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Valuation of credit default swaps via Bessel bridges*

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Abstract: A credit default swap (CDS) is a financial contract in which the holder of the instrument will be compensated in the event of a loan default. When available, CDS's are used to monitor the credit risk of countries and companies. In this work we develop a closed form procedure to value a CDS in the case in which the so-called "credit rate index" is modelled as a Bessel bridge of arbitrary order. In particular, these processes seem to capture the nature of a defaultable asset in the sense that they remain strictly positive before default, and thus enrich the existing literature in this field.

Keywords: Credit default swap, Bessel bridge, hitting time, defaultable bond

JEL Classification: G0, G1

Resumen: Un credit default swap (CDS) es un contrato financiero en el cual el tenedor del instrumento será compensado en el caso de suspensión de pagos de un crédito. Cuando están disponibles, los CDS son utilizados para monitorear el riesgo crediticio de países y compañías. En este trabajo desarrollamos, en forma cerrada, un procedimiento para valuar un CDS en el caso en que el llamado "índice de riesgo crediticio" es modelado como un puente de Bessel de orden arbitrario. En particular, estos procesos parecen capturar la naturaleza de un activo con riesgo de impago en el sentido de que permanece estrictamente positivo antes de la suspensión de pagos y, por lo tanto, enriquece la literatura existente en este campo.

Palabras Clave: Credit default swap, puente de Bessel, tiempo de paro, bono en suspensión de pagos

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1 Introduction

Credit default swaps were first introduced by Blythe Masters in 1994, with the purpose of incorporating the probability of default of a given creditor in the valuation of financial contracts. In turn, a common way to model the time of default is using the first passage time probability of Brownian motion below some barrier. This approach, used to value CDS's, goes back at least as far as [4] and has subsequently been studied in [9, 5, 22, 8, 11] to mention just a few.

This work is two-folded. On the one-hand, we develop a new procedure to value a CDS by modeling the so-called credit rate index of any given creditor as a δ -dimensional Bessel bridge. By doing this, we ensure that default happens exactly at time τ , and not before. This contrasts, with the standard model of a Brownian bridge where default could happen at any time between 0 and τ . On the other hand, we derive a closed form expression for a defaultable zero-coupon bond. Once again, the Bessel bridge plays a fundamental role in our derivation. In particular, these processes seem to capture the nature of a defaultable asset in the sense that they remain strictly positive before default [3].¹

The paper is organized as follows. We start, in Section 2, with a general description of a CDS as well as by describing the probability of default of the so-called credit-index process. In particular, the probability of default of the credit-index process will be studied in detail in the subsequent sections. In Section 3 we recall what a Bessel bridge is and give the necessary results to characterize it using stochastic differential equations up to the time it hits zero. This is important from a financial stand point, since default can be interpreted as the first time that a price process (or equity process) hits zero. In Section 4 we use an h -transform in a class of diffusions in order to study the hitting-time problem, and we apply the results to the Bessel bridge. That is, we derive the density of the first time in which a creditor defaults. In Sections 5 we additionally carry out particular space transformations to take a new point of view of the original problem, which helps to adapt the ideas of Section 4 to another class of diffusions. In fact, these results shed light into a previously unknown correspondence between the CIR and

¹When δ is a positive integer, recall that the Bessel process describes the dynamics of the Euclidean norm of a δ -dimensional Brownian motion (BM). On the other hand, a Bessel bridge is described as a Bessel process conditioned to reach a specific point at some time $T > 0$. In this work, we first describe a technique for calculating the density of the first time that a δ -dimensional Bessel bridge hits a given level $b \in \mathbb{R}$. Next, we identify a class of diffusion processes for which first hitting-time densities can be calculated in a similar fashion as for the Bessel bridges.

The problem of finding the first hitting-time density of diffusions may be traced back at least to Schrödinger [38]. Exact densities of hitting times for Brownian motion have been found in the case of reaching a linear boundary [13, 14], a square root boundary [7, 10, 37, 12, 39], and a parabolic boundary [17, 35, 23]. Consult also [31, 28] to see integral equations coming from the first passage time problem. In this context, one very well studied diffusion is the Bessel process [6, 18, 26, 27, 36, 19]; in particular, for Bessel bridges see [15, 20]. An application in financial mathematics of Bessel bridges can be found in [11].

Vasicek processes. In Section 6 we develop explicitly the valuation procedure of a defaultable bond. We end up in Section 7 with some comments and conclusions.

2 Credit default swaps

A credit default swap is an insurance scheme where the buyer of this protection pays a flow of settlements (or a continuous spread c), until the time of maturity s , as long as default does not occur. In the case in which default takes place, the buyer delivers a bond on the underlying defaulting asset in exchange of its face value. Without loss of generality, it is standard to normalize the notional value of the bond to 1. Thus, the protection seller's contingent payment is generally expressed as $1 - R$, where R is the so-called recovery rate, for $R \in (0, 1)$.

In order to find the price of a CDS, let us first introduce the following concepts: Given that X is the credit-index process (or in general any defaultable asset), *time of default* T_α is the time at which X reaches a critical level α for the first time:

$$T_\alpha := \inf\{t \geq 0 | X_t = \alpha\}.$$

Thus, the probability that asset X defaults before time t , is

$$H(t) := \mathbb{P}(T_\alpha \leq t).$$

Hence for a CDS contract written on an underlying X , assuming that premium payments are made at times t_i and the available maturities are $T_j = t_{k(j)}$, $j = 1, \dots, n$. We have that for contract j there is an upfront premium π_j^0 and a running premium rate π_j^1 (with accrual factors δ_i). Denote by $p(0, t_j)$ the price at time zero of a zero coupon risk-free bond with maturity t_j . Assuming the risk-less bond price to be independent of credit-worthiness of the underlying, the 'fair-premium' (π_j^0, π_j^1) satisfies

$$\pi_j^0 + \pi_j^1 \sum_{i=0}^k \delta_i p(0, t_i) H(t_i) = (1 - R) \sum_{i=0}^k p(0, t_i) [H(t_{i-1}) - H(t_i)].$$

In this work we find closed form expressions for the function H in the case in which in the credit-worthiness process is modelled as a δ -dimensional Bessel bridge. One of the main reasons that justifies the use of Bessel bridges to model process X is that it will remain strictly positive until default, as opposed to (for instance) a Brownian bridge. A more detailed description of the so-called Bessel bridges will be provided throughout the remainder of this work.

3 Preliminaries

- In this paper we consider a probability space over $\Omega := C([0, \infty))$ endowed with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, that satisfies the usual hypotheses, and that supports a Brownian motion W . As done

in Definition 3 in [16], we define the squared Bessel process Z with dimension $\delta \in \mathbb{R}$ and starting at $Z_0 := a \in \mathbb{R}$ as the unique strong solution of

$$dZ_t = \delta dt + 2\sqrt{|Z_t|}dW_t, \quad Z_0 = a.$$

Now, let us define the $\delta \in \mathbb{R}$ -dimensional Bessel process by

$$Y_t := \text{sgn}(Z_t)\sqrt{|Z_t|}, \quad \text{where } \text{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

starting at $Y_0 = \text{sgn}(a)\sqrt{|a|}$. It is also said that Y has index $\nu := \frac{\delta}{2} - 1$.

• If $\delta > 0$ one can deduce from the Appendix A.1 in [16] that Y satisfies the following stochastic differential equation (SDE) up to time $\tau_0 := \inf\{s > 0 : Y_s = 0\}$:

$$(1) \quad dY_t = \frac{\delta - 1}{2} \frac{1}{Y_t} dt + dW_t, \quad Y_0 := a > 0, \quad t \in [0, \tau_0).$$

It is known that for $\delta \geq 2$, $\tau_0 = \infty$ almost surely. Moreover, from Section 3 in [16], it turns out that when $\delta < 0$ and $a > 0$, the Bessel process Y is solution of

$$(2) \quad dY_t = \frac{-\delta - 1}{2} \frac{1}{Y_t} dt + dW_t, \quad Y_0 := a > 0,$$

whenever $t \in [0, \tau_0)$, see also Remark 5.1 below. And for $\delta \in \mathbb{R}$ and $a < 0$, the square Bessel process can be seen as the negative of a square Bessel process starting at $-a > 0$ with the same dimension δ . Thus, in this case, the Bessel process Y is such that $-Y$ is solution of (1) starting at $-a$. All these considerations allow us to use equation (1) to analyze Y for general $\delta, a \in \mathbb{R}$, at least up to the time it hits zero.

• Let $T > 0$. The process $X := \{X_s, s \in [0, T]\}$ will denote the δ -Bessel bridge with $X_0 := a \in \mathbb{R}$ and $X_T = c \in \mathbb{R}$. Loosely speaking, X is the process Y conditioned to take the value c at time T . Following [32, p.463], let us rigorously define the process X . Let P denotes the probability measure on Ω that defines the Bessel process. For $u \in \mathbb{R}$ and measurable subsets $A \subset \Omega$, it is known (see [25]) that there exists a probability kernel $u \times A \mapsto \eta_u(A)$ such that

$$(3) \quad P(A) = \int_{\mathbb{R}} \eta_u(A) \mu(du),$$

where μ is the distribution of Y_T . The following expression is an intuitive idea of what η_u is,

$$\eta_u(A) = P(A | Y_T = u).$$

With $u = c$, the probability measure η_c on Ω , denoted Q , defines a stochastic process called the Bessel bridge X of dimension δ starting at a and such that it finishes at c at time T .

To introduce our first result, Theorem 3.2, we recall the following facts.

Remark 3.1. *The density of the Bessel process with index $\nu := \delta/2 - 1 \geq -1$ and initial state $x > 0$ [32, p.446] is given by*

$$(4) \quad p_t(x, y) := \frac{1}{t} \frac{y^{\nu+1}}{x^\nu} e^{-\frac{x^2+y^2}{2t}} I_\nu\left(\frac{xy}{t}\right), \quad t > 0,$$

where $I_\nu(x)$ is the modified Bessel function (with index ν) of the first kind defined as

$$(5) \quad I_\nu(x) := \sum_{k=1}^{\infty} \frac{(x/2)^{2k+\nu}}{k! \Gamma(\nu + x + 1)}.$$

In the next theorem, we apply Itô's formula to Bessel processes Y of dimension $\delta > 0$. For $\delta \in (0, 1)$, Y is not semimartingale, except before the first time it reaches zero. The following result characterizes the Bessel bridge with dimension $\delta > 0$; in the literature, this is usually done only for $\delta \geq 2$ (see e.g. [32, p.468]). Moreover, from the discussion above, using the next theorem we can derive SDE to work with Bessel bridge with $\delta < 0$.

Theorem 3.2. *i) Fix $\delta > 0$, $a > 0$, and $c = 0$, and let $Z_t := h(t, Y_t)/h(0, a)$, where*

$$(6) \quad h(t, x) := \frac{T^{\delta/2}}{(T-t)^{\delta/2}} e^{-\frac{x^2}{2(T-t)}}.$$

Then for $t < T$ and $A \in \mathcal{F}_t$

$$Q(A) = \int_A Z_t dP.$$

ii) The process X satisfies the following SDE when $t \in [0, \tau_0)$,

$$(7) \quad dX_t = \left(\frac{\delta - 1}{2X_t} - \frac{X_t}{T-t} \right) dt + dW_t, \quad X_0 = a > 0,$$

where $\tau_0 := \inf\{s > 0 : X_s = 0\}$

To prove Theorem 3.2, we need the following lemma.

Lemma 3.3. *Fix $\delta > 0$ and $c > 0$. Let Y be the δ -Bessel process with measure P , and X the Bessel bridge defined by measure Q in Theorem 3.2. Then, for $0 < t < T$,*

$$(8) \quad \frac{dQ}{dP} \Big|_{\mathcal{F}_t} = \frac{T}{T-t} \frac{\exp\left\{-\frac{Y_t^2+c^2}{2(T-t)}\right\}}{\exp\left\{-\frac{a^2+c^2}{2t}\right\}} \frac{a^\nu I_\nu\left(\frac{cY_t}{T-t}\right)}{Y_t^\nu I_\nu\left(\frac{ac}{T}\right)},$$

with I_ν as in Remark 3.1.

Proof. Let $\{I_k^{(n)}\}_{k=1}^n$, for $n = 1, 2, \dots$, be a sequence of disjoint partitions of \mathbb{R} such that $\lim_{n \rightarrow \infty} I_k^{(n)}$ is a single point in \mathbb{R} for each k . Then, appealing to equation (3), we can write

$$\int_{\mathbb{R}} \eta_u(A) \mu(du) = P(A) = \sum_{k=1}^n P(A, Y_T \in I_k^{(n)}).$$

Since this is valid for each $n = 1, 2, \dots$, we have that

$$\int_{\mathbb{R}} \eta_u(A) \mu(du) = \lim_{n \rightarrow \infty} \sum_{k=1}^n P(A|Y_T \in I_k^{(n)}) P(Y_T \in I_k^{(n)}).$$

We can then conclude that

$$\eta_u(A) = \lim_{n \rightarrow \infty} P(A|Y_T \in I_k^{(n)}).$$

Having this, we can now proceed as follows. Let $A \in \mathcal{F}_t$, with $t < T$. Let I_n be a sequence of intervals such that $c \in I_n$ and $\lim_{n \rightarrow \infty} I_n = \{c\}$. Appealing to the theory of derivatives of measures (see Chapter 7 of [34]) and using the Markov property we have

$$\begin{aligned} Q(A) &= \lim_{n \rightarrow \infty} P(A|Y_T \in I_n) = \lim_{n \rightarrow \infty} \frac{P(A, Y_T \in I_n)}{P(Y_T \in I_n)} \\ &= \lim_{n \rightarrow \infty} \frac{E[P(A, Y_T \in I_n|Y_t)]}{P(Y_T \in I_n)} \\ &= \lim_{n \rightarrow \infty} \int_A \frac{P(Y_T \in I_n|Y_t)}{P(Y_T \in I_n)} dP. \end{aligned}$$

But

$$\lim_{n \rightarrow \infty} \frac{P(Y_T \in I_n|Y_t)}{P(Y_T \in I_n)} = \frac{p_{T-t}(Y_t, c)}{p_T(a, c)},$$

with $p_t(x, y)$ as in (4). Therefore, after appealing to theorem of bounded convergence, we can confirm that the above limit is precisely (8). \square

Proof. (of Theorem 3.2)

From Lemma 3.3, letting $c \rightarrow 0$ in (8), we obtain i). This is indeed true because, by (5),

$$\lim_{c \rightarrow 0} \frac{a^v I_v(xc/(T-t))}{x^v I_v(ac/T)} = \left(\frac{T}{T-t} \right)^v.$$

To prove ii), define Z as in i). It is known that Y is a semimartingale for $\delta \geq 1$. And for $\delta \in (0, 1)$, as pointed out in [24], process Y is a semimartingale up to the time it hits zero. This allows us to apply Itô's formula to process Z , which gives rise to the SDE

$$dZ_t = -Z_t \frac{Y_t}{T-t} dW_t, \quad Z_0 = 1, \quad t < \tau_0.$$

Finally, an application of Girsanov's theorem, see for instance Section 3.3.5 in [21], yields the desired result. \square

At this point, since in the literature there is available statistical knowledge on the stopping time $\inf\{s > 0 : Y_s = b\}$, one might use Theorem 3.2 to find information about $\inf\{s > 0 : X_s = b\}$, which is precisely what we are going to do below. However, we want to take a more general perspective in order to cover a larger class of diffusion processes.

It should be remembered that the hitting time is in direct connection with the so-called running maximum of the stochastic process. Thus, one could see that when dealing with the distributions of the former we are also dealing with the distributions of the latter. Refer to [29] to see distributions of running maximum of Bessel bridges.

4 First hitting time of Bessel bridges I

The function h in (6) is a solution of a specific partial differential equation (PDE). In fact one can see that h is the so-called Doob's h -transform to go from the process Y to the process X (see [33] for an introduction to h -transforms).

The idea now is to work with a class of processes satisfying certain SDEs. It is known that harmonic functions with respect to some Markov process might be used to construct an h -transform of the process. We do so in the following result.

Theorem 4.1. *Let $S \subset \mathbb{R}$ be an interval and let $\alpha : S \rightarrow \mathbb{R}$ be a function such that the following SDE has a unique strong solution, see [21],*

$$(9) \quad dY_t = \alpha(Y_t)dt + dW_t, \quad Y_0 = a \in S, \quad t \in [0, \tau_0),$$

where $\tau_0 \leq \infty$ is a stopping time with respect to Y , and which can take any value in $[0, \infty)$ with positive probability. Also, let $T > 0$ be fixed, and assume that there exists a positive solution $h : [0, T] \times S \rightarrow \mathbb{R}$ of the PDE

$$-h_t(s, y) = \frac{1}{2}h_{xx}(s, y) + \alpha(y)h_x(s, y), \quad y \in S, \quad s \in [0, T].$$

Then, if $Z_t := h(t, Y_t)/h(0, a)$ with $t < \tau_0$, the following defines a probability measure

$$(10) \quad Q(A) := E[Z_t I_A] \text{ for all } A \in \mathcal{F}_{\tau_0}.$$

And under Q the process Y is solution of the SDE

$$(11) \quad dX_t = \left[\alpha(X_t) + \frac{h_x(t, X_t)}{h(t, X_t)} \right] dt + dW_t, \quad X_0 = a, \quad t \in [0, \tau_0).$$

To be more explicit, under Q in (10), the process Y is denoted X . We will write E_P or E_Q to emphasize under which measure one is calculating an expectation. Below we will give an example that fits into this theorem.

Proof. By Itô's formula and the hypotheses (the constant $h(0, a)$ can be dismissed for a moment)

$$\begin{aligned} dZ_t &= h_x(t, Y_t) dY_t + h_t(t, Y_t) dt + \frac{1}{2} h_{xx}(t, Y_t) (dY_t)^2 \\ &= h_x(t, Y_t) \alpha(Y_t) dt + h_x(t, Y_t) dW_t + h_t(t, Y_t) dt + \frac{1}{2} h_{xx}(t, Y_t) dt \\ &= Z_t \frac{h_x(t, Y_t)}{h(t, Y_t)} dW_t, \end{aligned}$$

with $t \in [0, \tau_0)$ and $Z_0 = 1$. This means that Z is a positive martingale for $t < \tau_0$ and with $Z_0 = 1$. Thus, Q is well defined and (10) holds. Furthermore, Z satisfies the SDE $Z_t = 1 + \int_0^t Z_s dM_s$ for $t < \tau_0$, and where

$$M_t := \int_0^t \frac{h_x(s, Y_s)}{h(s, Y_s)} dW_s.$$

So, Z is the Doléans-Dade exponential

$$\exp \left\{ \int_0^t \frac{h_x(s, Y_s)}{h(s, Y_s)} dW_s - \frac{1}{2} \int_0^t \frac{h_x^2(s, Y_s)}{h^2(s, Y_s)} ds \right\}.$$

Hence the new dynamics (11) comes from a change of measures (see e.g. [33] or [30]). \square

Example 4.2. *We can corroborate that the Bessel bridge fits into the context of Theorem 4.1. Indeed if*

$$\alpha(x) := \frac{\delta - 1}{2x},$$

then the function

$$(12) \quad h(t, x) := \frac{T}{(T-t)^{\delta/2}} \exp \left\{ -\frac{x^2}{2(T-t)} \right\}$$

is the desired solution to the parabolic PDE

$$-h_t(t, x) = \frac{1}{2} h_{xx}(t, x) + \alpha(x) h_x(t, x), \quad x \in [0, \infty), t \in [0, T].$$

Moreover, one can check that the Bessel bridge X is recovered from the Bessel process Y ; in symbols:

$$\begin{aligned} (P) \quad dY_t &= \alpha(Y_t) dt + dW_t, \\ (Q) \quad dX_t &= \left(\alpha(X_t) + \frac{h_x(t, X_t)}{h(t, X_t)} \right) dt + dW_t, \\ Q &= \frac{h(t, Y_t)}{h(0, a)} P \text{ on } \mathcal{F}_t. \end{aligned}$$

In this case h_x/h simplifies considerably.

As a consequence of Theorem 4.1, we may find the distribution of $\inf\{s > 0 : X_s = b\}$ by knowing the distribution of $\inf\{s > 0 : Y_s = b\}$.

Corollary 4.3. *Under the conditions of Theorem 4.1, for any $a \in S$, let τ be a stopping time with respect to X such that $\tau \leq \tau_0$ almost surely. Then*

$$(13) \quad Q(\tau < t) = E_P [Z_t I_{\{\tau < t\}}], \quad t < T.$$

Proof. One can see that $\{\tau < t\} \in \mathcal{F}_{\tau_0}$. □

We can now continue with our program of finding the distribution of $\tau := \inf\{s > 0 : X_s = b\}$. There are expressions for the distribution of τ under P , that is for the distribution of $\inf\{s > 0 : Y_s = b\}$; we wish to use those expressions to find $Q(\tau < t)$.

Theorem 4.4. *Under the conditions of Theorem 4.1. Define $\tau := \inf\{s > 0 : X_s = b\}$ and suppose that this is such that the condition of Corollary 4.3 holds. Then*

$$(14) \quad Q(\tau < t) = \int_0^t \frac{h(s, b)}{h(0, a)} P(\tau \in ds), \quad t < T.$$

Proof. Using Corollary 4.3, we have that

$$\begin{aligned} Q(\tau < t) &= E_Q [I_{\{\tau < t\}}] \\ &= E_P \left[\frac{h(t, Y_t)}{h(0, a)} I_{\{\tau < t\}} \right] \\ &= \int_0^\infty E_P \left[\frac{h(t, Y_t)}{h(0, a)} I_{\{\tau < t\}} | \tau = s \right] P(\tau \in ds) \\ &\quad \text{(applying the optional sampling theorem)} \\ &= \int_0^t \frac{h(s, b)}{h(0, a)} P(\tau \in ds), \end{aligned}$$

where we have used the fact that $\tau = u$ implies $Y_u = b$. □

Example 4.5. *We can now join pieces to find the first hitting-time density. Let X be the δ -Bessel bridge with $\delta \in \{1, 3\}$, and such that $X_0 = a > 0$ and $X_T = 0$. If $0 < b < a$, using formula (14) above and (15) below, we have for $\tau := \inf\{s > 0 : X_s = b\}$ that*

$$Q(\tau \in dt) = \frac{h(t, b)}{h(0, a)} \left(\frac{b}{a} \right)^{\nu + |\nu|} \frac{a - b}{\sqrt{2\pi t^3}} e^{-\frac{(a-b)^2}{2t}}, \quad t \leq T,$$

where h is given in (12) and $\nu := \delta/2 - 1$.

Remark 4.6. *According to Theorem 2.2 of [18], for $\delta = 1$ or $\delta = 3$, and if $0 < b < a$, the distribution of the first time that a δ -Bessel process Y starting at a hits b is given by*

$$(15) \quad P(\tau \leq t) = \left(\frac{b}{a} \right)^{\nu + |\nu|} \int_0^t \frac{a - b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} ds.$$

Also, if $\nu - 1/2$ is an integer but $|\nu| \neq 1/2$ and again $0 < b < a$, then

$$(16) \quad \begin{aligned} P(\tau \leq t) &= \left(\frac{b}{a}\right)^{\nu+|\nu|} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} ds \\ &\quad - \left(\frac{b}{a}\right)^\nu \sum_{j=1}^{N_\nu} \frac{K_\nu(az_j/b)}{z_j K_{\nu+1}(z_j)} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s} + \frac{z_j(a-b)\sqrt{t}}{b\sqrt{s}}} ds, \end{aligned}$$

where K_ν is the modified Bessel function of the second kind (see Section 9.6 in [1]), N_ν is the number of zeros of the function K_ν , and $\{z_j, j = 1, \dots, N_\nu\}$ are the zeros of K_ν (which are different from each other).

Example 4.7. Here we give a formula for the first time that a Bessel bridge hits a line with positive slope. Let Y be a δ -Bessel process with $\delta > 0$ starting at $Y_0 := a > 0$, and let $\tau := \inf\{s > 0 : Y_s = b + cs\}$ with $b, c > 0$. Following Theorem 5.1 of [2],

$$(17) \quad P(\tau \in dt) := \frac{e^{(c/2b)(b^2-a^2)+tc^2/2}}{(1+tc/b)^{\nu+2}} \sum_{j=1}^{\infty} \frac{a^{-\nu} z_j J_\nu(z_j a/b)}{b^{2-\nu} J_\nu(z_j)} e^{-z_j^2 \frac{t}{2b(b+ct)}}$$

where J_ν is the Bessel function of the first kind (see Section 9.1 in [1]), for $t \geq 0$. Now, let X be the δ -Bessel bridge such that $X_0 := a$ and $X_T := 0$. Following the reasoning of Example 4.5, that is, taking into account Example 4.2 and Theorem 4.4, we arrive at an expression for the density of

$$\inf\{s > 0 : X_s = b + cs\},$$

given by

$$Q(\tau \in dt) = \frac{h(t, b)}{h(0, a)} P(\tau \in dt), \quad t \in [0, T],$$

where $P(\tau \in dt)$ is precisely (17).

5 First hitting time of Bessel bridges II

When dealing with a Bessel bridge, we find relevant to modify the hitting-time problem by making space-transformations of the SDEs that yield probably better-behaved equations, and this is precisely the content of this section. We also realized that with these transformations one can connect the original problem of Bessel bridges with one of Bessel processes with negative dimension $\delta < 0$.

Let us explain the idea. Let X be the Bessel bridge with dimension $\delta > 0$, and let Y be the Bessel process with dimension $4 - \delta < 4$, both starting at $a \in \mathbb{R}$ and such that $X_T = 0$. Let us apply Itô's formula (up to the time the processes hit zero) to the transformations $X^{\delta-2}$ and

$Y^{\delta-2}$. Then

$$\begin{aligned} dX_t^{\delta-2} &= (\delta-2)X_t^{\delta-3}dX_t \\ &\quad + \frac{1}{2}(\delta-2)(\delta-3)X_t^{\delta-4}(dX_t)^2 \\ &= \left((\delta-2)^2X_t^{\delta-4} - \frac{\delta-2}{T-t}X_t^{\delta-2} \right) dt \\ &\quad + (\delta-2)X_t^{\delta-3}dW_t. \end{aligned}$$

Hence, if $U := X^{\delta-2}$, then $X = U^{\frac{1}{\delta-2}}$ and

$$(18) \quad \begin{aligned} dU_t &= \left((\delta-2)^2U_t^{\frac{\delta-4}{\delta-2}} - \frac{\delta-2}{T-t}U_t \right) dt \\ &\quad + (\delta-2)U_t^{\frac{\delta-3}{\delta-2}}dW_t. \end{aligned}$$

Similarly, $V := Y^{\delta-2}$, i.e. V is the $(4-\delta)$ -Bessel process raised to the power $\delta-2$, satisfies the SDE

$$(19) \quad dV_t = (\delta-2)V_t^{\frac{\delta-3}{\delta-2}}dW_t.$$

Notice that if $\inf\{s > 0 : U_s = d\}$ and $\inf\{s > 0 : V_s = d\}$ are related somehow, then so are $\inf\{s > 0 : X_s = b\}$ and $\inf\{s > 0 : Y_s = b\}$. This is indeed the case due to the Theorem 5.2 below, whose proof follows the same line of reasoning of Theorem 4.1. First we note the following.

Remark 5.1. *To study Bessel bridges of dimension $\delta > 4$ we are using Bessel processes of dimension $\delta-4$. It is shown in [16, section 3], see pages 329 and 330, that if a δ -Bessel process, with $\delta < 0$, starts above zero, then it will become negative in finite time; however, prior to this moment it behaves as a $4-\delta$ Bessel process.*

Theorem 5.2. *Let $S \subset \mathbb{R}$ be an interval and $\sigma : [0, \infty) \times S \rightarrow [0, \infty)$ be a function such that the following SDE has a unique strong solution,*

$$(20) \quad dV_t = \sqrt{\sigma(t, V_t)}dW_t, \quad V_0 = a \in S, \quad t \in [0, \tau_0),$$

with $\tau_0 \leq \infty$ allowed to be a random variable. Assume as well that there exists a positive solution $h : [0, T] \times S \rightarrow \mathbb{R}$ of the following PDE

$$-h_t(s, y) = \frac{1}{2}\sigma(s, y)h_{xx}(s, y), \quad y \in S, \quad s \in [0, T].$$

Then $Z_t := h(t, V_t)/h(0, a)$ defines a new probability measure

$$(21) \quad Q(A) := E[Z_t I_A] \text{ for all } A \in \mathcal{F}_{\tau_0},$$

under which the process V is solution of the SDE

$$(22) \quad dU_t = \sigma(t, U_t) \frac{h_x(t, U_t)}{h(t, U_t)} dt + \sqrt{\sigma(t, U_t)} dW_t, \quad U_0 = a, \quad t \in [0, \tau_0).$$

Proof. By Itô's formula and the hypotheses (the constant $h(0, a)$ can be dismissed for a moment)

$$\begin{aligned} dZ_t &= h_x(t, V_t) dV_t + h_t(t, V_t) dt + \frac{1}{2} h_{xx}(t, V_t) (dV_t)^2 \\ &= h_x(t, V_t) dW_t + h_t(t, V_t) dt + \frac{1}{2} h_{xx}(t, V_t) \sigma(t, V_t) dt \\ &= Z_t \frac{h_x(t, V_t)}{h(t, V_t)} dW_t, \end{aligned}$$

with $t \in [0, \tau_0)$ and $Z_0 = 1$. This means that Z is a positive martingale for $t < \tau_0$ and with $Z_0 = 1$. Thus, Q is well defined and (10) holds. Furthermore, Z satisfies the SDE $Z_t = 1 + \int_0^t Z_s dM_s$ for $t < \tau_0$, and where

$$M_t := \int_0^t \frac{h_x(s, Y_s)}{h(s, Y_s)} dW_s.$$

So, Z is the Doléans-Dade exponential

$$\exp \left\{ \int_0^t \frac{h_x(s, Y_s)}{h(s, Y_s)} dW_s - \frac{1}{2} \int_0^t \frac{h_x^2(s, Y_s)}{h^2(s, Y_s)} ds \right\}.$$

Hence the new dynamics (11) comes from a change of measures (see e.g. [33] or [30]). □

Under Q the process V will be denoted by U .

Example 5.3. Let us put in action Theorem 5.2 to deal with equations (18) and (19), with $\tau_0 := \inf\{s > 0 : V_s = 0\}$.

For the solution V of (19), the associated PDE is

$$(23) \quad -h_t(t, x) = \frac{1}{2} (\delta - 2)^2 x^{2\frac{\delta-3}{\delta-2}} h_{xx}(t, x),$$

and the solution we are interested in is

$$(24) \quad h(t, x) = x(T-t)^{-\frac{\delta}{2}} \exp \left\{ -\frac{x^{\frac{2}{\delta-2}}}{2(T-t)} \right\}.$$

Then

$$\frac{h_x(t, x)}{h(t, x)} = \frac{1}{x} - \frac{x^{\frac{2}{\delta-2}-1}}{(\delta-2)(T-t)}.$$

We also have that

$$(25) \quad \sigma(x) = (\delta-2)^2 x^{2\frac{\delta-3}{\delta-2}},$$

and so

$$\sigma(x) \frac{h_x(t, x)}{h(t, x)} = (\delta - 2)^2 x^{\frac{\delta-4}{\delta-2}} - \frac{\delta-2}{T-t} x.$$

It follows that the dynamics of U in equation (18) can be expressed as

$$dU_t = \sigma(U_t) \frac{h_x(t, U_t)}{h(t, U_t)} dt + \sqrt{\sigma(U_t)} dW_t,$$

which is the new dynamics under Q .

Remark 5.4. One can see that under the hypotheses of Theorem 5.2 the conclusions in Corollary 4.3 and Theorem 4.4 remain valid, and the proofs are actually the same. That is, if $\tau := \inf\{s > 0 : U_s = b\}$ for some $b \in S$ with $\tau \leq \tau_0$ almost surely, then

$$Q(\tau < t) = E_P [Z_t I_{\{\tau < t\}}]$$

and, moreover

$$(26) \quad Q(\tau < t) = \int_0^t \frac{h(s, b)}{h(0, a)} P(\tau \in ds),$$

when $t \leq T$. Here, $P(\tau \in ds)$ is the law of τ under P , which ends up being $\inf\{s > 0 : V_s = b\}$.

We are now in position to find the distribution of $\inf\{s > 0 : X_s = b\}$ for $\delta \neq 2$, which is carried out by finding the distribution of $\inf\{s > 0 : U_s = b^{\delta-2}\}$. Since U is related to V by means of Theorem 5.2, we can then use formula (26). At the end, we can use the fact that

$$\inf\{s > 0 : V_s = b^{\delta-2}\} = \inf\{s > 0 : Y_s = b\}$$

together with the first hitting-time distribution of the Bessel process Y (which is found in the literature; see [6, 27, 18]). Let us present two examples of this idea in the coming proposition and example.

Remark 5.5. According to [26] (see also equation (2.5) in [18]), for $\delta \in \mathbb{R}$ and $0 < b < a$, the Laplace transform of the first time that a δ -Bessel process Y starting at a hits b is given by

$$(27) \quad E[e^{-\theta\tau}] = \frac{a^{-\nu} K_\nu(a\sqrt{2\theta})}{b^{-\nu} K_\nu(b\sqrt{2\theta})},$$

where $K_\nu(x)$ is the modified Bessel function of the second kind and $\nu := \delta/2 - 1$.

Proposition 5.6. *Let V be the solution of (19) with $V_0 = a > 0$, and let $\tau_V := \inf\{s > 0 : V_s = d\}$ with $0 < d < a$. Then its Laplace transform is*

$$(28) \quad E_Q \left[e^{-\theta \tau_V} \right] = \sqrt{\frac{a}{d}} \frac{K_{\frac{2-\delta}{2}} \left(\sqrt{2\theta} y^{\frac{1}{\delta-2}} \right)}{K_{\frac{2-\delta}{2}} \left(\sqrt{2\theta} d^{\frac{1}{\delta-2}} \right)},$$

for $\theta > 0$.

Proof. The result follows from formula (27) due to the equality

$$\tau_V = \inf\{s > 0 : Y_s = d^{1/(\delta-2)}\}.$$

Find more details in [26]. □

Example 5.7. *Take X to be the Bessel bridge of dimension $\delta = 5$ with $X_0 = a$. Thus according to (18) the process $U := X^3$ is solution of the SDE*

$$dU_t = \left(9U_t^{1/3} - \frac{3U_t}{T-t} \right) dt + 3U_t^{2/3} dW_t, \quad U_0 = a^3, \quad t < T.$$

On the other hand, we consider the process V defined in (19), which is the cube of a Bessel process with dimension -1 , solution of

$$dV_t = 3V_t^{2/3} dW_t, \quad V_0 = a^3, \quad t < T.$$

Using (26) and (24) we have that

$$\begin{aligned} Q(\tau < t) &= Q(\inf\{s > 0 : U_s = b^3\} < t) \\ &= \int_0^t \frac{h(s, b^3)}{h(0, a^3)} P(\inf\{s > 0 : V_s = b^3\} \in ds) \\ &= \int_0^t \frac{h(s, b^3)}{h(0, a^3)} P(\inf\{s > 0 : Y_s = b\} \in ds) \\ &= \int_0^t \frac{h(s, b^3)}{h(0, a^3)} P(\tau \in ds). \end{aligned}$$

Since $\nu - 1/2 = -2$, and if in addition $0 < b < a$, we are then in the situation of equation (16), and the density $P(\inf\{s > 0 : Y_s = b\} \in ds)$ can be written explicitly; in this case it is known that K_ν has only one zero $z_1 = -1$, so that $N_\nu = 1$.

Using Leibnitz' rule, we can write down the explicit density by taking the derivative of (16):

$$\begin{aligned}
P(\tau \in dt) &= \left(\frac{b}{a}\right)^{v+|v|} \frac{a-b}{\sqrt{2\pi t^3}} e^{-\frac{(a-b)^2}{2t}} \\
&- \left(\frac{b}{a}\right)^v \sum_{j=1}^{N_v} \frac{K_v(az_j/b)}{z_j K_{v+1}(z_j)} \frac{a-b}{\sqrt{2\pi t^3}} e^{-\frac{(a-b)^2}{2t} + \frac{z_j(a-b)}{b}} \\
&- \left(\frac{b}{a}\right)^v \sum_{j=1}^{N_v} \frac{K_v(az_j/b)}{z_j K_{v+1}(z_j)} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} \frac{z_j(a-b)}{2b\sqrt{st}} e^{-\frac{(a-b)^2}{2s} + \frac{z_j(a-b)\sqrt{t}}{b\sqrt{s}}} ds.
\end{aligned}$$

Therefore the density of the first time that the 5-Bessel bridge X hits level b (with $0 < b < a$) is given by

$$(29) \quad Q(\tau \in dt) = \frac{h(t, b^3)}{h(0, a^3)} P(\tau \in dt).$$

With h as in (24). Some numerical simulations to observe the form of the density of Q yield Figure 1.

6 Counterparty risk via Bessel bridges

In previous sections we studied hitting times of Bessel bridges, here we would like to give some words on a possible application of Bessel bridges in mathematical finance, specifically in the context of credit risk.

It is known that the short rate model r , see [3, p.374], specifies the zero-coupon bond price as

$$(30) \quad p(t, s) := E \left[e^{-\int_t^s r_u du} \middle| \mathcal{F}_t \right].$$

For instance, let $b, c > 0$ and suppose that r has the following dynamics (Vasicek or Ornstein-Uhlenbeck process)

$$(31) \quad dr_t = (b - cr_t)dt + \sigma dW_t + dX_t, \quad t > 0,$$

where W is a Wiener process and X is a stochastic process which is called in this context the *counterparty credit-index process*. In order to model the financial phenomena, process X is considered to take a specific value at time s , and this consideration represents the so-called *time of default*, see [11]. In particular, one may assume that X is modelled with a δ -Bessel bridge.

From stochastic calculus we have that

$$r(s) = e^{-cs} r(0) + \int_0^s e^{-c(s-u)} b du + \int_0^s \sigma e^{-c(s-u)} dW_u + \int_0^s e^{-c(s-u)} dX_u.$$

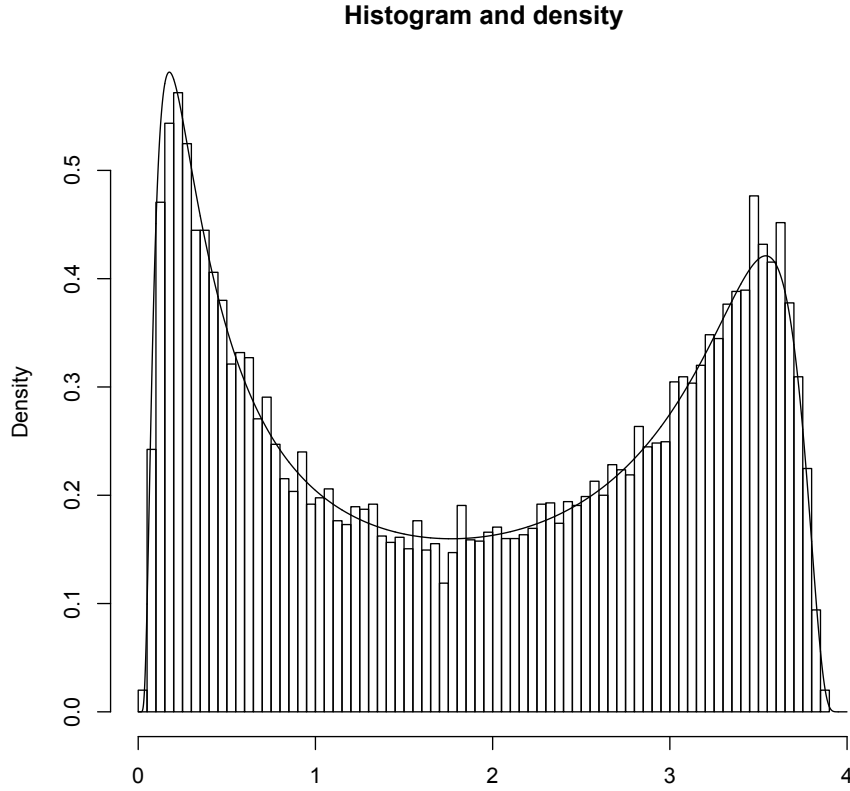


FIGURE 1. The graph was made in the statistical software R. The line represents the density of the first time that a 5-Bessel bridge, started at $X_0 = 10^{1/3}$ and finishing in zero at time $T = 4$, hits level $b = 3^{1/3}$, that is, using expression (29). The histogram is made out of 20,000 simulations of the described 5-Bessel bridge hitting the level.

Moreover, taking W and X independent, it turns out that to calculate the zero coupon bond price (30), see e.g. [3, p.382], one needs to calculate the following expectation

$$E \left[e^{-ce^{-cs} - \int_0^s ce^{-c(s-u)} X_u du} \right].$$

Knowing the density of the hitting times of X may be used to study the time of default.

7 Conclusions

In this paper we have given explicit expressions for the hitting-time densities of a class of Bessel bridges with $\delta \in \mathbb{R}$. The main tools used have been Doob's h -transform, some space transformations, and the optional sampling theorem (as has been done for the Brownian bridge). Our basic approach was described in Section 4. To broaden the range of applications we have

developed the technique in such a way that one can recycle it for other processes, namely those that solve specific SDEs.

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