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Banco de Mexico

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## On a new class of barrier options\*

Gerardo Hernández del Valle<sup>y</sup>  
Banco de Mexico

**Abstract:** A barrier option is a financial derivative which includes an activation (or deactivation) clause within a standard vanilla option. For instance, a copper mining company could secure to sell in at least  $K$  dollars each ton of copper during the next year, by buying  $M$  European put options. However, it could purchase a less expensive derivative (a barrier option) which includes a clause which deactivates the contract if ever the price of copper is below  $B$  dollars (for  $B < K$ ) during the life of the contract. In practice, such barrier does not have to remain constant during the life of the contract. However, pricing a barrier derivative is only known for barriers that are represented by linear, quadratic, and square root functions. In this work we propose a new methodology for pricing barrier options that include a larger family of barriers not previously studied in the literature.

**Keywords:** Pearcey function, boundary crossing, heat equation, Rayleigh equation, option pricing, boundary options.

**JEL Classification:** G10, G12, G13.

**Resumen:** Una opción con frontera es un derivado financiero que incluye una cláusula de activación (o desactivación) dentro de una opción estándar. Por ejemplo, una compañía minera de cobre podría asegurar la venta de cada tonelada de cobre en al menos  $K$  dólares, comprando  $M$  opciones put Europeas. Sin embargo, podría comprar un derivado menos caro (una opción con frontera) que incluye una cláusula que desactiva el contrato si alguna vez el precio del cobre está por debajo de  $B$  dólares (para  $B < K$ ) durante la vida del contrato. En la práctica, estas barreras no tienen que permanecer constantes durante la vida del contrato. Sin embargo, el valor de derivados con fronteras sólo son conocidos para barreras que están representadas por funciones lineales, cuadráticas y raíz cuadrada. En este trabajo proponemos una nueva metodología para valorar opciones con barrera que incluyen a una familia más amplia de fronteras que no han sido previamente estudiadas en la literatura.

**Palabras Clave:** Función de Pearcey, cruce de fronteras, ecuación del calor, ecuación de Rayleigh, Valuación de derivados, opciones con barreras.

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# 1 Introduction

Suppose there is a financial contract which will be deactivated if ever the price of an asset  $S$  (modelled as a Brownian motion) reaches a prescribed boundary  $f$ . For instance, in Figure 1, the dashed line represents the evolution of the price of  $S_t$ , for  $t \in [0, 10]$ . In turn, the hard line represents a linear boundary which deactivates a contract if it is ever reached by the price process  $S$ . In this particular case, the boundary is reached at approximately  $t = 5$ . An example of a financial contract of this type is the so-called barrier option, which in some cases, it can be priced *exactly* by first solving the moving boundary  $f$  problem of the heat equation  $h$  (see for instance [7]):

$$(1) \quad h_t(t, x) = \frac{1}{2} h_{xx}(t, x),$$

and such that

$$h(t, f(t)) = 0 \quad \forall t \geq 0.$$

In fact, when then boundary  $f$  is linear (see Figure 1) one function  $h$  which solves (1) is

$$(2) \quad h(t, x) = \frac{x}{\sqrt{2\pi t^3}} \exp\left\{-\frac{x^2}{2t}\right\} + b \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\},$$

with  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ , and  $b \in \mathbb{R}$ . This is true since  $h$ , in (2), is a linear combination of the fundamental solution of (1) and its first derivative with respect to the space variable  $x$ . Alternatively, to see that  $h$  indeed satisfies the moving *linear* boundary condition  $f$ , let  $x = -bt$ . Hence, for any  $a \in \mathbb{R}$ , and setting

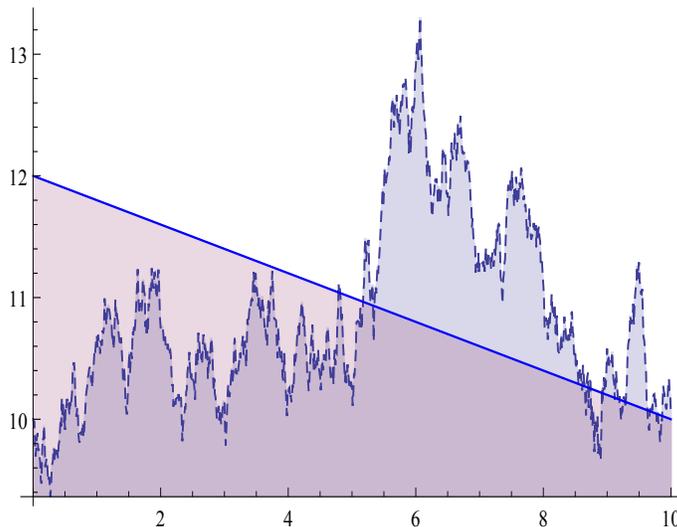


FIGURE 1. The dashed line represents the price of some asset  $S$ . In turn, the hard line represents a boundary which will deactivate a contract the first time it is reached. In this example the contract was activated at approximately time  $t = 5$ . This is a random time, since it would have been impossible to foresee the outcome.

$x = a - bt$  in (2) we obtain

$$h(t, a - bt) = \frac{a}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(a - bt)^2}{2t}\right\}, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

We note that the right-hand side of this identity is in fact the density of the first time that a Brownian motion hits a linear boundary [7, p. 196]. In practice the previous expression is used in: (a) the valuation of *barrier options* (see [2]); (b) in the quantification of counterparty risk (see [3]); (c) in general in physical problems.

The literature only has closed form solutions, for the valuation of boundary options, in the case in which the boundary is linear, quadratic or square root. In this work we will propose a new family of moving boundaries  $f$  for which we will find its corresponding function  $h$ . In financial jargon, this is equivalent to saying that for a class of boundaries  $f$  we are able to find the (approximate) price of its corresponding contract  $h$ . To this end we will study a particular function  $h$  the so-called *Pearcey integral* for which we will find a sequence of functions  $f_n$  which solve problem (1).<sup>1</sup>

The main contributions of this work are, on the one hand, finding a new family of boundaries for which the barrier can be priced. On the other, advancing in the direction of developing a rather simple and straightforward methodology to find explicit solutions of the time-varying boundary problem for the heat equation, hence in the valuation of boundary options.<sup>2</sup>

The paper is organized as follows. In Section 2 we introduce the Airy function of order 4 which will be our basic ingredient, since each of its zeros will lead to a new boundary (hence a new option). Next in Section 3 we define the Pearcey integral and describe its connection with the Airy function of order 4, this is achieved by convoluting the kernel of Brownian motion with the Airy function. This is useful because geometrically this is equivalent to constructing a time dependent spring. In Section 4 we derive a Rayleigh-type equation, whose solution kills the Pearcey function. That is, we solve the moving boundary problem. The techniques described in Section 4 are illustrated with examples in Section 5. In Section 6 we derive a function which asymptotically solves the moving boundary problem for the Pearcey integral. This approximation can be helpful in the numerical solution of the Rayleigh equation. We conclude in Section 8, with some final remarks and applications.

## 2 Generalized Airy function of order 4

With respect to the zeros of Fourier integrals, Pólya proved [12] that all the zeros of

$$(3) \quad \int_{-\infty}^{\infty} e^{-u^{2m} + izu} du, \quad \text{for } m = 1, 2, 3 \dots$$

<sup>1</sup>The Pearcey integral was first evaluated numerically by Pearcey [11] in his investigation of the electromagnetic field near a cusp. The integral appears also in optics [1], in the asymptotics of special functions [6], in probability theory [15], as the generating function of heat (and hence Hermite) polynomials of order  $4k$  for  $k \in \mathbb{N}$  [13]. It also falls into the category of functions considered by Pólya [12], that is functions with countably many zeros. For the numerical evaluation of the zeros of the Pearcey integral see for instance [6], this will be important since this zeros will correspond to the initial value of each function  $f_n(0)$ .

<sup>2</sup>In this regard we note that there exist techniques to study the latter aforementioned problem in terms of solutions to integral equations [4]. We recall that solutions in terms of integral equations, in general can only be evaluated numerically. In turn, our approach leads to solutions in terms of ODEs.

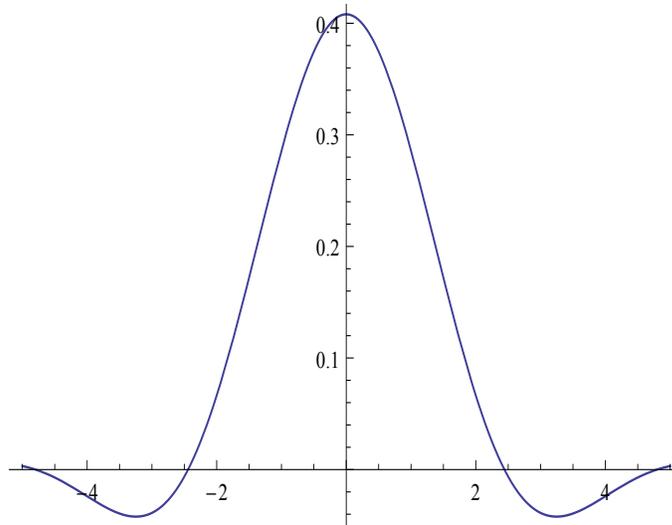


FIGURE 2. We plot function  $\phi$ , defined in (5). We observe the function is even and its first zero is at  $\pm 2.44197$ .

are real and infinitely many for  $m > 1$ . In turn, the generalized Airy function  $\phi$  of order 4 can be expressed as a solution of the following ODE

$$(4) \quad \begin{aligned} \phi^{(3)} &= x\phi, \\ \phi^{(j)} &= (j-3)\phi^{(j-4)} + x\phi^{(j-3)}, \quad \text{for } j > 3. \end{aligned}$$

One can prove, for instance applying the Fourier transform to (4) and solving the resulting equation, that  $\phi$  is a particular case of (3) when  $m = 2$ , namely

$$(5) \quad \phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left\{ ixy - \frac{y^4}{4} \right\} dy.$$

Furthermore function  $\phi$  is symmetric, with countably many zeros in the real line—hence oscillatory—and tends to zero as it increases to  $\pm\infty$ , see Figure 2. Regarding the zeros of (5), there exist asymptotic estimates which are derived by means of the method of steepest descent [14].

### 3 The Pearcey integral

Paris [9, 10] analyzed the asymptotic behavior of

$$P'_n(X, Y) = \int_{-\infty}^{\infty} e^{i(u^{2n} + Xu^n + Yu)} du, \quad n \in \mathbb{N}, n \geq 2,$$

which by rotation of the path of integration ( $u = te^{\frac{\pi i}{4n}}$ ) and use of Jordan's lemma (see [14]) can be expressed as

$$P'_n(X, Y) = P_n(x, y) = e^{\frac{\pi i}{4n}} \int_{-\infty}^{\infty} e^{-t^{2n} - xt^n + iyt} dt,$$

with  $x = Xe^{-\frac{\pi i}{4}}$  and  $y = Ye^{\frac{\pi i}{4n}}$ . In particular, the Pearcey integral, which solves (1), is the case  $n = 2$ . More explicitly, we have the following.

**Definition 3.1.** [11] *The Pearcey integral is defined as*

$$(6) \quad P'_2(X, Y) = \int_{-\infty}^{\infty} e^{i(u^4 + Xu^2 + Yu)} du.$$

In this work we study instead the following Fourier integral

$$(7) \quad v(t, x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left\{ i\lambda x - \frac{1}{2}\lambda^2 t - \frac{\lambda^4}{4} \right\} d\lambda,$$

because the zeros of  $v$ , for  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ , are expressed in terms of a continuously differentiable function  $f$ , as opposed to (6). See [14].

**Remark 3.2.** *We observe that the function  $v$  in (7) is the convolution between the kernel of standard Brownian motion and the generalized Airy function of order 4, in equation (4).*

## 4 Zeros of the Pearcey function

**Remark 4.1.** *Throughout this work, the  $n$ -th partial differentiation with respect to the space variable  $x$  of any given function  $v(t, x)$  is denoted as  $v^{(n)}$ .*

In this section we find the function  $f$  for which the Pearcey function is zero for every  $t \geq 0$ . The idea is to exploit, on the one hand, the differential form of the Airy function of order 4, defined in (4), and on the other to use the fact that the Pearcey function solves the heat equation (1).

The main result is the following.

**Theorem 4.2.** *Suppose that  $v$  is as in (7),  $\phi$  solves (4),  $\xi$  is such that  $\phi(\xi) = 0$ , and  $f$  is a solution to the following Rayleigh-type ODE*

$$(8) \quad f''(t) = 2[f'(t)]^3 - \frac{1}{2}t f'(t) - \frac{1}{4}f(t),$$

with  $f(0) = \xi$  and  $f'(0) = -\phi^{(2)}(\xi)/[2\phi^{(1)}(\xi)]$ . Then, for every  $t \geq 0$ , we have

$$v(t, f(t)) = 0.$$

*Proof.* Given that  $\phi$  is as in (5), its Fourier transform equals

$$\tilde{\phi}(\lambda) = \exp \left\{ -\frac{\lambda^4}{4} \right\}.$$

Furthermore, if we apply the Fourier transform directly to the ODE (4) we have that

$$(i\lambda)^3 \tilde{\phi} = i \frac{d}{d\lambda} \tilde{\phi}.$$

Since this expression is already in Fourier domain, we convolve the previous expression with the heat kernel as follows

$$\begin{aligned} \int e^{i\lambda x - \frac{1}{2}\lambda^2 t} (i\lambda)^3 \tilde{\phi} d\lambda &= \int i e^{i\lambda x - \frac{1}{2}\lambda^2 t} \frac{d}{d\lambda} \tilde{\phi} d\lambda \\ &= \int i e^{i\lambda x - \frac{1}{2}\lambda^2 t} d\tilde{\phi}. \end{aligned}$$

This in turn, and by direct application of the integration by parts formula, yields

$$\begin{aligned} v^{(3)} &= - \int \tilde{\phi} i(ix - \lambda t) e^{i\lambda x - \frac{1}{2}\lambda^2 t} d\lambda, \\ (9) \qquad &= xv + tv^{(1)}, \end{aligned}$$

as well as

$$(10) \qquad v^{(4)} = v + xv^{(1)} + tv^{(2)},$$

after differentiation with respect to  $x$ .

Next given that there exists an  $f$ , see Pólya [12], such that the following holds for all  $t$

$$v(t, f(t)) = 0,$$

we differentiate (Leibniz integral rule)  $v$  with  $x = f(t)$ , defined in (7), with respect to  $t$  to obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (i\lambda f'(t) - 1/2\lambda^2) \exp \left\{ i\lambda f(t) - \frac{1}{2}\lambda^2 t - \frac{\lambda^4}{4} \right\} d\lambda = 0,$$

which is equivalent to

$$(11) \qquad f'(t)v^{(1)}(t, f(t)) + \frac{1}{2}v^{(2)}(t, f(t)) = 0$$

and

$$(12) \qquad f''(t)v^{(1)} + f'(t)(f'(t)v^{(2)} + v^{(3)}) + \frac{1}{4}v^{(4)} = 0$$

after differentiation with respect to  $t$  twice. We note that equations (9) and (10) obtained from the Airy differential equation (4), as well as (11) and (12) obtained from the heat equation, involve derivatives of  $v$  up to order 4. What remains is to obtain (8) from these expressions. To this end, from (9) and (10) we first have

$$\begin{aligned} \frac{v^{(3)}(t, f(t))}{v^{(1)}(t, f(t))} &= t \\ \frac{v^{(4)}(t, f(t))}{v^{(1)}(t, f(t))} &= f(t) + t \frac{v^{(2)}(t, f(t))}{v^{(1)}(t, f(t))}. \end{aligned}$$

Next from (11) and (12) it follows that

$$\begin{aligned} f'(t) + \frac{1}{2} \frac{v^{(2)}(t, f(t))}{v^{(1)}(t, f(t))} &= 0 \\ f''(t) + f'(t) \left( f'(t) \frac{v^{(2)}(t, f(t))}{v^{(1)}(t, f(t))} + \frac{v^{(3)}(t, f(t))}{v^{(1)}(t, f(t))} \right) &= -\frac{1}{4} \frac{v^{(4)}(t, f(t))}{v^{(1)}(t, f(t))}. \end{aligned}$$

These identities yield

$$f''(t) + f'(t) (-2(f'(t))^2 + t) = -\frac{1}{4}(f(t) - 2tf'(t)).$$

This completes the proof of Theorem 4.2. □

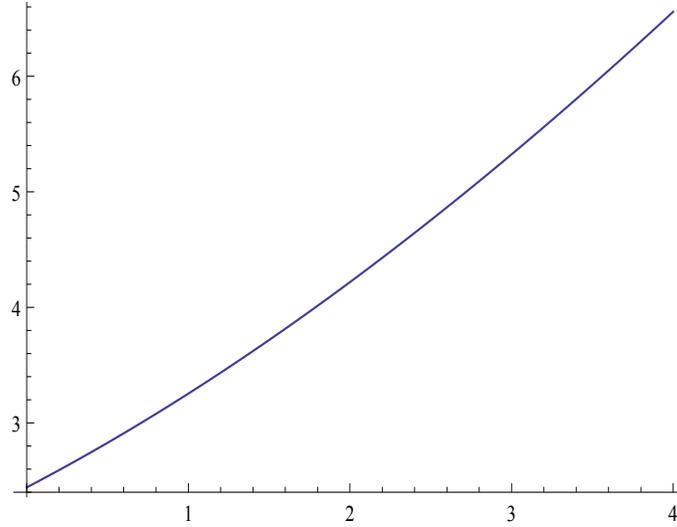


FIGURE 3. Numerical solution of (8) with  $f(0) = \xi = 2.44197$ , and  $f'(0) = 0.729925$ . The graph was plotted using Mathematica

## 5 Examples

To illustrate Theorem 4.2 we next present examples and numerical experiments.

**Numerical Example 5.1.** *One can show that at  $t = 0$  the following two identities hold for  $f$ :  $f(0) = \xi = 2.44197$  and  $f'(0) = 0.729925$ . Hence, from Theorem 4.2, we may plot the solution of (8) in Figure 3. The ODE was solved numerically using Mathematica.*

Next, we present some further examples of the methodology discussed in this section.

**Example 5.2.** *The Airy function of order 3 solves the following ODE*

$$\phi^{(2)}(x) = x\phi(x).$$

*Using the same argument as in the proof of Theorem 4.2 it follows that*

$$v^{(2)}(t, x) = xv(t, x) + tv^{(1)}(t, x).$$

*This last expression evaluated at  $f$  yields*

$$\frac{v^{(2)}}{v^{(1)}}(t, f(t)) = t.$$

*Finally, from the previous expression and (11) it follows that*

$$\begin{aligned} f'(t) &= -\frac{1}{2} \frac{v^{(2)}}{v^{(1)}}(t, f(t)) \\ &= -\frac{1}{2}t. \end{aligned}$$

*Thus, for some constant  $C$*

$$f(t) = C - \frac{1}{4}t^2.$$

See [16, p.126 ]. For applications of the Airy function in the first hitting time problem of Brownian motion up to a quadratic function see [8].

**Example 5.3.** Given the following ODE

$$\phi^{(2)}(x) = x\phi(x) + \phi^{(1)}(x),$$

and following the same line of reasoning as in the proof of Theorem 4.2 we have

$$v^{(2)}(t, x) = xv(t, x) + tv^{(1)}(t, x) + v^{(1)}(t, x).$$

From (11) we obtain

$$\frac{v^{(2)}}{v^{(1)}}(t, f(t)) = t + 1.$$

It follows that

$$f'(t) = -\frac{t+1}{2}$$

or

$$f(t) = -\frac{t}{2} - \frac{t^2}{4} + C.$$

For example if  $C = -2.58811$  the moving boundary problem of the heat equation associated with  $\phi$  is solved.

**Example 5.4.** Given the following Bessel ODE

$$\phi''(x) = -\left(\frac{5}{2} - \frac{1}{4}x^2\right)\phi(x).$$

Similar calculations as in the previous examples yield

$$f(t) = \pm \frac{1}{2}\sqrt{t^2 - 4}.$$

**Example 5.5.** The derivative of the Airy function  $Ai'(x)$  solves

$$x\phi''(x) = \phi'(x) + x^2\phi(x),$$

which yields

$$-tv^{(3)}(t, x) + (t^2 - x)v^{(2)}(t, x) + (1 + 2xt)v^{(1)}(t, x) + (x^2 + t)v(t, x) = 0.$$

Alternatively, from Example 5.2 we also have that

$$v^{(3)}(t, x) - tv^{(2)}(t, x) - xv^{(1)}(t, x) - 2v(t, x) = 0.$$

Using the same arguments as those described in Section 4 leads to

$$(13) \quad 2f(t)f'(t) + (1 + tf(t)) = 0.$$

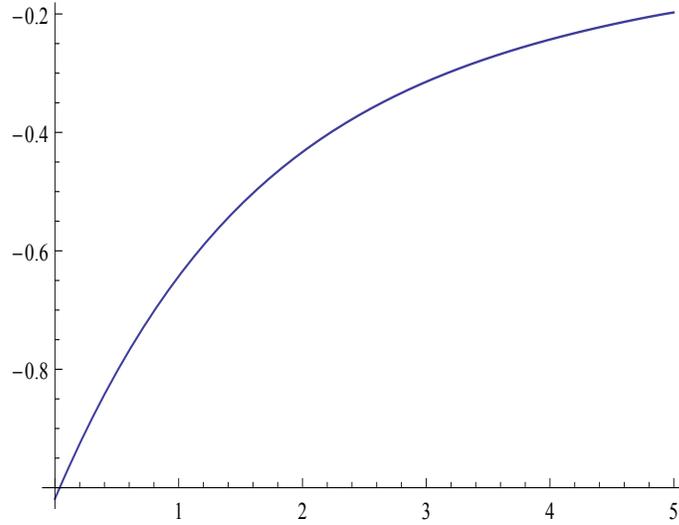


FIGURE 4. The Numerical solution of  $f$ , defined in Example 5.5, and such that  $f(0) = -1.01879$  was plotted with Mathematica

This is the Abel equation of the second kind and its solution can be expressed in terms of the Airy function of order 3  $Ai$ , as follows:

$$\frac{1}{2}tAi\left(\frac{t^2}{4} + f(t)\right) + Ai'\left(\frac{t^2}{4} + f(t)\right) = 0.$$

See Figure 4 for a numerical example with  $f(0) = -1.01879$ .

## 6 Zeros of the Pearcey function. An asymptotic approach

In this section we carry out an analysis in order to find an asymptotic solution to the moving boundary problem associated with the Pearcey integral. This result is useful when finding the numerical solution of the Rayleigh equation (8). The main result of this section is the following.

**Theorem 6.1.** *Suppose that  $v$  is as in (7),  $\xi$  is any zero of the Airy function of order 3, and*

$$(14) \quad f(t) = -2(t/3)^{3/2} + \xi(3t)^{1/6}, \quad t \geq 0.$$

Then

$$v(t, f(t)) \rightarrow 0,$$

as  $t \rightarrow \infty$ .

*Proof.* For brevity let us just consider the term within the brackets in (7), i.e.,

$$\exp\left\{i\lambda y - \frac{1}{2}\lambda^2 t - \frac{\lambda^4}{4}\right\}.$$

We add and subtract variable  $-\alpha^2 t/2$ ,

$$\exp \left\{ \lambda(iy + \alpha t) - \frac{1}{2}(\lambda + \alpha)^2 t - \frac{\lambda^4}{4} \right\} e^{\frac{1}{2}\alpha^2 t}.$$

Set  $u = \lambda + \alpha$  and rearrange terms to obtain

$$\begin{aligned} & \exp \left\{ u(iy + \alpha t) - \frac{1}{2}u^2 t - \frac{(u - \alpha)^4}{4} \right\} e^{-\frac{1}{2}\alpha^2 t - i\alpha y} \\ &= \exp \left\{ u(iy + \alpha t + \alpha^3) - \frac{1}{2}u^2(t + 3\alpha^2) + \alpha u^3 - \frac{u^4}{4} \right\} \\ & \quad e^{-\frac{\alpha^4}{4} - \frac{1}{2}\alpha^2 t - i\alpha y}. \end{aligned}$$

To get rid of the *heat* (or quadratic) term note that

$$(15) \quad t + 3\alpha^2 = 0, \quad \text{gives} \quad \alpha_{\pm} = \pm i\sqrt{\frac{t}{3}}.$$

That is,

$$\exp \left\{ ui \left[ y \pm \frac{2}{3^{3/2}} t^{3/2} \right] \pm \frac{i}{3^{1/2}} t^{1/2} u^3 - \frac{u^4}{4} \right\} e^{-\frac{\alpha^4}{4} - \frac{1}{2}\alpha^2 t - i\alpha y}.$$

Next, if we choose  $\alpha_+$ , as in (15),

$$\exp \left\{ ui \left[ y + \frac{2}{3^{3/2}} t^{3/2} \right] + \frac{i}{3^{1/2}} t^{1/2} u^3 - \frac{u^4}{4} \right\} e^{\frac{5}{36}t^2 + \sqrt{\frac{t}{3}}y},$$

and thus, from (7),

$$v(t, y) = e^{\frac{5}{36}t^2 + \sqrt{\frac{t}{3}}y} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ui \left[ y + \frac{2}{3^{3/2}} t^{3/2} \right] + i\sqrt{3t} \frac{u^3}{3} - \frac{u^4}{4}} du.$$

Now, let  $z = u(3t)^{1/6}$ ,  $u = z(3t)^{-1/6}$ , and  $(3t)^{-1/6} dz = du$ , which yields

$$(3t)^{-1/6} e^{\frac{5}{36}t^2 + \sqrt{\frac{t}{3}}y} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iz \frac{\left[ y + \frac{2}{3^{3/2}} t^{3/2} \right]}{(3t)^{1/6}} + i\frac{z^3}{3} - \frac{z^4}{4(3t)^{2/3}}} dz,$$

or equivalently

$$(3t)^{1/6} e^{-\frac{5}{36}t^2 - \sqrt{\frac{t}{3}}y} v(t, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iz \frac{\left[ y + \frac{2}{3^{3/2}} t^{3/2} \right]}{(3t)^{1/6}} + i\frac{z^3}{3} - \frac{z^4}{4(3t)^{2/3}}} dz.$$

Letting  $y = -2(t/3)^{3/2} + \xi(3t)^{1/6}$

$$(3t)^{1/6} e^{\frac{t^2}{12} - \xi \frac{t^{2/3}}{3^{1/3}}} v \left( t, -2 \left[ \frac{t}{3} \right]^{3/2} + \xi(3t)^{1/6} \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iz\xi + i\frac{z^3}{3} - \frac{z^4}{4(3t)^{2/3}}} dz.$$

Hence for arbitrary  $\xi$

$$\lim_{t \rightarrow \infty} \left\{ (3t)^{1/6} e^{\frac{t^2}{12} - \xi \frac{t^{2/3}}{3^{1/3}}} v \left( t, -2 \left[ \frac{t}{3} \right]^{3/2} + \xi(3t)^{1/6} \right) \right\} = \text{Ai}(\xi).$$

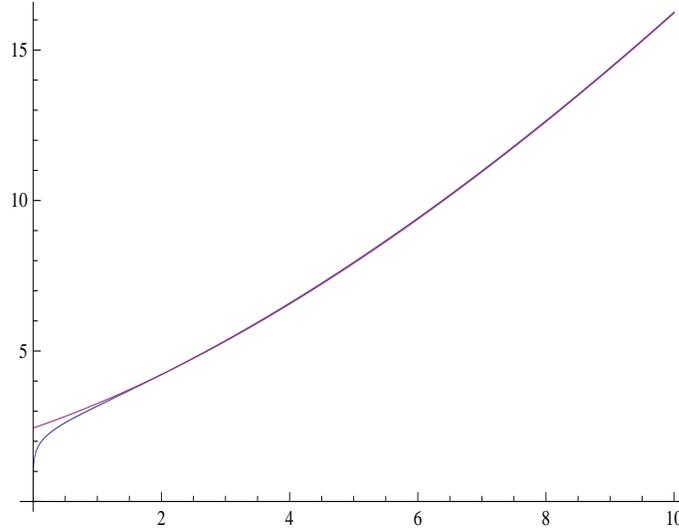


FIGURE 5. The red line is the real boundary  $f$ , the blue is as in (14), with  $\xi = -2.33811$ . The graph was plotted with Mathematica

In particular if  $\xi$  is a zero of the Airy function then

$$\begin{aligned} \lim_{t \rightarrow \infty} \left\{ (3t)^{1/6} e^{\frac{t^2}{12} - \xi \frac{t^{2/3}}{3^{1/3}}} v \left( t, -2 \left[ \frac{t}{3} \right]^{3/2} + \xi (3t)^{1/6} \right) \right\} &= \text{Ai}(\xi) \\ &= 0. \end{aligned}$$

□

**Numerical Example 6.2.** For instance, if  $\xi = -2.33811$  we have that  $f(t) = 2(t/3)^{3/2} + 2.33811(3t)^{1/6}$ . See Figure 5.

## 7 Possible applications and work in progress

Due to the stochastic and periodic nature of several economic variables, as for instance Mexico's general CPI or the Fruit and Vegetable annual inflation and assuming  $W$  is a random walk, these processes might be modelled as

$$(16) \quad X_t = \sum_{j=1}^n \beta_j \sin(\phi_j + 2\pi v_j t) + W_t, \quad t = 1, 2, \dots,$$

where the  $\beta_j$  and  $\phi_j$  represent respectively the amplitude and phase at a time given frequency  $v_j$ . In turn a continuous time approximation of (16) can be expressed in terms of the solution of an SDE of the form since

$$(17) \quad \begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \\ &\sim \\ \Delta X_j &= \mu(j, X_j)\Delta j + \sigma(j, X_j)\Delta W_j, \quad j = 0, 1, \dots, \end{aligned}$$

where functions  $\mu$  and  $\sigma$  are respectively:

$$\mu(t, X_t) = \sum_{j=1}^n 2\pi v_j \beta_j \cos(\phi_j + 2\pi v_j t), \quad \text{and} \quad \sigma(t, X_t) = 1,$$

and  $W$  is a Wiener process. A reasonable set of questions that could be asked could be for instance:

What is the probability that Fruit and Vegetable annual  
inflation will reach a certain threshold before the end of 2016?

What is probability that the general CPI will remain  
within a certain interval until the end of 2017?

As it turns out, to answer the previous questions it is necessary to understand the moving boundary problem of heat equation addressed in this work. More specific examples is still work in progress.

## 8 Concluding remarks

In this work we find the zeros of the Pearcey function, in terms of the solution of a Rayleigh-type equation. This goal is achieved by exploiting, on the one hand, the differential equation of an Airy function of order 4 and on the other by using the fact that the Pearcey function is a solution of the heat equation. As a by-product we develop a methodology, using straightforward techniques, to solve the moving boundary problem of the heat equation in the case in which the convolving function is a generalized Airy function. We expect that the techniques described within can be used in the construction of densities of the first hitting time problem of Brownian motion. The scope and applicability to the latter problem is still work in progress.

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