that the difference between output per worker across countries is mainly due to the differences in institution and government policies - the so-called social infrastructure. They used some instruments for social infrastructure that are weak (see Dmitriev (2013)). To address this issue, we increased the number of instruments from 4 to 18 . Our finding confirms the causal liked between social capital and output growth. The use of many instruments first increase the bias. When the regularization is introduced, this bias shrinks.

## 1 Introduction

The problem of weak instruments or weak identification has recently received considerable attention in both theoretical and applied econometrics ${ }^{1}$. Empirical examples include Angrist and Krueger (1991) who measure return to schooling, Eichenbaum, Hansen, and Singleton (1988) who consider consumption asset pricing models. Theoretical literature on weak instruments includes papers by Staiger and Stock (1997), Zivot, Startz, and Nelson (1998), Guggenberger and Smith (2005), Chao and Swanson (2005), Han and Phillips (2006), Hansen, Hausman, and Newey (2008), and Newey and Windmeijer (2009) among others ${ }^{2}$. Staiger and Stock (1997) proposed an asymptotic framework with local-to-zero parametrization of the coefficients of the instruments in the first-stage regression. They show that with the number of instruments fixed, the two-stage least squares (2SLS) and limited information maximum likelihood (LIML) estimators are not consistent and converge to nonstandard distributions. Subsequent work focused on situations where the number of instruments is large, using an asymptotic framework that lets the number of instruments go to infinity as a function of sample size. In these settings, the use of many moments can improve estimator accuracy. Unfortunately, usual Gaussian asymptotic approximation can be poor and IV estimators can be biased.

Carrasco (2012) and Carrasco and Tchuente (2013) proposed respectively regularized versions of 2SLS and LIML estimators for many strong instruments. The regularization permits to address the singularity of the covariance matrix resulting from many instruments. These papers use three regularization methods borrowed from inverse problem literature. The first estimator is based on Tikhonov (ridge) regularization, the second estimator is based on an iterative method called Landweber-Fridman (LF), the third estimator is based on the principal components associated with the largest eigenvalues. We extend these previous works to allow for the presence of a large number of

[^1]weak instruments or weak identification. We consider a linear model with homoskedastic error and allow for weak identification as in Hansen, Hausman, and Newey (2008) and Newey and Windmeijer (2009). This specification helps us to have different types of weak instruments sequences, including the many instruments sequence of Bekker (1994) and the many weak instruments sequence of Chao and Swanson (2005). We impose no condition on the number of moment conditions since our framework allows for an infinite countable or even a continuum of instruments. The advantage of regularization is that all available moments can be used without discarding any a priori. We show that regularized 2SLS and LIML are consistent in the presence of many weak instruments. If properly normalized, the regularized 2SLS and LIML are asymptotically normal and reach the semiparametric efficiency bounds. Therefore, their asymptotic variance is smaller than that of Hansen, Hausman, and Newey (2008) and Newey and Windmeijer (2009). All these methods involve a regularization parameter, which is the counterpart of the smoothing parameter that appears in the nonparametric literature. A data driven method was developed in Carrasco (2012) and Carrasco and Tchuente (2013) to select the best regularization parameter when the instruments are strong. We use these methods in our simulations for selecting the regularization parameter when the instruments are weak but we do not prove that these methods are valid in this case. A related paper is that of ? who propose a regularized jackknife instrumental variables estimator in a strong instruments setting where the design is not sparse.

Our Monte Carlo experiment shows that the leading regularized estimators (LF and T LIML) perform very well (are nearly median unbiased) even in the case of weak instruments.

The paper is organized as follows. Section 2 introduces four regularization methods we consider and the associated estimators. Section 3 derives the asymptotic properties of the estimators. Section 4 discusses efficiency and related results. Section 5 presents Monte Carlo experiments. Section 6 considers an application to the effect of social infrastructure on per capita income. Section 7 concludes. The proofs are collected in Appendix.

## 2 Presentation of the regularized 2SLS and LIML estimators

This section presents the weak instruments setup and the estimators used in this paper. Estimators studied here are the regularized 2SLS and LIML estimators introduced in Carrasco (2012) and Carrasco and Tchuente (2013). They can be used with many or even a continuum of instruments. This work extends previous works by allowing for weak instruments as in Hansen, Hausman, and Newey (2008).

Our model is inspired by Hausman, Newey, Woutersen, Chao, and Swanson (2012). The model is

$$
\left\{\begin{array}{r}
y_{i}=W_{i}^{\prime} \delta_{0}+\varepsilon_{i}, \\
W_{i}=\gamma_{i}+u_{i},
\end{array}\right.
$$

$i=1,2, \ldots, n$. The parameter of interest $\delta_{0}$ is a finite dimensional $p \times 1$ vector.
$E\left(u_{i} \mid x_{i}\right)=E\left(\varepsilon_{i} \mid x_{i}\right)=0 ; E\left(\varepsilon_{i}^{2} \mid x_{i}\right)=\sigma_{\varepsilon}^{2} . y_{i}$ is a scalar and $x_{i}$ is a vector of exogenous variables. Some rows of $W_{i}$ may be exogenous, with the corresponding rows of $u_{i}$ being zero. $\gamma_{i}=E\left(W_{i} \mid x_{i}\right)$ is a $p \times 1$ vector of reduced form values with $E\left(\gamma_{i} \varepsilon_{i}\right)=0 . \gamma_{i}$ is the optimal instrument which is typically unknown. The estimation is based on a set of instruments $Z_{i}=Z\left(\tau ; x_{i}\right)$, indexed by $\tau \in S$. The index $\tau$ may be an integer or may take its values in an interval. Examples of $Z_{i}$ are the following.

- when $x_{i}$ is a large $L \times 1$ vector, then one can select $Z_{i}=x_{i}$. In this case, $S=$ $\{1,2, \ldots . L\}$ thus we have $L$ instruments.
- assume that $x_{i}$ is a scalar and $Z\left(\tau ; x_{i}\right)=\left(x_{i}\right)^{\tau-1}$ with $\tau \in S=\mathbb{N}$, we obtain an infinite countable sequence of instruments.
- assume that $x_{i}$ is a vector and $Z\left(\tau ; x_{i}\right)=\exp \left(i \tau^{\prime} x_{i}\right)$ where $\tau \in S=\mathbb{R}^{\operatorname{dim}\left(x_{i}\right)}$, we obtain a continuum of moment.

To simplify the presentation, we will present the estimators in the case where $Z_{i}$ is a $L \times 1$ vector of instruments where $L$ is some large integer. The theoretical results of Section 3 are proved for an arbitrary $L$ which may be finite or infinite (case with a countable sequence of or a continuum of instruments). In all cases, $L$ does not
depend on $n$. The presentation of the estimators in the case with an infinite number of instruments is left in Appendix A.

This model allows for $\gamma_{i}$ to be a linear or a non linear combination of $Z_{i}$. The model also allows for $\gamma_{i}$ to approximate the reduced form. For example, we could let $\gamma_{i}$ be a vector of unknown functions of $x_{i}$ and $Z_{i}$ could be power functions of $x_{i}$ or interactions between elements of $x_{i}$. Adding extra instruments is a way to boost the concentration parameter as illustrated in the application in Section 6.

The estimate $\delta$ is based on the orthogonality condition.

$$
E\left[\left(y_{i}-W_{i}^{\prime} \delta\right) Z_{i}\right]=0
$$

where the vector of instruments $Z_{i}$ has dimension $L$.

$$
\text { Let } W=\left(\begin{array}{c}
W_{1}^{\prime} \\
W_{2}^{\prime} \\
\cdot \\
\cdot \\
W_{n}^{\prime}
\end{array}\right) n \times p \text { and } u=\left(\begin{array}{c}
u_{1}^{\prime} \\
u_{2}^{\prime} \\
\cdot \\
\cdot \\
u_{n}^{\prime}
\end{array}\right) n \times p \text {. }
$$

Let $\mathbf{Z}$ denote the $n \times L$ matrix having rows corresponding to $Z_{i}^{\prime}$. Denote $\psi_{j}$ the eigenvectors of the $n \times n$ matrix $\mathbf{Z Z} \mathbf{Z}^{\prime} / n$ associated with eigenvalues $\lambda_{j}$. Recall that two-stage least squares (2SLS) and LIML estimators involve a projection matrix

$$
P=\mathbf{Z}\left(\mathbf{Z}^{\prime} \mathbf{Z}\right)^{-1} \mathbf{Z}^{\prime}
$$

The matrix $\mathbf{Z}^{\prime} \mathbf{Z}$ may become nearly singular when $L$ gets large. Moreover, $\mathbf{Z}^{\prime} \mathbf{Z}$ is singular whenever $L \geq n$. To cover these cases, we will consider a regularized version of the inverse of the matrix $\mathbf{Z}^{\prime} \mathbf{Z}$. For an arbitrary $n \times 1$ vector $v$, we define the $n \times n$ $\operatorname{matrix} P^{\alpha}$ as

$$
P^{\alpha} v=\frac{1}{n} \sum_{j=1}^{n} q\left(\alpha, \lambda_{j}^{2}\right)\left(v^{\prime} \psi_{j}\right) \psi_{j}
$$

where $q\left(\alpha, \lambda_{j}^{2}\right)$ is a weight that takes different forms depending on the regularization schemes. We consider three types of regularization:

- The Tikhonov (T) regularization: $q\left(\alpha, \lambda_{j}^{2}\right)=\frac{\lambda_{j}^{2}}{\lambda_{j}^{2}+\alpha}$.
- The Landweber-Fridman (LF) regularization: $q\left(\alpha, \lambda_{j}^{2}\right)=\left[1-\left(1-c \lambda_{j}^{2}\right)^{1 / \alpha}\right]$, where $c$ is a constant such that $0<c<1 /\left\|Z^{\prime} Z / n\right\|^{2}$ and $\left\|Z^{\prime} Z / n\right\|$ denotes the largest eigenvalue of $Z^{\prime} Z / n$.
- The Spectral Cut-off (SC): $q\left(\alpha, \lambda_{j}^{2}\right)=I\left(\lambda_{j}^{2} \geq \alpha\right)$.

Note that all these regularization techniques involve a tuning parameter $\alpha$. The case $\alpha=0$ corresponds to the case without regularization, $q\left(\alpha, \lambda_{j}^{2}\right)=1$. Then, we obtain

$$
P^{0}=P=\mathbf{Z}\left(\mathbf{Z}^{\prime} \mathbf{Z}\right)^{-1} \mathbf{Z}^{\prime}
$$

Consider regularized k-class estimators defined as follows:

$$
\hat{\delta}_{\nu}=\left(W^{\prime}\left(P^{\alpha}-\nu I_{n}\right) W\right)^{-1} W^{\prime}\left(P^{\alpha}-\nu I_{n}\right) y .
$$

where $\nu$ is either a constant term or a random variable. The case where $\nu=0$ corresponds to regularized 2SLS estimator studied in Carrasco (2012):

$$
\hat{\delta}=\left(W^{\prime} P^{\alpha} W\right)^{-1} W^{\prime} P^{\alpha} y
$$

and the case $\nu=\nu_{\alpha}=\min _{\delta} \frac{(y-W \delta)^{\prime} P^{\alpha}(y-W \delta)}{(y-W \delta)^{\prime}(y-W \delta)}$ corresponds to the regularized LIML studied in Carrasco and Tchuente (2013). We denote $\hat{\delta}$ the regularized 2SLS estimator and $\hat{\delta}_{L}$ the regularized LIML estimators.

We study both 2SLS and LIML because LIML may have some advantages over 2SLS. For example when the number of instruments, $L$, increases with the sample size, $n$, so that $L / n \rightarrow c$ (with $c$ constant), the standard 2SLS estimator is not consistent whereas standard LIML estimator is consistent.

## 3 Asymptotic properties

Carrasco (2012) and Carrasco and Tchuente (2013) focused on strong instruments. They found that regularized 2SLS and LIML estimators are asymptotically normal and attain the semiparametric efficiency bound. Here, we extend Carrasco (2012) and Carrasco and Tchuente (2013) results to the case of many weak instruments.

The weakness of the instruments is measured by the concentration parameter. For $p=1$, the concentration parameter is equal to

$$
C P=\frac{\sum_{i=1}^{n} \gamma_{i}^{2}}{E\left(v_{i}^{2}\right)}
$$

When the instruments are weak, $C P$ converges to a constant and the parameter $\delta$ is not identified. This is the weak IV described by Staiger and Stock (1997). This case is not considered here. We will maintain the assumption that CP diverges. It may diverge at the $n$ rate (strong instruments) or at a slower rate (many weak IV asymptotics). By adding more instruments in the first stage equation:

$$
W=Z \Pi+V
$$

the concentration parameter

$$
C P=\frac{\Pi^{\prime} Z^{\prime} Z \Pi}{E\left(v_{i}^{2}\right)}
$$

does not decrease and actually increases if these instruments contain non trivial information. Hence, adding more instruments is a way to boost the concentration parameter. Where do you get these new instruments? If you already have exogenous instruments, it is possible to interact them as it has been done for the estimation of return to schooling (Angrist and Krueger (1991)) or take higher order power of the same instruments as in Dagenais and Dagenais (1997). In the case of panel data, the use of lag variables is usually a source of many instruments. We provide an empirical application of the use of many weak instruments in Section 6.

## Assumption 1:

(i) There exists a $p \times p$ matrix $S_{n}=\tilde{S}_{n} \operatorname{diag}\left(\mu_{1 n}, \ldots, \mu_{p n}\right)$ such that $\tilde{S}_{n}$ is bounded, the smallest eigenvalue of $\tilde{S}_{n} \tilde{S}_{n}^{\prime}$ is bounded away from zero; for each $j$, either $\mu_{j n}=\sqrt{n}$ (strong identification) or $\mu_{j n} / \sqrt{n} \rightarrow 0$ (weak identification), $\mu_{n}=\min _{1 \leq j \leq p} \mu_{j n} \rightarrow \infty$ and $\alpha \rightarrow 0$.
(ii) There exists a function $f_{i}=f\left(x_{i}\right)$ such that $\gamma_{i}=S_{n} f_{i} / \sqrt{n}$ and $\mu_{n} S_{n}^{-1} \rightarrow S_{0}$. $\sum_{i=1}^{n}\left\|f_{i}\right\|^{4} / n^{2} \rightarrow 0, \sum_{i=1}^{n} f_{i} f_{i}^{\prime} / n$ is bounded and uniformly nonsingular.

These conditions allow for many weak instruments. If $\mu_{j n}=\sqrt{n}$ this leads to asymptotic theory like in Kunitomo (1980), Morimune (1983), and Bekker (1994), but here we use regularization parameter instead of having an increasing sequence of instruments. For $\mu_{n}^{2}$ growing slower than $n$, the convergence rate will be slower that $\sqrt{n}$, leading to an asymptotic approximation as that of Chao and Swanson (2007). This is the case where we have many instruments without strong identification. Assumption 1 also allows for some components of the reduced form to give only weak identification (corresponding to $\mu_{j n} / \sqrt{n} \rightarrow 0$ which allows the concentration parameter to grow slower than $\sqrt{n}$ ), and other components (corresponding to $\mu_{j n}=\sqrt{n}$ ) to give strong identification for some coefficients of the reduced form. In particular, this condition allows for fixed constant coefficients in the reduced form. This specification of weak instruments can also be viewed as a generalization of Chao and Swanson (2007) but differs from that of Antoine and Lavergne (2012) who define the identification strength through the conditional moments that flatten as the sample size increases. To illustrate Assumption 1, let us consider the following example.

Example 1: Assume that $p=2, \tilde{S}_{n}=\left(\begin{array}{cc}1 & 0 \\ \pi_{21} & 1\end{array}\right)$, and $\mu_{j n}= \begin{cases}\sqrt{n}, & j=1 \\ \mu_{n}, & j=2\end{cases}$ with $\mu_{n} / \sqrt{n} \rightarrow 0$.

Then for $f\left(x_{i}\right)=\left(f_{1 i}^{\prime}, f_{2 i}^{\prime}\right)^{\prime}$ the reduced form is

$$
\gamma_{i}=\binom{f_{1 i}}{\pi_{21} f_{1 i}+\frac{\mu_{n}}{\sqrt{n}} f_{2 i}}
$$

We also have

$$
\mu_{n} S_{n}^{-1} \rightarrow S_{0}=\left(\begin{array}{cc}
0 & 0 \\
-\pi_{21} & 1
\end{array}\right)
$$

## Assumption 2:

(i) The operator $K$ is nuclear.
(ii) The $a$ th row of $\gamma$, denoted $\gamma_{a}$, belongs to the closure of the linear span of $\{Z(. ; x)\}$ for $a=1, \ldots, p$.
(iii) $E\left(Z\left(., x_{i}\right) f_{i a}\right)$ belong to the range of $K$.

Condition (i) refers to the covariance operator $K$ defined in Appendix A. $K$ is nuclear provided its trace is finite, see for instance Carrasco, Florens, and Renault (2007). This assumption is trivially satisfied if $L$ is finite but may or may not be satisfied when $L$ is infinite. This assumption implies in particular that the smallest eigenvalues decrease to zero sufficiently fast. For this to be true, the $Z_{i}$ have to be correlated with each other. If $E\left(Z_{i} Z_{i}^{\prime}\right)=I_{L}$ as in Assumption 5 of Newey and Windmeijer (2009), all the eigenvalues of the operator $K$ equal 1 and hence $K$ is not nuclear when $L$ goes to infinity. To see whether Condition (i) is realistic, we examine the properties of the sample counterpart of $K$, namely $K_{n}=Z^{\prime} Z / n$, in three applications: the return to schooling using 240 instruments from Angrist and Krueger (1991) (see also Carrasco and Tchuente (2013)), the elasticity of intertemporal substitution (see Carrasco and Tchuente (2013)), and the application on the effect of institutions on growth (see Section 6 of this paper). In the table below, we report the smallest eigenvalue, the largest eigenvalue, the condition number (which is the ratio of the largest eigenvalue on the smallest eigenvalue) and the trace of $Z^{\prime} Z / n$ in two cases: raw data and standardized instruments. In the standardized case, the instruments are divided with their standard deviation. This standardization has no impact on 2SLS and LIML estimators which are scale invariant. However, our estimators are not scale invariant and standardization may improve the results. Such standardizations are customary whenever regularizations are used, see for instance De Mol, Giannone, and Reichlin (2008) and ?. We observe that in all applications, the smallest eigenvalue is close to zero so the instruments are strongly correlated ${ }^{3}$. The condition number - which is scale invariant - is an indicator on how ill-posed the matrix $K_{n}$ is. The higher the condition number, the more imprecise the inverse of $K_{n}$ will be. The smallest possible condition number is 1 (which corresponds to the identity matrix). Here, the condition numbers are all very large which suggests that regularization will be helpful to improve the reliability of the estimate of $K^{-1}$. The trace of $K_{n}$ appears to be finite throughout the applications.

Condition (ii) guarantees that the optimal instrument $f$ can be approached by a sequence of instruments. It is similar to Assumption 4 in Hansen, Hausman, and Newey

[^2]Table 1: Properties of $Z^{\prime} Z / n$

|  | Largest <br> eigenvalue | Smallest <br> eigenvalue | Condition <br> number | Trace |
| :---: | :---: | :---: | :---: | :---: |
| Angrist and Krueger | 1.35 | 0.0000107 | 126168.22 | 5.05 |
| Angrist and Krueger | 5.93 | 0.0012 | 4941.66 | 244.47 |
| standardized | 1550 | $1.41 \times 10^{-13}$ | $1.09929 \times 10^{16}$ | 1550 |
| EIS | 11.8 | $2.35 \times 10^{-5}$ | $5.06 \times 10^{5}$ | 11.89 |
| EIS standardized | $474 \times 10^{7}$ | $9.47 \times 10^{-6}$ | $5.00528 \times 10^{14}$ | $4.78 \times 10^{9}$ |
| Institutions | 28.9 | 0.000116 | 249137.93 | 43.58 |

(2008). Condition (iii) is a technical assumption which can also be found in Carrasco (2012). Assumptions 2(ii) and (iii) are needed only for efficiency.

Proposition 1. (Asymptotic properties of regularized 2SLS with many weak instruments)

Assume $\left\{y_{i} ; W_{i} ; x_{i}\right\}$ are iid, $E\left(\varepsilon_{i}^{2} \mid X\right)=\sigma_{\varepsilon}^{2}$, $\alpha$ goes to zero as $n$ goes to infinity. Moreover, Assumptions 1 and 2 are satisfied. Then, the T, LF, and SC estimators of 2SLS satisfy:

1. Consistency: $S_{n}^{\prime}\left(\hat{\delta}-\delta_{0}\right) / \mu_{n} \rightarrow 0$ in probability as $n$, n $\alpha^{\frac{1}{2}}$ and $\mu_{n}$ go to infinity.
2. Asymptotic normality:
$S_{n}^{\prime}\left(\hat{\delta}-\delta_{0}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma_{\varepsilon}^{2}\left[E\left(f_{i} f_{i}^{\prime}\right)\right]^{-1}\right)$ as $n$, $n \alpha$ and $\mu_{n}$ go to infinity, where $E\left(f_{i} f_{i}^{\prime}\right)$ is a nonsingular $p \times p$ matrix.

Proof In Appendix.
The first point of Proposition 1 implies the consistency of the estimator, namely $\left(\hat{\delta}-\delta_{0}\right) \rightarrow 0$ (see the proof of Theorem 1 in ?). Moreover, Proposition 1 shows that the three estimators have the same asymptotic distribution. Instead of restricting the number of instruments (which may be very large or infinite), we impose restrictions on the regularization parameter which goes to zero. This insures us that all available and valid instruments are used in an efficient way even if they are weak. To obtain consistency, the condition on $\alpha$ is $n \alpha^{\frac{1}{2}}$ go infinity, whereas for the asymptotic normality, we need $n \alpha$ go to infinity. This means that $\alpha$ is allowed to go to zero at a slower rate. However, this rate does not depend on the weakness of the instruments.

Interestingly, our regularized 2SLS estimators reach the semiparametric efficiency bound. This result will be further discussed in Section 4.

We are now deriving the asymptotic properties of the regularized LIML with many weak instruments.

Proposition 2. (Asymptotic properties of regularized LIML with many weak instruments)

Assume $\left\{y_{i} ; W_{i} ; x_{i}\right\}$ are iid, $E\left(\varepsilon_{i}^{2} \mid X\right)=\sigma_{\varepsilon}^{2}, E\left(\varepsilon_{i}^{4} \mid X\right)<\infty, E\left(u_{b i}^{4} \mid X\right)<\infty, \alpha$ goes to zero as $n$ goes to infinity. Moreover, Assumptions 1 and 2 are satisfied. Then, the $T, L F$, and $S C$ estimators of LIML with weak instruments satisfy:

1. Consistency: $S_{n}^{\prime}\left(\hat{\delta}_{L}-\delta_{0}\right) / \mu_{n} \rightarrow 0$ in probability as $n$, $\mu_{n}$ and $\mu_{n}^{2} \alpha$ go to infinity.
2. Asymptotic normality:
$S_{n}^{\prime}\left(\hat{\delta}_{L}-\delta_{0}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma_{\varepsilon}^{2}\left[E\left(f_{i} f_{i}^{\prime}\right)\right]^{-1}\right)$ as $n, \mu_{n}$ and $\mu_{n}^{2} \alpha$ go to infinity where $E\left(f_{i} f_{i}^{\prime}\right)$ is a nonsingular $p \times p$ matrix.

Proof In Appendix.
Again, Proposition 1 implies the consistency of the estimator, namely $\left(\hat{\delta}-\delta_{0}\right) \rightarrow 0$. Interestingly we obtain the same asymptotic distribution as in the many strong instruments case (with a slower rate of convergence). We also find the same speed of convergence as in Hansen, Hausman, and Newey (2008) and Newey and Windmeijer (2009). For the consistency and asymptotic normality, $\mu_{n}^{2} \alpha$ needs to go to infinity, which means that the regularization parameter should go to zero at a slower rate than the concentration parameter. The asymptotic variance of regularized LIML corresponds to the lower bound and is smaller than that obtained in Hansen, Hausman, and Newey (2008). We believe that the reason, why Hansen, Hausman, and Newey (2008) obtain a larger asymptotic variance than us, is that they use the number of instruments as regularization parameter. As a result, they can not let $L$ grow fast enough to reach efficiency. Our estimator involves the extra tuning parameter $\alpha$ which is selected so that extra terms in the variance vanish asymptotically. Moreover, we assume that the set of instruments is sufficiently rich to span the optimal instrument (Assumption 2(ii)).

## Example 1:(cont)

$$
S_{n}^{\prime}\left(\hat{\delta}-\delta_{0}\right)=\binom{\sqrt{n}\left[\left(\hat{\delta}_{1}-\delta_{01}\right)+\pi_{21}\left(\hat{\delta}_{2}-\delta_{02}\right)\right]}{\mu_{n}\left(\hat{\delta}_{2}-\delta_{02}\right)} \text { is jointly asymptotically normal. }
$$

The linear combination $\left(\hat{\delta}_{1}-\delta_{01}\right)+\pi_{21}\left(\hat{\delta}_{2}-\delta_{02}\right)$ converges at rate $\sqrt{n}$. This is the /coefficient of $f_{i 1}$ in the reduced form equation for $y_{i}$. And the estimator of the coefficients $\delta_{2}$ of $W_{i 2}$ variables converges at rate $\frac{1}{\mu_{n}}$.

Now, as in Newey and Windmeijer (2009), we consider a t-ratios for a linear combination $c^{\prime} \delta$ of the parameter of interest. The following proposition is a corollary of Proposition 2.

Proposition 3. Under the assumptions of Proposition 2 and if we assume that there exist $r_{n}, c$ and $c^{*} \neq 0$ such that $r_{n} S_{n}^{-1} c \rightarrow c^{*}$ and $S_{n}^{\prime} \hat{\Phi} S_{n} / n \rightarrow \Phi$ in probability with $\Phi=\sigma_{\varepsilon}^{2}\left[E\left(f_{i} f_{i}^{\prime}\right)\right]^{-1}$.
Then,

$$
\frac{c^{\prime}\left(\hat{\delta}_{L}-\delta_{0}\right)}{\sqrt{c^{\prime} \hat{\Phi} c}} \xrightarrow{d} \mathcal{N}(0,1)
$$

as $n$ and $\mu_{n}^{2} \alpha$ go to infinity.
This result allows us to form confidence intervals and test statistics for a single linear combination of parameters in the usual way.

## 4 Efficiency and Related Literature

### 4.1 Efficiency

If the optimal instrument $\gamma_{i}$ were known, the estimator would be solution of

$$
\frac{1}{n} \sum_{i=1}^{n} \gamma_{i}\left(y_{i}-W_{i}^{\prime} \delta\right)=0 .
$$

Hence,

$$
\begin{aligned}
\hat{\delta}= & \left(\sum_{i=1}^{n} \gamma_{i} W_{i}^{\prime}\right)^{-1} \sum_{i=1}^{n} \gamma_{i} y_{i}, \\
\hat{\delta}-\delta_{0}= & \left(\sum_{i=1}^{n} \gamma_{i} W_{i}^{\prime}\right)^{-1} \sum_{i=1}^{n} \gamma_{i} \varepsilon_{i} \\
= & \left(S_{n} \frac{\sum_{i=1}^{n} f_{i} f_{i}^{\prime}}{n} S_{n}^{\prime}+S_{n} \frac{\sum_{i=1}^{n} f_{i} u_{i}}{\sqrt{n}}\right)^{-1} S_{n} \frac{\sum_{i=1}^{n} f_{i} \varepsilon_{i}}{\sqrt{n}}, \\
S_{n}\left(\hat{\delta}-\delta_{0}\right)= & \left(\frac{\sum_{i=1}^{n} f_{i} f_{i}^{\prime}}{n}+\frac{\sum_{i=1}^{n} f_{i} u_{i}}{\sqrt{n}} S_{n}^{\prime-1}\right)^{-1} \frac{\sum_{i=1}^{n} f_{i} \varepsilon_{i}}{\sqrt{n}} \\
& \xrightarrow{d} \mathcal{N}\left(0, \sigma_{\varepsilon}^{2}\left[E\left(f_{i} f_{i}^{\prime}\right)\right]^{-1}\right) .
\end{aligned}
$$

So the lowest asymptotic variance that can be obtained is $\sigma_{\varepsilon}^{2}\left[E\left(f_{i} f_{i}^{\prime}\right)\right]^{-1}$. We will refer to this as the semiparametric efficiency bound ${ }^{4}$.

In Carrasco (2012, Section 2.4), it was shown that the regularized 2SLS estimator coincides with a 2 SLS estimator that uses a specific nonparametric estimator, $\hat{\gamma}_{i}$, of $\gamma_{i}:$

$$
\hat{\delta}=\left(\sum_{i=1}^{n} \hat{\gamma}_{i}^{\prime} W_{i}\right)^{-1} \sum_{i=1}^{n} \hat{\gamma}_{i}^{\prime} y_{i} .
$$

This may explain why for the regularized 2SLS estimator, the conditions on $\alpha$ are not related to $\mu_{n}$ whereas, in the case of LIML, the rate of convergence of $\alpha$ depends on how weak the instruments are.

### 4.2 Related Literature

In the literature on many weak instruments, the asymptotic behavior of estimators depends on the relation between the number of moment conditions $L$ and sample size $n$. For the CUE, $L$ and $n$ need to satisfy $L^{2} / n \rightarrow 0$ for consistency and $L^{3} / n \rightarrow 0$ for asymptotic normality. Under homoskedasticity, Stock and Yogo (2005) require $L^{2} / n \rightarrow 0$. Hansen, Hausman, and Newey (2008) allowed $L$ to grow at the same rate as $n$, but restricted $L$ to grow slower than the square of the concentration parameter,

[^3]for the consistency of LIML and FULL. Andrews and Stock (2006) require $L^{3} / n \rightarrow 0$ when normality is not imposed.

Caner and Yildiz (2012) in a recent work consider a Continuous Updating Estimator (CUE) with many weak moments under nearly singular design. They show that the nearly singular design affects the form of asymptotic covariance matrix of the estimator compared to that of Newey and Windmeijer (2009). Our work is also related to Hausman, Lewis, Menzel, and Newey (2011) who modify the continuous updating estimator (CUE) by introducing two tuning parameters which perform a Tykhonovtype regularization. They show that their estimator has finite moments when the regularization parameters are positive. On the other hand, their estimator is shown to be asymptotically equivalent to the conventional CUE under many weak asymptotics when the regularization parameters go to zero. There are two main differences with our approach. First, they introduce two tuning parameters instead of one. Second, they restrict the number of moments as in Newey and Windmeijer (2009), whereas we allow for the number of instruments to exceed the sample size.

Belloni, Chen, Chernozhukov, and Hansen (2012) propose to use an alternative regularization named lasso in the IV context. This regularization imposes a $l_{1}$ type penalty on the first stage coefficient. Assuming that the first stage equation is approximately sparse, they show that the postlasso estimator reaches the asymptotic efficiency bound.

Just as 2SLS is not consistent if $L$ is too large relative to $n$, LIML estimator is not feasible if $L>N$ because the matrix $Z^{\prime} Z$ is not invertible. Therefore, some form of regularization needs to be implemented to obtain consistent estimators when the number of instruments is really large. Regularization has also the advantage to deliver an asymptotically efficient estimator.

Table 2 gives an overview of the assumptions used in the main papers on many weak instruments.

Table 2: Comparison of different IV asymptotics

## Number of instruments

Conventional
Phillips (1989)
Staiger and Stock (1997)
Bekker (1994)
Han and Phillips (2006)

Chao and Swanson (2005)
Hansen et al. (2008)
Newey and Windmeijer (2009)
Carrasco (2012)

Belloni et al. (2012)

Fixed $L$
Fixed $L, \operatorname{Cov}(W, x)=0$
Fixed $L, \operatorname{Cov}(W, x)=O\left(n^{-1 / 2}\right)$
$L / n \rightarrow c<1, \mu_{n}^{2}=O(n)$
$L \rightarrow \infty$ and $\frac{L}{n c_{n}} \rightarrow c$ $c_{n} \mu_{n}$ constant or zero
$\frac{L}{\mu_{n}^{2}} \rightarrow 0$ or $\frac{L^{1 / 2}}{\mu_{n}^{2}} \rightarrow 0$
(I) $\frac{L}{\mu_{n}^{2}}$ bounded or (II) $\frac{L}{\mu_{n}^{2}} \rightarrow \infty \quad \sum z_{i} z_{i}^{\prime} / n$ nonsingular $L \rightarrow \infty, \frac{L}{\mu_{n}^{2}}$ bounded, $\frac{L^{3^{n}}}{n} \rightarrow 0$
No condition on $L$, possibly continuum strong instruments
$\log (L)=o\left(n^{1 / 3}\right)$, strong instruments

Compactness of covariance matrix Approximately sparse first stage equation

## 5 Monte Carlo study

We now carry out a Monte Carlo simulation for the simple linear IV model where the disturbances and instruments have a Gaussian distribution and the instruments are independent from each other as in Newey and Windmeijer (2009). The parameters of this experiment are the correlation coefficient $\rho$ between the structural and reduced form errors, the concentration parameter $(C P)$, and the number of instruments $L$.

The data generating process is given by:

$$
\begin{gathered}
y_{i}=W_{i}^{\prime} \delta_{0}+\varepsilon_{i}, \\
W_{i}=x_{i}^{\prime} \pi+u_{i}, \\
\varepsilon_{i}=\rho u_{i}+\sqrt{1-\rho^{2}} v_{i}, \\
u_{i} \sim \mathcal{N}(0,1), v_{i} \sim \mathcal{N}(0,1), x_{i} \sim \mathcal{N}\left(0, I_{L}\right) \\
\pi=\sqrt{\frac{C P}{L n}} \iota_{L}
\end{gathered}
$$

where $\iota_{L}$ is an $L$-vector of ones. The sample size is $n=500$. The instruments are
$Z_{i}=x_{i}$ and the number of instruments $L$ equals 15,30 , and 50 . Note that this setting is not favorable for us because the eigenvalues of the matrix $Z^{\prime} Z / n$ are all equal to 1. If $L$ were infinite, the matrix $Z^{\prime} Z / n$ would become an infinite dimensional identity matrix which is not nuclear. Therefore, our basic assumption of compactness of the operator $K$ would not satisfied if we would let $L$ go to infinity. However, here $L$ being no larger than $50, K$ is nuclear.

In the simulations, $\rho=0.5$ and $\delta_{0}=0.1$. The values of $C P$ equal $4,8,35$, and 65 .
The estimators we proposed in this paper depend on a regularization (smoothing) parameter $\alpha$ that needs to be selected. In the simulations, we use a data-driven method for selecting $\alpha$ based on an expansion of the MSE and proposed in Carrasco (2012) and Carrasco and Tchuente (2013). These selection criteria were derived assuming strong instruments and may not be valid in presence of weak instruments. Providing a robust to weak instruments selection procedure is beyond the scope of this paper.

We report the median bias (Med.bias), the median of the absolute deviation of the estimator from the true value (Med.abs), the difference between the 0.1 and 0.9 quantiles (dis) of the distribution of each estimator, and the coverage rate (Cov.) of a nominal $95 \%$ confidence interval for unfeasible instrumental variable regression (IV), regularized two-stage least squares (T2SLS (Tikhonov), L2SLS (Landweber Fridman), P2SLS (Principal component)), LIML and regularized LIML (TLIML (Tikhonov), LLIML (Landweber Fridman), PLIML (Principal component)) and Donald and Newey's (2001) 2SLS and LIML (D2SLS and DLIML). For confidence intervals, we compute the coverage probabilities using the following estimate of asymptotic variance as in Donald and Newey (2001) and Carrasco (2012).

$$
\hat{V}(\hat{\delta})=\frac{(y-W \hat{\delta})^{\prime}(y-W \hat{\delta})}{n}\left(\hat{W}^{\prime} W^{-1}\right)^{-1} \hat{W}^{\prime} \hat{W}\left(W^{\prime} \hat{W}\right)^{-1}
$$

where $\hat{W}=P^{\alpha} W$ for 2SLS and $\hat{W}=\left(P^{\alpha}-\nu I_{n}\right) W$ for LIML. Note that the formulaes for the confidence intervals is the same as for strong instruments (see Carrasco and Tchuente (2013)).

Table 3 reports simulation results. We use different strength (measured by the concentration parameter) of instruments and number of instruments. We investigate the case of very weak instruments for example, when $C P=4$ and $L=50$, the first stage

Table 3: Simulations results with $C P=4,8,35$ and $65 ; n=500 ; L=15,30$ and $50 ; 1000$ replications.

|  |  |  | T2SLS | L2SLS | P2SLS | D2SLS | TLIML | LLIML | PLIML | DLIML | IV |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{L}=15$ | $\mathrm{CP}=4$ | Med.bias | 0.3939 | 0.0215 | 0.3567 | 0.4093 | 0.1521 | 0.0212 | 0.3652 | 0.4016 | 0.0212 |
|  |  | Med.abs | 0.3949 | 0.3474 | 0.5334 | 0.5633 | 0.5616 | 0.3474 | 0.5191 | 0.5208 | 0.3476 |
|  |  | Disp | 0.5563 | 1.5397 | 2.6328 | 2.4639 | 3.2144 | 1.5882 | 1.8150 | 1.9161 | 1.5407 |
|  |  | Cov | 0.5000 | 0.9510 | 0.7750 | 0.7750 | 0.4510 | 0.9510 | 0.7250 | 0.7130 | 0.9520 |
|  | $\mathrm{CP}=8$ | Med.bias | 0.3244 | 0.0030 | 0.3036 | 0.3385 | 0.0388 | 0.0085 | 0.2664 | 0.2873 | 0.0031 |
|  |  | Med.abs | 0.3263 | 0.2502 | 0.3917 | 0.4080 | 0.3671 | 0.2556 | 0.3820 | 0.3864 | 0.2506 |
|  |  | Disp | 0.5051 | 1.0218 | 1.5854 | 1.7167 | 1.9271 | 1.0512 | 1.3442 | 1.3465 | 1.0210 |
|  |  | Cov | 0.5600 | 0.9510 | 0.7420 | 0.7380 | 0.5670 | 0.9480 | 0.7450 | 0.7210 | 0.9500 |
|  | $\mathrm{CP}=35$ | Med.bias | 0.1447 | 0.0011 | 0.1584 | 0.1940 | -0.0022 | 0.0034 | 0.0323 | 0.0400 | 0.0012 |
|  |  | Med.abs | 0.1518 | 0.1215 | 0.1938 | 0.2123 | 0.1370 | 0.1198 | 0.1312 | 0.1313 | 0.1214 |
|  |  | Disp | 0.3465 | 0.4550 | 0.4974 | 0.4297 | 0.5100 | 0.4603 | 0.5036 | 0.4920 | 0.4551 |
|  |  | Cov | 0.7640 | 0.9540 | 0.7770 | 0.7430 | 0.8510 | 0.9490 | 0.8490 | 0.8640 | 0.9540 |
|  | CP65 | Med.bias | 0.0891 | 0.0008 | 0.0953 | 0.1198 | 0.0007 | 0.0024 | 0.0112 | 0.0061 | 0.0009 |
|  |  | Med.abs | 0.1029 | 0.0879 | 0.1222 | 0.1343 | 0.0920 | 0.0885 | 0.0915 | 0.0920 | 0.0881 |
|  |  | Disp | 0.2785 | 0.3240 | 0.3359 | 0.3039 | 0.3453 | 0.3402 | 0.3530 | 0.3473 | 0.3240 |
|  |  | Cov | 0.8310 | 0.9550 | 0.8340 | 0.8120 | 0.8950 | 0.9450 | 0.8910 | 0.8980 | 0.9550 |
| $\mathrm{L}=30$ | $\mathrm{CP}=4$ | Med.bias | 0.4518 | 0.0283 | 0.3622 | 0.4197 | 0.2262 | 0.0370 | 0.4006 | 0.4160 | 0.0280 |
|  |  | Med.abs | 0.4518 | 0.3819 | 0.5737 | 0.6374 | 0.6524 | 0.3569 | 0.4872 | 0.5266 | 0.3822 |
|  |  | Disp | 0.3989 | 1.5903 | 2.8445 | 3.2928 | 3.7210 | 1.5973 | 1.6529 | 1.9753 | 1.5888 |
|  |  | Cov | 0.1920 | 0.9600 | 0.7870 | 0.7780 | 0.2870 | 0.9560 | 0.7330 | 0.7010 | 0.9600 |
|  | $\mathrm{CP}=8$ | Med.bias | 0.4006 | -0.0093 | 0.3113 | 0.3802 | 0.0711 | -0.0047 | 0.3251 | 0.3742 | -0.0096 |
|  |  | Med.abs | 0.4006 | 0.2628 | 0.4665 | 0.5419 | 0.4506 | 0.2469 | 0.4360 | 0.4652 | 0.2624 |
|  |  | Disp | 0.3898 | 1.0192 | 2.4811 | 2.6333 | 2.4185 | 1.0549 | 1.5180 | 1.7457 | 1.0184 |
|  |  | Cov | 0.2410 | 0.9540 | 0.7740 | 0.7180 | 0.3750 | 0.9520 | 0.7410 | 0.7190 | 0.9540 |
|  | $\mathrm{CP}=35$ | Med.bias | 0.2333 | -0.0043 | 0.2216 | 0.2681 | 0.0071 | -0.0026 | 0.0942 | 0.1337 | -0.0046 |
|  |  | Med.abs | 0.2333 | 0.1205 | 0.2617 | 0.2912 | 0.1513 | 0.1186 | 0.1733 | 0.1948 | 0.1203 |
|  |  | Disp | 0.2942 | 0.4364 | 0.6190 | 0.6097 | 0.6229 | 0.4590 | 0.5995 | 0.6845 | 0.4363 |
|  |  | Cov | 0.4700 | 0.9580 | 0.6930 | 0.6270 | 0.7140 | 0.9570 | 0.7820 | 0.7370 | 0.9580 |
|  | $\mathrm{CP}=65$ | Med.bias | 0.1594 | -0.0030 | 0.1582 | 0.2026 | 0.0039 | -0.0021 | 0.0349 | 0.0341 | -0.0032 |
|  |  | Med.abs | 0.1597 | 0.0883 | 0.1868 | 0.2186 | 0.1020 | 0.0874 | 0.1023 | 0.1028 | 0.0883 |
|  |  | Disp | 0.2515 | 0.3156 | 0.4219 | 0.4517 | 0.3925 | 0.3324 | 0.3888 | 0.3957 | 0.3156 |
|  |  | Cov | 0.5950 | 0.9590 | 0.7370 | 0.6750 | 0.8280 | 0.9580 | 0.8320 | 0.8270 | 0.9590 |
| $\mathrm{L}=50$ | $\mathrm{CP}=4$ | Med.bias | 0.4595 | 0.4609 | 0.4388 | 0.4710 | 0.2363 | 0.2708 | 0.4659 | 0.4772 | 0.0104 |
|  |  | Med.abs | 0.4595 | 0.4609 | 0.6686 | 0.6988 | 0.6457 | 0.6732 | 0.5557 | 0.5801 | 0.3617 |
|  |  | Disp | 0.3213 | 0.3631 | 3.1782 | 3.5704 | 4.2559 | 4.0867 | 2.0354 | 2.1010 | 1.7490 |
|  |  | Cov | 0.0580 | 0.1350 | 0.7830 | 0.7970 | 0.2390 | 0.2500 | 0.7220 | 0.6950 | 0.9570 |
|  | $\mathrm{CP}=8$ | Med.bias | 0.4287 | 0.4222 | 0.3933 | 0.4166 | 0.1141 | 0.1303 | 0.4048 | 0.4250 | -0.0042 |
|  |  | Med.abs | 0.4287 | 0.4222 | 0.5661 | 0.6089 | 0.5061 | 0.5014 | 0.4839 | 0.5253 | 0.2548 |
|  |  | Disp | 0.3146 | 0.3538 | 2.7942 | 3.1653 | 3.1462 | 3.2328 | 1.7514 | 1.8235 | 1.0971 |
|  |  | Cov | 0.0730 | 0.1440 | 0.7670 | 0.7730 | 0.2920 | 0.3120 | 0.6930 | 0.6830 | 0.9510 |
|  | $\mathrm{CP}=35$ | Med.bias | 0.2933 | 0.2797 | 0.2631 | 0.2948 | -0.0024 | -0.0083 | 0.1546 | 0.2108 | -0.0020 |
|  |  | Med.abs | 0.2933 | 0.2797 | 0.2993 | 0.3414 | 0.1754 | 0.1801 | 0.2204 | 0.2554 | 0.1185 |
|  |  | Disp | 0.2641 | 0.2813 | 0.9371 | 1.2226 | 0.7222 | 0.7513 | 0.7612 | 0.8481 | 0.4712 |
|  |  | Cov | 0.1770 | 0.2550 | 0.6710 | 0.6390 | 0.5920 | 0.6070 | 0.7080 | 0.6820 | 0.9520 |
|  | $\mathrm{CP}=65$ | Med.bias | 0.2158 | 0.2038 | 0.1967 | 0.2431 | -0.0055 | -0.0062 | 0.0602 | 0.0670 | -0.0014 |
|  |  | Med.abs | 0.2158 | 0.2038 | 0.2244 | 0.2735 | 0.1100 | 0.1145 | 0.1215 | 0.1240 | 0.0856 |
|  |  | Disp | 0.2247 | 0.2385 | 0.5012 | 0.6035 | 0.4410 | 0.4383 | 0.4394 | 0.4815 | 0.3408 |
|  |  | Cov | 0.3110 | 0.3920 | 0.7070 | 0.6510 | 0.7320 | 0.7550 | 0.7740 | 0.7400 | 0.9470 |

F-statistic equals $\frac{C P}{L}+1=1.08$.
We observe that
(a) The performances of the regularized estimators increase with the strength of instruments but decrease with the number of instruments. Providing regularization parameter selection procedure robust to weak instruments would certainly improve these results.
(b) The bias of regularized LIML is quite a bit smaller than that of regularized 2SLS.
(c) The bias of our regularized estimators are smaller than those of the corresponding Donald and Newey's estimators. On the other hand, DN estimator has often better coverage.
(d) LF LIML and T-LIML estimators have very low median bias even in the case of relatively weak instruments $(C P=8)$.
(e) The coverage of our estimators deteriorates when the instruments are weak.

## 6 Empirical application: Institution and Growth

This section revisits Hall and Jones (1999) empirical work. Hall and Jones (1999) argue that the difference between output per worker across countries is mainly due to the differences in institution and government policies - the so-called social infrastructure. They write "Countries with corrupt government officials, severe impediments to trade, poor contract enforcement, and government interference in production will be unable to achieve levels of output per worker anywhere near the norms of western Europe, northern America, and eastern Asia." To quantify the effect of social infrastructure on per capita income, they use two-stage least squares (2SLS) with four instruments: the fraction of population speaking English at birth (EnL), the fraction of population speaking one of the five major European languages at birth (EuL), the distance from the equator ${ }^{5}$ (latitude, $L t$ ) and Romer and Frankel (1999) geography-predicted trade

[^4]intensity $(F R)$. The linear IV regression model is given by:
$$
y=c+\delta S+\varepsilon,
$$
where $y$ is an $n \times 1$ vector of $\log$ income per capita, $S$ is an $n \times 1$ vector which is the proxy for social infrastructure, $\theta$ is an $L \times 1$ vector, $c$ and $\delta$ are scalars. Dmitriev (2013) points out the fact that the instruments ${ }^{6} X=[E n L, E u L, L t, F R]$ are weak. To address this issue, we increased the number of instruments from 4 to 18 . The 18 instruments in our regression are derived from $X$ and are given by ${ }^{7}$
$Z=\left[X, X .^{2}, X .^{3}, X(:, 1) * X(:, 2), X(:, 1) * X(:, 3), X(:, 1) * X(:, 4), X(:, 2) * X(:, 3), X(:\right.$ $, 2) * X(:, 4), X(:, 3) * X(:, 4)]$ where all instruments are divided by their standard deviation.

The use of many instruments increased the concentration parameter from $\hat{\mu}_{n}^{2}=28.6$ to $\hat{\mu}_{n}^{2}=51.48$. However, it also increased the condition number of the $Z^{\prime} Z$ matrix from $1.08 e+04$ for 4 instruments to $2.48 e+05$ for 18. The regularized 2SLS and LIML correct the bias due to the use of many instruments. This enables us to have better points estimates.

We use a sample of 79 countries for which no data were imputed ${ }^{8}$. The results are reported in Table 4 below. Robust to heteroskedasticity standard errors are given in parentheses. They are computed using the formula of Carrasco and Tchuente (2013):

$$
\hat{V}(\hat{\delta})=\left(\hat{W}^{\prime} W\right)^{-1} \hat{W}^{\prime} \widehat{\Omega} \hat{W}\left(W^{\prime} \hat{W}\right)^{-1}
$$

where $\hat{W}=P^{\alpha} W$ for 2SLS, $\hat{W}=\left(P^{\alpha}-\nu I_{n}\right) W$ for LIML, and $\widehat{\Omega}$ is the diagonal matrix with $i$ th diagonal element equal to $\hat{\varepsilon}_{i}^{2}=\left(y_{i}-W_{i}^{\prime} \hat{\delta}\right)^{2}$.

Our findings suggest that "social infrastructure" has a significant causal effect on long-run economic performance throughout the world. The use of many instruments first increase the bias as illustrated by the fact that the distance between 2SLS and LIML is larger when 18 instruments are used. When the regularization is introduced,

[^5]Table 4: Institutions and growth

| 2SLS (4) | 2SLS (18) | T2SLS | L2SLS | P2SLS |
| ---: | ---: | ---: | ---: | ---: |
| $4.6612(0.7027)$ | $4.0124(0.5041)$ | $4.2916(0.338)$ | $\alpha=0.01$ | Number of iterations 1000 | Number of eigenvalues 15

NB: We report 2SLS and LIML for 4 and 18 instruments. For LIML with 18 instruments, we report the many instrument robust standard error of Hansen, Hausman, and Newey (2008) in parentheses. The regularized estimators are computed for 18 instruments. For the regularized estimators, the heteroskedasticity robust standard errors are given in parentheses.
this gap shrinks. For instance, the regularized LF, LIML and 2SLS are very close, this may be due to bias correction. But, for the PC regularization, the gap remains wide. The reason may be due to the lack of factor structure in the instruments set.

## 7 Conclusion

This paper illustrates the usefulness of regularization techniques for estimation in the many weak instruments framework. We derived the properties of the regularized 2SLS and LIML estimators in the presence of many or a continuum of moments that may be weak. We show that if well normalized the regularized 2SLS and LIML are consistent and reach the semiparametric efficiency bound. Our simulations show that the leading regularized estimators (LF and T of LIML) perform well.

In this work, we restricted our investigation to 2SLS and LIML with weak instruments. It would be interesting, for future research, to study the behavior of regularized version of other k-class estimators, such as FULL (Fuller (1977)) and bias adjusted 2SLS or other estimators as generalized method of moments or generalized empirical likelihood, in presence of many weak instruments. This will help us to have results that can be compared with those of Newey and Windmeijer (2009) and Hansen, Hausman, and Newey (2008). Another topic of interest is the use of our regularization tools to provide version of robust tests for weak instruments as Anderson-Rubin tests, that can be used with a large number or a continuum of moment conditions.

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## A General notation

Here we consider the general case where the estimation is based on a sequence of instruments $Z_{i}=Z\left(\tau ; x_{i}\right), \tau \in S$. Let $\pi$ be a positive measure on $S$. We denote $L^{2}(\pi)$ the Hilbert space of square integrable functions with respect to $\pi$.

We define the covariance operator $K$ of the instruments as

$$
\begin{gathered}
K: L^{2}(\pi) \rightarrow L^{2}(\pi) \\
(K g)\left(\tau_{1}\right)=\int E\left(Z\left(\tau_{1} ; x_{i}\right) \overline{Z\left(\tau_{2} ; x_{i}\right)}\right) g\left(\tau_{2}\right) \pi\left(\tau_{2}\right) d \tau_{2}
\end{gathered}
$$

where $\overline{Z\left(\tau_{2} ; x_{i}\right)}$ denotes the complex conjugate of $Z\left(\tau_{2} ; x_{i}\right)$.
$K$ is assumed to be a nuclear operator. Let $\lambda_{j}$ and $\phi_{j}, j=1,2, \ldots$ be respectively, the eigenvalues (ranked in decreasing order) and orthonormal eigenfunctions of $K . K$ can be estimated by $K_{n}$ defined as:

$$
\begin{gathered}
K_{n}: L^{2}(\pi) \rightarrow L^{2}(\pi) \\
\left(K_{n} g\right)\left(\tau_{1}\right)=\int \frac{1}{n} \sum_{i=1}^{n} Z\left(\tau_{1} ; x_{i}\right) \overline{Z\left(\tau_{2} ; x_{i}\right)} g\left(\tau_{2}\right) \pi\left(\tau_{2}\right) d \tau_{2} .
\end{gathered}
$$

If the number of moment conditions is infinite then the inverse of $K_{n}$ needs to be regularized because it is not continuous. By definition (see Kress, 1999, page 269), a regularized inverse of an operator $K$ is

$$
R_{\alpha}: L^{2}(\pi) \rightarrow L^{2}(\pi)
$$

such that $\lim _{\alpha \rightarrow 0} R_{\alpha} K \varphi=\varphi, \forall \varphi \in L^{2}(\pi)$.
Three different types of regularization schemes are considered: Tikhonov (T), Landwerber Fridman (LF), Spectral cut-off (SC) or Principal Components (PC). They are defined as follows:

## 1. Tikhonov(T)

This regularization scheme is related to the ridge regression.

$$
\begin{gathered}
\left(K^{\alpha}\right)^{-1}=\left(K^{2}+\alpha I\right)^{-1} K \\
\left(K^{\alpha}\right)^{-1} r=\sum_{j=1}^{\infty} \frac{\lambda_{j}}{\lambda_{j}^{2}+\alpha}\left\langle r, \phi_{j}\right\rangle \phi_{j}
\end{gathered}
$$

where $\alpha>0$ is the regularization parameter. A fixed $\alpha$ would result in a loss of efficiency. For the estimator to be asymptotically efficient, $\alpha$ has to go to zero at a certain rate which will be determined later on. This regularization is closely related to ridge regularization. Ridge regularization was first used in regression in a context where there were too many regressors. The aim was then to stabilize the inverse of $X^{\prime} X$ by replacing $X^{\prime} X$ by $X^{\prime} X+\alpha I$. However, this was done at the expense of a bias relative to OLS estimator. In the IV regression, the 2SLS estimator has already a bias and the use of many instruments usually increases its bias. The selection of an appropriate ridge parameter for the first step regression helps to reduce this bias. This explains why, in the IV case, ridge regularization is useful.

## 2. Landweber Fridman (LF)

This method of regularization is iterative. Let $0<c<1 /\|K\|^{2}$ where $\|K\|$ is the largest eigenvalue of K (which can be estimated by the largest eigenvalue of $K_{n}$ ). $\hat{\varphi}=\left(K^{\alpha}\right)^{-1} r$ is computed using the following algorithm:

$$
\left\{\begin{array}{l}
\hat{\varphi}_{l}=\left(1-c K^{2}\right) \hat{\varphi}_{l-1}+c K r, \quad l=1,2, \ldots, \frac{1}{\alpha}-1 \\
\hat{\varphi}_{0}=c K r
\end{array}\right.
$$

where $\frac{1}{\alpha}-1$ is some positive integer. We also have

$$
\left(K^{\alpha}\right)^{-1} r=\sum_{j=1}^{\infty} \frac{\left[1-\left(1-c \lambda_{j}^{2}\right)^{\frac{1}{\alpha}}\right]}{\lambda_{j}}\left\langle r, \phi_{j}\right\rangle \phi_{j} .
$$

## 3. Spectral cut-off (SC)

This method consists in selecting the eigenfunctions associated with the eigenvalues greater than some threshold. The aim is to select those who have greater contribution.

$$
\left(K^{\alpha}\right)^{-1} r=\sum_{\lambda_{j}^{2} \geq \alpha} \frac{1}{\lambda_{j}}\left\langle r, \phi_{j}\right\rangle \phi_{j}
$$

for $\alpha>0$.
This method is similar to principal components (PC) which consists in using the first eigenfunctions:

$$
\left(K^{\alpha}\right)^{-1} r=\sum_{j=1}^{1 / \alpha} \frac{1}{\lambda_{j}}\left\langle r, \phi_{j}\right\rangle \phi_{j}
$$

where $\frac{1}{\alpha}$ is some positive integer. It is equivalent to projecting on the first principal components of $W$. Interestingly, this approach is used in factor models where $W_{i}$ is assumed to depend on a finite number of factors (see Bai and Ng (2002), Stock and Watson (2002)) As the estimators based on PC and SC are identical, we will use PC and SC interchangeably.

These regularized inverses can be rewritten in common notation as:

$$
\left(K^{\alpha}\right)^{-1} r=\sum_{j=1}^{\infty} \frac{q\left(\alpha, \lambda_{j}^{2}\right)}{\lambda_{j}}\left\langle r, \phi_{j}\right\rangle \phi_{j}
$$

where for $\mathrm{T}: q\left(\alpha, \lambda_{j}^{2}\right)=\frac{\lambda_{j}^{2}}{\lambda_{j}^{2}+\alpha}$,
for LF: $q\left(\alpha, \lambda_{j}^{2}\right)=\left[1-\left(1-c \lambda_{j}^{2}\right)^{1 / \alpha}\right]$,
for SC: $q\left(\alpha, \lambda_{j}^{2}\right)=I\left(\lambda_{j}^{2} \geq \alpha\right)$, for PC $q\left(\alpha, \lambda_{j}^{2}\right)=I(j \leq 1 / \alpha)$.
In order to compute the inverse of $K_{n}$ we have to choose the regularization parameter $\alpha$. Let $\left(K_{n}^{\alpha}\right)^{-1}$ be the regularized inverse of $K_{n}$ and $P^{\alpha}$ a $n \times n$ matrix defined as in Carrasco (2012) by $P^{\alpha}=T\left(K_{n}^{\alpha}\right)^{-1} T^{*}$ where

$$
T: L^{2}(\pi) \rightarrow \mathbb{R}^{n}
$$

$$
T g=\left(\begin{array}{c}
\left\langle Z_{1}, g\right\rangle \\
\left\langle Z_{2}, g\right\rangle \\
\cdot \\
\cdot \\
\left\langle Z_{n}, g\right\rangle
\end{array}\right)
$$

and

$$
\begin{gathered}
T^{*}: \mathbb{R}^{n} \rightarrow L^{2}(\pi) \\
T^{*} v=\frac{1}{n} \sum_{i=1}^{n} Z_{i} v_{i}
\end{gathered}
$$

such that $K_{n}=T^{*} T$ and $T T^{*}$ is an $n \times n$ matrix with typical element $\frac{\left\langle Z_{i}, Z_{j}\right\rangle}{n}$. Let $\hat{\phi}_{j}, \hat{\lambda}_{1} \geq \hat{\lambda}_{2} \geq \ldots>0, j=1,2, \ldots$ be the orthonormalized eigenfunctions and eigenvalues of $K_{n} \cdot \hat{\lambda}_{j}$ are consistent estimators of $\lambda_{j}$ the eigenvalues of $T T^{*}$. We then have $T \hat{\phi}_{j}=\sqrt{\lambda_{j}} \psi_{j}$ and $T^{*} \psi_{j}=\sqrt{\lambda_{j}} \hat{\phi}_{j}$.
For $v \in \mathbf{R}^{n}, P^{\alpha} v=\sum_{j=1}^{\infty} q\left(\alpha, \lambda_{j}^{2}\right)\left\langle v, \psi_{j}\right\rangle \psi_{j}$. It follows that for any vectors $v$ and $w$ of $\mathbf{R}^{n}$ :

$$
\begin{align*}
v^{\prime} P^{\alpha} w & =v^{\prime} T\left(K_{n}^{\alpha}\right)^{-1} T^{*} w \\
& =\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} \sum_{i=1}^{n} Z_{i}(.) v_{i},\left(K_{n}^{\alpha}\right)^{-1 / 2} \frac{1}{n} \sum_{i=1}^{n} Z_{i}(.) w_{i}\right\rangle . \tag{1}
\end{align*}
$$

## B Proofs

## Proof of Proposition 1:

We first prove the consistency of our estimator.
Let $g_{n}=\frac{1}{n} \sum_{i=1}^{n} Z_{i} W_{i}=S_{n}\left[\frac{1}{n} \sum_{i=1}^{n} Z_{i} f_{i}\right] / \sqrt{n}+\frac{1}{n} \sum_{i=1}^{n} Z_{i} u_{i}=S_{n} g_{n 1} / \sqrt{n}+g_{n 2}$ (remember that $g_{n}$ is a function indexed by $\tau$ and $Z_{i}$ is also a function of $\tau$, such a representation can handle both countable and continuum of instruments). Note that $g_{n 2}=\frac{1}{n} \sum_{i=1}^{n} Z_{i} u_{i}=o_{p}(1), \sqrt{n} g_{n 2}=O_{p}(1)$ and $S_{n} / \sqrt{n}$ is bounded by Assumption 1(i).

$$
\hat{\delta}-\delta_{0}=\left(W^{\prime} P^{\alpha} W\right)^{-1} W^{\prime} P^{\alpha} \varepsilon
$$

We have $S_{n}^{\prime}\left(\hat{\delta}-\delta_{0}\right) / \mu_{n}=\left[S_{n}^{-1} W^{\prime} P^{\alpha} W S_{n}^{-1^{\prime}}\right]^{-1}\left[S_{n}^{-1} W^{\prime} P^{\alpha} \varepsilon / \mu_{n}\right]$ and by construction ${ }^{9}$ of $P^{\alpha}$ :

$$
\begin{aligned}
W^{\prime} P^{\alpha} W= & n\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n},\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n}^{\prime}\right\rangle \\
= & S_{n}\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1},\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1}^{\prime}\right\rangle S_{n}^{\prime} \\
& +S_{n}\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1},\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 2}^{\prime}\right\rangle \sqrt{n} \\
& +\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 2},\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1}^{\prime}\right\rangle S_{n}^{\prime} \sqrt{n} \\
& +\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 2},\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 2}^{\prime}\right\rangle n . \\
S_{n}^{-1} W^{\prime} P^{\alpha} W S_{n}^{-1^{\prime}}= & \left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1},\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1}^{\prime}\right\rangle \\
& +\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1},\left(K_{n}^{\alpha}\right)^{-1 / 2} \sqrt{n} g_{n 2}^{\prime}\right\rangle S_{n}^{-1^{\prime}} \\
& +S_{n}^{-1}\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} \sqrt{n} g_{n 2},\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1}^{\prime}\right\rangle \\
& +S_{n}^{-1}\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} \sqrt{n} g_{n 2},\left(K_{n}^{\alpha}\right)^{-1 / 2} \sqrt{n} g_{n 2}^{\prime}\right\rangle S_{n}^{-1^{\prime}} .
\end{aligned}
$$

Hence,

$$
S_{n}^{-1}\left[W^{\prime} P^{\alpha} W\right] S_{n}^{-1^{\prime}}=\left\langle\left(K_{n}^{\alpha}\right)^{-\frac{1}{2}} g_{n 1},\left(K_{n}^{\alpha}\right)^{-\frac{1}{2}} g_{n 1}^{\prime}\right\rangle+o_{p}(1) .
$$

At this stage, we can apply the same proof as that of Proposition 1 of Carrasco (2012) which shows that

$$
\left\langle\left(K_{n}^{\alpha}\right)^{-\frac{1}{2}} g_{n 1},\left(K_{n}^{\alpha}\right)^{-\frac{1}{2}} g_{n 1}^{\prime}\right\rangle \rightarrow\left\langle g_{1}, g_{1}^{\prime}\right\rangle_{K}
$$

[^6]in probability as $n$ and $n \alpha^{\frac{1}{2}}$ go to infinity, with $\left\langle g_{1}, g_{1}^{\prime}\right\rangle_{K}$ a $p \times p$ matrix with ( $a, b$ ) element $\left\langle K^{-\frac{1}{2}} E\left(Z\left(., x_{i}\right) f_{i a}\right), K^{-\frac{1}{2}} E\left(Z\left(., x_{i}\right) f_{i b}\right)\right\rangle$ which is assumed to be nonsingular.
\[

$$
\begin{aligned}
\frac{S_{n}^{-1} W^{\prime} P^{\alpha} \varepsilon}{\mu_{n}}= & \frac{n S_{n}^{-1}}{\mu_{n}}\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n},\left(K_{n}^{\alpha}\right)^{-1 / 2} \frac{1}{n} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right\rangle \\
= & \frac{1}{\mu_{n}}\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1},\left(K_{n}^{\alpha}\right)^{-1 / 2} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right\rangle \\
& +\frac{\mu_{n} S_{n}^{-1}}{\mu_{n}^{2}}\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} \sqrt{n} g_{n 2},\left(K_{n}^{\alpha}\right)^{-1 / 2} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right\rangle \\
= & o_{p}(1)
\end{aligned}
$$
\]

because $\mu_{n} S_{n}^{-1} \rightarrow S_{0}$ by Assumption 1(ii) and $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}=O_{p}(1)$. This proves the consistency of the regularized 2SLS.

For the asymptotic normality we write

$$
S_{n}^{\prime}\left(\hat{\delta}-\delta_{0}\right)=\left[S_{n}^{-1} W^{\prime} P^{\alpha} W S_{n}^{\prime-1}\right]^{-1}\left[S_{n}^{-1} W^{\prime} P^{\alpha} \varepsilon\right]
$$

We then have

$$
\begin{aligned}
S_{n}^{-1} W^{\prime} P^{\alpha} \varepsilon= & n S_{n}^{-1}\left\langle\left(K_{n}^{\alpha}\right)^{-1} g_{n}, \frac{1}{n} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right\rangle \\
= & \left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1},\left(K_{n}^{\alpha}\right)^{-1 / 2} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right\rangle \\
& +S_{n}^{-1}\left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} \sqrt{n} g_{n 2},\left(K_{n}^{\alpha}\right)^{-1 / 2} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right\rangle \\
= & \left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1},\left(K_{n}^{\alpha}\right)^{-1 / 2} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right\rangle \\
& +o_{p}(1) .
\end{aligned}
$$

Moreover,

$$
\begin{align*}
& \left\langle\left(K_{n}^{\alpha}\right)^{-1 / 2} g_{n 1},\left(K_{n}^{\alpha}\right)^{-1 / 2} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right\rangle  \tag{2}\\
= & \left\langle\left(K_{n}^{\alpha}\right)^{-1} g_{n 1}-K^{-1} g_{1}, \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right\rangle  \tag{3}\\
+ & \left.\left\langle K^{-1} g_{1}, \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right)\right\rangle .
\end{align*}
$$

The first term is negligible since
$\left.\left\langle\left(K_{n}^{\alpha}\right)^{-1} g_{n 1}-K^{-1} g_{1}, \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right)\right\rangle \leq\left\|\left(K_{n}^{\alpha}\right)^{-1} g_{n 1}-K^{-1} g_{1}\right\|\left\|\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right\|=o_{p}(1) O_{p}(1)$.
By the functional central limit theorem, we obtain the following result
$\left\langle K^{-1} g_{1}, \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{i} \varepsilon_{i}\right\rangle \rightarrow \mathcal{N}\left(0, \sigma_{\varepsilon}^{2}\left\langle g_{1}, g_{1}^{\prime}\right\rangle_{K}\right)$ as $n$ and $n \alpha$ go to infinity.
We then apply the continuous mapping theorem and Slutzky's theorem to show that $S_{n}^{\prime}\left(\hat{\delta}-\delta_{0}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma_{\varepsilon}^{2}\left\langle g_{1}, g_{1}^{\prime}\right\rangle_{K}^{-1}\right)$.
By assumption, $g_{1 a}=E\left(Z\left(., x_{i}\right) f_{i a}\right)$ belong to the range of $K$. Let $L^{2}(Z)$ be the closure of the space spanned by $\{Z(x, \tau), \tau \in I\}$ and $g_{1}$ is an element of this space. If $f_{i} \in L^{2}(Z)$ we can compute the inner product in the RKHS and show that

$$
\left\langle g_{1 a}, g_{1 b}\right\rangle_{K}=E\left(f_{i a} f_{i b}\right)
$$

This can be seen by applying Theorem 6.4 of Carrasco, Florens, and Renault (2007). It follows that

$$
S_{n}^{\prime}\left(\hat{\delta}-\delta_{0}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma_{\varepsilon}^{2}\left[E\left(f_{i} f_{i}^{\prime}\right)\right]^{-1}\right)
$$

This completes the proof of Proposition 1.

## Proof of Proposition 2:

To prove this proposition, we need three lemmas. The first lemma corresponds to lemma A0 of Hansen, Hausman, and Newey (2008).

Lemma 1: Under assumption 1 if $\left\|S_{n}^{\prime}\left(\hat{\delta}_{L}-\delta_{0}\right) / \mu_{n}\right\|^{2} /\left(1+\left\|\hat{\delta}_{L}\right\|^{2}\right) \xrightarrow{P} 0$ then $\| S_{n}^{\prime}\left(\hat{\delta}_{L}-\right.$ $\left.\delta_{0}\right) / \mu_{n} \| \xrightarrow{P} 0$.

Proof: The proof of this lemma is the same as in Hansen, Hausman, and Newey (2008).

Lemma 2: Let us assume that there exists a constant $C$ such that $E\left(\left\|\varepsilon_{i}\right\|^{4} \mid X\right) \leq C$ and $E\left(\left\|u_{a i}\right\|^{4} \mid X\right) \leq C$ for all $i$. Then,

$$
\begin{aligned}
\operatorname{Var}\left(\varepsilon^{\prime} P^{\alpha} u_{a}\right) & \leq C\left(\sum_{j} q_{j}^{2}\right), \\
\varepsilon^{\prime} P^{\alpha} u_{a}-E\left(\varepsilon^{\prime} P^{\alpha} u_{a} \mid X\right) & =O\left(\left(\sum_{j} q_{j}^{2}\right)^{\frac{1}{2}}\right), \\
\frac{\varepsilon^{\prime} P^{\alpha} \varepsilon}{\mu_{n}^{2}} & =O_{p}\left(\frac{1}{\alpha \mu_{n}^{2}}\right)=o_{p}(1) .
\end{aligned}
$$

## Proof:

For notational simplicity, we suppress the conditioning on $X$. Let $E\left(\varepsilon_{i}^{2}\right)=\sigma_{\varepsilon}^{2}, E\left(\varepsilon_{i} u_{a i}\right)=$ $\sigma_{\varepsilon u_{a}}$ and $E\left(u_{a i}^{\prime} u_{a i}\right)=\sigma_{u_{a}}^{2}$,

$$
\operatorname{Var}\left(\varepsilon^{\prime} P^{\alpha} u_{a}\right)=E\left(\varepsilon^{\prime} P^{\alpha} u_{a} u_{a}^{\prime} P^{\alpha} \varepsilon\right)-E\left(\varepsilon^{\prime} P^{\alpha} u_{a}\right) E\left(u_{a}^{\prime} P^{\alpha} \varepsilon\right)
$$

Using the spectral decomposition of $P^{\alpha}$, we have

$$
\begin{aligned}
E\left(\varepsilon^{\prime} P^{\alpha} u_{a} u_{a}^{\prime} P^{\alpha} \varepsilon\right) & =\frac{1}{n^{2}} \sum_{j, l} q_{j} q_{l} E\left\{\left(\varepsilon^{\prime} \psi_{l}\right)\left(u_{a}^{\prime} \psi_{l}\right)^{\prime}\left(\varepsilon^{\prime} \psi_{j}\right)\left(u_{a}^{\prime} \psi_{j}\right)\right\} \\
& =\frac{1}{n^{2}} \sum_{j, l} q_{j} q_{l} E\left\{\sum_{i} \varepsilon_{i} u_{a i}^{\prime} \psi_{l i}^{2} \sum_{b} \varepsilon_{b} u_{a b} \psi_{j b}^{2}\right. \\
& +\sum_{c} \varepsilon_{c} u_{a c}^{\prime} \psi_{l c} \psi_{j c} \sum_{d} \varepsilon_{d} u_{a d} \psi_{j d} \psi_{l d} \\
& \left.+\sum_{c} \varepsilon_{c}^{2} \psi_{l c} \psi_{j c} \sum_{d} u_{a d}^{\prime} u_{a d} \psi_{j d} \psi_{l d}\right\} \\
& =\left(\sum_{j} q_{j}\right)^{2} \sigma_{\varepsilon u_{a}}^{\prime} \sigma_{\varepsilon u_{a}}+\left(\sigma_{\varepsilon u_{a}}^{\prime} \sigma_{\varepsilon u_{a}}+\sigma_{\varepsilon}^{2} \sigma_{u_{a}}^{2}\right) \sum_{j} q_{j}^{2}
\end{aligned}
$$

by the fact that $\left(u_{a i}, \varepsilon_{i}\right)$ are independent across $i$ and the eigenvectors are orthonormal.

$$
\begin{aligned}
E\left(\varepsilon^{\prime} P^{\alpha} u_{a}\right) & =\frac{1}{n} \sum_{l} q_{l} E\left\{\left(\sum_{k} u_{a k}^{\prime} \psi_{l k}\right)\left(\sum_{i} \varepsilon_{i} \psi_{l i}\right)\right\} \\
& =\frac{1}{n} \sum_{l} q_{l} n \sigma_{\varepsilon u_{a}}^{\prime} \\
& =\sigma_{\varepsilon u_{a}}^{\prime}\left(\sum_{j} q_{j}\right)
\end{aligned}
$$

Thus

$$
\operatorname{Var}\left(\varepsilon^{\prime} P^{\alpha} u_{a}\right)=\left(\sigma_{\varepsilon u_{a}}^{\prime} \sigma_{\varepsilon u_{a}}+\sigma_{\varepsilon}^{2} \sigma_{u_{a}}^{2}\right) \sum_{j} q_{j}^{2} \leq C\left(\sum_{j} q_{j}^{2}\right) .
$$

The second conclusion follows by Markov inequality.

$$
\begin{aligned}
E\left(\varepsilon^{\prime} P^{\alpha} \varepsilon\right) & =\operatorname{tr}\left(P^{\alpha} E\left(\varepsilon \varepsilon^{\prime}\right)\right) \\
& =\sigma_{\varepsilon}^{2}\left(\sum_{j} q_{j}\right)=O_{p}(1 / \alpha)
\end{aligned}
$$

Using the result for $\varepsilon^{\prime} P^{\alpha} u_{a}$ with $\varepsilon$ in place of $u_{a}$, we obtain

$$
\operatorname{Var}\left(\varepsilon^{\prime} P^{\alpha} \varepsilon\right) \leq C\left(\sum_{j} q_{j}^{2}\right)
$$

It follows that $\left(\varepsilon^{\prime} P^{\alpha} \varepsilon-E\left(\varepsilon^{\prime} P^{\alpha} \varepsilon\right)\right) / \mu_{n}^{2}=O_{p}\left(\left(\sum_{j} q_{j}^{2}\right)^{1 / 2} / \mu_{n}^{2}\right)=o_{p}\left(\sum_{j} q_{j} / \mu_{n}^{2}\right)$. Hence, the third equality holds.

Lemma 3: Let $\hat{A}=\frac{f^{\prime} P^{\alpha} f}{n}$ and $\hat{B}=\frac{\bar{W}^{\prime} \bar{W}}{n}$ with $\bar{W}=[y, W]$, there exist two constants $C$ and $C^{\prime}$ such that $\hat{A} \geq C I_{p}$ and $\|\hat{B}\| \leq C^{\prime}$.

Proof: By the definition of $P^{\alpha}$, we have (see Equation (1)):

$$
\hat{A}=\frac{f^{\prime} P^{\alpha} f}{n}=\left\langle\left(K_{n}^{\alpha}\right)^{-\frac{1}{2}} f_{n},\left(K_{n}^{\alpha}\right)^{-\frac{1}{2}} f_{n}^{\prime}\right\rangle
$$

with

$$
f_{n}=\frac{1}{n} \sum_{i} Z_{i} f_{i} .
$$

By Lemma 5(i) of Carrasco (2012) and the law of large numbers,

$$
\frac{f^{\prime} P^{\alpha} f}{n}=\frac{f^{\prime} f}{n}+o_{p}(1)=E\left(f_{i}^{\prime} f_{i}\right)+o_{p}(1)
$$

as $\alpha$ goes to zero. Because $E\left(f_{i}^{\prime} f_{i}\right)$ is positive definite, there exists a constant $C$ such that

$$
\hat{A} \geq C I_{p}
$$

with probability one.
We have $\bar{W}=[y, W]=W D_{0}+\varepsilon e$ where $D_{0}=\left[\delta_{0}, I\right], \delta_{0}$ is the true value of the parameter and $e$ is the first unit vector.

$$
\begin{aligned}
\hat{B} & =\frac{\bar{W}^{\prime} \bar{W}}{n} \\
& =D_{0}^{\prime} S_{n} \frac{f^{\prime} f}{n} S_{n}^{\prime} D_{0} / n+D_{0}^{\prime} S_{n} \frac{f^{\prime} u}{n} D_{0} / \sqrt{n}+D_{0}^{\prime} S_{n} \frac{f^{\prime} \varepsilon}{n} e / \sqrt{n} \\
& +D_{0}^{\prime} \frac{u^{\prime} f}{n} S_{n}^{\prime} D_{0} / \sqrt{n}+D_{0}^{\prime} \frac{u^{\prime} u}{n} D_{0}+D_{0}^{\prime} \frac{u^{\prime} \varepsilon}{n} e \\
& +e^{\prime} \frac{\varepsilon^{\prime} f}{n} S_{n}^{\prime} D_{0} / \sqrt{n}+e^{\prime} \frac{\varepsilon^{\prime} u}{n} D_{0}+e^{\prime} \frac{\varepsilon^{\prime} \varepsilon}{n} e
\end{aligned}
$$

Using the law of large numbers, we can conclude that $\|\hat{B}\| \leq C^{\prime}$, where $C^{\prime}$ is a constant, with probability one.

## Proof of consistency

Let us consider

$$
\begin{gathered}
\hat{Q}(\delta)=\frac{(y-W \delta)^{\prime} P^{\alpha}(y-W \delta) / \mu_{n}^{2}}{(y-W \delta)^{\prime}(y-W \delta) / n} \\
\hat{\delta}_{L}=\operatorname{argmin} Q(\delta)
\end{gathered}
$$

For $\delta=\delta_{0}, \hat{Q}\left(\delta_{0}\right)=\frac{\varepsilon^{\prime} P^{\alpha} \varepsilon / \mu_{n}^{2}}{\varepsilon^{\prime} \varepsilon / n}$. With probability one $\varepsilon^{\prime} \varepsilon / n>C$ and by lemma 2

$$
\varepsilon^{\prime} P^{\alpha} \varepsilon / \mu_{n}^{2}=o_{p}(1)
$$

Hence $\hat{Q}\left(\delta_{0}\right)=o_{p}(1)$.
Since $0 \leq \hat{Q}\left(\hat{\delta}_{L}\right) \leq \hat{Q}\left(\delta_{0}\right)$ it is easy to see that $\hat{Q}\left(\hat{\delta}_{L}\right)=o_{p}(1)$.
Let us show that

$$
\mu_{n}^{-2}(y-W \delta)^{\prime} P^{\alpha}(y-W \delta) \geq C\left\|S_{n}^{\prime}\left(\delta-\delta_{0}\right) / \mu_{n}\right\|^{2}
$$

Let $D(\delta)=\mu_{n}^{-2}(y-W \delta)^{\prime} P^{\alpha}(y-W \delta)=\mu_{n}^{-2}\left(1,-\delta^{\prime}\right) \bar{W}^{\prime} P^{\alpha} \bar{W}\left(1,-\delta^{\prime}\right)^{\prime}$. Moreover, $D(\delta)=\mu_{n}^{-2}\left(1,-\delta^{\prime}\right) D_{0}^{\prime} S_{n} \frac{f^{\prime} P^{\alpha} f}{n} S_{n}^{\prime} D_{0}\left(1,-\delta^{\prime}\right)^{\prime}+o_{p}(1)=\mu_{n}^{-2}\left(1,-\delta^{\prime}\right) D_{0}^{\prime} S_{n} E\left(f f^{\prime}\right) S_{n}^{\prime} D_{0}\left(1,-\delta^{\prime}\right)^{\prime}+$ $o_{p}(1)$. It follows from lemma 3 that

$$
D(\delta) \geq C\left\|S_{n}^{\prime}\left(\delta-\delta_{0}\right) / \mu_{n}\right\|^{2}
$$

We also have that $(y-W \delta)^{\prime}(y-W \delta) / n=\left(1,-\delta^{\prime}\right) \hat{B}\left(1,-\delta^{\prime}\right)^{\prime}$. Hence,

$$
\frac{\left\|S_{n}^{\prime}\left(\hat{\delta}_{L}-\delta_{0}\right) / \mu_{n}\right\|^{2}}{\left(1+\left\|\hat{\delta}_{L}\right\|^{2}\right)} \leq C \hat{Q}\left(\hat{\delta}_{L}\right) .
$$

Then by Lemma 1 we have $S_{n}^{\prime}\left(\hat{\delta}_{L}-\delta_{0}\right) / \mu_{n} \rightarrow 0$ in probability as $n$ and $\mu_{n}^{2} \alpha$ go to infinity. This proves the consistency of LIML with many weak instruments.

Now let us prove the asymptotic normality.

## Proof of asymptotic normality

Denote $A(\delta)=(y-W \delta)^{\prime} P^{\alpha}(y-W \delta) / 2, B(\delta)=(y-W \delta)^{\prime}(y-W \delta)$ and

$$
\Lambda(\delta)=\frac{A(\delta)}{B(\delta)}
$$

We know that the LIML is $\hat{\delta}_{L}=\operatorname{argmin} \Lambda(\delta)$.
We calculate the gradient and Hessian $\Lambda_{\delta}(\delta)=B(\delta)^{-1}\left[A_{\delta}(\delta)-\Lambda(\delta) B_{\delta}(\delta)\right]$,

$$
\Lambda_{\delta \delta}(\delta)=B(\delta)^{-1}\left[A_{\delta \delta}(\delta)-\Lambda(\delta) B_{\delta \delta}(\delta)\right]-B(\delta)^{-1}\left[B_{\delta}(\delta) \Lambda_{\delta}^{\prime}(\delta)-\Lambda_{\delta}(\delta) B_{\delta}^{\prime}(\delta)\right] .
$$

Then by the mean-value theorem applied to the first-order condition $\Lambda_{\delta}(\hat{\delta})=0$, we have:

$$
S_{n}^{\prime}\left(\hat{\delta}_{L}-\delta_{0}\right)=-\left[S_{n}^{-1} \Lambda_{\delta \delta}(\tilde{\delta}) S_{n}^{-1^{\prime}}\right]^{-1}\left[S_{n}^{\prime} \Lambda_{\delta}\left(\delta_{0}\right)\right]
$$

where $\tilde{\delta}$ is the mean-value. By the consistency of $\hat{\delta}_{L}, \tilde{\delta} \rightarrow \delta_{0}$.
It then follows that

$$
\begin{aligned}
B_{\delta}(\tilde{\delta}) / n & =-2 \sum_{i} W_{i} \tilde{\varepsilon}_{i} / n, \\
& =-2 \sum_{i}\left(\gamma_{i}+u_{i}\right) \tilde{\varepsilon}_{i} / n \\
& =-2 S_{n} / \sqrt{n}\left(\sum_{i} f_{i} \tilde{\varepsilon}_{i} / n\right)-2\left(\sum_{i} u_{i} \tilde{\varepsilon}_{i} / n\right) \\
& =-2 \sigma_{u \varepsilon}+o_{p}(1)
\end{aligned}
$$

under the assumption that $S_{n} / \sqrt{n}$ is bounded, with $\tilde{\varepsilon}_{i}=\left(y_{i}-W_{i}^{\prime} \tilde{\delta}\right)$ and $\sigma_{u \varepsilon}=E\left(u_{i} \varepsilon_{i}\right)$.

$$
B(\tilde{\delta}) / n \xrightarrow{P} \sigma_{\varepsilon}^{2}, B_{\delta}(\tilde{\delta}) / n \xrightarrow{P}-2 \sigma_{u \varepsilon}
$$

$$
\Lambda(\delta)=\frac{(y-W \delta)^{\prime} P^{\alpha}(y-W \delta) / 2 n}{(y-W \delta)^{\prime}(y-W \delta) / n}
$$

For $\delta=\delta_{0}, \Lambda\left(\delta_{0}\right)=\frac{\varepsilon^{\prime} P^{\alpha} \varepsilon / 2 n}{\varepsilon^{\prime} \varepsilon / n}$. With probability one, $\varepsilon^{\prime} \varepsilon / n>C$, and by Lemma 2 and $\mu_{n}^{2} \leq n$,

$$
\varepsilon^{\prime} P^{\alpha} \varepsilon / n=o_{p}(1)
$$

We have $\Lambda\left(\delta_{0}\right)=o_{p}(1)$. Therefore, $\Lambda(\tilde{\delta}) \xrightarrow{P} 0$. By the first order condition, we also have

$$
\Lambda_{\delta}(\tilde{\delta}) \xrightarrow{P} 0
$$

$$
B_{\delta \delta}(\tilde{\delta})=2 W^{\prime} W / n \xrightarrow{P} 2 E\left(W_{i} W_{i}^{\prime}\right), A_{\delta \delta}(\tilde{\delta}) / n=W^{\prime} P^{\alpha} W / n
$$

We can then conclude that $\Lambda_{\delta \delta}(\tilde{\delta})=n B^{-1}(\tilde{\delta})\left[A_{\delta \delta}(\tilde{\delta}) / n\right]+o_{p}(1)$. Hence

$$
\begin{aligned}
n \tilde{\sigma}_{\varepsilon}^{2} \Lambda_{\delta \delta}(\tilde{\delta}) & =W^{\prime} P^{\alpha} W \\
& =S_{n}\left\langle\left(K_{n}^{\alpha}\right)^{-\frac{1}{2}} g_{n 1},\left(K_{n}^{\alpha}\right)^{-\frac{1}{2}} g_{n 1}^{\prime}\right\rangle S_{n}^{\prime}+o_{p}(1) \\
& =S_{n} H S_{n}^{\prime}+o_{p}(1)
\end{aligned}
$$

with $H=E\left(f\left(x_{i}\right) f\left(x_{i}\right)^{\prime}\right)$ and $\tilde{\sigma}_{\varepsilon}^{2}=(y-W \widetilde{\delta})^{\prime}(y-W \widetilde{\delta}) / n$.
Hence

$$
n \tilde{\sigma}_{\varepsilon}^{2} S_{n}^{-1} \Lambda_{\delta \delta}(\tilde{\delta}) S_{n}^{-1^{\prime}}=H+o_{p}(1)
$$

Let $\hat{\phi}=\frac{W^{\prime} \varepsilon}{\varepsilon^{\prime} \varepsilon}, \phi=\frac{\sigma_{u \varepsilon}}{\sigma_{\varepsilon}^{2}}$ and $v=u-\varepsilon \phi^{\prime}$. It is useful to remark that $v^{\prime} P^{\alpha} \varepsilon=O_{p}(1 / \sqrt{\alpha})$ using Lemma 2 with $v$ in place of $u$ and $E\left(u_{i} v_{i}\right)=0$. Moreover, $\hat{\phi}-\phi=O_{p}(1 / \sqrt{n})$ by the central limit theorem and the delta method. Hence, $(\hat{\phi}-\phi) \varepsilon^{\prime} P^{\alpha} \varepsilon=O_{p}(1 / \alpha \sqrt{n})$.

Furthermore, $f^{\prime}\left(I-P^{\alpha}\right) \varepsilon / \sqrt{n}=O_{p}\left(\Delta_{\alpha}^{2}\right)=o_{p}(1)$ by Lemma 5(ii) Carrasco (2012) with $\Delta_{\alpha}=\operatorname{tr}\left(f^{\prime}\left(I-P^{\alpha}\right)^{2} f / n\right)=O_{p}\left(\alpha^{\min (\beta, 2)}\right)=o_{p}(1)$. We have

$$
\begin{aligned}
-n \tilde{\sigma}^{2} S_{n}^{-1} \Lambda_{\delta}\left(\delta_{0}\right) & =S_{n}^{-1}\left(W^{\prime} P^{\alpha} \varepsilon-\varepsilon^{\prime} P^{\alpha} \varepsilon \frac{W^{\prime} \varepsilon}{\varepsilon^{\prime} \varepsilon}\right) \\
& =f^{\prime} \varepsilon / \sqrt{n}-f^{\prime}\left(I-P^{\alpha}\right) \varepsilon / \sqrt{n}+S_{n}^{-1} v^{\prime} P^{\alpha} \varepsilon-S_{n}^{-1}(\hat{\phi}-\phi) \varepsilon^{\prime} P^{\alpha} \varepsilon \\
& =f^{\prime} \varepsilon / \sqrt{n}+o_{p}(1)+S_{n}^{-1} O_{p}(1 / \sqrt{\alpha})+S_{n}^{-1} O_{p}(1 / \alpha \sqrt{n}) \\
& =f^{\prime} \varepsilon / \sqrt{n}+o_{p}(1) \xrightarrow{d} \mathcal{N}\left(0, \sigma_{\varepsilon}^{2} H\right)
\end{aligned}
$$

as $n, \alpha \mu_{n}^{2}$ go to infinity under the assumption $\mu_{n} S_{n}^{-1} \rightarrow S_{0}$.
The conclusion follows from Slutzky's theorem.

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[^0]:    ${ }^{*}$ Carrasco gratefully acknowledges financial support from SSHRC.
    ${ }^{\dagger}$ School of Economics, Email: g.tchuente@kent.ac.uk

[^1]:    ${ }^{1}$ Hahn and Hausman (2003) define weak instruments, by two features: (i) two-stage least squares (2SLS) analysis is badly biased toward the ordinary least-squares (OLS) estimate, and alternative unbiased estimators such as limited-information maximum likelihood (LIML) may not solve the problem; and (ii) the standard (first-order) asymptotic distribution does not give an accurate framework for inference. Weak instrument may also be an important cause of the finding that the often-used test of over identifying restrictions (OID test) rejects too often.
    ${ }^{2}$ Section 4 discusses related literature in more details.

[^2]:    ${ }^{3} \mathrm{~A}$ word of caution: when the number of instruments is large enough relative to the sample size, the sample covariance matrix $Z^{\prime} Z / n$ will be near singular or singular which does not mean that the smallest eigenvalue of $K$ is not bounded away from 0 in the population. Moreover, eigenvalues are not scale invariant.

[^3]:    ${ }^{4}$ We do not provide a formal proof that this bound is the semiparametric bound. This proof is beyond the scope of the present paper. We refer the interested readers to Newey (1990), ?, and ?.

[^4]:    ${ }^{5}$ The distance from the equator is measured as the absolute value of latitude in degrees divided by 90 to place it on a 0 to 1 scale.

[^5]:    ${ }^{6}$ This correspond to the specification (iv) of Dmitriev (2013).
    ${ }^{7} X .{ }^{k}=\left[X_{i j}^{k}\right], X(:, k)$ is the $k^{t h}$ column of $X$ and $X(:, k) * X(:, l)$ is a vector of interactions between columns $k$ and $l$.
    ${ }^{8}$ The data were downloaded from Charles Jones' webpage: http://www.stanford.edu/ ${ }^{\sim}$ chadj/HallJones400.asc

[^6]:    ${ }^{9}$ Let $g$ and $h$ be two $p$ vectors of functions of $L^{2}(\pi)$. By a slight abuse of notation, $\left\langle g, h^{\prime}\right\rangle$; denotes the matrix with elements $\left\langle g_{a}, h_{b}\right\rangle a, b=1, \ldots, p$

