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Research Report

Search and matching frictions and business cycle fluctuations in Bulgaria: Technical Appendix

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1 Households' problem

The problem can be simplified as

$$\begin{aligned}
\mathcal{L} &= E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \ln C_t + \phi \ln(1-N_t) \\
& \quad -\lambda_t [C_t + K_{t+1} - (1-\delta)K_t + b_0 S_t^{\eta}(1-N_t) - (1-\tau^l) r_t K_t - (1-\tau^l) w_t N_t - \Pi_t - G_t^{tr}] \\
&+ \mu_t [(1-\psi)N_t + p_t S_t (1-N_t) - N_{t+1}] \Bigg\}, \qquad (1.1)
\end{aligned}$$

where λ_t, μ_t are the Lagrangean multiplier in front of the households' budget constraint and the law of motion for employment.

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FOCs:

$$C_t : \frac{1}{C_t} = \lambda_t \quad (1.2)$$

$$K_{t+1} : \lambda_t = \beta E_t \lambda_{t+1}[(1-\tau^k)\alpha \frac{Y_{t+1}}{K_{t+1}} + (1-\delta)] \quad (1.3)$$

$$S_t : \lambda_t b_0 \eta S_t^{\eta-1}(1-N_t) = \mu_t p_t (1-N_t) \quad (1.4)$$

$$N_{t+1} : \beta E_t \lambda_{t+1} \left[(1-\tau^l) w_{t+1} + b_0 S_{t+1}^{\eta} - \frac{\phi}{1-N_{t+1}} \right] + \beta E_t \mu_{t+1}[(1-\psi) - p_{t+1} S_{t+1}] = \mu_t \quad (1.5)$$

$$\lim_{t\to\infty} \frac{1}{C_t} K_{t+1} = 0 \quad (1.6)$$

Substitute out the Lagrangean multiplier attached to the budget constraint, as well as the probability of a match

$$\frac{1}{C_t} b_0 \eta S_t^{\eta-1} = \mu_t p_t \tag{1.7}$$

$$\begin{aligned} -\frac{\beta\phi}{1-N_{t+1}} + \beta E_t \frac{1}{C_{t+1}}(1-\tau^l) w_{t+1} + \beta E_t \frac{1}{C_{t+1}} b_0 S_{t+1}^{\eta} \\ + \beta E_t \mu_{t+1}(1-\psi) - \beta \mu_{t+1} p_{t+1} S_{t+1} = \mu_t \end{aligned} \tag{1.8}$$

Note that

$$\mu_t = \frac{b_0 \eta S_t^{\eta-1}}{C_t p_t} \tag{1.9}$$

Rearrange terms

$$\beta E_t \frac{1}{C_{t+1}} \left[(1-\tau^l) w_{t+1} + b_0 S_{t+1}^{\eta} - \frac{\phi}{1-N_{t+1}} \right] + \beta E_t \mu_{t+1}[(1-\psi) - p_{t+1} S_{t+1}] = \mu_t \tag{1.10}$$

or

$$\beta E_t \frac{1}{C_{t+1}} \left[(1-\tau^l) w_{t+1} + b_0 S_{t+1}^{\eta} - \frac{\phi}{1-N_{t+1}} \right] + \beta E_t \frac{b_0 \eta S_{t+1}^{\eta-1}}{C_{t+1} p_{t+1}} [(1-\psi) - p_{t+1} S_{t+1}] = \frac{b_0 \eta S_t^{\eta-1}}{C_t p_t}$$

Multiply by p_t

$$p_t \beta E_t \frac{1}{C_{t+1}} \left[(1-\tau^l) w_{t+1} + b_0 S_{t+1}^{\eta} - \frac{\phi}{1-N_{t+1}} \right] + p_t \beta E_t \frac{b_0 \eta S_{t+1}^{\eta-1}}{C_{t+1} p_{t+1}} [(1-\psi) - p_{t+1} S_{t+1}] = \frac{b_0 \eta S_t^{\eta-1}}{C_t}$$

or

$$\frac{b_0 \eta S_t^{\eta-1}}{C_t} = p_t \beta E_t \left\{ \frac{1}{C_{t+1}} \left[(1-\tau^l) w_{t+1} + b_0 S_{t+1}^{\eta} \right] + \frac{\phi}{1-N_{t+1}} + \frac{b_0 \eta S_{t+1}^{\eta-1}}{C_{t+1} p_{t+1}} [(1-\psi) - p_{t+1} S_{t+1}] \right\}$$

2 Firm's problem

The problem of the firm is also one of constrained optimization nature:

$$\max_{\{K_t,V_t,N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_t \Big\{ A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - r_t K_t - aV_t - \nu_t[(1-\psi)N_t + q_t V_t - N_{t+1}] \Big\}, \tag{2.1}$$

where

$$\beta_t = \beta E_{t-1}\left(\frac{C_t}{C_{t-1}}\right) \tag{2.2}$$

is the stochastic discount factor.

FOCs:

$$K_t : \alpha \frac{Y_t}{K_t} = r_t \tag{2.3}$$

$$V_t : \nu_t q_t = -a \tag{2.4}$$

$$N_t : \beta_t \left[(1-\alpha)\frac{Y_{t+1}}{N_{t+1}} - w_{t+1} - \nu_{t+1}(1-\psi) \right] = -\nu_t \tag{2.5}$$

From the optimality condition for vacancies, express $\nu_t = -a/q_t$ and plug it into the FOC for employment to obtain

$$\beta_t \left[(1-\alpha)\frac{Y_{t+1}}{N_{t+1}} - w_{t+1} + \frac{a(1-\psi)}{q_{t+1}} \right] = \frac{a}{q_t} \tag{2.6}$$

or

$$\beta E_{t-1}\left(\frac{C_t}{C_{t-1}}\right) \left[(1-\alpha)\frac{Y_{t+1}}{N_{t+1}} - w_{t+1} + \frac{a(1-\psi)}{q_{t+1}} \right] = \frac{a}{q_t} \tag{2.7}$$

In the literature, this is also referred to as the job creation condition (JCC). In other words, this equation characterizes the firm's labor demand.

3 Steady-State

$$1 = \beta[(1-\tau^k)\alpha\frac{y}{k} + 1 - \delta] \tag{3.1}$$

$$\frac{a}{c} = \mu(1-\gamma)\frac{m}{v} \tag{3.2}$$

$$\frac{b_0\eta s^{\eta-1}(1-n)}{c} = \mu\gamma\frac{m}{s} \tag{3.3}$$

$$m = \psi n \tag{3.4}$$

$$u = 1 - n \tag{3.5}$$

$$m = v^{1-\gamma}[su]^{\gamma} \tag{3.6}$$

$$c + \delta k + b_0 s^{\eta}(1-n) + av = y \tag{3.7}$$

$$y = Ak^{\alpha}n^{1-\alpha} \tag{3.8}$$

$$\frac{\beta}{c}[(1-\tau^l)(1-\alpha)\frac{y}{n} + b_0 s^{\eta}] + \frac{\beta\phi}{1-n} = \mu[1 - \beta(1 - \psi - \gamma\frac{m}{1-n})] \tag{3.9}$$

$$p = \frac{m}{s(1-n)} = \left(\frac{\theta}{s}\right)^{1-\gamma} \tag{3.10}$$

$$\theta = \frac{m}{1-n} \tag{3.11}$$

$$q = \frac{m}{v} = \left(\frac{s}{\theta}\right)^{\gamma} \tag{3.12}$$

4 Log-linearization

4.1 Capital accumulation equation

$$k_{t+1} = i_t + (1-\delta)k_t \tag{4.1}$$

Take natural logarithms from both sides to obtain

$$\ln k_{t+1} = \ln[i_t + (1-\delta)k_t] \tag{4.2}$$

Totally differentiate with respect to time to obtain

$$\frac{d\ln k_{t+1}}{dt} = \frac{d\ln[i_t + (1-\delta)k_t]}{dt} \tag{4.3}$$

$$\hat{k}_{t+1} = \frac{d[i_t + (1-\delta)k_t]}{dt}\frac{1}{i + (1-\delta)k} \tag{4.4}$$

Using that in steady state $k = i + (1-\delta)k$, or $i = \delta k$

$$\hat{k}_{t+1} = \frac{di_t}{kdt} + (1-\delta)\frac{dk_t}{kdt} \tag{4.5}$$

$$\hat{k}_{t+1} = \delta\hat{i}_t + (1-\delta)\hat{k}_t \tag{4.6}$$

4.2 Employment dynamics equation

$$n_{t+1} = (1-\psi)n_t + m_t \tag{4.7}$$

Take natural logarithms from both sides to obtain

$$\ln n_{t+1} = \ln[(1-\psi)n_t + m_t] \tag{4.8}$$

Totally differentiate w.r.t. time to obtain

$$\frac{d\ln n_{t+1}}{dt} = \frac{d\ln[(1-\psi)n_t + m_t]}{dt} \tag{4.9}$$

$$\frac{dn_{t+1}}{dt}\frac{1}{n} = \frac{d[(1-\psi)n_t + m_t]}{dt}\frac{1}{(1-\psi)n + m} \tag{4.10}$$

Using that in steady state $n = (1-\psi)n + m$,

$$\frac{dn_{t+1}}{dt}\frac{1}{n} = (1-\psi)\frac{dn_t}{dt}\frac{1}{n} + \frac{dm_t}{dt}\frac{1}{n} \tag{4.11}$$

or

$$\frac{dn_{t+1}}{dt}\frac{1}{n} = (1-\psi)\frac{dn_t}{dt}\frac{1}{n} + \frac{dm_t}{dt}\frac{1}{n}\frac{m}{m} \tag{4.12}$$

Pass to log-deviations to obtain

$$n\hat{n}_{t+1} = (1-\psi)n\hat{n}_t + m\hat{m}_t \tag{4.13}$$

4.3 Matching function

$$m_t = v_t^{1-\gamma}[s_t(1-n_t)]^\gamma \tag{4.14}$$

Take natural logarithms from both sides to obtain

$$\ln m_t = (1-\gamma)\ln v_t + \gamma \ln s_t + \gamma \ln(1-n_t) \tag{4.15}$$

Totally differentiate w.r.t. time to obtain

$$\frac{d\ln m_t}{dt} = (1-\gamma)\frac{d\ln v_t}{dt} + \gamma\frac{d\ln s_t}{dt} + \gamma\frac{d\ln(1-n_t)}{dt} \tag{4.16}$$

Pass to log-deviations

$$\hat{m}_t = (1-\gamma)\hat{v}_t + \gamma\hat{s}_t - \frac{\gamma n}{1-n}\hat{n}_t \tag{4.17}$$

4.4 Log-linearizing transition probability from unemployment to employment

$$p_t = \frac{M_t}{S_t(1-N_t)} \tag{4.18}$$

$$\hat{p}_t = \hat{m}_t - \hat{s}_t + \frac{n}{1-n}\hat{n}_t \tag{4.19}$$

Alternatively

$$p_t = \left(\frac{\theta_t}{S_t}\right)^{1-\gamma} \tag{4.20}$$

or

$$\hat{p}_t = (1-\gamma)\hat{\theta}_t - (1-\gamma)\hat{s}_t \tag{4.21}$$

4.5 Log-linearizing market tightness

$$\theta_t = \frac{M_t}{1 - N_t} \tag{4.22}$$

$$\hat{\theta}_t = \hat{m}_t + \frac{n}{1-n}\hat{n}_t \tag{4.23}$$

4.6 Log-linearizing transition probability from an unfilled vacancy to a filled one

$$q_t = \frac{m_t}{v_t} \tag{4.24}$$

$$\hat{q}_t = \hat{m}_t - \hat{v}_t \tag{4.25}$$

Alternatively,

$$q_t = \left(\frac{S_t}{\theta_t}\right)^{\gamma} \tag{4.26}$$

or

$$\hat{q}_t = \gamma \hat{s}_t - \gamma \hat{\theta}_t \tag{4.27}$$

4.7 Production function

$$y_t = A_t k_t^{\alpha} n_t^{1-\alpha} \tag{4.28}$$

Take natural logarithms from both sides to obtain

$$\ln y_t = \ln A_t + \alpha \ln k_t + (1-\alpha)\ln n_t \tag{4.29}$$

Totally differentiate w.r.t. time to obtain

$$\frac{d\ln y_t}{dt} = \frac{d\ln A_t}{dt} + \alpha\frac{d\ln k_t}{dt} + (1-\alpha)\frac{d\ln n_t}{dt} \tag{4.30}$$

Pass to log-deviations to obtain

$$\hat{y}_t = \hat{A}_t + \alpha\hat{k}_t + (1-\alpha)\hat{n}_t \tag{4.31}$$

4.8 Euler equation

$$\frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}}[(1-\tau^k)\alpha\frac{y_{t+1}}{k_{t+1}} + (1-\delta)] \tag{4.32}$$

Take natural logarithms from both sides to obtain

$$\ln\frac{1}{c_t} = \ln\beta E_t \frac{1}{c_{t+1}}[\alpha\frac{y_{t+1}}{k_{t+1}} + (1-\delta)] \tag{4.33}$$

$$-\ln c_t = \ln\beta - \ln E_t c_{t+1} + \ln E_t[(1-\tau^k)\alpha\frac{y_{t+1}}{k_{t+1}} + (1-\delta)] \tag{4.34}$$

Totally differentiate w.r.t. time and simplify to obtain

$$-\frac{d\ln c_t}{dt} = -\frac{d\ln E_t c_{t+1}}{dt} + \frac{d\ln E_t[(1-\tau^k)\alpha\frac{y_{t+1}}{k_{t+1}} + (1-\delta)]}{dt} \tag{4.35}$$

or

$$-\hat{c}_t = -\hat{c}_{t+1} + \frac{dE_t[(1-\tau^k)\alpha\frac{y_{t+1}}{k_{t+1}} + (1-\delta)]}{dt}\frac{1}{\alpha\frac{y}{k} + (1-\delta)} \tag{4.36}$$

Noting that in equilibrium $\beta = (1-\tau^k)\alpha\frac{y}{k} + (1-\delta)$

$$-\hat{c}_t = -\hat{c}_{t+1} + \beta(1-\tau^k)\alpha\frac{dE_t\frac{y_{t+1}}{k_{t+1}}}{dt} \tag{4.37}$$

$$-\hat{c}_t = -\hat{c}_{t+1} + \beta(1-\tau^k)\alpha[\frac{dy_{t+1}k - dk_{t+1}y}{dtk^2}] \tag{4.38}$$

$$-\hat{c}_t = -\hat{c}_{t+1} + \beta(1-\tau^k)\alpha\frac{y}{k}\hat{y}_{t+1} - \beta(1-\tau^k)\alpha\frac{y}{k}\hat{k}_{t+1} \tag{4.39}$$

4.9 FOC vacancy rate

$$\frac{a}{c_t} = \mu_t(1-\gamma)\frac{m_t}{v_t} \tag{4.40}$$

Take natural logarithms from both sides to obtain

$$\ln\frac{a}{c_t} = \ln\mu_t(1-\gamma)\frac{m_t}{v_t} \tag{4.41}$$

$$\ln a - \ln c_t = \ln\mu_t + \ln(1-\gamma) + \ln m_t - \ln v_t \tag{4.42}$$

Totally differentiate w.r.t. time to obtain

$$\frac{d\ln a}{dt}-\frac{d\ln c_t}{dt}=\frac{d\ln\mu_t}{dt}+\frac{d\ln(1-\gamma)}{dt}+\frac{d\ln m_t}{dt}-\frac{d\ln v_t}{dt} \tag{4.43}$$

Simplify to obtain

$$-\frac{d\ln c_t}{dt}=\frac{d\ln\mu_t}{dt}+\frac{d\ln m_t}{dt}-\frac{d\ln v_t}{dt} \tag{4.44}$$

Pass to log-deviations

$$-\hat{c}_t=\hat{\mu}_t+\hat{m}_t-\hat{v}_t \tag{4.45}$$

4.10 FOC search effort

$$\frac{b_0\eta s_t^{\eta-1}(1-n_t)}{c_t}=\mu_t\gamma\frac{m_t}{s_t} \tag{4.46}$$

Take natural logarithms from both sides to obtain

$$\ln\frac{b_0\eta s_t^{\eta-1}(1-n_t)}{c_t}=\ln\mu_t\gamma\frac{m_t}{s_t} \tag{4.47}$$

$$\ln b_0+\ln\eta+(\eta-1)\ln s_t+\ln(1-n_t)-\ln c_t=\ln\mu_t+\ln\gamma+\ln m_t-\ln s_t \tag{4.48}$$

Totally differentiate w.r.t time and simplify to obtain

$$(\eta-1)\frac{d\ln s_t}{dt}+\frac{d\ln(1-n_t)}{dt}-\frac{d\ln c_t}{dt}=\frac{d\ln\mu_t}{dt}+\frac{d\ln m_t}{dt}-\frac{d\ln s_t}{dt} \tag{4.49}$$

$$(\eta-1)\hat{s}_t-\frac{n}{1-n}\hat{n}_t-\hat{c}_t=\hat{\mu}_t+\hat{m}_t-\hat{s}_t \tag{4.50}$$

or

$$\eta\hat{s}_t-\frac{n}{1-n}\hat{n}_t-\hat{c}_t=\hat{\mu}_t+\hat{m}_t \tag{4.51}$$

4.11 Dynamic FOC employment

$$[E_t\frac{\beta}{c_{t+1}}[(1-\alpha)\frac{y_{t+1}}{n_{t+1}}+b_0s_{t+1}^{\eta}]-\frac{\beta\phi}{1-n_{t+1}}]=\mu_t-\beta E_t\mu_{t+1}[1-\psi-\gamma\frac{m_{t+1}}{1-n_{t+1}}] \tag{4.52}$$

Take natural logarithms from both sides and totally differentiate w.r.t time to obtain

$$\frac{d[E_t\frac{\beta}{c_{t+1}}[(1-\alpha)\frac{y_{t+1}}{n_{t+1}}+b_0s_{t+1}^{\eta}]-\frac{\beta\phi}{1-n_{t+1}}]}{dt}=\frac{d\mu_t}{dt}-\frac{d\beta E_t\mu_{t+1}[1-\psi-\gamma\frac{m_{t+1}}{1-n_{t+1}}]}{dt} \tag{4.53}$$

$$\begin{aligned}&\frac{\beta}{c}(1-\alpha)\frac{y}{n}E_t\hat{c}_{t+1}+\frac{\beta}{c}(1-\alpha)\frac{y}{n}E_t\hat{y}_{t+1}-\frac{\beta}{c}(1-\alpha)\frac{y}{n}E_t\hat{n}_{t+1}\\ &=\mu\hat{\mu}_t-\beta\mu E_t\hat{\mu}_{t+1}[1-\psi-\gamma\frac{m}{1-n}]+\beta\gamma\mu\frac{m}{1-n}E_t\hat{m}_{t+1}+\beta\gamma\mu\frac{mn}{(1-n)^2}E_t\hat{n}_{t+1}\end{aligned} \tag{4.54}$$

4.12 Market clearing

$$y_t = c_t + k_{t+1} - (1-\delta)k_t + g_t + b_0 s_t^\eta (1-n_t) + av_t \tag{4.55}$$

Take natural logarithms from both sides and totally differentiate w.r.t to time to obtain

$$\frac{d}{dt}\ln y_t = \frac{d}{dt}\Big[c_t + k_{t+1} - (1-\delta)k_t + g_t + b_0 s_t^\eta (1-n_t) + av_t\Big] \tag{4.56}$$

$$y\hat{y}_t = c\hat{c}_t + \hat{k}k_{t+1} - (1-\delta)\hat{k}k_t + g\hat{g}_t + \eta b_0 s^\eta (1-n)\hat{s}_t - b_0 s^\eta n\hat{n}_t + av\hat{v}_t \tag{4.57}$$

4.13 Nash bargaining rule

$$w_t = \gamma\bigg((1-\alpha)\frac{y_t}{n_t} + \frac{av_t}{1-n_t}\bigg) + (1-\gamma)\bigg(-\frac{1-n_t}{\phi c_t} - b_0 s_t^\eta (1-n_t)\bigg) \tag{4.58}$$

Take natural logarithms from both sides and totally differentiate w.r.t to time to obtain

$$\frac{d}{dt}\ln w_t = \frac{d}{dt}\ln\bigg\{\gamma\bigg((1-\alpha)\frac{y_t}{n_t} + \frac{av_t}{1-n_t}\bigg) + (1-\gamma)\bigg(-\frac{1-n_t}{\phi c_t} - b_0 s_t^\eta (1-n_t)\bigg)\bigg\} \tag{4.59}$$

$$w\hat{w}_t = \frac{d}{dt}\bigg\{\gamma\bigg((1-\alpha)\frac{y_t}{n_t} + \frac{av_t}{1-n_t}\bigg) + (1-\gamma)\bigg(-\frac{1-n_t}{\phi c_t} - b_0 s_t^\eta (1-n_t)\bigg)\bigg\} \tag{4.60}$$

$$w\hat{w}_t = \gamma(1-\alpha)\frac{y}{n}\hat{y}_t - \gamma(1-\alpha)\frac{y}{n}\hat{n}_t + \frac{d}{dt}\bigg\{\bigg(\frac{av_t}{1-n_t}\bigg) + (1-\gamma)\bigg(-\frac{1-n_t}{\phi c_t} - b_0 s_t^\eta (1-n_t)\bigg)\bigg\}$$

$$w\hat{w}_t = \gamma(1-\alpha)\frac{y}{n}\hat{y}_t - \gamma(1-\alpha)\frac{y}{n}\hat{n}_t + \frac{av}{1-n}\hat{v}_t - \frac{avn}{(1-n)^2}\hat{n}_t + \frac{d}{dt}\bigg\{(1-\gamma)\bigg(-\frac{1-n_t}{\phi c_t} - b_0 s_t^\eta (1-n_t)\bigg)\bigg\}$$

$$\begin{aligned} w\hat{w}_t = \gamma(1-\alpha)\frac{y}{n}\hat{y}_t - \gamma(1-\alpha)\frac{y}{n}\hat{n}_t + \frac{av}{1-n}\hat{v}_t - \frac{avn}{(1-n)^2}\hat{n}_t + (1-\gamma)\frac{1-n}{\phi c}\hat{c}_t + (1-\gamma)\frac{n}{\phi c}\hat{n}_t \\ -\eta b_0 s^\eta (1-n)\hat{s}_t + b_0 s^\eta n\hat{n}_t \end{aligned}$$