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Countercyclical versus Procyclical Taylor Principles*

Jean-Bernard Chatelain[†] Kirsten Ralf[‡]

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Abstract

Assuming inflation is a forward variable in Taylor (1999) model, this paper finds opposite policy rule recommendations with countercyclical policy rule parameters (Taylor principle: inflation rule larger than one and bounded upwards) in the case of optimal policy under commitment versus pro-cyclical policy rule parameters (inflation rule parameter below zero) in the case of discretionary policy. For the observed high inertia of the Fed with tiny variations of the nominal policy rate within the range $[0\%, 4\%]$ during the great moderation, the cost of time-inconsistency is negligible for optimal policy without commitment. In this case, time-inconsistency cannot be the ultimate argument to reject counter-cyclical Taylor principle.

JEL classification numbers: C61, C62, E43, E44, E47, E52, E58.

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1 Introduction

What is the best monetary policy? This paper revisits the classic comparison between Ramsey optimal policy under commitment (Simaan and

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Cruz (1973), Kydland and Prescott (1980), Miller and Salmon (1985), Miller (1985), Backus and Driffill (1986), Ljungqvist and Sargent (2012, chapter 19)) versus time-consistent "discretionary" policy (Cohen and Michel (1988), Oudiz and Sachs (1985), Backus and Driffill (1986)), see a recent interesting comparison in Leith and Wren-Lewis (2013) among others. We seek comparable observationally and mathematically equivalent representations of the Taylor rules for each of the two policies. The focus is the range of values of the inflation Taylor rule parameter obtained by each of these two policies.

Taylor's (1999) model is the simplest possible dynamic model of monetary policy with an interest rule describing the Federal Reserve's account of its actions. The Fed's policy interest rate has a negative effect on current output gap. Current output gap increases next period's expected inflation. Cochrane (2011 and 2016) refers to Taylor's (1999) as the benchmark model of old Keynesian countercyclical monetary policy. Inflation and output are predetermined variables, but Taylor (1999) mentions that "*these equations summarize more complex forward-looking models*" (p.662).

We turn Taylor's (1999) model into a forward-looking model, where expected inflation is a forward looking variable. In the early eighties, inflation in the U.S. fell like a stone during Volcker's mandate. It is difficult to explain the magnitude of this fall by old-Keynesian adaptive expectations where expectations depends only on the past values of inflation. Defining inflation as a forward looking variable, we find the feasible sets of inflation rule parameter (Taylor principles) and expectation-driven jumps of inflation for credible optimal policy under commitment, for discretionary policy and for simple rules.

We find opposite Taylor principles with optimal policy under commitment (with a countercyclical inflation rule parameter) as compared to discretionary policy (with a pro-cyclical inflation rule parameter). This result *matters a lot* for policy maker's advice (Miles (2015)). *Procyclical* interest rule parameters are the opposite of the explicitly *countercyclical* stabilizing language in the Federal Reserve's account of its actions. In simple rule equilibrium with expected inflation as a forward variable, "*higher inflation leads the Fed to set interest rates in a way that produces even higher future inflation. For only one value of inflation today will inflation fail to explode, or, more generally leave a local region. Ruling out non-local equilibria, new-Keynesian modelers conclude that inflation today must jump to the unique value that leads to a locally bounded equilibrium path*" (Cochrane (2011), p.566). So far, Bank of England officials do not state to the non-academic public that they set interest rates based on pro-cyclical positive feedback except on a unique path chosen by the public and that this mechanism is robust and successful to tame inflation (Miles (2015)). This paper adds discretionary policy as an-

other theory of *pro-cyclical positive feedback* inflation Taylor rule parameter, besides simple rule equilibrium with expected inflation as a forward variable.

The mechanism of the *Fed's lack of credibility* for determining the price level with discretionary policy seems to differ from the simple rule mechanisms or narratives reviewed by Cochrane (2011). The Fed's lack of credibility implies that the private sector's agents protect themselves against Fed's time inconsistency by *forcing* the Fed to select a *pro-cyclical* inflation rule parameter. Hence, the Fed is *unable* to commit to an optimal sequence of future interest rates determined by a *countercyclical* inflation rule parameter that it will renege in the future. Because *all* the agents of the private sector acknowledge the *forever* failure of the Fed's credibility to commit to a announced sequence of policy interest rates in the future, the private sector and the Fed decide, in a time consistent Nash equilibrium, to *restrict* optimal policy making to a number of variables equal to the number of predetermined variables. A procyclical inflation rule parameter constrains the inflation path to be a recursive function of a number of linearly independent (non-multicollinear) stationary macroeconomic variables equal to the number of private sector's predetermined variables.

Cochrane (2011) mentions two theories for a *countercyclical negative feedback* inflation Taylor rule parameter: the fiscal theory of the price level (Leeper (1991), Leith and Leeper (2016)) and the old Keynesian theory which assume that expected inflation and prices are predetermined variables instead of being forward variables (Taylor (1999)). This paper adds a third theory: Ramsey optimal policy under commitment (e.g. Kydland and Prescott (1980)). With this theory, a *mathematical* result is that the inflation rule parameter is *identical* to the one found by optimal policy *as if* all private sector's forward variables were assumed to be predetermined.

The mechanism of the *Fed's credibility* for determining the price level with optimal policy under commitment seems to differ from simple rule mechanisms and narratives reviewed by Cochrane (2011). The private sector's believes in the policy maker's commitment not to renege the announced plan of future values of the policy interest rate. For a monetary policy transmission mechanism described by a given model of the private sector, in order to fasten the convergence of inflation to its long run target (maximizing welfare), *counter-cyclical* rule parameters allow the optimal inflation path to be a recursive function of a number of linearly independent (non-multicollinear) stationary macroeconomic variables equal to the number of *all* linearly independent predetermined *and forward* variables describing the private sector's behaviour in the model. Fed's negative feedback rule increased the number of linearly independent stationary variables in the economy.

The Fed selects an optimal initial anchor of inflation on predetermined

variables maximizing welfare. The policy interest rate at the date of the commitment is predetermined. There is no sunspot even though the optimal path is surrounded by an infinity of out-of-equilibrium inflation paths converging to the long-run inflation target.

By contrast, in simple rule equilibrium with forward variables, the policy maker's welfare criterion, which is instrumental in selecting the optimal anchor of inflation, is assumed away. The policy interest rate is a forward variable. This generates sunspots and the indeterminacy of initial inflation and of the initial price level.

The observed high inertia of the Fed with tiny variations of the nominal policy rate within the range [0%,4%] during the great moderation may suggest the Fed's preference for a high relative cost of changing the policy rate during this period. In this "minimum energy optimal control" case, our paper demonstrates that the cost of time-inconsistency is negligible (or of second order of magnitude) for optimal monetary policy. In this case, time-consistency is *no longer* the ultimate criterion for advising discretionary policy with a pro-cyclical inflation rule parameter instead of optimal policy under commitment with a countercyclical inflation rule parameter.

A commitment to a low volatility of policy interest rate with countercyclical negative feedback rule parameters is a policy recommendation which enhances the credibility of the Fed's time-consistent behavior for the private sector. This recommendation seems more in line with policy makers concerns (Miles (2015)) than pro-cyclical positive feedback rule parameters as a private sector's defense against the Fed's time-inconsistent behavior.

Section 2 presents equilibria hypothesis using Taylor's model. Section 3 finds solutions of these equilibria. Section 4 compares these equilibria with respect to Taylor principles and discusses the condition for negligible time-inconsistency issues. Section 5 proposes some extensions. The last section concludes.

2 Policymaker Equilibria with Taylor's (1999) model

The Taylor (1999) model assumes that the current output gap x_t depends negatively (parameter $-\delta$) on the current nominal policy rate i_t minus current inflation π_t and an additive identically and independently distributed normal component $\varepsilon_{x,t}$ of a random productivity shock $z_{x,t}$:

$$x_t = -\delta (i_t - \pi_t) + z_{x,t} \text{ where } \delta > 0$$

The productivity shock $z_{x,t}$, also called forcing variable, includes an autoregressive component (auto-correlation ρ):

$$z_{x,t} = \rho z_{x,t-1} + \varepsilon_{x,t} \text{ where } 0 < \rho < 1 \text{ and } \varepsilon_{x,t} \text{ i.i.d. normal } N(0, \sigma_x^2)$$

Future inflation π_{t+1} increases with current inflation and current output gap plus a random cost push shock $\varepsilon_{\pi,t}$, with possibly a non-zero covariance with the productivity shock:

$$\pi_{t+1} = \pi_t + \kappa x_t + \varepsilon_{\pi,t} \text{ where } \kappa > 0 \text{ and } \varepsilon_{\pi,t} \text{ i.i.d. normal } N(0, \sigma_\pi^2)$$

Eliminating output gap reduces the model to a two-equations system:

$$\pi_{t+1} = (1 + \sigma) \pi_t - \sigma i_t + z_t \text{ where } 0 < \sigma = -\kappa \delta \quad (1)$$

$$z_t = \rho z_{t-1} + \varepsilon_t \text{ where } 0 < \rho < 1 \text{ and } \varepsilon_t \text{ i.i.d. normal } N(0, \sigma^2 (\varepsilon_{x,t} + \varepsilon_{\pi,t})) \quad (2)$$

We change Taylor (1999) model assuming inflation is a forward variable with unknown initial condition in order to model rational expectations. As a consequence, we deal with jumps of expected inflation and the related issue of time-inconsistency of the policy-maker. The forcing variable z_t is assumed to be predetermined with given initial value z_0 and bounded with an autoregressive parameter strictly between zero and one (initial and final boundary conditions):

$$z_0 \text{ predetermined} \quad (3)$$

$$\lim_{t \rightarrow +\infty} \beta^t E_t(z_t) = 0. \quad (4)$$

The Fed determines the policy rate i_t and the path of inflation π_t minimizing a discounted quadratic loss function L_0^* with a discount factor β and relative weight $Q_{\pi z} \geq 0$ on the covariance of inflation with the forcing variable and a strictly positive relative cost of changing the policy rate $R > 0$ subject to the private sector's model (with an optimal value function $v(\pi_0, z_0)$):

$$v(\pi_0, z_0) = \max_{\{i_t, \pi_t\}} - \frac{1}{2} E_t \sum_{t=0}^{+\infty} \beta^t [\pi_t^2 + 2Q_{\pi z} \pi_t z_{\pi,t} + R i_t^2] \quad (5)$$

We compare dynamics, policy rules and Taylor principles of these equilibria ordered by their degree of optimality:

(1) Optimal policy under commitment (Simaan and Cruz (1973), Kydland and Prescott (1980), Miller and Salmon (1985), Miller (1985), Backus and Driffill (1986), Ljungqvist and Sargent (2012, chapter 19)): The Fed minimizes the loss function (5) subject to private sector's law of motion (1, 2) and boundary conditions (3,4) and to two initial and final transversality conditions for the forward variable inflation (6, 7). The Fed chooses the optimal initial and final value of inflation minimizing the optimal value of her loss function at the initial and the final date. The marginal value of Fed's optimal loss function (equal to the Lagrange multiplier of inflation $\mu_{\pi,t=0}$) is set to zero at the initial date and for its infinite horizon limit. Hence, the Lagrange multiplier of inflation is a second predetermined variable besides the forcing variable z_t . This implies that expected inflation is bounded:

$$\frac{\partial v(\pi_0, z_0)}{\partial \pi_0} = 0 = \mu_0 \text{ predetermined} \Leftrightarrow \pi_0 = \pi_0^* \quad (6)$$

$$\lim_{t \rightarrow +\infty} \frac{\partial v(\pi_t, z_t)}{\partial \pi_t} = 0 = \lim_{t \rightarrow +\infty} \beta^t \mu_t \Leftrightarrow \lim_{t \rightarrow +\infty} \pi_t = \lim_{t \rightarrow +\infty} \pi_t^* \quad (7)$$

For example, Bryson and Ho ((1975), p.55) explain these transversality conditions. *"If π_t is not prescribed at $t = 0$, it does not follow that $\delta\pi_0 = 0$. In fact, there will be an optimum value for π_0 and it will be such that $\delta v = 0$ for arbitrary small variations of π_0 around this value. For this to be the case, we choose $\frac{\partial v}{\partial \pi_0} = \mu_0 = 0$ (1) which simply says that small changes of the optimal initial value of the forward variables π_0 on the loss function is zero. We have simply traded one boundary condition: π_0 given, for another, (1). Boundary conditions such as (1) are sometimes called "natural boundary conditions" or transversality conditions associated with the extremum problem."* When using the Lagrange multiplier solution, the policy maker's Lagrange multipliers of private sector's forward inflation is *predetermined at the value zero*: $\mu_0 = 0$. Hence, the policy maker's Hamiltonian system includes a number of predetermined variables which is equal to the number of private sector's predetermined variables and the number of the policy maker's Lagrange multiplier of each of the private sector's forward variables. The number of the policy maker's predetermined variables is equal to the sum of the number of private sector's predetermined *and* forward variables.

(2) Time-consistent "discretionary" policy (Oudiz and Sachs (1985)): The Fed minimizes the loss function (5) subject to the private sector's law of motion (1, 2) and boundary conditions of the forcing variable (3, 4) and subject to two additional constraints: a private sector's inflation rule (8) and a policy rule (9). First, it is assumed that the private sector's inflation rule is a linear function of the predetermined variable with an optimal bounded

parameter N_D to find:

$$\pi_t = N_D z_t \quad (8)$$

Equation (8) implies that transversality conditions on expected inflation (6 and 7) are no longer useful. As seen in section 3, the initial transversality condition on the private sector's forward variable, inflation, (equation 6) sets an anchor of inflation proportional to the forcing variable only at the initial date. It has been eliminated by the private sector's inflation rule (equation 8), which sets an anchor of inflation proportional to the forcing variable for all future periods and not only for the initial date.

Second, the policy maker's interest rule is a linear function of the predetermined forcing variable with an optimal bounded forcing variable Taylor rule parameter $F_{z,D}$ to find, or, alternatively, an optimal bounded inflation Taylor rule parameter $F_{\pi,D}$ to find:

$$i_t = F_{z,D} z_t = F_{\pi,D} \pi_t \text{ with } F_{\pi,D} = F_{z,D} N_D^{-1} \quad (9)$$

Time-inconsistent discretionary policy without rules depending on a number of variables equal to the number of predetermined variables may exist.

(3) Simple rule: The Fed does not minimize the loss function (5). One only seeks determinacy sets of rule parameters $F_{z,S}$ and $F_{\pi,S}$ for the *ad hoc* rational expectations linear system including the private sector's law of motion (1, 2), boundary conditions of the forcing variable (3, 4), private sector inflation rule (8) and policy rule (10) instead of (9):

$$i_t = F_{z,S} z_t + F_{\pi,S} \pi_t \quad (10)$$

(4) Laissez-faire equilibrium: It is a corner solution of the simple rule equilibrium where the rule parameters are equal to zero: $F_{z,S} = F_{\pi,S} = 0$:

$$i_t = 0 \quad (11)$$

The following sections details the solutions of each of these equilibria.

3 Equilibrium Solutions

3.1 Optimal policy under commitment

The Fed chooses optimal policy while taking expected inflation law of motion as constraints. Her Lagrangian includes Lagrange multipliers $2\beta^{t+1}\mu_{t+1}$.

$$\mathcal{L} = - \sum_{t=0}^{+\infty} \beta^t \left[\begin{array}{c} \pi_t^2 + 2Q_{\pi z} \pi_t z_t + R i_t^2 \\ + 2\beta \mu_{t+1} [(1 + \sigma) \pi_t + z_t - \sigma i_t - \pi_{t+1}] \end{array} \right] \quad (12)$$

Because of the certainty equivalence principle for determining optimal policy in the linear quadratic regulator including additive normal random shocks (Simon (1956)), the expectations of random variables ε_t are set to zero and do not appear in the Lagrangian. In what follows, let us denote $\pi_{t+1} = E_t \pi_{t+1}$. The first order conditions are with respect to the Fed's target (forward inflation π_{t+1}) and with respect to the Fed's instrument (policy interest rate i_t):

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \beta (1 + \sigma) \mu_{t+1} = \mu_t - \pi_t - Q_{\pi z} z_t \quad (13)$$

$$\frac{\partial \mathcal{L}}{\partial i_t} = 0 \Rightarrow \beta (-\sigma) \mu_{t+1} = -R i_t. \quad (14)$$

Substitute the Fed's Lagrange multiplier of inflation by the Fed's interest rate in the Fed's Euler equation $\frac{\partial \mathcal{L}}{\partial \pi_t}$. The *Fed's interest rate Euler equation* links recursively the future value of Fed's interest rate to its current value, because of the Fed's relative costs of changing interest rate $R > 0$ in its loss function:

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \left(1 + \frac{1}{\sigma}\right) R i_t + \pi_t + Q_{\pi z} z_t = \frac{R}{\beta \sigma} i_{t-1} \quad (15)$$

The Hamiltonian system and its boundary conditions can be alternatively written by an Euler equation including the policy rate i_t or by an Euler equation including the Lagrange multiplier on inflation μ_t :

$$\left\{ \begin{array}{l} z_t = \rho z_{t-1} + \varepsilon_t \\ E_t \pi_{t+1} = (1 + \sigma) \pi_t - \sigma i_t + z_t \\ \left(1 + \frac{1}{\sigma}\right) R i_t + \pi_t + Q_{\pi z} z_t = \frac{R}{\beta \sigma} i_{t-1} \\ \mu_{t+1} = \frac{R}{\beta \sigma^2} i_t \\ 0 = \lim_{t \rightarrow +\infty} \beta^t \mu_{\pi, t} \\ \frac{\partial L^*}{\partial \pi_0} = 0 = \mu_0 \text{ and } z_0 \text{ given} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} z_t = \rho z_{t-1} + \varepsilon_t \\ E_t \pi_{t+1} = (1 + \sigma) \pi_t - \frac{\beta \sigma^2}{R} \mu_{t+1} + z_t \\ \beta (1 + \sigma) \mu_{t+1} = c - \pi_t - Q_{\pi z} z_t \\ i_t = \frac{\beta \sigma^2}{R} \mu_{t+1} \\ 0 = \lim_{t \rightarrow +\infty} \beta^t \mu_{\pi, t} \\ \frac{\partial L^*}{\partial \pi_0} = 0 = \mu_0 \text{ and } z_0 \text{ given} \end{array} \right. \quad (16)$$

In the appendix, it is shown that the Hamiltonian system when $z_t = 0$ including inflation and the Lagrange multiplier of inflation has the usual saddle point equilibrium property: one eigenvalue is stable and the other eigenvalue is unstable. Hence, the full system including inflation, the Lagrange multiplier of inflation (or the policy rate), and the stationary dynamics of the forcing variable $z_t \neq 0$ with stable eigenvalue ρ has two stable eigenvalue and one unstable eigenvalue. Using the infinite horizon transversality conditions,

the relevant solution is the one that stabilizes the state-costate vector for any initialization of inflation π_0 and of the exogenous shock z_0 in a stable subspace of dimension two within a space of dimension three (z_t, π_t, μ_t) or (z_t, π_t, i_t) of the Hamiltonian system (Anderson *et al.* (1996)). We seek a characterization of the Lagrange multiplier μ_t on inflation minimizing the loss function L_t^* at all dates t such that it is a linear function of the two other variables in order to remain in the stable invariant subspace of dimension two of the Hamiltonian system:

$$\mu_t = -P_\pi \pi_t - P_{\pi z} z_t = \frac{\partial v(\pi_t, z_t)}{\partial \pi_t}. \quad (17)$$

The parameter P_π of the optimal value v_0 is the solution of a Riccati equation. The parameter $P_{\pi z}$ of the optimal value v_0 is the solution of a Sylvester equation (Anderson *et al.* (1996)). They depend on the monetary policy transmission mechanism (σ, ρ, β) (the discount factor is set to one to simplify matters) and on the Fed's preferences $(R, Q_{\pi z})$. They do not depend on the variance of the shock nor on initial conditions of inflation and of the forcing variable. Assuming $\beta = 1$ for simplicity and without loss of generality in what follows, we obtain (see appendix):

$$P_\pi \left(\begin{matrix} \sigma, R \\ - \\ + \end{matrix} \right) = \frac{1}{2} + R \left(\frac{1}{2} + \frac{1}{\sigma} \right) - \sqrt{\left(\frac{1}{2} + R \left(\frac{1}{2} + \frac{1}{\sigma} \right) \right)^2 + \frac{R}{\sigma^2}} \quad (18)$$

$$P_{\pi z} \left(\begin{matrix} \sigma, R, \rho, Q_{\pi z} \\ - \\ + \end{matrix} \right) = \frac{Q_{\pi z} + (1 + \sigma - \sigma F_\pi) P_\pi}{1 - (1 + \sigma - \sigma F_\pi) \rho}. \quad (19)$$

Alternatively, we seek a characterization of the Fed's policy rate such that it is a linear function of the two other variables in order to remain in the stable invariant subspace of dimension two of the Hamiltonian system, with the optimal rule parameters $F_{\pi, C}$ and $F_{z, C}$ solution of the discounted augmented linear quadratic regulator. They depend on the monetary policy transmission mechanism (σ, ρ, β) and on the Fed's preferences $(R, Q_{\pi z})$:

$$i_t = F_{\pi, C} \pi_t + F_{z, C} z_t \quad (20)$$

$$F_{\pi, C} \left(\begin{matrix} \sigma, R \\ - \\ - \end{matrix} \right) = \frac{\sigma P_\pi + \sigma^2 P_\pi}{R + \sigma^2 P_\pi} = \frac{\frac{1}{\sigma} + 1}{1 + \frac{R}{\sigma^2 P_\pi}} > 1 \quad (21)$$

$$F_{z, C} \left(\begin{matrix} \sigma, R, \rho, Q_{\pi z} \\ - \\ - \end{matrix} \right) = \frac{\frac{1}{\sigma} + \frac{\rho P_{\pi z}}{\sigma P_\pi}}{1 + \frac{R}{\sigma^2 P_\pi}}. \quad (22)$$

The rule parameters $F_{\pi,C}$ and $F_{z,C}$ and the Lagrange multiplier (and optimal value function) parameters P_π and $P_{\pi z}$ *do not depend* on the variance covariance of normal additive shocks (Simon (1956) certainty equivalence). *They do not depend on the initial conditions of inflation π_0 and of the forcing variable z_0 .* Hence, these optimal parameters $F_{\pi,C}$, $F_{z,C}$, P_π , $P_{\pi z}$ *do not change* if the initial condition of inflation is predetermined at the value $\bar{\pi}_0$ (old Keynesian optimal policy) or if inflation is a forward variable to be optimally anchored at the value $\pi_0^* \neq \bar{\pi}_0$ (optimal policy under commitment). Hence, old Keynesian optimal policy rule parameters $F_{\pi,C}$, $F_{z,C}$, P_π , $P_{\pi z}$ are *identical* to the ones of optimal policy under commitment. Hence, Taylor principles defining sets of feasible values of the inflation rule parameter are the *same*, if inflation is predetermined or if inflation is forward-looking.

Optimal policy under commitment differs from old Keynesian optimal policy with respect to the initial value of inflation, which is no longer given. The optimal initial anchor (or jump) of inflation on the forcing variable is found minimizing the policy maker's loss function, with the Lagrange multiplier of inflation μ_0 predetermined to the value zero:

$$\mu_0 = P_\pi \pi_0 + P_{\pi z} z_0 = 0 \Rightarrow \pi_0^* = -\frac{P_{\pi z}}{P_\pi} z_0. \quad (23)$$

Alternatively, the value of the policy rate $i_{-1} = \frac{\beta \sigma^2}{R} \mu_0 = 0$ is *predetermined at zero at the date before the optimization*. This models a structural break of the Fed's credible commitment to a new sequence of future interest rates on date zero. There is no past promises to keep on the previously announced sequence of policy rate. The value of the policy rate i_0 at the date of the optimization is given by:

$$i_0 = F_{\pi,C} \left(-\frac{P_{\pi z}}{P_\pi} \right) z_0 + F_{z,C} z_0. \quad (24)$$

The Lagrange multiplier of inflation is *predetermined at zero* at the date $t = 0$ of optimization, the interest rate is *predetermined at zero* at the date $t = -1$ before the start of the Fed's credible commitment to a new sequence of interest rates. Both players, including the private sector, know that the Lagrange multiplier of inflation (or the policy interest rate) is *predetermined*, because it is the costate of a forward variable (inflation) for which an optimal initial anchor has to be decided using a transversality condition. The Fed is assumed not to optimize again using $t > 0$ initial transversality condition, which would break the announced sequence of interest rates at date $t = 0$. On future periods, the Lagrange multiplier of inflation is usually different

from zero:

$$\mu_t = P_\pi \pi_t + P_{\pi z} z_t \neq 0 \Rightarrow \pi_t \neq -\frac{P_{\pi z}}{P_\pi} z_t. \quad (25)$$

However, the particular case of a zero interest rate at date zero and all future dates $i_t = 0$ is obtained for Fed's maximal inertia preference ($R \rightarrow +\infty$), with $-\frac{P_{\pi z}}{P_\pi} = -\frac{F_{z,C}}{F_{\pi,C}} = N \neq 0$, where N is the value of the laissez-faire jump (see laissez-faire section and appendix). In this case, the optimal jump is identical to the laissez-faire jump, but both negative feedback policy rule parameters are different from zero: $F_{\pi,C} = -N \cdot F_{z,C} \neq 0$. The Lagrange multiplier of inflation is zero at all dates, so that maximal inertia monetary policy is always time-consistent.

$$\mu_t = \left(P_\pi \left(-\frac{P_{\pi z}}{P_\pi} \right) + P_{\pi z} \right) z_t = 0 \quad (26)$$

A representation of optimal policy under commitment dynamics is given by the private sector law of motion (1,2), the optimal policy rule which is the substitute of the Fed's Euler interest rate once the infinite horizon transversality condition (7) has been taken into account and the optimal initial anchor of inflation (23):

$$(S_\pi) \left\{ \begin{array}{l} z_t = \rho z_{t-1} + \varepsilon_t \\ E_t \pi_{t+1} = (1 + \sigma) \pi_t - \sigma i_t + z_t \\ i_t = F_{\pi,C} \pi_t + F_{z,C} z_t \\ \pi_0^* = -\frac{P_{\pi z}}{P_\pi} z_0 \text{ with } z_0 \text{ given} \\ \mu_t = -P_\pi \pi_t - P_{\pi z} z_t \\ \text{for all dates } t \geq 0. \end{array} \right. \Leftrightarrow (S_\mu) \left\{ \begin{array}{l} z_t = \rho z_{t-1} + \varepsilon_t \\ E_t(-P_\pi^{-1} \mu_{t+1} - P_\pi^{-1} P_{\pi z} z_{t+1}) = \\ (1 + \sigma)(-P_\pi^{-1} \mu_t - P_\pi^{-1} P_{\pi z} z_t) - \sigma i_t + z_t \\ i_t = -F_{\pi,C} P_\pi^{-1} \mu_t + (F_{z,C} - F_{\pi,C} P_\pi^{-1} P_{\pi z}) z_t \\ \text{with } F_\mu = -F_{\pi,C} P_\pi^{-1} \\ \text{with } F_{z,\mu} = F_{z,C} - F_{\pi,C} P_\pi^{-1} P_{\pi z} \\ \mu_0 = -P_\pi \pi_0 - P_{\pi z} z_0 = 0 \text{ with } z_0 \text{ given} \\ \pi_t = -P_\pi^{-1} \mu_t - P_\pi^{-1} P_{\pi z} z_t \\ \text{for all dates } t \geq 0. \end{array} \right. \quad (27)$$

Ljungqvist and Sargent (2012, chapter 19) begin with the system (S_π) and finally use the mathematically equivalent system (S_μ) for optimal policy. They found the system (S_μ) after linear substitution of inflation by its Lagrange multiplier and the forcing variable using the linear equation $\pi_t = -P_\pi^{-1} \mu_t - P_\pi^{-1} P_{\pi z} z_t$ in all the equations of the system (S_π) . The system of equations (S_π) with a policy rule function of private sectors of private sector's predetermined and forward variables (z_t, π_t) and the system of equations (S_μ) with a policy rule function of predetermined variables (z_t, μ_t) are *mathematically and observationally equivalent*. The initial transversality condition

$\mu_0 = -P_\pi \pi_0 - P_{\pi z} z_0 = 0$ with $P_\pi \neq 0$ in the system (S_μ) is *mathematically and observationally equivalent* to $\pi_0^* = -\frac{P_{\pi z}}{P_\pi} z_0$ in the system (S_π) .

For the Fed, an *implementable* policy rule is the one described in system (S_π) because Fed's interest rate depends on inflation π_t , for which official measures are available in national accounts. The rule in system (S_μ) is *not implementable* by the Fed, because Fed's interest rate depends on Fed's Lagrange multiplier of inflation, for which official measures are not available in national accounts. This rule has no clear interpretation because the interest rate i_t depends on the Lagrange multiplier of inflation μ_t , but the Lagrange multiplier μ_t is also a substitute of the interest rate i_t (cf. equations 16).

Because of the *mathematical equivalence* between both systems of equations, it is *impossible* to prove mathematically that the representation of the policy rule in the system (S_π) *cannot be implemented* whereas, *simultaneously*, the representation of the policy rule in the system (S_μ) *can be implemented*. Violating this mathematical equivalence amounts to assume that economic agents participating to this Stackelberg dynamic game endorse *mathematical illiteracy*.

If one intends to prove the existence of multiple initial conditions of forward variables (sunspots) with the representation of the optimal policy rule $i_t = F_{\pi,C} \pi_t + F_{z,C} z_t$ and simultaneously the uniqueness of initial conditions (no sunspots) with the representation of the optimal policy rule $i_t = F_\mu \mu_t + F_{z,\mu} z_t$, *one is forced to assume the following statement which is mathematically inconsistent*: "For optimal policy under commitment, the equation $\pi_0^* = -P_\pi^{-1} P_{\pi z} z_0$ should be *excluded* in system (S_π) and the *mathematically equivalent* equation $\mu_0 = -P_\pi \pi_0 - P_{\pi z} z_0 = 0$ with P_π invertible should be *included* in the system (S_μ) ".

Ljungqvist and Sargent (2012, chapter 19), following von zur Muehlen's suggestion, computed an observationally equivalent rule equation which includes a lagged dependent instrument. They use linear substitutions of the inflation Lagrange multiplier by a lagged interest rate and a lagged forcing variable within the system of equations (S_μ) in this alternative representation of the policy rule to be included in the system (S_μ) or (S_π) :

$$i_t = F_{i(-1),C} i_{t-1} + F_{z,i(-1),C} z_t + F_{z(-1),i(-1),C} z_{t-1}. \quad (28)$$

There are three rule parameters in this representation of the rule. By contrast, the mathematically and observationally equivalent rules in (S_μ) or (S_π) include only two rule parameters. This suggests an identification restriction such that the lagged interest rate rule parameter $F_{i(-1),C}$ is a function of two other rule parameters $F_{z,i(-1),C}$ and $F_{z(-1),i(-1),C}$. A rise in interest rate persistence due to an increase of the cost of changing the policy

rate (R) leads to a non-linear decline of the "long run" policy rule parameters $F_{\pi,C}$ and $F_{z,C}$ in (S_π) . Simulations detail this result in the following sections. For these reasons, we set aside this "short run" representation of the policy rule as a function of the lagged value of the interest rate and of the lagged value of the forcing variable.

The recursive dynamics after substitution of the optimal policy rule function of the initial conditions is given by:

$$\begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1 - \sigma F_{z,C}}{\rho - (1 + \sigma - \sigma F_{\pi,C})} \end{pmatrix} \rho^t z_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 + \sigma - \sigma F_{\pi,C})^t \underbrace{\left(-\frac{P_{\pi z}}{P_\pi} z_0 - \frac{1 - \sigma F_{z,C}}{\rho - (1 + \sigma - \sigma F_{\pi,C})} z_0 \right)}_{\neq 0 \text{ but tends to zero when } R \rightarrow +\infty \text{ or } R \rightarrow 0} \quad (29)$$

The optimal inflation rule parameter $F_{\pi,C} \left(\sigma, R \right)$ decreases with the monetary transmission parameter σ and with the Fed's relative weight R on the volatility of the policy rate. When varying the Fed's preference $0 < R < +\infty$ for a given monetary policy transmission parameter $\sigma > 0$, the inflation rule parameter $F_{\pi,C} \left(\sigma, R \right)$ varies within the following "Taylor principle" set $]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}[$: the inflation rule parameter is larger than one. The inflation growth factor λ_π is a linear decreasing function of the inflation rule parameter $F_{\pi,C}$. It varies between zero (for the relative cost of changing the interest rate tending to zero: $R \rightarrow 0$) and the inverse of the laissez-faire growth factor $1 + \sigma$ (for the relative cost of changing the interest rate tending to infinity: $R \rightarrow +\infty$):

$$\begin{aligned} 0 < R < +\infty \text{ and } \sigma > 0 &\Rightarrow 1 + \frac{1}{1 + \sigma} < F_{\pi,C} < 1 + \frac{1}{\sigma} \\ &\Rightarrow 0 < \lambda_\pi = 1 + \sigma - \sigma F_{\pi,C} < \frac{1}{1 + \sigma} < 1. \end{aligned}$$

There are two stable eigenvalues ρ and $\lambda_\pi = 1 + \sigma - \sigma F_{\pi,C}$. Expected inflation and the expected forcing variable (π_t, z_t) dynamics of optimal policy under commitment is a converging sink with an initial optimal anchor (jump) of inflation. The number of stable roots (two) for the saddle point equilibrium of this Hamiltonian system is equal to the number (two) of predetermined variables: (z_t, μ_t) or (z_t, i_t) in the three dimensions space (π_t, z_t, μ_t) or (π_t, z_t, i_t) .

Both players, including the private sector, know that the Lagrange multiplier of inflation (or the policy interest rate) is *predetermined*, because it is the costate of a forward variable (inflation). The Hamiltonian system with boundary conditions (S_π) including the representation of the feedback rule $i_t = F_{\pi,C}\pi_t + F_{z,C}z_t$ satisfies Blanchard and Kahn (1980) determinacy condition. The number of stable roots (two) for the saddle point equilibrium of this Hamiltonian system which is equal to the number (two) of predetermined variables: (z_t, μ_t) or (z_t, i_t) in the three dimensions space (π_t, z_t, μ_t) or (π_t, z_t, i_t) .

3.2 Time Consistent Discretionary policy

Oudiz and Sachs's (1985) time consistent discretionary policy model assumes that the private sector does not believe in the policy maker's commitment. Both the private sector and the policy maker know that their best response rules only depend on predetermined variable at all periods with time-invariant rule parameters N_D and $F_{z,D}$ to be chosen:

$$\pi_t = N_D z_t \quad (30)$$

$$i_t = F_{z,D} z_t. \quad (31)$$

Substituting the private sector's inflation rule (8) and policy rule (9) in the inflation law of motion (1) and comparing it with the forcing variable law of motion (2) leads to the following relation between N_D and $F_{z,D}$:

$$\begin{aligned} E_t z_{t+1} &= \frac{1}{N_D} (z_t + (1 + \sigma)N_D z_t - \sigma F_{z,D} z_t) = \left(1 + \sigma + \frac{1 - \sigma F_{z,D}}{N_D}\right) z_t = \rho z_t \\ \Rightarrow N_D &= N - N\sigma F_{z,D} \text{ with } N = \frac{1}{\rho - (1 + \sigma)}. \end{aligned} \quad (32)$$

Substituting the private sector's inflation rule (8) and policy rule (9) in the loss function (3) leads to the optimal program:

$$\max_{\{F_{z,D}, N_D\}} -\frac{1}{2} (N_{z,D}^2 + 2Q_{\pi z} N_D + R F_{z,D}^2) \frac{1}{1 - \beta \rho^2} z_0^2 \quad (33)$$

$$N_D = N - N\sigma F_{z,D} \text{ with } N = \frac{-1}{(1 + \sigma) - \rho} \quad (34)$$

with solutions:

$$0 < R < +\infty \Rightarrow \quad (35)$$

$$0 < F_{z,D} = \frac{1}{\sigma} \frac{1 + \frac{Q_{\pi z}}{N}}{1 + \frac{R}{\sigma^2 N^2}} < \frac{1 + \frac{Q_{\pi z}}{N}}{\sigma} \quad (36)$$

$$-Q_{\pi z} < N_D = N - N \frac{1 + \frac{Q_{\pi z}}{N}}{1 + \frac{R}{\sigma^2 N^2}} < N. \quad (37)$$

Discretionary policy for all dates $t > 0$ has the two following representations related to two mathematically and observationally equivalent systems of equations:

$$(S_{D,z}) \left\{ \begin{array}{l} z_t = \rho z_{t-1} + \varepsilon_t \\ \pi_t = N_D z_t \\ N_D = \frac{1 - \sigma F_{z,D}}{\rho - (1 + \sigma)} \\ i_t = F_{z,D} z_t \\ z_0 \text{ given} \end{array} \right. \iff (S_{D,\pi}) \left\{ \begin{array}{l} \pi_t = \rho \pi_{t-1} + N_D \varepsilon_t \\ z_t = \frac{1}{N_D} \pi_t \\ N_D = \frac{-1}{(1 + \sigma - \sigma F_{\pi,D}) - \rho} \\ i_t = F_{\pi,D} \pi_t \text{ with } F_{\pi,D} = \frac{F_{z,D}}{N_D} \\ z_0 \text{ given} \end{array} \right. \quad (38)$$

Discretionary policy inflation π_t is exactly negatively correlated with the auto-regressive forcing variable z_t ($N_D < 0$), with a coefficient of determination equal to one. The policy rate i_t is exactly negatively correlated with inflation π_t ($F_{\pi,D} < 0$) or exactly positively correlated with the auto-regressive forcing variable z_t ($F_{z,D} > 0$) with coefficients of determination equal to one.

For the Fed, an *implementable* policy rule is the one described in system $(S_{D,\pi})$ because Fed's interest rate i_t depends on inflation π_t , for which official measures are available in national accounts. The rule in system $(S_{D,z})$ is *not implementable* by the Fed, because Fed's interest rate depends on an auto-regressive cost-push shock z_t for which official measures are not available in national accounts, as this residual variable z_t changes with the specification of the model of the private sector.

Because of the *mathematical equivalence* between both systems of equations, it is *impossible* to prove mathematically that the representation of the policy rule in the system $(S_{D,\pi})$ *cannot be implemented* whereas, *simultaneously*, the representation of the policy rule in the system $(S_{D,z})$ *can be implemented*. Violating this mathematical equivalence amounts to assume that economic agents participating to this Nash dynamic game endorse *mathematical illiteracy*.

The discretionary inflation rule parameter $F_{\pi,D}$ varies in the following "Taylor principle set" which is strictly negative:

$$0 < R < +\infty \Rightarrow \frac{1}{\sigma N} \left(-\frac{N}{Q_{\pi z}} - 1 \right) < F_{\pi,D} = \frac{1}{N} \frac{F_{z,D}}{1 - \sigma F_{z,D}} < 0. \quad (39)$$

The discretionary Taylor principle is such that the inflation parameter of discretionary policy is strictly negative when varying the Fed's preferences (R) for the relative cost of changing the policy rate:

$$\left\{ \begin{array}{l} F_{\pi,D} \in]-\infty, 0[\\ \sigma > 0 \\ 0 < N_D < N \\ 0 < R < +\infty \end{array} \right\} \iff \left\{ \begin{array}{l} \lambda_{\pi,D} = 1 + \sigma - \sigma F_{\pi,D} \in]1 + \sigma, +\infty[\\ \sigma > 0 \\ 0 < N_D < N \\ 0 < R < +\infty \end{array} \right\} \quad (40)$$

In the time-consistent equilibrium, the eigenvalues are ordered as follows:

$$0 < \rho < 1 < 1 + \sigma \leq \lambda_{\pi,D} = 1 + \sigma - \sigma F_{\pi,D}. \quad (41)$$

In the time-consistent equilibrium, a lower cost R of changing the policy interest rate implies *an increase of positive feedback* with a larger inflation rule parameter $F_{\pi,D}$. This increases the unstable eigenvalue $\lambda_{\pi,D}$ and this increases the interest rate i_t . This decreases inflation through its negative effect *on the slope* $\frac{-1}{\lambda_{\pi,D} - \rho}$ *of the eigenvectors of the stable eigenvalue* ρ . This slope is a negative function of the unstable eigenvalue $\lambda_{\pi,D}$. *It is not simple to explain these linear algebra mechanics using economic intuition.* Because of the negative correlation between the interest rate and inflation predicted by the model, naive researchers may be tempted to cut a long story short and describe this operation of monetary policy as negative feedback. Such reasoning may face a strong charge because it involves a serious misunderstanding of the causal logic of the system $(S_{D,\pi})$. The inflation rule parameter $F_{\pi,D}$ tends to minus infinity, inflation π_t decreases to zero:

$$\begin{aligned} \begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} &= \begin{pmatrix} 1 \\ \frac{-1}{(1 + \sigma - \sigma F_{\pi,D}) - \rho} \end{pmatrix} \rho^t z_0 \\ &+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 + \sigma - \sigma F_{\pi,D})^t \underbrace{\left(\frac{1}{(1 + \sigma - \sigma F_{\pi,D}) - \rho} z_0 + \pi_0 \right)}_{=0} \end{aligned}$$

The counter-cyclical positive feedback effect of the inflation rule parameter $F_{\pi,D}$ leads to an explosive eigenvalue $\lambda_{\pi,D} = 1 + \sigma - \sigma F_{\pi,D}$ without any effect on equilibrium inflation. Because the private sector's agents do not trust policy maker's commitment to a sequence of future interest rates, they force the initial inflation jump $\pi_0 = \frac{-1}{(1 + \sigma - \sigma F_{\pi,D}) - \rho} z_0$ which is surrounded by out-of-equilibrium explosive inflationary or deflationary paths.

3.3 Simple rule

Simple rule with forward variables models assume that the private sector does not believe in the policy maker's commitment. The private sector policy rule depends only on predetermined variables at all periods as in Oudiz and Sachs (1985) discretionary policy (equation (8)):

$$\pi_t = N_S z_t. \quad (42)$$

The policy maker is myopic: she does not notice that her commitment is not credible for the private sector. Her simple policy rule depends on forward and predetermined variables:

$$i_t = F_{z,S} z_t + F_{\pi,S} \pi_t. \quad (43)$$

Combining private sector's rule with policy maker's simple rule implies that the policy maker's policy rule is observationally equivalent to representations of the policy rule which depends only on a number of variables equal to the number of predetermined variables (equal to one in this model). The policy maker is not credible to commit to a sequence of future interest rates modifying recursively the sequence of the expectations of inflation. The policy maker is not credible to expand the number of linearly independent stationary variables in the economy using a negative feedback policy rule:

$$\left\{ \begin{array}{l} \pi_t = N_S z_t \\ i_t = F_{z,S} z_t + F_{\pi,S} \pi_t \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} \pi_t = N_S z_t \\ i_t = (F_{z,S} + F_{\pi,S} N_S) z_t \end{array} \right. \quad (44)$$

$$\Longleftrightarrow \left\{ \begin{array}{l} z_t = N_S^{-1} \pi_t \\ i_t = \left(\frac{F_{z,S}}{N_S} + F_{\pi,S} \right) \pi_t. \end{array} \right. \quad (45)$$

A non-optimal simple rule equilibrium consists of equations (1,2,4,5,8,10). The combination of the private sector's inflation rule as a linear function of the forcing variable (8) and of a policy interest rule function of two variables implies a linear identification restriction on the interest policy rule, which can be written as a function of only one variable. For the Taylor principle, the usual practice is to consider the policy representation as a function of inflation (identification restriction $F_{z,S} = 0$):

$$i_t = F_{\pi,S} \pi_t. \quad (46)$$

There is one predetermined variable z_t and two forward variables: inflation π_t and the policy interest rate i_t . The policy rate is no longer a predetermined variable, as it is in optimal policy, because the initial transversality

condition on inflation (equation 6) has been eliminated by the private sector inflation rule (equation 8).

There is one stable eigenvalue (ρ) related to the predetermined exogenous forcing variable z_t . Then, the second eigenvalue should be unstable $0 < \rho < 1 < |1 + \sigma - \sigma F_\pi|$ according to Blanchard and Kahn's (1980) determinacy condition for rational expectations *ad hoc* linear system of equations. This implies the following Taylor principle set:

$$|\lambda_{\pi,S}| = |1 + \sigma - \sigma F_{\pi,S}| > 1 \Leftrightarrow F_{\pi,S} \in] -\infty, 1[\cup] 1 + \frac{2}{\sigma}, +\infty[. \quad (47)$$

Depending on the identifying restrictions on policy rule parameters, we have the following representation $S_{S,z}$ or $S_{S,\pi}$ of the rational expectations simple rule unique solution with the policy rule and the vector auto-regressive of order one (VAR(1)) function of the forcing variable z_t or of inflation π_t :

$$(S_{S,z}) \left\{ \begin{array}{l} z_t = \rho z_{t-1} + \Sigma_\varepsilon \varepsilon_t \\ \pi_t = N_S z_t \\ N_S = \frac{1 - \sigma F_{z,S}}{\rho - (1 + \sigma)} \\ i_t = F_{z,S} z_t \\ z_0 \text{ given} \end{array} \right. \Leftrightarrow (S_{S,\pi}) \left\{ \begin{array}{l} \pi_t = \rho \pi_{t-1} + N_S \Sigma_\varepsilon \varepsilon_t \\ z_t = N_S^{-1} \pi_t \\ N_S = \frac{-1}{(1 + \sigma - \sigma F_{\pi,S}) - \rho} \\ i_t = F_{\pi,S} \pi_t \\ z_0 \text{ given.} \end{array} \right. \quad (48)$$

3.4 Laissez-faire equilibrium

Laissez-faire equilibrium is a simple rule equilibrium assuming the simple rule inflation parameter $F_{\pi,S}$ is zero:

$$i_t = F_{\pi,S} \pi_t = 0 \text{ with } F_{\pi,S} = 0. \quad (49)$$

The optimal laissez-faire rational expectations equilibrium path can be written for all dates $t > 0$ with the two following representations related to two mathematically and observationally equivalent systems of equations:

$$\left\{ \begin{array}{l} z_{t+1} = \rho z_t + \varepsilon_{t+1} \\ \pi_t = N z_t \\ N = \frac{-1}{(1 + \sigma) - \rho} < 0 \\ i_t = 0 \\ z_0 \text{ given} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \pi_{t+1} = \rho \pi_t + N \varepsilon_{t+1} \\ z_t = N^{-1} \pi_t \\ N = \frac{-1}{(1 + \sigma) - \rho} < 0 \\ i_t = 0 \\ z_0 \text{ given} \end{array} \right. \quad (50)$$

with the following constraint on the private sector's inflation rule parameter N :

$$0 < \rho < 1 \text{ and } \sigma > 0 \Rightarrow -\frac{1}{\sigma} < N = \frac{-1}{(1 + \sigma) - \rho} < \frac{-1}{1 + \sigma} < 0.$$

Laissez-faire inflation π_t is exactly negatively correlated with the auto-regressive forcing variable z_t ($N < 0$), with a coefficient of determination equal to one. The slope $N(\rho, \sigma)$ of the eigenvector of the stable exogenous eigenvalue ρ parameter decreases with the auto-correlation parameter ρ of the forcing variable. It increases with the (out of equilibrium) expected inflation growth rate $\sigma > 0$. One estimates the parameter ρ of an AR(1) model of inflation and the variance of the residuals $N^2 \Sigma_\varepsilon^2$ increases with Σ_ε^2 and with σ and decreases with ρ . If the forcing variable z_t is not observable, then one cannot regress $\pi_t = Nz_t$ and estimate directly the reduced form parameter N and the structural parameter σ . Then identifying Σ_ε^2 and σ requires an identification restriction.

4 Comparison of the different models

4.1 Taylor principles

Table 1 sums up distinct Taylor principles. To have a *countercyclical negative feedback* inflation Taylor rule parameter F_π , two predetermined variables are needed. Besides the forcing variable, either inflation is predetermined or inflation is forward and the Lagrange multiplier of inflation is predetermined (alternatively the policy interest rate) for optimal policy. Optimal policy under commitment is an *active* (or *reactive*) policy, with countercyclical credible anchor and the monitoring of private sector expectations dynamics. To have a *pro-cyclical positive feedback* inflation Taylor rule parameter, only one predetermined variable is required. This is the case of discretionary policy and of simple rule. Because the private sector does not believe that the policy maker is credible for not reneging commitment, the policy maker is *unable* to monitor private sector's inflation expectations. For this reason, a pro-cyclical positive feedback Taylor rule parameter is a *passive* discretionary monetary policy. The controllable eigenvalue λ_π linearly decreases with the inflation rule parameter: $\lambda_\pi = 1 + \sigma - \sigma F_\pi$.

There is a large gap between the largest value of the inflation rule parameter F_π with discretionary policy (zero) to its smallest value with optimal policy strictly larger than one: $1 + \frac{1}{1+\sigma}$. It is not feasible to have a continuous shift of the inflation parameter F_π of the policy interest rule from time-consistent discretionary policy to optimal policy under commitment.

Figure 1 represents the relation between the eigenvalue λ_π and the inflation rule parameter F_π for each policy.

Table 1: Taylor principles

Policy	Predetermined inflation π_t	Forward inflation π_t
Optimal	2 predetermined: z_t, π_t 1 forward: i_t $R \in]0, +\infty[$ $F_\pi \in]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}[$ $\lambda_\pi \in]0, \frac{1}{1+\sigma}[$ countercyclical active policy	2 predetermined: z_t, i_t or μ_t 1 forward: π_t $R \in]0, +\infty[$ $F_{\pi,C} \in]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}[$ $\lambda_{\pi,C} \in]0, \frac{1}{1+\sigma}[$ countercyclical active policy
Discretionary	2 predetermined: z_t, π_t 1 forward: i_t $R \in]0, +\infty[$ $F_\pi \in]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}[$ $\lambda_\pi \in]0, \frac{1}{1+\sigma}[$ countercyclical active policy	1 predetermined: z_t 2 forward: π_t, i_t $R \in]0, +\infty[$ $F_{\pi,D} \in]-\infty, 0[, F_z = 0$ $\lambda_{\pi,D} \in]1 + \sigma, +\infty[$ pro-cyclical passive policy
Simple rule	2 predetermined: z_t, π_t 1 forward: i_t $\lambda_\pi \in]-1, 1[$ $F_\pi \in]1, 1 + \frac{2}{\sigma}[$ countercyclical active policy	1 predetermined: z_t 2 forward: π_t, i_t $\lambda_{\pi,S} \in]1, +\infty[\cup]-\infty, -1[$ $F_{\pi,S} \in]-\infty, 1[\cup]1 + \frac{2}{\sigma}, +\infty[$ $F_{z,S} = 0$ pro-cyclical passive policy

4.2 Simple Rule as Reduced Form of Discretionary Policy

In table 2, numerical simulations track the relation between policy rules and the Fed's relative cost of changing the policy rate. A striking result is that a simple rule may correspond to the reduced form of a discretionary policy and allows to infer Fed's preference parameters under certain conditions.

In the simulations, the transmission mechanism parameters correspond to the large persistence usually found for inflation $\rho = 0.9$ and a modest marginal effect of the user cost channel of monetary policy: a rise of nominal policy rate of 1% leads to a modest fall of next period inflation $-\sigma\% = -0.1\%$. The initial negative shock $z_0 = -10\%$ remains in the range of assuming that the private sector's transmission mechanism equations (1 and 2) are a linear approximation of non-linear models. The shock is chosen to be negative in

order to check when the nominal interest rate crosses the zero lower bound constraint set at -2% for a long run equilibrium inflation target set at 2% . More precisely, the response of inflation is positive. The negative correlation comes from the sign of the initial anchor of inflation on the forcing variable which is the sign of $N = \frac{-1}{(1+\sigma)-\rho} = -5 < 0$. Then, the feedback response of the policy rate is negative. The laissez-faire equilibrium leads to a very large initial jump of inflation $\pi_0 = \frac{-1}{(1+\sigma)-\rho} z_0 = 50\%$. For example, the initial jump could be reduced to the value $\pi_0 = 25\%$ in setting $z_0 = -5\%$ or in decreasing the inflation persistence $\rho = 0.8$ and in increasing the marginal effect of the policy rate on future inflation to $\sigma = 0.2$.

The policy preference includes a zero discount rate and a discount factor equal to one: $\beta = 1$, a relative weight of inflation variance standardized to one: $Q_{\pi\pi} = 1$, a modest relative weight on the covariance of inflation with the forcing variable $Q_{\pi z} = 10\%$ of the weight on inflation variance, and the relative weight of policy rate variance measuring Fed's inertia varies in the range $R \in]0, +\infty[$.

Table 2: Differences between optimal policy, discretionary policy and simple rule: $\sigma = 0.1$, $\rho = 0.9$, $\beta = 1$, $Q_{\pi\pi} = 1$, $Q_{\pi z} = 0.1$ for an initial negative shock $z_0 = -0.1$.

$F_{\pi,S}$	R	$F_{\pi,D}$	$F_{\pi,C}$	$\lambda_{\pi,S}$	$\lambda_{\pi,D}$	$\lambda_{\pi,C}$	$F_{z,C}$
$-\infty$	$\rightarrow 0$	$-\infty$	11	$+\infty$	$+\infty$	0	$0.99 \cdot F_{\pi,C}$
-81.7	10^{-3}	-81.7	10.09	9.267	9.267	0.091	10.8
-4.67	0.1	-4.67	3.70	1.567	1.567	0.730	9.98
-0.49	1	-0.49	2.26	1.149	1.149	0.874	9.66
-0.05	10	-0.05	1.95	1.105	1.105	0.905	9.56
0	$\rightarrow +\infty$	0	1.91	1.1	1.1	0.905	$5 \cdot F_{\pi,C}$
1	simple non-D	-	-	1	-	-	-
21	simple non-D	-	-	-1	-	-	-
$+\infty$	simple non-D	-	-	$-\infty$	-	-	-

Figure 2 plots rule parameters $F_{\pi,C}$, $F_{\pi,D}$, $F_{z,C}$ as non-linear functions of the relative cost of changing the policy rate R . Figure 3 plots eigenvalues $\lambda_{\pi,C}$ and $\lambda_{\pi,D}$ as non-linear functions of the relative cost of changing the policy rate R .

As seen in table 1, for optimal policy under commitment, the inflation rule parameter is larger than one, positive and bounded $F_{\pi,C} \in]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}[=]1.91, 11[$. When the cost of changing the policy rate R increases from zero to infinity, optimal monetary policy is less reactive: the inflation rule parameter *decreases* from 11 to 1.91. The relation between the controllable eigenvalue and the inflation inflation rule parameter is $\lambda_{\pi} = 1 + \sigma - \sigma F_{\pi}$. The eigenvalue λ_{π} increases from zero (fastest convergence) to the slowest optimal conver-

gence with the inverse $\lambda_\pi = \frac{1}{1+\sigma} = 0.905$ of the laissez-faire value of λ_π equal to $1 + \sigma$.

For discretionary policy, the inflation rule parameter $F_{\pi,D}$ is negative. When the cost of changing the policy rate R increases from zero to infinity, the inflation rule monetary policy *increases* from minus infinity to zero. The eigenvalue λ_π is always larger than one. It increases from $1 + \sigma$ (laissez-faire equilibrium) to infinity. The eigenvalue λ_π has only an impact on the paths which are out of discretionary equilibrium, with faster divergence when there is a smaller jump of inflation in the discretionary equilibrium.

Finally, there is always a second rule parameter for optimal policy under commitment $F_{\pi,C}$ which is strictly different from zero, with rule parameters decreasing from 11 to 9.56 when the cost of changing the policy rate R increases from zero to infinity.

The simulation allows to revisit the comparison between optimal policy, time-consistent policy and simple rule:

(1) For given policy transmission mechanism parameters (ρ, σ) , a given simple rule with negative inflation rule parameter $F_{\pi,S} \in]-\infty, 0[$ is always the *reduced* form of the rule of *discretionary* policy with inflation parameter $F_{\pi,D}$ with a unique Fed's preference parameter for the relative cost of changing the policy rate $R = R(F_{\pi,S})$. This result holds here because the simple rule includes a number of parameters which is equal to the number of identified parameters in a policy rule of discretionary policy, which is exactly the number of predetermined variables.

(2) Simple rules with forward inflation, such that their inflation rule parameter is strictly positive: $(F_{\pi,S} \in]0, 1[\cup]1 + \frac{2}{\sigma}, +\infty[)$ are never the reduced form of a rule of discretionary policy (rows "simple non-D" in table 2). As seen in table 3: $F_{\pi,S} \in]0, 1[$ increases inflation initial jump with respect to laissez-faire, as far as doubling the initial inflation jump. *This set of simple rule parameter values does not avoid "disasters" whereas discretionary policy and optimal policy under commitment avoid them.*

(3) Simple rules with forward inflation are never a reduced form of an optimal policy under commitment with counter-cyclical inflation rule parameter $F_{\pi,S} \in]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}[$. The private sector *never believes* that the Fed is credible to commit to an optimal policy. Hence, the sequence of interest rates is based on a procyclical positive feedback simple rule.

(4) The relation between the Fed's preference parameter for inertia (relative cost of changing the policy rate R) and the Taylor rule parameter is highly non-linear with two asymptotes. It also depends on transmission mechanism parameters (ρ, σ) . Announcing the Fed's preferences *and* rule parameters is useful.

These results suggest that the comparison between simple rule and time-

consistent policy is *not relevant* when a simple rule corresponds to the reduced form of a discretionary policy rule with corresponding Fed's preferences parameters. This occurs under three conditions: (A) the monetary transmission channel and the Fed's preferences parameters do not change during the period under study, (B) the number of identified parameters of a simple rule (taking into account identification restrictions) is exactly equal to the number of predetermined variables, (C) the simple rule parameters belongs to the feasible set of rule parameters of discretionary policy when varying the Fed's preferences.

4.3 Conditions for a Negligible Cost of Time Inconsistency

For large or small Fed's preferences for inertia (R), in particular, *for the observed high inertia of the Fed with tiny variations of the nominal policy rate within the range [0%,4%] during the great moderation, the cost of time-inconsistency and, hence, the requirement of commitment are negligible (or of a second order of magnitude) for optimal policy.* Optimal policy is then a near-time-consistent policy. A near-time-consistent optimal policy and a time-consistent sub-optimal discretionary policy are to be chosen by policy makers. Their key difference is opposite countercyclical versus pro-cyclical inflation rule parameters and the size of the stable subspace of the economy.

Table 3 presents welfare, initial inflation and policy rate values for optimal policy under commitment, discretionary policy and simple rules, when varying the relative cost of changing the interest rate in the range $R \in]0, +\infty[$. Figures represent welfare (figure 4), initial inflation (figure 5) and initial interest rate and zero lower bound constraints (figure 6) as function of the relative cost of changing the policy interest rate R .

Table 3: Optimal policy, discretionary policy and simple rule: transmission mechanism: $\sigma = 0.1$, $\rho = 0.9$, Preferences: $\beta = 1$, $Q_{\pi\pi} = 1$, $Q_{\pi z} = 0.1$ for $z_0 = -10\%$.

$F_{\pi,S}$	R	$\frac{v_D^*}{v_{LF}^*}$	$\frac{v_C^*}{v_{LF}^*}$	$\frac{\pi_{0,,D,S}^*}{\pi_{0,LF}^*}$	$\frac{\pi_{0,C}^*}{\pi_{0,LF}^*}$	$\frac{i_{0,D,S}}{z_0}$	$\frac{i_{0,C}}{z_0}$
$-\infty$	$\rightarrow 0$	0	0	0	0	$10 = \frac{1}{\sigma}$	$10 = \frac{1}{\sigma}$
-81.7	10^{-3}	0.03	0.03	0.02	0.04	9.76	8.81
-4.67	0.1	0.28	0.23	0.30	0.44	7	4.27
-0.49	1	0.78	0.70	0.80	0.82	1.96	1.26
-0.05	10	0.98	0.98	0.98	0.98	0.24	0.17
0	$\rightarrow +\infty$	1	1	1	1	0	0
1	simple	-	-	2	-	$10 = \frac{1}{\sigma}$	-
21	simple	-	-	-0.1	-	11.05	-
$+\infty$	simple	-	-	0	-	$10 = \frac{1}{\sigma}$	-

The third and fourth column present the Fed's loss function for commitment v_C^* and for discretion v_D^* divided by the laissez-faire loss function v_{LF}^* . An increase of the relative cost R of changing the policy interest rate implies a decrease of welfare (figure 4), an increase of initial inflation (figure 5) and a decrease of initial interest rate (figure 5) for commitment and for discretion. For intermediate values of Fed's relative cost R of changing the policy interest rate, welfare and initial inflation are *higher* for commitment as compared to discretion. Initial interest rate is *lower* for commitment as compared to discretion. For very low or very high Fed's inertia ($R \rightarrow 0$ or $R \rightarrow +\infty$), welfare and the initial values of inflation and of the interest rate are identical for commitment and for discretion.

The zero lower bound variations of interest rate within the range $[0\%, 4\%]$ for shocks as high as 10% on the forcing variable z_0 are related to values of R larger than 1 for optimal policy and values of R *larger than* 5 for discretionary policy. The zero lower bound constraint with a low long-run inflation target at 2% implies Fed's preferences with a high inertia of the policy rate. Proponents of sub-optimal time-consistent policy satisfying the zero lower bound constraint notice that, for R *larger than* 5, the difference between optimal welfare and discretionary welfare is negligible (figure 4). Restricting very small shocks to at most 1% deviations of the forcing variable z_0 increases the range of values of interest rate and of the relative cost R of changing the policy rate compatible with the zero lower bound constraint (figure 6). Considering only these tiny shocks decreases by 10 the difference between the loss function of commitment and of discretion, which remains negligible.

The path of optimal policy under commitment evolves in a two-dimensional stable subspace of stationary variables (figure 8, phase diagram). The optimal anchor of inflation on the forcing variable z_0 *at the initial date* selects one of the stable paths in the two-dimensional stable subspace. Negative feedback stabilizes the eigenvalue λ_π which can be lower than the eigenvalue

ρ related to the forcing variable. In this case, after identical initial jumps of inflation, optimal policy accelerates the decrease of inflation as compared to a path constrained to remain on a straight line, declining at the rate ρ , in the phase diagram and converging to the long run equilibrium, at the origin of the phase diagram. This is reflected in the curvature of the optimal path in figure 8.

Discretionary policy path evolves on a straight line, in a one-dimensional stable subspace of stationary variables. This reduction of dimensions is due to the Fed's lack of credibility, so that the private sector defends itself in forcing the Fed to use positive feedback rules. This is reflected in the proportional anchor of forward inflation on the predetermined forcing variable z_t *at all dates*. For given Fed's preferences, a path of time-consistent discretionary policy is an approximation in a stable subspace of dimension one (a straight line) of the path of optimal policy evolving in a stable subspace of dimension two (a plane). Optimal policy declining rate is a linear combination of two eigenvalues $\lambda_{\pi,C} < \rho < 1$. This implies a faster convergence to equilibrium than discretionary policy declining rate which is $\rho < 1$. Discretionary policy implies a lower initial jump of inflation than optimal policy, knowing the optimal policy outperforms discretionary policy during future periods. This is shown in the impulse responses over time (figure 7). As both paths have to satisfy long-term final transversality conditions, both of them end in the origin.

This is matched with a higher interest rate at the beginning of the period for discretionary policy as compared to optimal policy. *It is tempting to attribute to negative feedback the negative correlation of a fall of inflation along with a rise of interest rate.* But the underlying mechanism is based on positive feedback, with very poor economic insight. This fall of inflation is explained by a smaller slope (in absolute value) of the eigenvectors of the stable eigenvalue $\rho < 1$. This slope $\frac{-1}{\lambda_{\pi,D}-\rho}$ is equal to the inverse of the difference between the stable and the unstable eigenvalues: $\rho < 1 < \lambda_{\pi,D}$. A decrease of inflation corresponds to a smaller slope in absolute value. This occurs for an increase of the unstable eigenvalue $\lambda_{\pi,D}$ corresponding to *an increase of the positive feedback effect*. This positive feedback effect is caused by a decrease of the negative value of the inflation rule parameter $F_{\pi,D}$.

Phase diagram 9 compares old Keynesian optimal policy to optimal policy under commitment. Both optimal policies share identical policy rule parameters. They differ with respect to initial inflation which is predetermined for the old Keynesian model and optimally decided using initial transversality condition for optimal policy under commitment. In phase diagram 9, for the old Keynesian model, a predetermined inflation is chosen arbitrarily to

be the same value than in laissez-faire equilibrium: $\pi_0 = 50\%$. The optimal initial jump allows to decrease inflation below 50%. This increases the speed of convergence of inflation and increases welfare.

The initial inflation jump π_0 (figure 10) and the eigenvalue λ_π (figure 1) are plotted as a function of the inflation rule parameters F_π . These three variables depend on the cost of changing the policy rate R . Correlations between the inflation rule parameter and the inflation initial jump have opposite signs for optimal policy under commitment and for time-consistent discretionary policy.

- An increase of the inflation rule parameter of optimal policy under commitment from 1.9 to 11 leads to a decrease of the initial inflation jump from the initial value of inflation of laissez-faire to instantaneous adjustment of inflation. Phase diagrams of figures 11,12,13 also plots two of the out-of-equilibrium paths that may be chosen by an evil agent in robust control (Hansen and Sargent (2008) and section 5.3) with initial deviation from the initial inflation jump of $\pm 5\%$. These paths are converging faster and faster to the optimal path when decreasing the cost of changing the policy interest rate. This suggests an decrease of the costs of misspecification chosen by an evil agent.

- A decrease of the inflation rule parameter of discretionary policy from zero to minus infinity leads to a decrease of the jump from the value of laissez-faire to instantaneous adjustment of inflation. Phase diagrams of figures 14,15,16 also plots two of the out-of-equilibrium paths that may be chosen by an evil agent in robust control (Hansen and Sargent (2008) and section 5.3) with initial deviation from the initial inflation jump of $\pm 5\%$. These paths are diverging faster and faster to the optimal path when decreasing the cost of changing the policy interest rate. This suggests an increase of the costs of misspecification chosen by an evil agent.

- Inflation simple rule parameters between zero and one overshoot inflation up to doubling the value of the initial shock when reaching one. Inflation simple rule parameters over 21 overshoot inflation down with an opposite sign of the jump with respect to Laissez-faire. This overshooting is also seen in impulse response functions (figure 8) and phase diagram (figure 9).

- With old Keynesian models, there is no jump. But only a subset of stabilizing values of the rule parameters are optimal (between 1.95 and 11), the remaining values between 1 and 1.95 and between 11 and 21 could only be used in simple rule old Keynesian models with inflation as a predetermined variable (Taylor (1999)).

- Timeless perspective assumes that the optimal jump occurred for example 20 periods before, even if the shock z_0 occurs now. This is equivalent to consider as the current inflation jump the value of inflation 20 periods

ahead. Inflation jump is minimal and very close to the long run value of inflation. Timeless perspective optimal policy is equivalent to optimal policy with a very low cost R of changing the policy rate and maximal volatility of the policy rate. In this case, optimal policy and discretionary policy are equivalent, so time-consistency does not matter. There is no cost of reneging commitment. A main issue is that timeless perspective may be applied to high cost R of changing the policy rate, which is also time-consistent with minimal volatility of the policy rate. With optimal policy under commitment, a minimal volatility of the policy rate is not compatible with a minimal initial jump of inflation. With *ad hoc* timeless perspective hypothesis, it is possible.

Phase diagrams in figures 17, 18, 19 is another way to visualize how the time-inconsistency problem turn to be negligible for large or close to zero relative cost of changing the policy interest rate. Figures 17,18,19 plot phase diagrams of $t = 0$ optimal policy under commitment, $t = 3$ optimal policy under commitment in the plane with inflation and the forcing variable.

(1) For intermediate values of the cost of changing the policy rate (figure 18), the straight line joining the optimal jump to the origin would be the path chosen if the policy maker optimizes again at each period and if nonetheless the private sector believes the policy maker is credible. It is defined as $\pi_t = -P_{\pi z} P_{\pi}^{-1} z_t$. This straight line is not equal to the discretionary path, because the discretionary path leads to another value of the initial jump $\pi_t = N_{z,D} z_t$. A second curve corresponds to the $t = 3$ -optimal policy. The policy maker will anchor initially inflation at $\pi_3 = -P_{\pi z} P_{\pi}^{-1} z_3$ and renege her commitment from the path of $t = 0$ optimal policy. After several periods, the $t = 3$ -optimal policy path joins the $t = 0$ -optimal policy path.

(2) For very large or very small cost of changing the policy rate ($R = 10$, figure 17, or $R = 10^{-3}$, figure 19), $t = 0$ optimal policy and $t = 3$ optimal policy follows nearly the same straight line, so that reneging commitment implies negligible or second order cost of time inconsistency. The paths of optimal policy under commitment and of discretionary policy are the same, but their rule countercyclical versus pro-cyclical rule parameters are different.

5 Extensions

This section discusses how this paper's methods and results may be extended in certain areas by many researchers. The method uses comparable representations of policy rules functions of the current variables of the private sector, chosen among the infinite set of mathematically and observationally equivalent rules of a given dynamic equilibrium. An algorithm to solve optimal policy under commitment with mathematically and observationally equivalent

lent solutions than existing algorithm can also be used (Chatelain and Ralf (2016)). Beyond Taylor's (1999) specific model, these methods and results may be extended to any micro-founded dynamic stochastic general equilibrium (DSGE) model, with welfare derived from private sector's agents' sum of utilities, with a linear quadratic approximation.

5.1 Counter-cyclical versus Pro-cyclical Rules Parameters: General Case

The opposite result of countercyclical rule parameters with optimal policy under commitment versus pro-cyclical rule parameters are *not at all* specific to Taylor's (1999) model. This result is at the root of the strictly larger dimension of the invariant stable subspace of the Hamiltonian system of optimal policy with respect to the one of time-consistent discretionary policy. This result also applies to any linear recursive system of the private sector with quadratic preferences of the policy maker. This is a general result for any linear quadratic Stackelberg game versus time-consistent Nash dynamic game. This result applies to models with micro-foundations of the private sector, such as the new-Keynesian model (Chatelain and Ralf (2016)). This result applies to models with a policy maker's quadratic loss function approximated from the private sector's non-quadratic utilities. This result applies to policy maker's loss function including not only inflation but also output gap and financial instability indicators, as in macro-prudential DSGE models.

Proposition: When varying policy maker's preferences \mathbf{Q} and \mathbf{R} in the sets such that $\mathbf{Q} \geq \mathbf{0}$ and $\mathbf{R} > \mathbf{0}$, the global set of policy rule parameters \mathbf{F}_C for the representation of the rule function of current controllable private sectors variables (case when all forcing exogenous variables are initially set to zero) for optimal policy under commitment (with stable subspace of dimension equal to the number of private sector's predetermined and forward variables) versus \mathbf{F}_D for discretionary policy (with stable subspace of dimension equal to the number of private sector's predetermined variables) are always distinct.

Proof. This is a corollary of Wonham (1967) pole placement theorem for a controllable system. Assuming that the pair (\mathbf{A}, \mathbf{B}) is controllable, there exist a feedback matrix \mathbf{F} such as the eigenvalues of the matrix $\mathbf{A} + \mathbf{BF}$ can be placed in arbitrary locations (Wonham (1967)). Assume there are n predetermined variables and m forward variables, and A is a square transition matrix of dimension $n + m$. The feedback matrices \mathbf{F}_C of optimal policy under commitment which places $n + m$ (stable) eigenvalues in the unit disk of the complex plane are *never* equal to the feedback matrices \mathbf{F}_D of dis-

cretionary policy which places n (stable) eigenvalues in the unit disk of the complex plane and m (unstable) eigenvalues in the unit disk complement of the complex plane. Figure 1 is a scalar example of this result.

5.2 Multiple Taylor principle sets

The sets of values of the inflation rule parameter (Taylor principles) with Taylor's (1999) model cannot be robust to changes of the private sector's model nor to distinct definitions of policy maker and private sector dynamic game equilibria. Alternative sets of values of the inflation rule parameter are found for:

- (1) Different private sector's monetary policy transmission mechanisms, in particular with opposite signs of the effects of current policy instruments on future periods policy targets.
- (2) Different dynamic game equilibrium definitions, leading either to procyclical policy rule parameters (simple rule with forward variables, discretionary policy) or countercyclical policy rule parameters (old Keynesian simple and optimal policy without forward variables, optimal policy under commitment).

5.3 Robustness and the heterogeneity of beliefs

Giordani and Söderlind (2004, appendix D, proposition 1) demonstrated that the simple rule equilibrium is not robust to misspecification, using the Hansen and Sargent (2008) definition of robust control. *"An evil agent who is able to commit will choose a non-stationary (ever increasing or decreasing) "non zero deviation" in the law of motion, which makes the loss function unbounded... The misspecification feared is then a trend increase (or decrease) of inflation"* [p.2388]. Not only the policy maker who commits to a simple rule is not credible to monitor expectations, but also, he is *"defenceless against the evil agent"* when fearing misspecification of the monetary policy transmission channel.

Another definition of robustness is to search for a simple instrument rule that performs at least moderately well (avoiding disasters) in a variety of models. But simulations using a variety of forward-looking models are always solved such that an evil agent *never* selects an out-of-equilibrium trend increase of inflation (disasters). As mentioned by Giordani and Söderlind (2004), the alternative definition of robustness across a variety of existing models may be sometimes *less* robust than Hansen and Sargent's (2008) definition of robust control.

Giordani and Söderlind (2004) proposed a robust discretionary equilibrium. It is based on the assumption that the evil agent excludes the possibility that there are heterogeneous beliefs on the credibility of policy makers. Matthes (2015) estimates such a proportion for the U.S. before and after Volcker was Chairman of the Fed. If ever there is at least a small proportion of private sectors' agents who believe in the credibility of the policy maker, their inflation rule may depend on past or future values of inflation: $\pi_t = N_S z_{t-1} + M \pi_{t-1}$. In this case, it is likely that the aggregate evil agent is able to commit and to choose a non-stationary trend increase of inflation. In this case, the policy maker's robust control response is to shift from Giordani and Söderlind (2004) robust discretionary policy to Giordani and Söderlind (2004) robust optimal policy under commitment.

Robust optimal control assumes that the policy maker minimizes its loss function taking into account the worst of misspecification chosen by an evil agent (cf. *advocatus diaboli* in Catholic church) defined in a given set. Simple rules with forward variables and discretionary equilibrium are not robust by design and by default of policy maker's credibility, unless the given set of misspecifications chosen by an evil agent faces large restrictions. This may explain why cautious real world policy maker are reluctant to follow the procyclical positive feedback rule parameters argument, advocated by academics without skin in the game.

5.4 Multiple Time-Consistent Equilibria versus Unique Optimal Policy

In this paper, discretionary policy has a unique equilibrium because there is no endogenous and controllable predetermined variable. As soon as there is at least one endogenous predetermined variable (the stock of savings invested in public debt) and one forward variable (inflation) which are *both controllable* by one or two instrument (interest rate and/or taxes), there are multiple equilibria *when varying policy rule parameters* for simple rules and for discretionary policy (Leeper (1991), Leeper and Leith (2016)). There are at least two distinct eigenvalues under control of the policy makers. One of them should be stable and the other one unstable, because there is one predetermined variable and one forward variable.

For example, in Leeper (1991), either the "inflation eigenvalue" is unstable and the "public debt eigenvalue" is stable or there is the fiscal theory of the price level: the "inflation eigenvalue" is stable and the "public debt eigenvalue" is unstable. This corresponds to two distinct one-dimensional stable subspaces related to two distinct non-collinear eigenvectors. This corresponds

to multiple solutions of Riccati equation which also corresponds to sunspots multiple equilibria when varying rule parameters \mathbf{F} . If all eigenvalues are distinct, the number of equilibria is equal to the number of combinations $\binom{n_p}{n_c}$ of selecting n_p stable eigenvalues (for n predetermined variables) in a larger set of the total number $n_c > n_p$ endogenous and controllable predetermined and forward variables (Freiling (2002), p.253, remark b).

By contrast, with optimal policy under commitment, setting aside exogenous predetermined variables and their eigenvalues, the remaining number of stable eigenvalues is *exactly* equal to number of the private sector's controllable predetermined *and forward* variables. The number of equilibria is equal to the number of combinations for selecting n_c stable eigenvalue in a set of n_c potentially stable eigenvalue: $\binom{n_c}{n_c} = 1$ when varying rule parameters \mathbf{F} . There are no multiple equilibria à la Leeper (1991) when varying policy rule parameters \mathbf{F} for optimal policy under commitment. With the assumptions for the controllable part of the economy that policymaker's preferences are such that $\mathbf{Q} \geq \mathbf{0}$, $\mathbf{R} > \mathbf{0}$ and of infinite horizon transversality condition, Kalman (1960) demonstrated the uniqueness of the rule parameters \mathbf{F} of the linear quadratic regulator, for a representation of the rule which depends on the current values of the private sector's variables.

If there is a unique equilibrium of optimal policy under commitment at date $t = 0$, there is another unique optimal equilibrium of optimal policy under commitment for each future date $t > 0$. Simaan and Cruz's (1973) and Kydland and Prescott's (1980) time-inconsistency corresponds to multiple equilibria over time. When the cost of changing the policy instruments \mathbf{R} tends to zero or tends to infinity, these distinct equilibria over time tend to be all identical. Simaan and Cruz's (1973) and Kydland and Prescott's (1980) time inconsistency tends to be a second order or negligible issue.

5.5 Optimal Policy Requires Identification Restrictions of Autoregressive Components of Shocks

Optimal policy under commitment and discretionary policy under commitment have often been estimated (see Matthes (2015) for the most recent estimates). Both policies differ with respect to the identification and the parsimony of assumptions of auto-regressive moving average (ARMA) components in shocks added for estimating DSGE models. The number of endogenous stationary variables is equal to the number of predetermined variables in discretionary policy and simple rule. It is equal to the number of predetermined and forward variables in optimal policy under commitment. This increases

the dimension of the minimal size vector auto-regressive (VAR) component of the rational expectations system to estimate.

With optimal policy under commitment, it is necessary to add identification restrictions on auto-regressive components in shocks for each of the forward variables. The number of variables with a recursive relation is equal to the number of variables which are all stationary. For example, expected inflation depends on lagged inflation and other variables, because of the Fed's credible commitment to a sequence of interest rate. The lagged dependent variable (inflation) and the auto-regressive component in the cost-push shock face an identification issue, as they both intend to fit persistence of the inflation time series. This identification issue is identical to the one put forward on the estimation of Taylor rules by Feve, Matheron and Poilly (2007). It is an option to restrict the auto-regressive component of shocks to be equal to zero.

Discretionary policy and simple rule with forward variables models predict that there is a smaller number of linearly independent time series (equal to the number of predetermined variables) than the observed number of time series. Perfect linear correlations of forward prices and flows of quantities with endogenous predetermined stocks (wealth, capital, debt) are unlikely. Auto-regressive and moving average exogenous components are added in shocks for each of the forward variables in order to fit the data while saving the theoretical model from being rejected by statistical tests of goodness of fit. It is no longer an option to restrict the auto-regressive component of shocks to be equal to zero.

Finally, both commitment and discretionary policy predict that all macro-economic time series involved in the model are stationary. U.S. inflation from 1973 to 1980, U.S. housing prices from 2001 to 2007, U.S. public debt over GDP ratio from 1980 to now were likely to be non-stationary.

6 Conclusion

The main results of this paper are:

- (1) There are distinct Taylor principles for optimal policy under commitment with negative feedback policy rule parameters and discretionary policy with positive feedback policy rule parameters.
- (2) Time-inconsistency turns to be a negligible issue when Central Banks' preferences imply a low volatility of policy instruments.
- (3) Under certain conditions, a simple rule corresponds to the reduced form of a discretionary policy for the corresponding policy maker's preferences.

Although the paths of both policies and hence the welfare are nearly identical for very inertial or very reactive interest rate volatility of the Central Bank (large or negligible relative cost of changing the interest rate with respect to the cost of deviating from inflation target), the policy rule recommendations are the opposite: pro-cyclical rule parameters with discretionary policy and counter-cyclical rule parameters with optimal policy under commitment.

These results stimulate further research for finding distinct sets of policy rule parameters functions of the private sector's variables for optimal policy under commitment and for discretionary policy for any DSGE model of the private sector, for the evaluation of the costs of time-inconsistency, and for comparing the policy recommendations of simple-rule, discretionary and optimal policy.

References

- [1] Anderson E.W., Hansen L.P., McGrattan E.R. and Sargent T.J. [1996]. Mechanics of Forming and Estimating Dynamic Linear Economies. in Amman H.M., Kendrick D.A. and Rust J. (editors) *Handbook of Computational Economics*, Elsevier, Amsterdam, 171-252.
- [2] Blanchard O.J. and Kahn C. [1980]. The solution of linear difference models under rational expectations. *Econometrica*, 48, pp. 1305-1311.
- [3] Bryson A.E. and Ho Y.C. [1975]. *Applied Optimal Control*, John Wiley and Sons, New York.
- [4] Backus D. and Driffill J. [1986]. The consistency of optimal policy in stochastic rational expectations models. Discussion paper 124, CEPR, London.
- [5] Chatelain J.B. and Ralf K. [2016]. A New Algorithm for Solving Ramsey Optimal Policy under Commitment. Working Paper.
- [6] Chatelain J.B. and Ralf K. [2016]. Countercyclical versus Procyclical Policy Rules: The new Keynesian model revisited. Working Paper.
- [7] Cochrane J.H. [2011]. Determinacy and identification with Taylor Rules. *Journal of Political Economy*, 119(3), pp. 565-615.
- [8] Cochrane J.H. [2016]. Do Higher Interest Rates Raise or Lower Inflation? Unpublished manuscript, University of Chicago Booth School of Business.

- [9] Cohen D. and Michel P. [1988]. How Should Control Theory be Used to Calculate a Time-Consistent Government Policy? *Review of Economic Studies*, 55, 263-274.
- [10] Fève P., Matheron J. and Poilly C. [2007]. Monetary Policy Dynamics in the Euro Area. *Economics Letters*, 96(1), 97-102.
- [11] Freiling G. [2002]. A Survey of Nonsymmetric Riccati Equations. *Linear Algebra and its Applications*, 96(1), 97-102.
- [12] Giordani and Söderlind [2004]. Solution of Macromodels with Hansen-Sargent Robust Policies: some extensions. *Journal of Economic Dynamics and Control*, 12, 2367-2397.
- [13] Hansen L.P. and Sargent T. [2008]. *Robustness*, Princeton University Press, Princeton.
- [14] Kalman R.E. [1960]. Contributions to the Theory of Optimal Control. *Boletín de la Sociedad Matemática Mexicana*, 5, pp.102-109.
- [15] Kydland F. and Prescott E.C. [1980]. Dynamic Optimal Taxation, Rational Expectations and Optimal Control. *Journal of Economic Dynamics and Control*, 2, pp.79-91.
- [16] Leeper D. [1991]. Equilibria under "active" and "passive" Monetary and Fiscal Policies. *Journal of Monetary Economics*. 27(1): 129-147.
- [17] Leeper D. and Leith C. [2016]. Understanding Inflation as a Joint Monetary-Fiscal Phenomenon. *Handbook of Macroeconomics*. vol. 2, J.B. Taylor and Uhlig H. editors. in Press.
- [18] Leith C. and Wren-Lewis S. [2013]. Fiscal Sustainability in a New Keynesian model. *Journal of Money Credit and Banking*. 55(7), pp.1253-1276.
- [19] Ljungqvist L. and Sargent T.J. [2012]. *Recursive Macroeconomic Theory*. 3rd edition. The MIT Press. Cambridge, Massachussets.
- [20] Matthes C. [2015]. Figuring out the FED - Beliefs about policymakers and gains from transparency. *Journal of Money, Credit and Banking*, 47(1), 1-29.
- [21] Miles D. [2015]. What Can Monetary Policy Do. *Bank of England Speech*, 22nd january, Edinburgh. pp. 1-21.

- [22] Miller M. [1985]. Monetary Policy Stabilization in an Open Economy, *Scottish Journal of Political Economy*, 32(3), 220-233.
- [23] Miller M. and Salmon M. [1985]. Dynamic Games and the Time Inconsistency of Optimal Policy in Open Economies, *Economic Journal*, 95 supplement: conference papers, 124-137.
- [24] Oudiz G. and Sachs J. [1985]. International Policy Coordination in Dynamic Macroeconomic Models, in W.H. Buiter and R.C. Marston (eds), *International Economic Policy Coordination*, Cambridge, Cambridge University Press.
- [25] Simon H.A. [1956]. Dynamic Programming under Uncertainty with a Quadratic Criterion Function. *Econometrica*, 24(1), 74-81.
- [26] Simaan M. and Cruz J.B. [1973]. Additional Aspects of the Stackelberg Strategy in Nonzero-Sum Games. *Journal of Optimization Theory and Applications*. 11(6), pp.613-626.
- [27] Taylor J.B. [1999]. The Robustness and Efficiency of Monetary Policy Rules as Guidelines for Interest Rate Setting by the European Central Bank. *Journal of Monetary Economics*. 43, pp.655-679.
- [28] Wonham W.N. [1967]. On pole assignment in multi-input controllable linear system. *IEEE transactions on automatic control*. 12(6), pp 660-665.

7 Appendix: Augmented Linear Quadratic Regulator

For the augmented discounted quadratic regulator, write the system in matrix form:

$$\begin{pmatrix} z_{t+1} \\ \pi_{t+1} \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 1 & 1 + \sigma \end{pmatrix} \begin{pmatrix} z_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} 0 \\ -\sigma \end{pmatrix} i_t + \begin{pmatrix} \sigma_\varepsilon \\ 0 \end{pmatrix} \varepsilon_t \quad (51)$$

To match Anderson et al. (1996) notations, define elements of matrices **A** and **B** as:

$$\begin{pmatrix} \rho & 0 \\ 1 & 1 + \sigma \end{pmatrix} = \begin{pmatrix} A_{zz} & 0 \\ A_{\pi z} & A_{\pi\pi} \end{pmatrix}, \begin{pmatrix} 0 \\ -\sigma \end{pmatrix} = \begin{pmatrix} 0 \\ B_\pi \end{pmatrix} \quad (52)$$

$$A_{zz} = \rho, A_{\pi z} = 1, A_{\pi\pi} = 1 + \sigma, B_\pi = -\sigma \quad (53)$$

In section 3, optimal rule parameters \mathbf{F}_π and \mathbf{F}_z and weights \mathbf{P}_π and \mathbf{P}_z of the optimal value of the loss function are found using linear substitutions in the Hamiltonian system (Anderson et al. (1996)) leading to there formulas for the augmented linear quadratic regulator (we assume $\beta = 1$):

$$\mathbf{P}_\pi = \mathbf{Q}_{\pi\pi} + \beta \mathbf{A}'_{\pi\pi} \mathbf{P}_\pi \mathbf{A}_{\pi\pi} - \beta \mathbf{A}'_{\pi\pi} \mathbf{P}_\pi \mathbf{B}_\pi \left(\mathbf{R} + \beta \mathbf{B}'_\pi \mathbf{P}_\pi \mathbf{B}_\pi \right)^{-1} \beta \mathbf{B}'_\pi \mathbf{P}_\pi \mathbf{A}_{\pi\pi} \quad (54)$$

$$\mathbf{F}_{\pi,C} = \left(\mathbf{R} + \beta \mathbf{B}'_\pi \mathbf{P}_\pi \mathbf{B}_\pi \right)^{-1} \beta \mathbf{B}'_\pi \mathbf{P}_\pi \mathbf{A}_{\pi\pi} \quad (55)$$

$$\mathbf{P}_z = \mathbf{Q}_{\pi z} + \beta (\mathbf{A}_{\pi\pi} + \mathbf{B}_\pi \mathbf{F}_{\pi,C})' \mathbf{P}_\pi \mathbf{A}_{\pi z} + \beta (\mathbf{A}_{\pi\pi} + \mathbf{B}_\pi \mathbf{F}_{\pi,C})' \mathbf{P}_z \mathbf{A}_{zz} \quad (56)$$

$$\mathbf{F}_{z,C} = \left(\mathbf{R} + \beta \mathbf{B}'_\pi \mathbf{P}_\pi \mathbf{B}_\pi \right)^{-1} \beta \mathbf{B}'_\pi (\mathbf{P}_\pi \mathbf{A}_{\pi\pi} + \mathbf{P}_z \mathbf{A}_{zz}) \quad (57)$$

In what follows, it is shown that the Hamiltonian system includes a stable subspace of dimension 2 in a three dimension space defined by the three variables (z_t, π_t, μ_t) . The Fed's Lagrangian system includes stationary autoregressive forcing shock law of motion, expected inflation law of motion (the two equations describing the private sector) and the first order condition on Fed's interest rate (or on Fed's Lagrange multiplier on inflation) linking its current optimal value to its next period optimal value (*the Euler equation of Fed's interest rate*):

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta \frac{\sigma^2}{R} \\ 0 & 0 & \beta(1+\sigma) \end{pmatrix} \begin{pmatrix} z_{t+1} \\ \pi_{t+1} \\ \mu_{t+1} \end{pmatrix} &= \begin{pmatrix} \rho & 0 & 0 \\ 1 & 1+\sigma & 0 \\ -Q_{\pi z} & -1 & 1 \end{pmatrix} \begin{pmatrix} z_t \\ \pi_t \\ \mu_t \end{pmatrix} \Rightarrow \\ \begin{pmatrix} z_{t+1} \\ \pi_{t+1} \\ \mu_{t+1} \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 + \frac{\sigma^2}{R(1+\sigma)} Q_{\pi z} & 1 + \sigma + \frac{\sigma^2}{R(1+\sigma)} & -\frac{\sigma^2}{R(1+\sigma)} \\ -\frac{1}{\beta(1+\sigma)} Q_{\pi z} & -\frac{1}{\beta(1+\sigma)} & \frac{1}{\beta(1+\sigma)} \end{pmatrix}}_{=\mathbf{M}^a} \begin{pmatrix} z_t \\ \pi_t \\ \mu_t \end{pmatrix} \end{aligned}$$

Using the infinite horizon transversality conditions, the relevant solution is the one that stabilizes the state-costate vector for any initialization of inflation π_0 and of the exogenous shock z_0 in a stable subspace of dimension two within a space of dimension three (π_t, μ_t, z_t) of the Lagrange system (Anderson *et al.* (1996)). We seek a characterization of the Lagrange multiplier μ_t on inflation minimizing the optimal value function $v(\pi_t, z_t)$ at all date t of the form:

$$v(\pi_t, z_t) = -\frac{1}{2} (P_\pi \pi_t^2 + 2P_{\pi z} \pi_t z_t) \text{ with } P_\pi \neq 0 \quad (58)$$

$$\frac{\partial v(\pi_t, z_t)}{\partial \pi_t} = -P_\pi \pi_t - P_{\pi z} z_t = \mu_t \quad (59)$$

such that the resulting sequence (z_t, π_t, μ_t) is in the stable subspace of the augmented matrix \mathbf{M}^a . When the forcing sequence is initialized at zero $z_0 = 0$, the matrix P_π is such that all vectors of the stable subspace of the matrix \mathbf{M} related to the controllable part of the system can be represented as $(\pi_t, P_\pi \pi_t)$. First, consider the solution when the forcing sequence is initialized at zero $z_0 = 0$ in order to find the policy instrument optimal response parameter F_π to the endogenous variable. Second, compute the policy instrument optimal response parameter F_z to the non-controllable forcing variable. For the first step, the optimal system is not augmented:

$$\begin{pmatrix} \pi_{t+1} \\ \mu_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 + \sigma + \frac{\sigma^2}{R(1+\sigma)} & -\frac{\sigma^2}{R(1+\sigma)} \\ -\frac{1}{\beta(1+\sigma)} & \frac{1}{\beta(1+\sigma)} \end{pmatrix}}_{=\mathbf{M}} \begin{pmatrix} \pi_t \\ \mu_t \end{pmatrix}$$

The Fed's Lagrange system includes one eigenvalue λ_1 with absolute value below one and the other eigenvalue is such that $\lambda_2 = 1/\lambda_1$ because the matrix \mathbf{M} is symplectic (Anderson et al. (1996)). The Jordan transform and the transversality condition at the final date leads to Blanchard and Kahn's (1980) unique stable solution ($P_{\lambda_1^{-1}}$ is the slope of the eigenvectors of the unstable eigenvalue $1/\lambda_1$):

$$\begin{pmatrix} \pi_{t+1} \\ \mu_{t+1} \end{pmatrix} = \begin{pmatrix} 1 \\ P_\pi \end{pmatrix} \lambda_1^t \left(\frac{\mu_0 - P_{\lambda_1^{-1}} \pi_0}{P_\pi - P_{\lambda_1^{-1}}} \right) + \begin{pmatrix} 1 \\ P_{\lambda_1^{-1}} \end{pmatrix} \lambda_1^{-t} \left(\frac{P_\pi \pi_0 - \mu_0}{P_\pi - P_{\lambda_1^{-1}}} \right) \quad (60)$$

$$\lim_{t \rightarrow +\infty} \beta^{t+1} \mu_{t+1} = 0, \text{ transversality condition for } t \rightarrow +\infty. \quad (61)$$

$$\begin{pmatrix} \pi_{t+1} \\ \mu_{t+1} \end{pmatrix} = \begin{pmatrix} 1 \\ P_\pi \end{pmatrix} (1 + \sigma - \sigma F_\pi)^t \pi_0 \text{ with } \mu_t = P_\pi \pi_t \text{ for all } t \in \mathbb{N}.$$

The policy maker's transversality condition at the final date (infinite horizon) rules out diverging paths driven by the unstable eigenvalue λ_1^{-1} . Hence, the optimal path is of dimension one, driven by converging powers of the unique stable eigenvalue λ_1^t , along the stable arm of the two dimensions saddle point equilibrium.

Finally, one obtains stable dynamics function of initial conditions with the representation using these two private sector's variables (z_t, π_t) among the set of three variables (z_t, π_t, i_t) or (z_t, π_t, μ_t) :

$$\begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \left(\begin{pmatrix} \rho & 0 \\ 1 & 1 + \sigma \end{pmatrix} + \begin{pmatrix} 0 \\ -\sigma \end{pmatrix} \begin{pmatrix} F_z^* & F_\pi^* \end{pmatrix} \right) \begin{pmatrix} z_t \\ \pi_t \end{pmatrix} \quad (62)$$

$$\begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 1 - \sigma F_z^* & 1 + \sigma - \sigma F_\pi^* \end{pmatrix} \begin{pmatrix} z_t \\ \pi_t \end{pmatrix} \quad (63)$$

$$\begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1 - \sigma F_z^*}{\rho - (1 + \sigma - \sigma F_\pi^*)} & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & 1 + \sigma - \sigma F_\pi^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1 - \sigma F_z^*}{\rho - (1 + \sigma - \sigma F_\pi^*)} & 1 \end{pmatrix} \begin{pmatrix} z_t \\ \pi_t \end{pmatrix} \quad (64)$$

$$\begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1 - \sigma F_z^*}{\rho - (1 + \sigma - \sigma F_\pi^*)} & 1 \end{pmatrix} \begin{pmatrix} \rho^t & 0 \\ 0 & (1 + \sigma - \sigma F_\pi^*)^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1 - \sigma F_z^*}{\rho - (1 + \sigma - \sigma F_\pi^*)} & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ \pi_0 \end{pmatrix} \quad (65)$$

$$\begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1 - \sigma F_z^*}{\rho - (1 + \sigma - \sigma F_\pi^*)} & 1 \end{pmatrix} \rho^t z_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 + \sigma - \sigma F_\pi^*)^t \left(-\frac{1 - \sigma F_z^*}{\rho - (1 + \sigma - \sigma F_\pi^*)} z_0 + \pi_0 \right) \quad (66)$$

Properties of Fed's Maximal Inertia (minimum energy optimal control in linear quadratic framework):

It corresponds to a relative cost of changing the interest rate tending to infinity, which is equivalent to set the weight of the variance of inflation and of the covariance of inflation with the forcing variable to zero ($\mathbf{Q} = \mathbf{0}$) and set $R = 1$ (or any strictly positive value $R > 0$):

$$\mathcal{L} = - \sum_{t=0}^{+\infty} \beta^t \left[+2\beta \mu_{t+1} [(1 + \sigma) \pi_t + z_t - \sigma i_t - \pi_{t+1}] \right] \quad (67)$$

The first order conditions are with respect to the Fed's target (forward inflation π_{t+1}) is decoupled from inflation and with respect to the Fed's instrument (policy interest rate i_t):

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \beta (1 + \sigma) \mu_{t+1} = \mu_t \quad (68)$$

$$\frac{\partial \mathcal{L}}{\partial i_t} = 0 \Rightarrow \beta (-\sigma) \mu_{t+1} = -i_t \quad (69)$$

The *Fed's interest rate Euler equation* links recursively the future value of Fed's interest rate to its current value, because of the Fed's relative costs of changing interest rate $R = 1$ in its loss function:

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \left(1 + \frac{1}{\sigma}\right) i_t = \frac{1}{\beta \sigma} i_{t-1} \quad (70)$$

The Hamiltonian system and its boundary conditions can be alternatively written the Euler equation including the policy rate or with the Euler equation with the Lagrange multiplier on inflation (setting $\beta = 1$):

$$\left\{ \begin{array}{l} z_t = \rho z_{t-1} + \varepsilon_t \\ E_t \pi_{t+1} = (1 + \sigma) \pi_t - \sigma i_t + z_t \\ i_t = \frac{1}{\beta(1+\sigma)} i_{t-1} = \left(\frac{1}{\beta(1+\sigma)}\right)^t i_0 \\ \mu_{t+1} = \frac{1}{\beta \sigma} i_t \\ 0 = \lim_{t \rightarrow +\infty} \beta^t \mu_t \\ \frac{\partial L^*}{\partial \pi_0} = 0 = \mu_0 \text{ and } z_0 \text{ given} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} z_t = \rho z_{t-1} + \varepsilon_t \\ E_t \pi_{t+1} = (1 + \sigma) \pi_t - \frac{\beta \sigma^2}{R} \mu_{t+1} + z_t \\ \mu_{t+1} = \frac{1}{\beta(1+\sigma)} \mu_t = \left(\frac{1}{\beta(1+\sigma)}\right)^{t+1} \mu_0 \\ i_t = \beta \sigma \mu_{t+1} \\ 0 = \lim_{t \rightarrow +\infty} \beta^t \mu_t \\ \frac{\partial L^*}{\partial \pi_0} = 0 = \mu_0 \text{ and } z_0 \text{ given} \end{array} \right. \quad (71)$$

Because of the initial transversality condition and because the Lagrange multiplier is decoupled from inflation and the forcing variable, $\mu_t = 0 = i_t$ at all dates. For $z_t = 0$ and setting $\beta = 1$, one has:

$$\begin{pmatrix} \pi_{t+1} \\ \mu_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 + \sigma & -\frac{\sigma^2}{1+\sigma} \\ 0 & \frac{1}{1+\sigma} \end{pmatrix}}_{=\mathbf{M}} \begin{pmatrix} \pi_t \\ \mu_t \end{pmatrix}$$

There is one stable eigenvalue $\frac{1}{1+\sigma}$ and one unstable eigenvalue $1 + \sigma$ in \mathbf{M} . The eigenvalues $\frac{1}{1+\sigma}$ and ρ and related eigenvectors are kept to define the stable subspace after control. For the representation of the dynamics in the stable subspace using inflation and forcing variable as basis vectors (62), the resulting eigenvalue $\lambda_\pi = 1 + \sigma - \sigma F_{\pi,C}$ after control is equal to $\frac{1}{1+\sigma}$.

$$\lim_{R \rightarrow +\infty} \lambda_\pi = \frac{1}{1 + \sigma} = \lim_{R \rightarrow +\infty} 1 + \sigma - \sigma F_{\pi,C} \quad (72)$$

First, this implies that the rule parameter is above one and tends to:

$$\lim_{R \rightarrow +\infty} F_\pi = 1 + \frac{1}{1 + \sigma}$$

Second, this implies that the optimal jump is equal to laissez-faire jump (remind that $Q_{\pi z} = 0$ for maximal inertia):

$$\lim_{R \rightarrow +\infty} \frac{-P_{\pi z}}{P_\pi} = \frac{1}{\rho - \frac{1}{1+\sigma-\sigma F_\pi}} = \frac{1}{\rho - (1 + \sigma)} = N \quad (73)$$

Third, this implies again that the initial interest rate is zero, because the optimal jump is equal to Laissez-faire jump (where $i_0 = 0$), this implies that the initial interest rate is equal to zero (this was noticed from Fed's Euler equation and the initial transversality condition). This implies that the forcing variable rule parameter is equal to the opposite of the Laissez-faire jump times the inflation rule parameter:

$$i_0 = \left(F_{\pi,C} \left(-\frac{P_{\pi z}}{P_{\pi}} \right) + F_{z,C} \right) z_0 = 0 \Rightarrow \frac{-F_{z,C}}{F_{\pi,C}} = \frac{-P_{\pi z}}{P_{\pi}} = N \Rightarrow F_{z,C} = -N F_{\pi,C} \quad (74)$$

Fourth, one checks that the Lagrange multiplier of inflation is zero at all dates, so that maximal inertia monetary policy is always time-consistent.

$$\mu_t = \left(P_{\pi} \left(-\frac{P_{\pi z}}{P_{\pi}} \right) + P_{\pi z} \right) z_t = 0 \cdot z_t = 0 \quad (75)$$

Figure 1: "Inflation" eigenvalue (vertical axis) function of inflation rule parameter (horizontal axis).

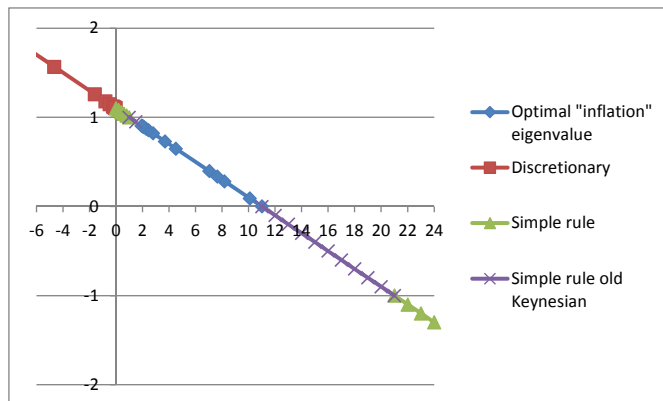


Figure 2: Optimal and discretionary policy rule parameters function of the relative cost of changing policy rate R

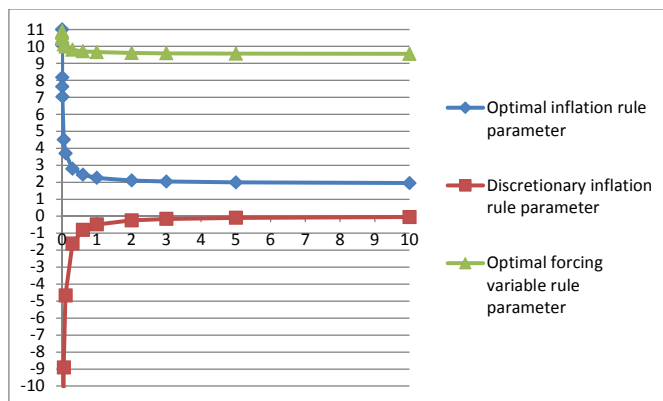


Figure 3: Optimal and discretionary eigenvalue function of the relative cost of changing policy rate R

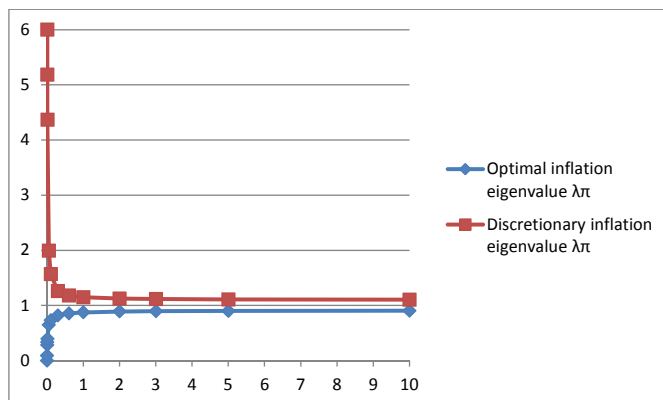


Figure 4: Welfare gains in percentage of laissez-faire welfare as a function of the relative cost of changing policy rate R:

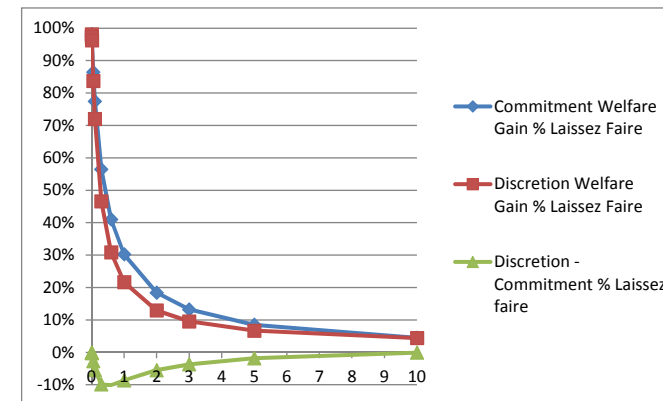


Figure 5: Initial inflation jump function / laissez faire jump, function of the relative cost of changing policy interest rate R

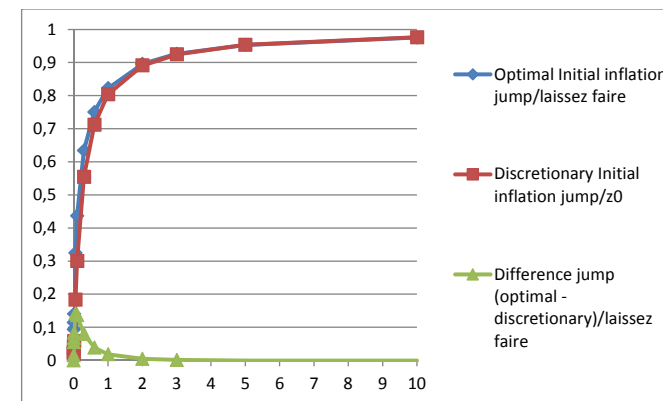


Figure 6: Initial interest rate/initial forcing variable function of the relative cost of changing policy interest rate R

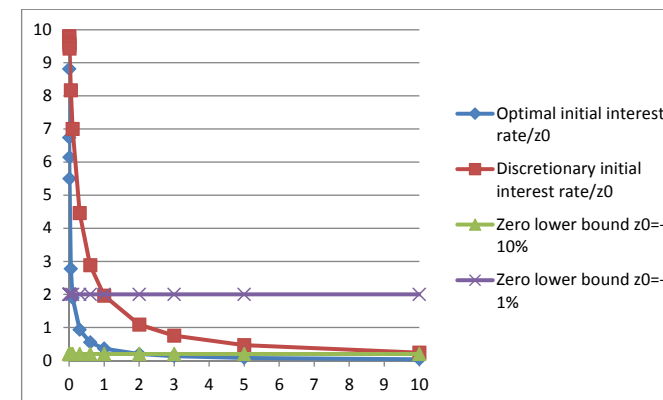


Figure 7: Inflation (vertical axis) impulse response function after -10% shock on forcing variable (time: horizontal axis).

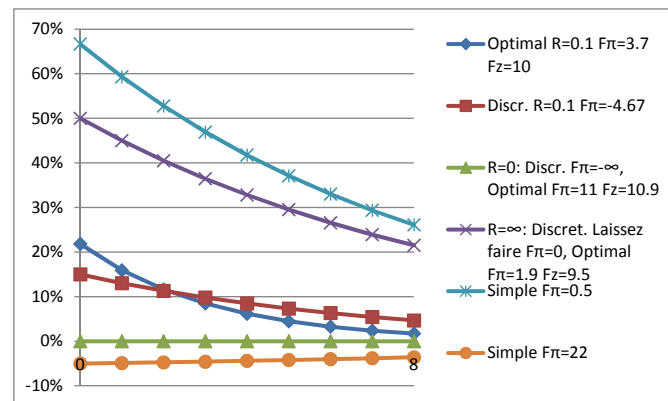


Figure 8: Phase diagrams: inflation (vertical axis) after -10% shock on forcing variable (horizontal axis).

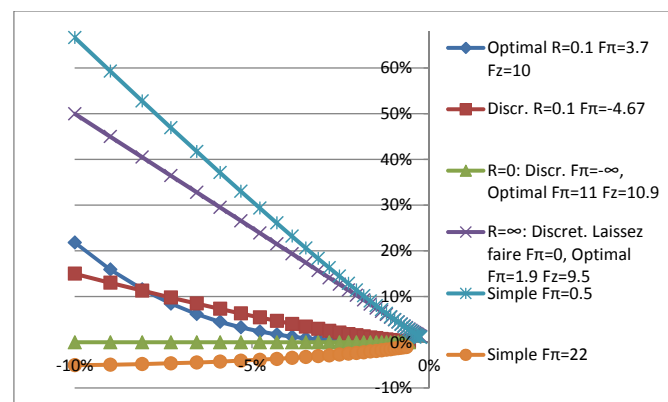


Figure 9: Inflation function of forcing variable: old Keynesian initial predetermined inflation 50% vs Commitment optimal jump of initial inflation for different relative cost of changing the policy rate R .

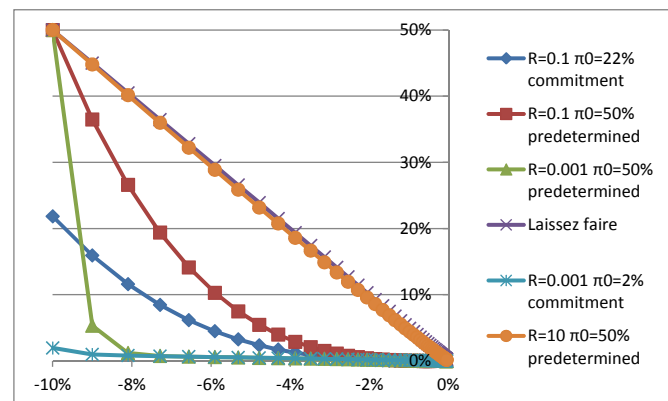
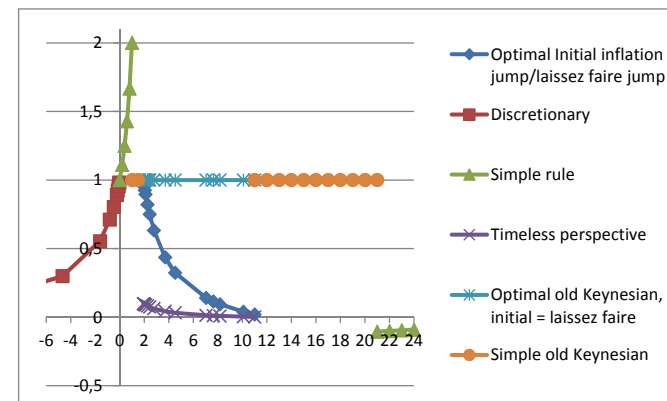


Figure 10: Initial inflation jump π_0 function of the inflation rule parameter F_π .



Optimal policy: Expected inflation π_t (vertical axis) following a -10% autoregressive (0.9) shock z_t (horizontal axis).

Figure 11. Maximal cost of changing interest rate $R=10$. Rule: inflation $F_\pi=1.95$, shock: $F_z=9.56$, Initial inflation $\pi_0=0.49$.

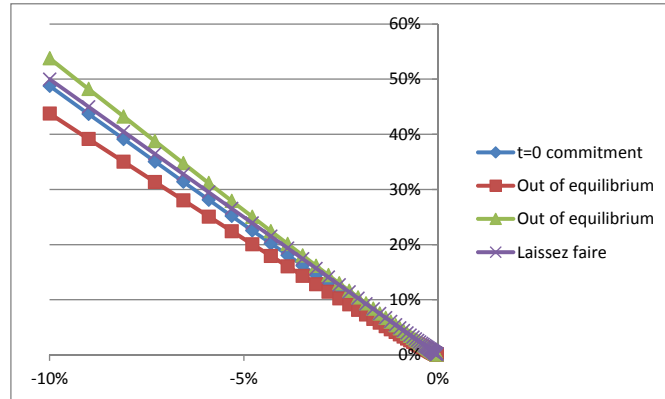


Figure 12 Moderate cost of changing interest rate $R=0.1$. Rule: inflation $F_\pi=3.7$, shock: $F_z=9.98$. Initial inflation $\pi_0=0.21$.

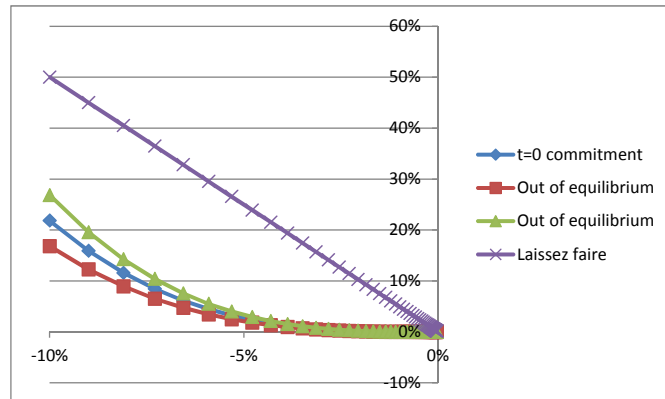
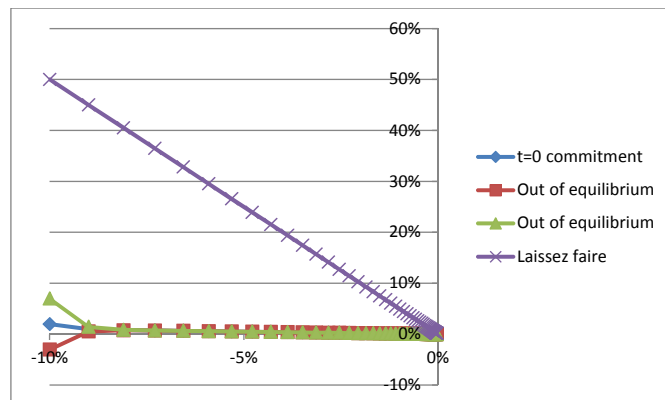


Figure 13 Negligible cost of changing interest rate $R=0.001$. Rule: inflation $F_\pi=10.09$, shock: $F_z=10.8$. Initial inflation 0.02 .



Discretionary policy: Expected inflation (vertical axis), -10% shock with autoregressive (0.9) shock (horizontal axis),

Figure 14. Very large cost $R=10$. Rule: inflation $F_{\pi,D}=-0.05$, shock: $F_{z,D}=0$. Initial inflation 0.49 .

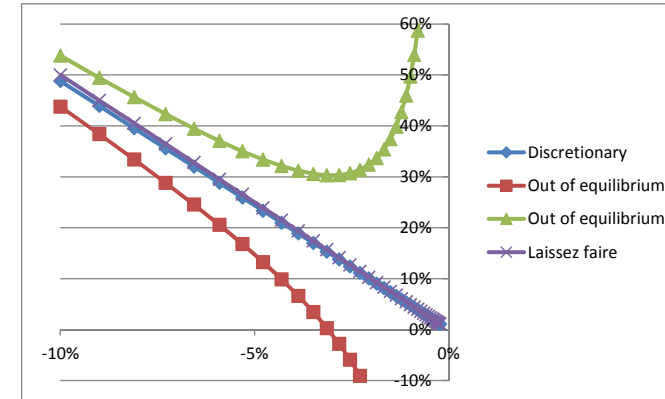


Figure 15. Moderate cost $R=0.1$. Rule: inflation $F_{\pi,D}=-4.7$, shock: $F_{z,D}=0$. Initial inflation 0.15 .

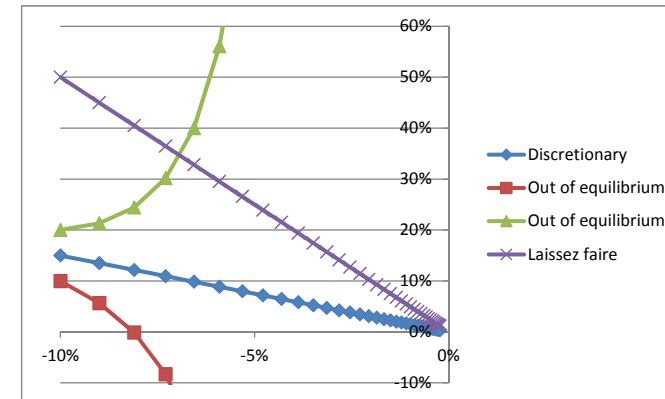
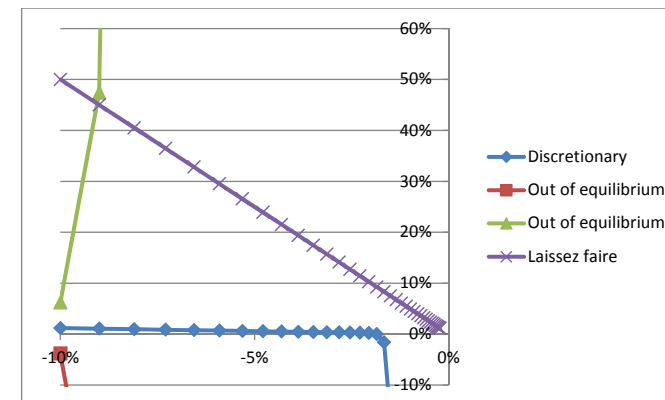


Figure 16. Negligible cost $R=0.001$. Rule: inflation $F_{\pi,D}=-81.7$, shock: $F_{z,D}=0$. Initial inflation 0.012 .



Optimal policy: Expected inflation π_t (vertical axis) following a -10% autoregressive (0.9) shock z_t (horizontal axis).

Figure 17. Maximal cost of changing interest rate $R=10$. Rule: **inflation $F_\pi=1.95$, shock: $F_z=9.56$** , Initial inflation $\pi_0=0.49$.

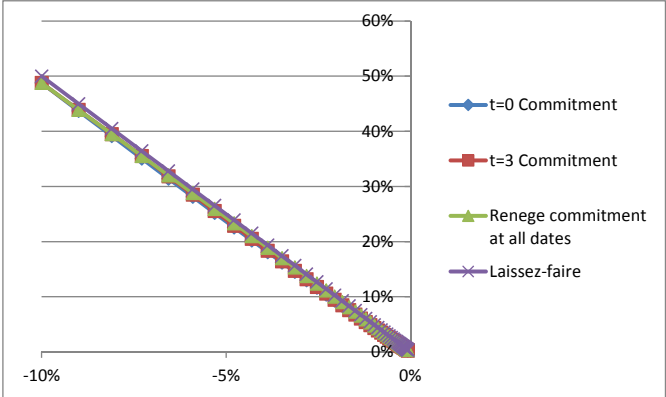


Figure 18 Moderate cost of changing interest rate $R=0.1$. Rule: **inflation $F_\pi=3.7$, shock: $F_z=9.98$** , Initial inflation $\pi_0=0.21$.

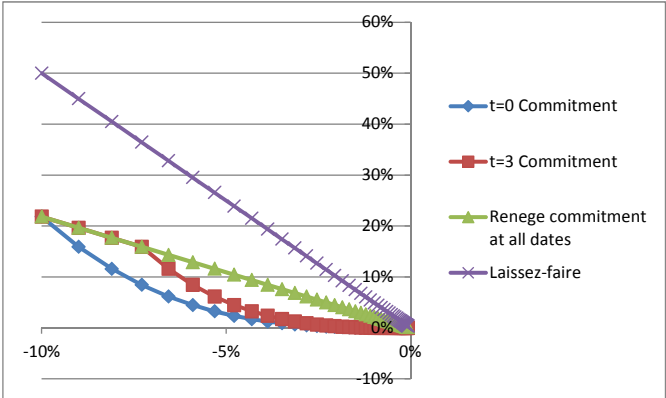


Figure 19 Negligible cost of changing interest rate $R=0.001$ Rule: **inflation $F_\pi=10.09$, shock: $F_z=10.8$** , Initial inflation 0.02 .

