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On the pricing of defaultable bonds and Hitting times of Ito processes*

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Abstract: The main aim of this work is to price defaultable bonds. In order to achieve this goal we link first hitting densities of Brownian motion with functionals of controlled diffusions. From a practical point of view examples of diffusions with this property are: Brownian motion with linear drift, the 3D Bessel process, the 3D Bessel bridge, and the Brownian bridge, just to mention a few. In turn, these processes are used in finance and economics since they may fall within the category of controlled processes, and/or mean reverting processes.

Keywords: Bond valuation, Ito processes, hitting times

JEL Classification: C60, G0, G1

Resumen: El objetivo principal de este documento es valorar bonos con riesgo de impago. Con el fin de alcanzar nuestra meta relacionamos densidades del primer tiempo de llegada de movimiento Browniano con funcionales de difusiones controladas. Desde un punto de vista práctico, algunos ejemplos de procesos con esta propiedad son: movimiento Browniano con deriva, el proceso 3D de Bessel, el puente 3D de Bessel, el puente Browniano, por mencionar solo a algunos. Alternativamente, estos procesos son utilizados en aplicaciones financieras y económicas ya que forman parte de los procesos catalogados como controlados y/o a la categoría de procesos con reversión a la media.

Palabras Clave: Valuación de bonos, procesos de Ito, cruce de frontera

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1 Introduction

The present paper deals with the pricing of defaultable bonds. That is, bonds which are contingent on some defaultable claims. In order to achieve our goal we model the short rate model as an affine process, which is standard in the literature, and the defaultable part as a controlled Ito process. The latter assumption is useful since it will allow us to price defaultable bonds assuming a portion of its debtors will default at some future time s .

In order to achieve our goal we link functionals of controlled Ito processes with the probability of first hitting densities of Brownian motion. In particular, in this work, the controlled diffusion is the so-called 3-D Bessel bridge and the short rate is of Vasicek type. However within the document we provide very general results regarding the probabilistic properties of controlled processes which may allow in the future for a wider span of possibilities in the pricing of defaultable bonds.

The main result, developed in Section 2, represents the price of defaultable bonds as a product of a standard zero coupon bond, times a functional of a 3-D Bessel bridge. In turn, the latter can be computed by solving numerically a Volterra integral equation. That is, the price can be computed to any desired degree of accuracy.

From the technical point of view the main results presented are: we derive the densities of the first time that the aforementioned Ito processes reach moving boundaries. Furthermore we assume that the moving boundaries are real valued and twice continuously differentiable. That is, the boundaries may be time varying. To this end we distinguish two cases: (a) the case in which the process has unbounded state space before absorption, and (b) the case in which the process has bounded state space before absorption. An example of the second case is the density of the first time that a 3-D Bessel bridge started at $y > 0$, and absorbed at zero at time s , hits a fixed level a , where $y < a$. That is, the 3-D Bessel bridge lives on $(0, a)$ before being absorbed at either level a , or at level 0 at time s , see for instance Hernandez-del-Valle (2012). In particular, as an example of the usefulness of the techniques derived within this document and in particular of the 3-D Bessel bridge we will model and price credit default swaps.

From a mathematical standpoint the main contribution of this work is to advance in producing a general technique to derive hitting time densities using classical tools, such as Doob's h -transform, and the optional sampling theorem, in a rather straightforward way. From a financial perspective the techniques described within may allow for a broader scope of possibilities in the pricing of defaultable bonds.

The paper is organized as follows: In Section 2 we present and price defaultable bonds. The remainder of the paper develops the necessary technical tools. In Section 3 an h -transform and general notation are introduced. Namely we use an h -transform together with Girsanov's theorem to relate a class of diffusions with standard Brownian motion. Next, in Section 4, the

unbounded state space case is discussed. That is, we study the case of processes whose state space is *not* a closed set. Section 5, deals with problems where the processes live on a bounded interval, *i.e.* the state space is a closed subset of \mathbb{R} . Next, in Section 6, we define and describe the relationship between classes \mathcal{B}^1 and \mathcal{B}^2 . In Section 2 we present an application in finance which uses some of the ideas described within this paper. We conclude in Section 7 with some final comments and remarks.

2 Defaultable bonds via 3-D Bessel bridges.

In this section we price a defaultable bond. To this end we model the short rate as a Vasicek process plus a defaultable term, which in this paper turns out to be a 3-D Bessel bridge. Provided the previous assumptions, the pricing depends upon finding the expected value of a functional of the 3-D Bessel bridge process. This choice is due to the fact that a 3-D Bessel bridge describes the evolution of a process whose trajectories remain strictly positive until absorption at time s , which is in accordance with the assumptions of a defaultable asset.

As it turns out, we will show that the valuation problem can be transformed into finding the first hitting time density of Brownian motion for some specific exponential boundary. Thus, the bond can be expressed in terms of a Volterra integral equation, see Peskir (2001), which can be solved numerically.

Before we proceed we should highlight that the the Vasicek process could be, in principle, interchanged for any other affine process used in the bond-pricing literature, see Björk (2009). Furthermore the defaultable term could be modeled in terms of some of the controlled diffusions studied in detail in the remainder of the paper.

It is known that the short rate model r , [see pp. 374–381 in Björk (2009)], specifies the zero-coupon bond price at time t and expiration date s as

$$(1) \quad p(t, s) = \mathbb{E} \left[e^{-\int_t^s r(u) du} \middle| \mathcal{F}_t \right], \quad \text{for } 0 \leq t < s < \infty,$$

where \mathcal{F} is the filtration generated by r . In particular, suppose that r is modelled as an Ornstein-Uhlenbeck diffusion plus a “credit-index process” X conditioned to default at time s [see Pistorius and Davis (2010)]. That is, r has the following dynamics

$$(2) \quad dr = (b - cr)dt + \sigma dW + dX,$$

where $b > 0$ describes the long term mean level, $c > 0$ the speed of reversion, σ the volatility of the process, and where W is a standard Brownian motion and X is modelled as a positive process with continuous paths, and which reaches zero for the first time at expiration date s , hence as a 3-D Bessel bridge.

It follows, from (2), that the integral form representation of r is

$$r(s) = e^{-cs}r(0) + \int_0^s e^{-c(s-u)}bdu + \int_0^s \sigma e^{-c(s-u)}dW_u + \int_0^s e^{-c(s-u)}dX_u.$$

Notice that if W and X are independent, one can find the density of r at the time of default s , by simulation.

Next, in order to price the zero-coupon bond p , we use the integration by parts formula to show that it will be the product of the Vasicek affine term structure P^V [see p. 382 in Bjork (2009)]

$$\begin{aligned} P^V(t, s) &:= A(t, s)e^{-B(t, s)r(t)}, \quad \text{where} \\ B(t, s) &:= \int_t^s e^{-c(u-t)}du, \quad \text{and} \\ A(t, s) &:= \exp \left[\left(\frac{b}{c} - \frac{\sigma^2}{2c^2} \right) (B(t, s) - s + t) - \frac{\sigma^2}{4c} B^2(t, s) \right]. \end{aligned}$$

multiplied by

$$(3) \quad \mathbb{E} \left[e^{-ce^{-cs} - \int_0^s ce^{-c(s-u)}X_u du} \right].$$

That is

$$(4) \quad p(0, s) = P^V(0, s) \cdot \mathbb{E} \left[e^{-ce^{-cs} - \int_0^s ce^{-c(s-u)}X_u du} \right].$$

In summary, in our model, the pricing of a defaultable bond turns out to be the product of a standard zero-coupon bond multiplied by a functional of the defaultable credit-index process X . Hence pricing can be achieved by simulation, however throughout the following lines we provide an explicit representation of (3) in terms of a solution to a Volterra integral equation. The latter is accomplished by linking (3) to a first hitting problem of standard Brownian motion.

Next we present the main result of this paper [for a more detailed derivation of the following result see Hernandez-del-Valle (2013)] :

Lemma 2.1. *Given that*

$$(5) \quad h(t, x) = \frac{x}{\sqrt{2\pi t^3}} \exp \left\{ -\frac{x^2}{2t} \right\}$$

$$(6) \quad \begin{aligned} \tilde{W}_t &= W_t - \int_0^t e^{-c(s-u)}du \\ f'(t) &= e^{-c(s-t)} \end{aligned}$$

and

$$\begin{aligned}
T_s &:= \inf \left\{ t \geq 0 \mid B_t = a + \frac{1}{c} e^{-c(s-t)} \right\}, \quad t, s, c > 0 \\
&= \inf \left\{ t \geq 0 \mid B_t = a + \frac{e^{-cs}}{c} + \int_0^t e^{-c(s-u)} du \right\} \\
&= \inf \left\{ t \geq 0 \mid B_t - \int_0^t e^{-c(s-u)} du = a + \frac{e^{-cs}}{c} \right\} \\
(7) \quad &= \inf \left\{ t \geq 0 \mid \tilde{B}_t = a + \frac{e^{-cs}}{c} \right\},
\end{aligned}$$

it follows that the price of the defaultable bond $p(0, s)$ defined in (4), equals $P^V(0, s)$ times:

$$(8) \quad \tilde{\mathbb{E}}[e^{-\int_0^\tau f''(u) \tilde{X}_u du}] = \frac{e^{-f'(0)a_0 - 1/2 \int_0^\tau (f'(u))^2 du} h(\tau, a_0)}{\mathbb{P}(T_s \in d\tau)/d\tau},$$

where \mathbb{P} stands for probability.

Proof. If we let \tilde{W} be as in (6) and

$$\begin{aligned}
f'(t) &= e^{-c(s-t)} \\
f''(t) &= ce^{-c(s-t)}.
\end{aligned}$$

It follows by Girsanov's theorem

$$\tilde{Z}_\tau = \exp \left\{ - \int_0^\tau f'(u) d\tilde{B}_u - \frac{1}{2} \int_0^\tau (f'(u))^2 du \right\}.$$

The latter together with $a_0 = f(0)$, and T_s as in (7) yields

$$\begin{aligned}
\mathbb{P}(T_s < t) &= \tilde{\mathbb{E}}[\tilde{Z}_\tau \mathbb{I}_{(T < t)}] \\
&= \int_0^t e^{-f'(\tau)a_0 - 1/2 \int_0^\tau (f'(u))^2 du} \tilde{\mathbb{E}}[e^{\int_0^\tau f''(u) \tilde{B}_u du} | T = \tau] h(\tau, a_0) d\tau \\
&= \int_0^t e^{-f'(0)a_0 - 1/2 \int_0^\tau (f'(u))^2 du} \tilde{\mathbb{E}}[e^{-\int_0^\tau f''(u) \tilde{X}_u du}] h(\tau, a_0) d\tau
\end{aligned}$$

where h is as in (5) and \tilde{X} is a 3-D Bessel bridge.

From the previous identity we are now ready to identify (3) by taking derivative in the previous expression. This yields

$$\mathbb{P}(T_s \in d\tau)/d\tau = e^{-f'(0)a_0 - 1/2 \int_0^\tau (f'(u))^2 du} \tilde{\mathbb{E}}[e^{-\int_0^\tau f''(u) \tilde{X}_u du}] h(\tau, a_0)$$

which in turn leads to (8) as claimed. \square

Remark 2.2. Identity (8) allows one to link functionals of 3-D Bessel bridges with first hitting densities of Brownian motion. In particular, under mild assumptions, these densities can

be expressed in terms of solutions to Volterra integral equations which in turn can be solved numerically.

Remark 2.3. In order to verify numerically that our solution is correct it is useful to remember the following inequalities

$$\exp \left\{ - \int_0^\tau f''(u) \tilde{\mathbb{E}}_{0,a_0}[\tilde{X}_u] du \right\} \leq \tilde{\mathbb{E}}_{0,a_0} \left[e^{-\int_0^\tau f''(u) \tilde{X}_u du} \right] \leq 1.$$

which are a direct consequence of Jensen's inequality.

We are now ready to use (8) to value a defaultable bond (1). In order to do so we will have to make some assumptions:

Example 2.4. For simplicity let the parameters of the Vasicek process be $a = c = 1$. Thus $a_0 = 1 + e^{-s}$ and by Jensen's inequality

$$\exp \left\{ -e^s \int_0^s e^u \mathbb{E}_{0,a_0}[X_u] du \right\} \leq \tilde{\mathbb{E}}_{0,a_0} \left[e^{-e^{-s} \int_0^s e^u \tilde{X}_u du} \right] \leq 1.$$

Furthermore,

$$\tilde{\mathbb{E}}_{0,a_0} \left[e^{-\int_0^s e^{-(s-u)} \tilde{X}_u du} \right] = \frac{\mathbb{P}(T_s \in ds)/ds \cdot e^{-s(1+e^{-s})+1/4(1+e^{-2s})}}{h(s, 1+e^{-s})}$$

where $\mathbb{P}(T_s \in ds)/ds$ can be found by solving a Volterra integral equation. In Figure 1 the rate, defined in (8), appears as a hard line. It is be interpreted as the rate at which a standard zero-coupon is rescaled due to the effect of the defaultable part. Alternatively, the lower bound obtained in Remark 2.2 through Jensen's inequality, appears in the dotted line.

3 Preliminaries

In this section we present the notation and necessary tools used throughout the paper. Namely, a special case of Doob's h -transform [see Doob (1984)] and Girsanov's theorem. The first is used to relate the dynamics of SDEs whose local drift is a solution to Burgers equation with standard Brownian motion. On the other hand, the second is used to relate the hitting times of the latter with moving boundaries f .

Remark on notation 3.1. As in the analysis of diffusion processes, PDEs with derivatives with respect to variables (t, x) are called backward equations, whereas PDEs with derivatives in (s, y) are called forward equations.

Furthermore, throughout this work (i) $B = \{B_t, \mathcal{F}_t\}_{t \geq 0}$ stands for one-dimensional standard Brownian motion. (ii) For a given function, say w , partial differentiation with respect to a given variable, say x , will be denoted as w_x .

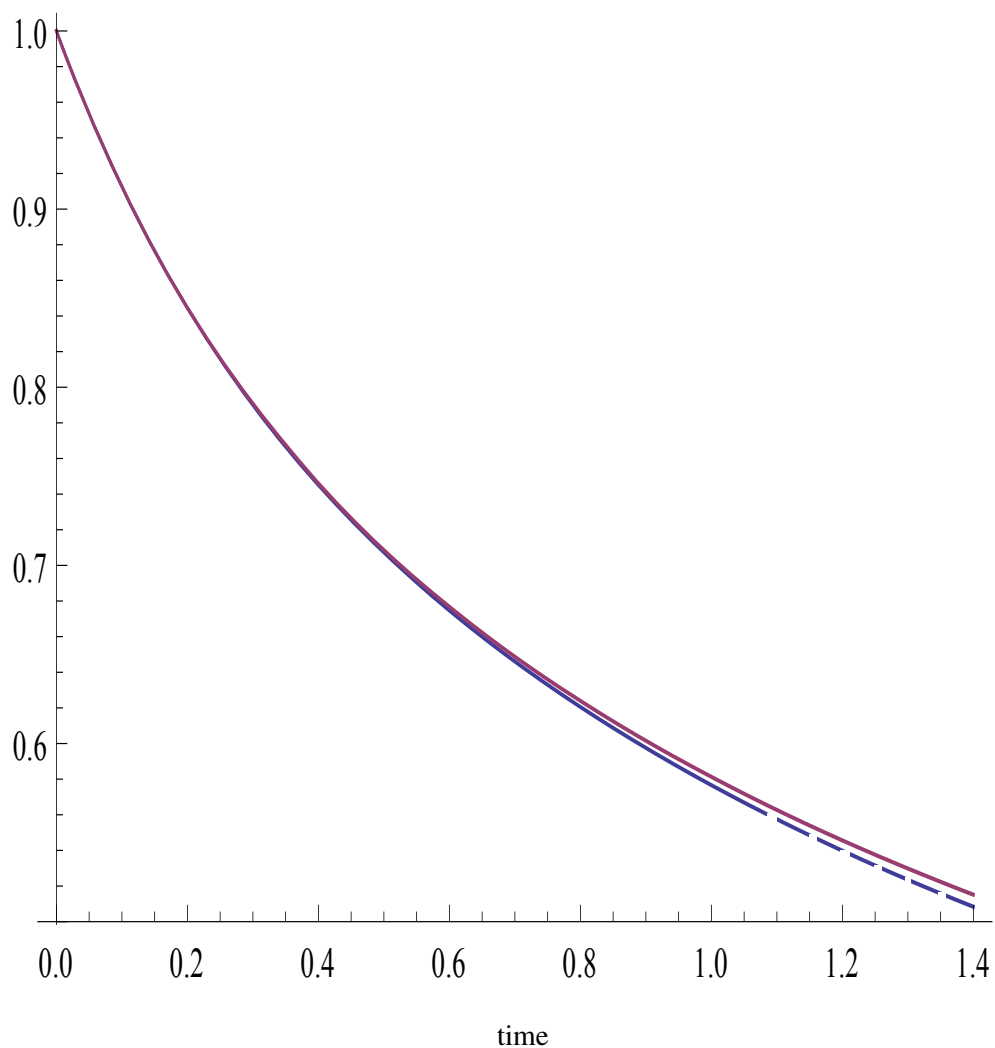


FIGURE 1. The dotted line is the lower bound obtained from Jensen's inequality. The hard line represents the true value of the defaultable portion (3) of the bond, with $a = c = 1$.

Remark 3.2 (Optimal Control). *The processes described within this document can be considered as controlled diffusion processes with dynamics*

$$(9) \quad dX_t = \mu(t)dt + \sqrt{D}dW_t \quad 0 \leq t \leq s$$

where μ is the control variable. In fact, if we define a Lagrangian function in the form

$$\mathcal{L}_D(\mu) = -\frac{1}{2D}\mu^2.$$

One can show, using dynamic programming, that an optimal control μ^* which minimizes the cost functional

$$J(t, x, \mu) := \mathbb{E}_{t,x} \int_t^\tau \mathcal{L}_D(\mu(s)) ds + \psi(\tau, X(\tau)),$$

leads to a solution of Burgers equation. See Hongler (2004).

Theorem 3.3. Let h be of class $C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ as well as a solution to the backward heat equation

$$(10) \quad -h_t = \frac{1}{2} D h_{xx}.$$

Furthermore, consider processes X , and Y which respectively satisfy (at least in the weak sense), the following equations (each under their corresponding measures \mathbb{P} , and \mathbb{Q}),

$$(11) \quad (\mathbb{P}) \quad dX_t = D \frac{h_x(t, X_t)}{h(t, X_t)} dt + \sqrt{D} dB_t$$

$$(12) \quad (\mathbb{Q}) \quad dY_t = \sqrt{D} dB_t.$$

Moreover, suppose that f is real valued and integrable. Then the following identity holds

$$(13) \quad \mathbb{E}_{t,x}^{\mathbb{P}} [f(X_\tau)] = \mathbb{E}_{t,x}^{\mathbb{Q}} \left[\frac{h(\tau, Y_\tau)}{h(t, x)} f(Y_\tau) \right].$$

Proof. For the proof, see Theorem 2.10 in Hernandez-del-Valle (2011). \square

We recall Girsanov's theorem (Ito's lemma) in the case in which the drift is a deterministic function of time.

Lemma 3.4. Let $f(\cdot)$ be a real-valued differentiable function, h a solution of the one-dimensional backward heat equation, and processes Z and S have the following dynamics

$$dZ_t = f'(t) dt + dB_t$$

$$dS_t = -f'(t) S_t dB_t$$

for $0 \leq t < \infty$, under some measure \mathbb{Q} . Then

$$S \cdot h(\cdot, Z)$$

is a \mathbb{Q} -martingale.

Proof. From Ito's lemma

$$\begin{aligned} dh(t, Z_t) &= h_t(t, Z_t) dt + h_z(t, Z_t) f'(t) dt \\ &\quad + h_z(t, Z_t) dY_t + \frac{1}{2} h_{zz}(t, Z_t) dt \\ &= h_z(t, Z_t) f'(t) dt + h_z(t, Z_t) dY_t. \end{aligned}$$

Hence,

$$\begin{aligned}
d[S_t \cdot h(t, Z_t)] &= f(t, Z_t) dS_t + S_t dh(t, Z_t) \\
&\quad + dh(t, Z_t) \cdot dS_t \\
&= h(t, Z_t) [-f'(t) S_t dY_t] \\
&\quad + S_t h_z(t, Z_t) f'(t) dt + S_t h_z(t, Z_t) dY_t \\
&\quad - h_z(t, Z_t) S_t f'(t) dt \\
&= [S_t h_z(t, Z_t) - f'(t) X_t h(t, Z_t)] dY_t.
\end{aligned}$$

□

4 Unbounded state space

In this section, we specialize to the case in which the process X , with dynamics as in (14), has unbounded state space before absorption. That is, the state space is *not* a closed set of \mathbb{R} . In particular we find the density of the first time that X hits a real-valued and twice continuously differentiable function f .

This will be achieved by a consecutive and direct application of Doob's h -transform, Girsanov's theorem and the optional sampling theorem. However, one should take into the account the possibility that the state space of the process is a half open set of \mathbb{R} . Examples of such processes are the 3-D Bessel and 3-D Bessel bridge respectively.

Remark 4.1. *Henceforth let h be a solution to the one-dimensional backward heat equation (10). Let process X have the following dynamics*

$$(14) \quad dX_t = \frac{h_x(t, X_t)}{h(t, X_t)} dt + dB_t, \quad X_0 = y \in \begin{cases} \mathbb{R} \\ \mathbb{R}^+ \end{cases}, \quad \text{or}$$

for $0 \leq t < \infty$. And assume the drift h_x/h satisfies the Ito conditions.

We will use as well the following:

Definition 4.2. *Given a constant $a \in \mathbb{R}$ and a real-valued, twice continuously differentiable function $f(\cdot)$ —which we refer to as a “moving boundary”— B is a standard Brownian motion, and X is a solution to (14) we define the following stopping times*

$$(15) \quad \begin{aligned} T &:= \inf \left\{ t \geq 0 \mid X_t = a + \int_0^t f'(u) du \right\} \quad y \neq a \\ T^B &:= \inf \left\{ t \geq 0 \mid B_t = a + \int_0^t f'(u) du \right\}. \end{aligned}$$

Furthermore, let $p_f^B(\cdot)$ be the density of T^B , that is, the density of the first time that a one-dimensional Wiener process reaches a deterministic moving boundary f . For a detailed historical and technical account of the problem of finding $p_f^B(\cdot)$ as a solution of an integral equation see Peskir (2001).

The main result of this section is the following:

Theorem 4.3. *Suppose that X has dynamics as in (14), T is as in (15), and $a \neq y$. Then*

$$\mathbb{P}_y(T \in du) = \frac{h(u, a + \int_0^u f'(v)dv)}{h(0, y)} p_f^B(u) du,$$

for $u \geq 0$.

Proof. For T as in (15), it follows from Theorem 3.3 that

$$\begin{aligned} \mathbb{P}_y(T < t) &= \mathbb{E}_y [\mathbb{I}_{(T < t)}] \\ &= \mathbb{E}_y^{\mathbb{Q}} \left[\frac{h(t, Y_t)}{h(0, Y_0)} \mathbb{I}_{(T < t)} \right]. \end{aligned}$$

From Girsanov's theorem, and given that \tilde{Y} is a $\tilde{\mathbb{Q}}$ -Wiener process we have

$$= \mathbb{E}_y^{\tilde{\mathbb{Q}}} \left[e^{-\int_0^t f'(u) d\tilde{Y}_u - \frac{1}{2} \int_0^t (f'(u))^2 du} \frac{h(t, \tilde{Y}_t + \int_0^t f'(u) du)}{h(0, y)} \mathbb{I}_{(T < t)} \right].$$

Finally, from Lemma 3.4 and the optional sampling theorem

$$\begin{aligned} &= \mathbb{E}_y^{\mathbb{Q}} \left[e^{-\int_0^T f'(u) d\tilde{Y}_u - \frac{1}{2} \int_0^T (f'(u))^2 du} \frac{h(T, a + \int_0^T f'(u) du)}{h(0, y)} \mathbb{I}_{(T < t)} \right] \\ &= \int_0^t \frac{h(u, a + \int_0^u f'(v) dv)}{h(0, y)} p_f^B(u) du. \end{aligned}$$

□

Examples of processes which satisfy equation (14) and that have unbounded domain before hitting the boundary f are:

Examples 4.4. (i) *Standard Brownian motion, where $h(t, x) = c$.*

(ii) *Brownian motion with linear drift, where*

$$h(t, x) = e^{\pm \lambda x + \frac{1}{2} \lambda (s-t)}, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

(iii) *Brownian bridge, where*

$$h(t, x) = \frac{1}{\sqrt{2\pi(s-t)}} e^{-\frac{x^2}{2(s-t)}}, \quad (t, x) \in [0, s] \times \mathbb{R}.$$

(iv) *3D Bessel process, where $h(t, x) = x$.*

(v) *3D Bessel bridge, where*

$$h(t,x) = \frac{x}{\sqrt{2\pi(s-t)^3}} e^{-\frac{x}{2(s-t)}}, \quad (t,x) \in [0,s] \times \mathbb{R}^+.$$

Remark 4.5. *(On numerical procedure) Throughout the remainder of the paper the simulations used are as follows: (1) We first discretize the corresponding SDE, (2) The Gaussian random noise is generated with the pre established library in R.*

Example 4.6. *In Figures 2 and 3 the theoretical density and distribution of the first time that a Brownian bridge started at $y = 1$, absorbed at time $s = 3$ at level $c = 0$ hits the linear barrier*

$$f(t) = 2 - t, \quad t \geq 0$$

is compared with $n = 5500$ simulations. See Durbin and Williams (1992).

Example 4.7. *In Figures 4 and 5, the theoretical densities and distributions of the first time that a 3-D Bessel process started at $y = 3$ and absorbed at time $s = 4$, reaches level $a = 1$. For a general overview of the 3D Bessel bridge, see Revuz and Yor (2005). The case in which a level is reached from below is in general studied in Pitman and Yor (1999) or Hernandez-del-Valle (2012).*

5 Bounded domain

In this section, we specialize to the case in which the process X , with dynamics as in (14), has bounded state space before absorption. The main technical difference with the problem discussed in the previous section, is that one has to take into account the fact that the process X lives within a bounded set. In order to deal with this fact, let us first recall the following facts:

Given that B is a one-dimensional Wiener process started at y and

$$(16) \quad T_0^B := \inf\{t \geq 0 | B_t = 0\}$$

$$(17) \quad T_a^B := \inf\{t \geq 0 | B_t = a\}, \quad 0 < y < a.$$

The following identities hold

$$\begin{aligned} \mathbb{P}_y(T_a^B \wedge T_0^B \in dt) &:= \frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} \left[(2na + y) \exp \left\{ -\frac{(2na + y)^2}{2t} \right\} \right. \\ &\quad \left. + (2na + a - y) \exp \left\{ -\frac{(2na + a - y)^2}{2t} \right\} \right] dt, \\ \mathbb{P}_{t,y}(T_a^B \in s, T_0^B > s) &:= \frac{1}{\sqrt{2\pi(s-t)^3}} \sum_{n=-\infty}^{\infty} \left[(2na + a - y) \right. \\ &\quad \left. \times \exp \left\{ -\frac{(2na + a - y)^2}{2(s-t)} \right\} \right], \\ \mathbb{P}_y(T_0^B \in dt) &:= \frac{y}{\sqrt{2\pi t^3}} \exp \left\{ -\frac{y^2}{2t} \right\} dt. \end{aligned}$$

See for instance Chapter 2, Section 8 in Karatzas and Shreve (1991).

The main result of this section is the following:

Theorem 5.1. *Given that X has dynamics as in (14), T_0^B is as in (16), T_a^B is as in (17), and T is as in (15). We have for $0 \leq t \leq s$ and $0 < y \leq a$ that*

$$\mathbb{P}_y(T \in du) = \frac{h(u, a)}{h(0, y)} \left[\mathbb{P}_y(T_a^B \wedge T_0^B \in du) - \mathbb{P}_y(T_0^B \in du) \right],$$

where $0 \leq u \leq s$.

Proof. We follow the proof of Theorem 4.3. However, we must take into account the fact that the \mathbb{Q} -Wiener process Y is absorbed at zero. Given that T is as in (15)

$$\begin{aligned} \mathbb{P}_y(T < t) &= \mathbb{E}_y \left[\mathbb{I}_{(T < t)} \mathbb{I}_{(T_0 > t)} \right] \\ &= \mathbb{E}_y^{\mathbb{Q}} \left[\frac{h(t, Y_t)}{h(0, Y_0)} \mathbb{I}_{(T < t, T_0 > t)} \right] \\ &= \mathbb{E}_y^{\mathbb{Q}} \left[\frac{h(T, a)}{h(0, Y_0)} \mathbb{I}_{(T < t, T_0 > t)} \right], \end{aligned}$$

where the last line follows from the optional sampling theorem. Finally recall the identity

$$\begin{aligned} \mathbb{P}_y(T < t, T_0 > t) &= \mathbb{P}_y(T_0 > t) - \mathbb{P}_y(T > t, T_0 > t) \\ &= \mathbb{P}_y(T_0 > t) - \mathbb{P}_y(T \wedge T_0 > t) \\ &= \mathbb{P}_y(T \wedge T_0 < t) - \mathbb{P}_y(T_0 < t). \end{aligned}$$

□

Some examples of processes which fall within this category are for instance:

Examples 5.2. *Some examples are: (i) 3D Bessel process (reaching a fixed level from below) (ii) 3D Bessel bridge (reaching a fixed level from below), see Hernandez-del-Valle (2012). Regarding the first hitting probabilities of general Bessel processes see Wendel (1980), or Betz and Gzyl (1994a, 1994b).*

Example 5.3. *In Figure 6 the theoretical distribution of the first time that a 3D Bessel process reaches $a = 1.5$ from below, is compared with a simulation $n = 5500$ (see hard line).*

Example 5.4. *If we set*

$$h^a(s-t, x) := \mathbb{P}_{t,x}(T \in s, T_0 > s),$$

and define a process \tilde{Y} to be as in

$$\begin{aligned} d\tilde{Y}_t &= \frac{h_y^a(s-t, \tilde{Y}_t)}{h^a(s-t, \tilde{Y}_t)} dt + dB_t, \quad 0 < t < s \\ \tilde{Y}_s &= a. \end{aligned}$$

One can show that this process has state space $(0, a)$ for $t \in [0, s)$ [see Hernandez-del-Valle (2011)]. In Figure 7 the density and distribution of the Wiener process, started at $x = 1/2$, absorbed at zero and that reaches level $a = 2$ for the first time at $s = 2$ is plotted at $t = 1$ and $t = 7/8$.

6 Heat polynomials and Burgers equation

In this section we provide a characterization of the results described in Sections 4 and 5. In particular we show that the only linear combination of two solutions to Burgers equation, which are themselves solutions to Burgers equation is given in terms of the first derivative with respect to the state space x of the fundamental solution to the heat equation (10). The previous statement, in our context, links the density of the 3-D Bessel bridge, with both the densities of the 3-D Bessel process and Brownian bridge respectively.

To this end let us first introduce the following classification of SDEs.

Definition 6.1. *We will say that process X , which satisfies the following equation*

$$dX_t = \mu(t, X_t)dt + dB_t,$$

is of class \mathcal{B}^n , $n = 1, 2, \dots$, if its local drift μ can be expressed as

$$(18) \quad \mu(t, x) = \sum_{j=1}^n \frac{h_x^j(t, x)}{h^j(t, x)}.$$

Where each h^j is a solution to the backward heat equation (10).

Making use of this classification it follows that:

Remark 6.2. Processes X_1 , X_2 , and X_3 which respectively satisfy the following equations

$$(19) \quad \begin{cases} (\mathbb{P}^1) & dX_1(t) = \frac{1}{X_1(t)} dt + dB_t \\ (\mathbb{P}^2) & dX_2(t) = -\frac{X_2(t)}{s-t} dt + dB_t \\ (\mathbb{P}^3) & dX_3(t) = \left[\frac{1}{X_3(t)} - \frac{X_3(t)}{s-t} \right] dt + dB_t \end{cases}$$

are of class \mathcal{B}^1 . That is, the 3-D Bessel process X_1 , the Brownian bridge X_2 , and the 3-D Bessel bridge have a local drift which is a solution to Burgers equation. This statement is verified by using the following solutions to the heat equation correspondingly

$$(20) \quad \begin{cases} k(t, x) = x, & g(t, x) = \frac{1}{\sqrt{2\pi(s-t)}} \exp\left\{-\frac{x^2}{2(s-t)}\right\} \\ h(t, x) = \frac{x}{\sqrt{2\pi(s-t)^3}} \exp\left\{-\frac{x^2}{2(s-t)}\right\}, \end{cases}$$

together with the Cole-Hopf transform which relates Burgers equation with the heat equation.

Examples of processes which are not \mathcal{B}^1 are the following.

Example 6.3. The Bessel process of odd order $2n+1$, $n = 1, 2, \dots$ is of class \mathcal{B}^n .

Recall that the Bessel process of order $m \in \mathbb{N}$ is the solution to

$$dX_t = \frac{m-1}{2X_t} dt + dB_t,$$

if $m = 2n+1$

$$\begin{aligned} dX_t &= \frac{(2n+1)-1}{2X_t} dt + dB_t \\ &= \frac{n}{X_t} dt + dB_t \\ &= \left[\frac{k_x(t, X_t)}{k(t, X_t)} + \dots + \frac{k_x(t, X_t)}{k(t, X_t)} \right] dt + dB_t, \end{aligned}$$

where k is as in (20).

However there exists a least one important process which is both \mathcal{B}^1 and \mathcal{B}^2 .

Proposition 6.4. The 3-D Bessel bridge process X_3 , which solves (19. \mathbb{P}^3) is \mathcal{B}^1 and \mathcal{B}^2 .

Proof. it follows by verifying that for k , g , and h as in (20) the following identity holds

$$\frac{h_x}{h} = \frac{k_x}{k} + \frac{g_x}{g}.$$

□

In turn, this feature of process X_3 [*i.e.* a process that is both \mathcal{B}^1 and \mathcal{B}^2], leads to some interesting properties. [See Hernandez-del-Valle (2011).] The next natural question to ask is if there are more processes with such property.

In this direction we will show that process X_3 is the only element of both \mathcal{B}^1 and \mathcal{B}^2 in the case in which the generating functions w , of class \mathcal{B}^1 , are heat polynomials. [That is if w is a solution to the heat equation and w_x/w models the local drift of a process X]

Definition 6.5. (*Heat polynomials*) [Widder and Rosenbloom (1959)]. A heat polynomial $v_j(x, t)$ of degree j is defined as the coefficient of $z^n/n!$ in the power series expansion

$$(21) \quad e^{xz + \frac{1}{2}z^2t} = \sum_{n=0}^{\infty} v_n(x, t) \frac{z^n}{n!}.$$

An associated function $w_n(x, t)$ is defined as

$$(22) \quad w_n(x, t) = g(t, x)v_n(x, -t) (t/2)^{-n},$$

where g , as in (20), is the fundamental solution to the heat equation.

Remark 6.6. Observe that if processes X_1 , X_2 , and X_3 are as in (19) then their local drifts can be described in terms of heat polynomials, Definition 6.5. In the case of X_1 its corresponding polynomial is v_1 . Alternatively for X_2 and X_3 their corresponding polynomials are w_1 and w_2 respectively.

The main result of this section is the following.

Theorem 6.7. If process $X \in \mathcal{B}^1$. [That is X is a solution to

$$dX_t = \frac{h_x(t, X_t)}{h(t, X_t)} dt + dB_t$$

and h solves the backward heat equation $-h_t = \frac{1}{2}h_{xx}$.] And h is either a heat (21) or derived heat (22) polynomial. Then the only process which is also \mathcal{B}^2 is the 3-D Bessel bridge X_3 , which has dynamics as in (19. \mathbb{P}^3).

Proof. Given that v_n and w_n are as in (21) and (22) respectively; and letting w' stand for differentiation with respect to the first variable we have that

$$\frac{w'_0(x, t)}{w_0(x, t)} = -\frac{x}{t} \quad \text{and} \quad \frac{v'_1(x, t)}{v_1(x, t)} = \frac{1}{x}.$$

Next, since [see p. 225 in Rosenbloom and Widder (1959)]

$$w'_{n-1}(x, t) = -\frac{1}{2}w_n(x, t)$$

it follows that

$$\begin{aligned}
 \frac{w'_n(x,t)}{w_n(x,t)} &= -\frac{x}{t} + \frac{2n}{t} \frac{w_{n-1}(x,t)}{w_n(x,t)} \\
 &= -\frac{x}{t} - \frac{n}{t} \left[\frac{w_{n-1}(x,t)}{w'_{n-1}(x,t)} \right] \\
 (23) \qquad &= \frac{w'_0(x,t)}{w_0(x,t)} - \frac{n}{t} \left[\frac{w_{n-1}(x,t)}{w'_{n-1}(x,t)} \right].
 \end{aligned}$$

Alternatively from (22)

$$\begin{aligned}
 \frac{w_{n-1}}{w'_{n-1}} &= \frac{g(x,t)v_{n-1}(x,-t)(t/2)^{-n+1}}{-\frac{1}{2}w_n(x,t)} \\
 &= \frac{g(x,t)v_{n-1}(x,-t)(t/2)^{-n+1}}{-\frac{1}{2}g(x,t)v_n(x,-t)(t/2)^{-n}} \\
 &= -\frac{tv_{n-1}(x,-t)}{v_n(x,-t)}.
 \end{aligned}$$

This implies, from (23), that

$$\begin{aligned}
 \frac{w'_n(x,t)}{w_n(x,t)} &= \frac{w'_0(x,t)}{w_0(x,t)} - \frac{n}{t} \left[\frac{w_{n-1}(x,t)}{w'_{n-1}(x,t)} \right] \\
 &= \frac{w'_0(x,t)}{w_0(x,t)} - \frac{n}{t} \left[-\frac{tv_{n-1}(x,-t)}{v_n(x,-t)} \right] \\
 (24) \qquad &= \frac{w'_0(x,t)}{w_0(x,t)} + n \frac{v_{n-1}(x,-t)}{v_n(x,-t)}.
 \end{aligned}$$

However, since

$$v'_n(x,t) = nv_{n-1}(x,t)$$

[see equation (1.9) in Widder and Rosenbloom (1959)] we have, from (24), that

$$\begin{aligned}
 \frac{w'_n(x,t)}{w_n(x,t)} &= \frac{w'_0(x,t)}{w_0(x,t)} + \frac{nv_{n-1}(x,-t)}{v_n(x,-t)} \\
 &= \frac{w'_0(x,t)}{w_0(x,t)} + \frac{v'_n(x,-t)}{v_n(x,-t)}.
 \end{aligned}$$

In general, since $v_n(x,t)$ is a solution to the backward heat equation, then $v_n(x,-t)$ is a solution to the forward equation. This is true as long as $n > 1$. However if $n = 1$, $v_1(x,-t)$ is also a solution to the backward equation because it does not depend on t . In this case we have that

$$\frac{w'_1(x,t)}{w_1(x,t)} = \frac{x}{t} + \frac{1}{x},$$

as claimed. □

7 Concluding remarks

The main objective of this document is to price defaultable bonds, in order to achieve our goal we make use of standard affine pricing models and controlled diffusions. In particular we obtain an expression in terms of the Vasicek and 3-D Bessel bridge processes. A generalization of this result as well as further applications is work in progress.

From a technical point of view we study processes X , which have local drift modeled in terms of solutions to Burgers equation. In particular, we find the density of the first time that such processes hit a moving boundary. Next, we propose a classification of SDEs in terms of solutions of Burgers equation. We say that process X is of class \mathcal{B}^j if its local drift can be expressed as a sum of j solutions to Burgers equation.

We note that the 3-D Bessel process, the 3-D Bessel bridge, and the Brownian bridge are all \mathcal{B}^1 . However we show that the 3-D Bessel bridge is of class \mathcal{B}^2 as well. Furthermore, we show, that the 3-D Bessel bridge is the only process which satisfies this property if the solutions of Burgers equation is constructed by use of heat polynomials. A more detailed study of this classification is work in progress.

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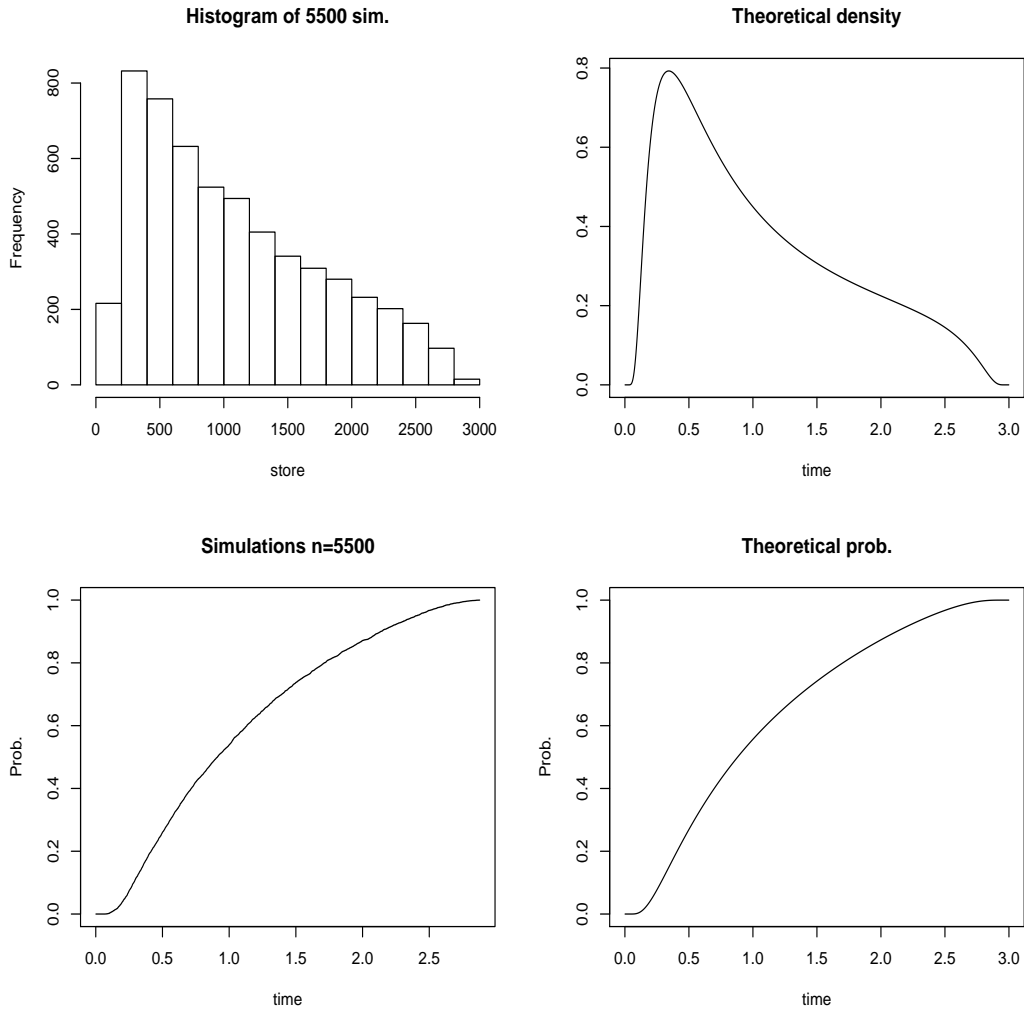


FIGURE 2. (Example 4.6). The graph is plotted in R. The upper left graph is the histogram of the (simulated) first time that a Brownian bridge started at $y = 1$, and absorbed at $c = 0$ at time $s = 3$, reaches the linear boundary $f(t) = 2 - t$. In the upper right frame we have its theoretical density. In the lower left we have the simulated distribution, and finally on its right we have its theoretical counterpart.

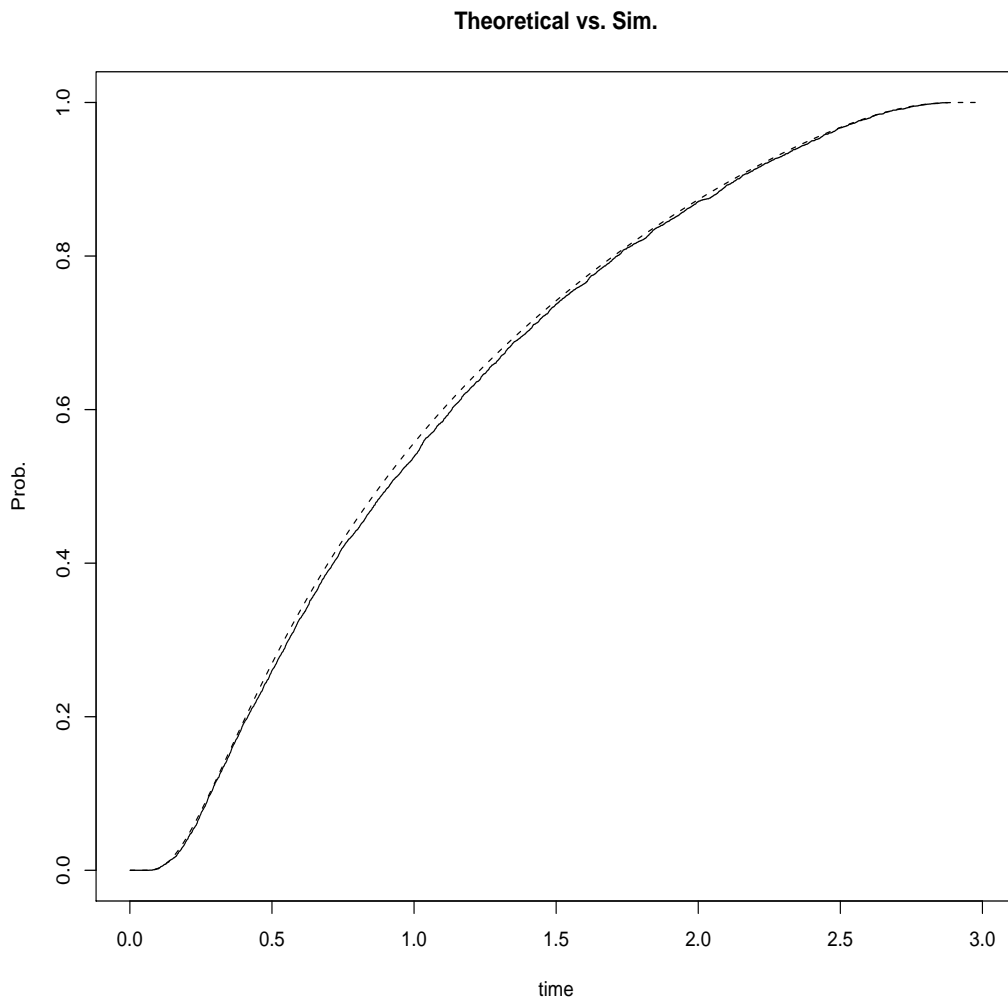


FIGURE 3. (Example 4.6). The graph is plotted in R. The dotted line is the theoretical probability. The hard line is a simulation with $n = 5500$.

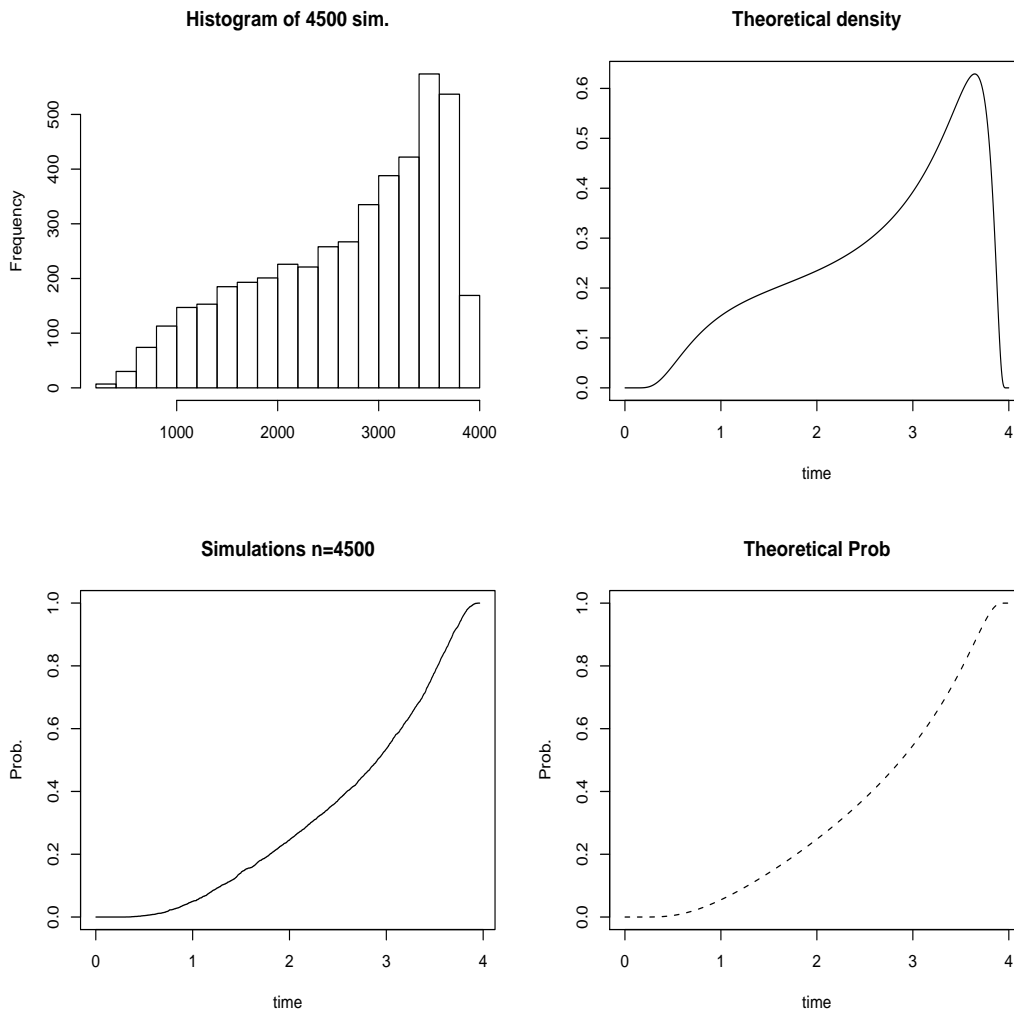


FIGURE 4. (Example 4.7). The graph is plotted in R. The upper left graph is the histogram of the (simulated) first time that a 3-D Bessel bridge started at $y = 3$, and absorbed at $s = 4$, reaches level $a = 1$. In the upper right frame we have the its theoretical density. In the lower left we have the simulated distribution, and finally on its right we have its theoretical counterpart.

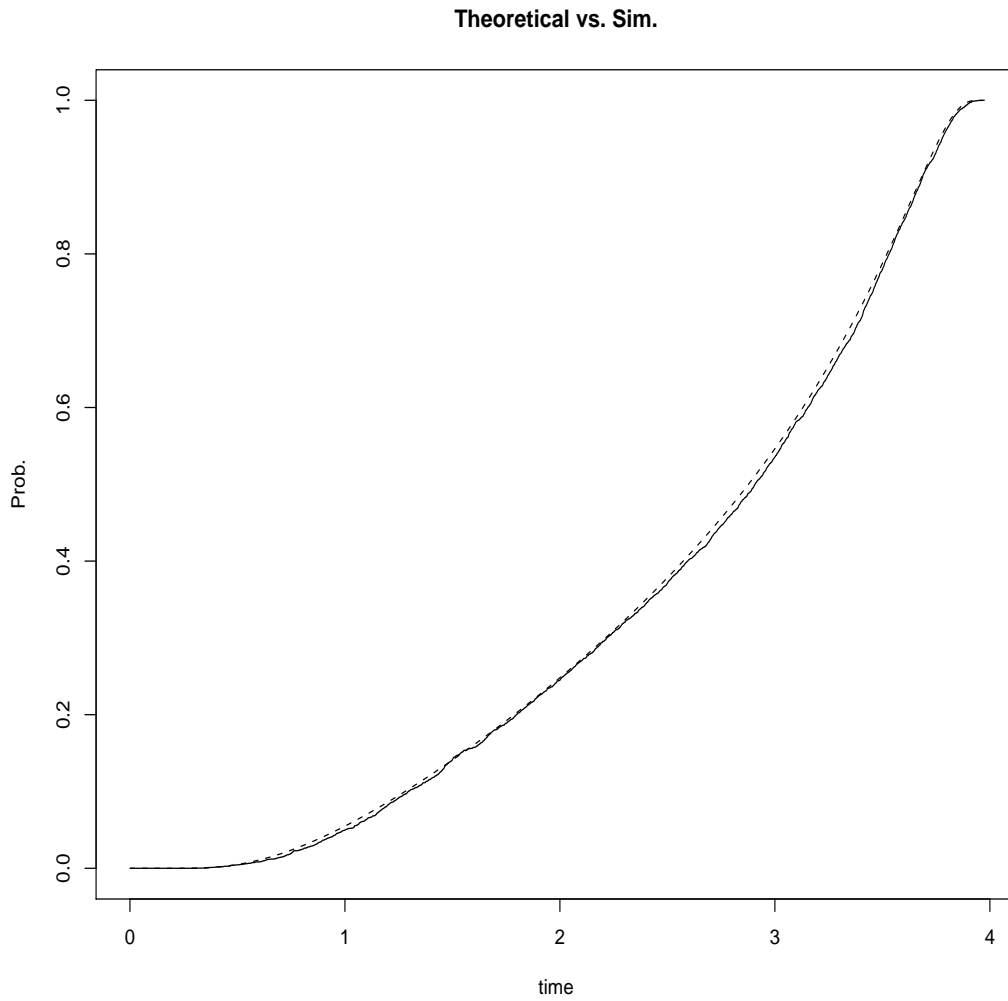


FIGURE 5. (Example 4.7). The graph is plotted in R. The dotted line is the theoretical probability. The hard line is a simulation with $n = 4500$.

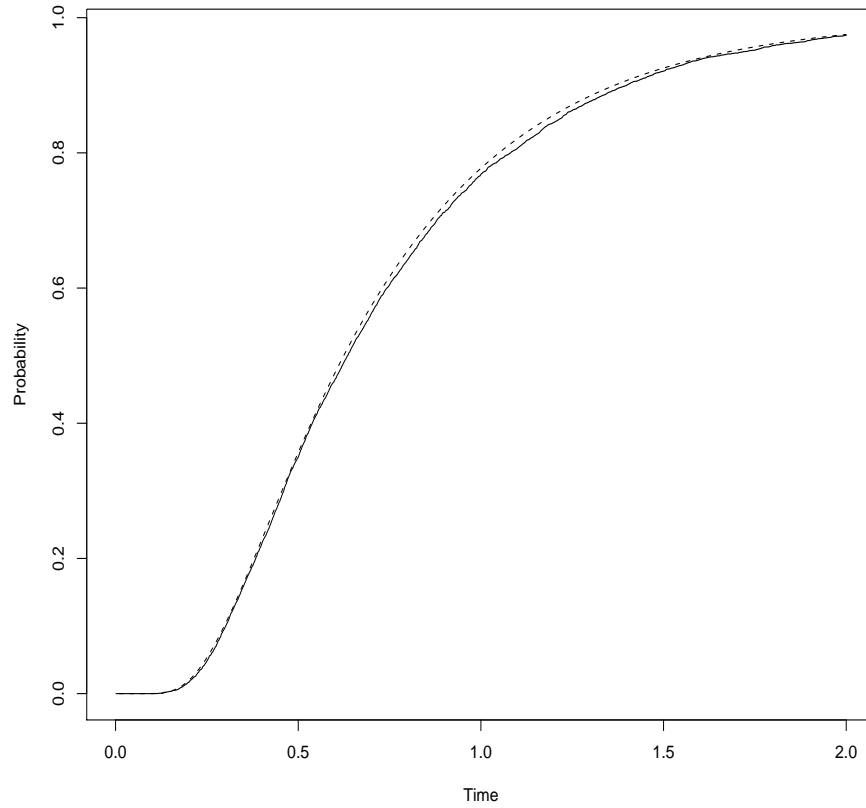


FIGURE 6. (Example 4.7). The graph is plotted in R. The dotted line is the theoretical probability with $a = 1.5$. The hard line is a simulation with $n = 5500$.

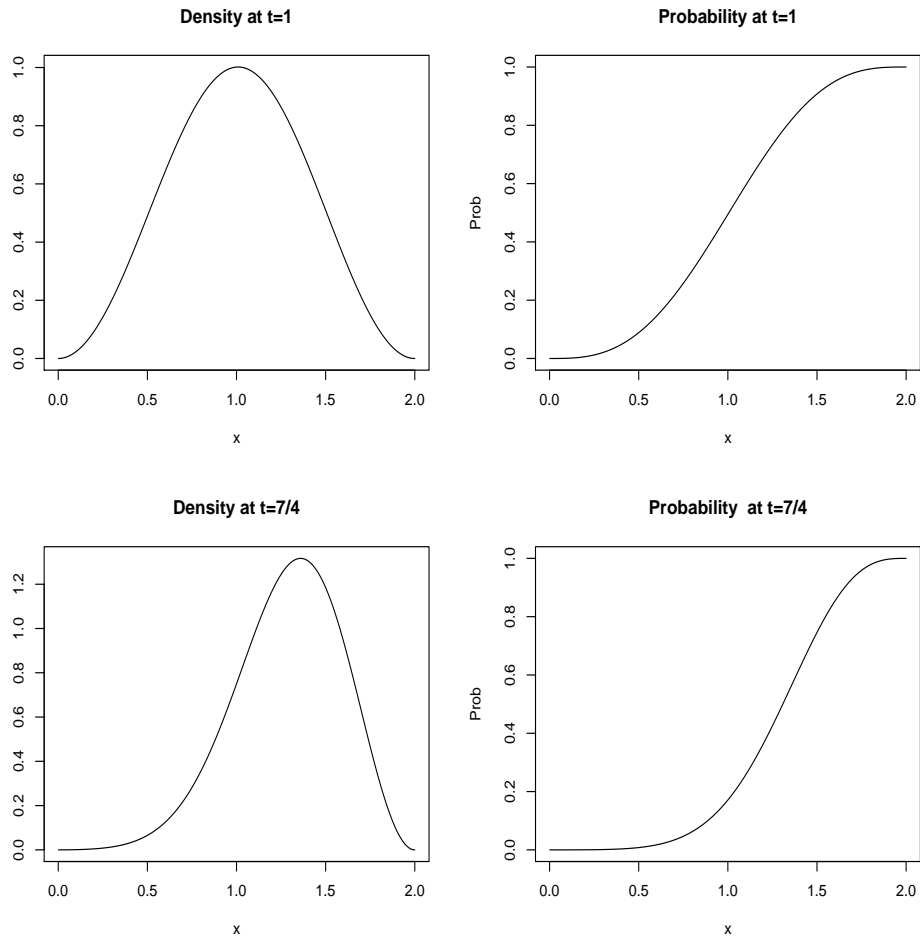


FIGURE 7. (Example 5.3). The graphs are plotted in R. The upper left graph is the density of a Wiener process (started at $x = 1/2$), absorbed at zero and that reaches $a = 2$ for the first time at $s = 2$; evaluated at $t = 1$. The upper right graph is the corresponding distribution at $t = 1$. In the lower left figure we have the density at time $t = 7/4$. Finally, the lower right panel is its corresponding distribution.