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Aging, Social Security Design, and Capital Accumulation

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Abstract

This paper analyzes the impact of aging on capital accumulation and welfare in a country with a sizable unfunded social security system. Using a two-period overlapping generation model with potentially endogenous retirement decisions, the paper shows that the type of aging, i.e. declining fertility or increasing longevity, and the type of unfunded social security system, i.e. defined contributions or defined benefits, are important in understanding this impact. Moreover, the analysis provides a refinement to common policy recommendations that favor eliminating mandatory early retirement regulations in aging societies: aging leads to a greater increase in welfare when it is driven by an increasing longevity and the retirement age is unregulated. In comparison, when aging is driven by a decreasing fertility rate, a mandatory retirement system fosters more savings and, thus, income and welfare, in a closed economy with unfunded pension system based on defined contributions.

JEL-Codes: H200, F420, H800.

Keywords: aging, public finance sustainability, social security.

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1 Introduction

Demographic aging poses a major challenge to all industrialized economies and a large number of developing countries. According to UN projections, total world population will increase by 40% and median age on the planet will increase by 7.8 years within the next four decades. Compared to the recent history, these changes represent a significant slow-down in population growth and a considerable acceleration in aging.

Although an increase in the average age is a common trend around the world, the factors that lead to such changes vary across countries. In the short and medium term these may be led by transitory events such as out-migration of young population, wars or aging of past baby-boom generations. However, in the long-term, it can be traced back to decreases in fertility rates and increases in longevity, albeit at different magnitudes of importance in different economies.

Economic implications of demographic aging are complex, and they are not always well understood in public discussions. Some consequences are clearly unfavorable: aging, whether it is driven by a decrease in fertility or an increase in longevity, pushes the old-age dependency rate up, *i.e.* a larger group of elderly people relative to economically active population. This will, in turn, increase the pressure on financial balances of unfunded pensions. In contrast, some consequences of aging are perceived to be positive: if the decline in fertility outweighs the increase in longevity, total population would decrease. This outcome may be welcomed by some on the basis of environmental concerns. Finally, there are some ambiguous consequences. An example is the effect of aging on capital accumulation, a key determinant of growth, which we investigate in this paper.

Studying the effect of aging on capital accumulation is particularly difficult when a large number of discretionary policy choices can affect the outcomes. In this paper, we use a two period overlapping generation model to show that the effect of aging on capital accumulation and welfare depend on: i) the type of aging, *i.e.* decreasing fertility or increasing longevity, ii) the type of unfunded social security system, *i.e.* defined contribution (DC) or defined benefit (DB), and finally, iii) the regulation of the retirement age, *i.e.* mandatory early retirement vs. *laissez-faire*.

To fix these ideas, we set up an economic environment where each individual lives two periods. The first period of her life has a unitary length, while the second one has a variable longevity. In the first period, the individual works and earns a wage equal to her marginal productivity net of social security contributions if any. This income is then devoted to consumption in the first period and saving for future consumption. In the second period, she works for a fraction of her remaining lifetime, where the work

duration is determined by the balance between the marginal income and disutility created by the work. In the case of mandatory retirement, however, the optimal retirement choice may be overwritten and she may be forced to retire earlier than she would like to. In the end, the second period consumption is equal to the earnings from within period work, savings from first period with interest earnings, and finally pension benefits from the unfunded social security system.

This structure enables us to elaborate on our main results by introducing a number of institutional and demographic factors *vis a vis* the standard Diamond case, where individuals do not work in the second period of their life time and there is no PAYG pension system. First, we allow work in the second period of life and investigate how a mandatory early retirement rule affects the outcome. Second, we consider different types of unfunded social security systems in order to see how incentives respond to changes in demography under different pension system obligations and entitlements. Third, and finally, we also investigate these effects under different aging profiles, i.e. fertility driven vs. longevity driven changes in age composition.

The main contribution of this study is, then, to show the incidence of these three factors on the effects of aging on capital accumulation and welfare. In the standard Diamond case, an increase in fertility decreases capital accumulation in the absence of a PAYG pension system, as capital is diluted with more workers. In our framework, this depressive effect is reinforced if the country has a DC pension system. In contrast, it is weakened or possibly reversed with DB pensions. Similar results are also derived with increasing longevity. A small increase in longevity has a fostering effect on capital accumulation in the standard case. Introducing PAYG pensions and possibility of work in the second period, however, diminishes and potentially reverses the fostering effect.

The economics literature comprises a large number of studies devoted to understanding the effects of demographic aging in different settings. These could be classified on the basis of numerous criteria: explicit recognition of the distinction between different sources of aging, e.g. longevity and fertility changes, consideration of different social security systems, and characterization of growth in exogenous or endogenous settings. We will not provide an exhaustive review of this large field. However, a subset of these studies which investigate how institutional factors and behavioral responses may affect the impact of aging on capital accumulation is more relevant for our purposes.

An interesting discussion on the effect of longevity increase on growth is provided by Bloom et.al. (2007). The authors point out that, in theory, improvements in healthy life expectancy should generate increases in the average age of retirement, with little effect on savings rates. In many countries, however, retirement incentives in social security pro-

grams prevent retirement ages from keeping pace with changes in life expectancy, leading to an increased need for life-cycle savings. Using a cross-country panel of macroeconomic data, then, the paper finds that increased longevity raises aggregate savings rates in countries with universal pension coverage and retirement incentives. Similarly, Bloom et. al. (2003) show that aging leads to more capital accumulation even if retirement is endogenous. Echevarria (2004) reaches the same conclusion. Kalemli-Ozcan *et al.* (2000) show that the positive effect of mortality decline on capital accumulation is made larger if education decisions are endogenous.

De la Croix and Licandro (1999) and Zhang et al. (2001, 2003) argue that the effect of increasing longevity depends on its initial level. For low levels of life expectancy the effect is positive but it can turn negative for high levels. Similarly, Miyazawa (2006) also shows that the effect of an increase in longevity on economic growth has a hump-shaped pattern. This is the result of a two-effect system. First, higher longevity increases the aggregate saving rate directly by increasing precautionary saving for the prolonged retirement and indirectly by increasing the accidental bequests (bequest-wage ratio is important because the higher income group has a higher propensity to save). Second, it reduces the frequency of accidental bequests, which implies that the population share of the higher income group decreases. This leads to a reduction in aggregate savings. The relative shares of these factors change over the aging horizon. This is also true for the income inequality (first positive, then negative). Kinugasa and Mason (2006) provide empirical support to shows that an increase of wealth across countries is likely with mortality decline.

Let us also mention the effect of aging on human capital and hence on growth. The idea that an increase in longevity can foster investment in education became well known after Ben-Porath (1967). Recently, Ludwig and Vogel (2010), using a two-period overlapping generation model similar to ours, look at the effect of longevity on human capital accumulation, but also extend the analysis for fertility. They show that whereas declining fertility stimulates education and capital accumulation, increasing fertility has an ambiguous effect on both.

Among the studies that link the impact of aging with social security systems, Ito and Tabata (2008) find that the unfunded social security system provides a sufficient mechanism to have such a hump shaped relationship between longevity and per capita output. Tabata (2014) looks at the effect of a shift from a DB to a DC PAYG pension on growth. He shows that this shift is growth enhancing and alleviates the cost of aging. Heijdra, B. and J. Mierau (2011) also compare the relative effects of DB and DC PAYG pensions on economic growth with aging. They show that the DC formula fares better than the DB one in facilitating growth. They also show that raising the retirement age as a response to

an increase in longevity dampens the growth gains. The analysis in this paper provides a comparison of several different institutional settings, i.e. different social security systems and retirement age policies, and types of aging in a unified framework. Therefore, it provides a consistent survey of the welfare effects of demographic aging under various conditions.

Overall, some of the theoretical results we develop in this paper are known in the existing literature. Others are novel including the surprisingly positive effect of mandatory early retirement on welfare. In addition to such novelties, a major contribution of this paper is to organize all these findings in a unified framework that can teach us what would be the ideal social security system when society is aging. We assume throughout our analysis that the pay-as-you-go (PAYG) pension system is given and that, given the assumption of dynamic inefficiency, this system is welfare-worsening. In other words we do not tackle the issue of shifting from a PAYG to a fully funded system of pensions.

The rest of the paper is organized as follows. In section 2, we present the basic model and the main results for an economy that consists of identical individuals with a defined contribution pension system. Section 3 is devoted to comparative statics where we investigate the change in capital accumulation and welfare for all pension systems, retirement schemes, and aging types. Fourth and fifth sections present static and dynamic simulations, respectively. Finally, the last section offers some concluding remarks.

2 Basic Model

We use a standard two-period overlapping generation model. An individual who belongs to generation t lives in two periods: t and $t + 1$. The first period of her life has a unitary length, while the second one has a length $\ell \leq 1$, where ℓ reflects variable longevity.

In the first period, the individual works and earns a wage, w_t , which is devoted to the first-period consumption, c_t , saving, s_t , and pension contribution, τ . In the second period, she works an amount of time $z_{t+1} \leq \ell \leq 1$ and earns $z_{t+1}w_{t+1}$. These earnings, together with the proceeds of savings $R_{t+1}s_t$ and the PAYG pension p , finance the second period consumption d_{t+1} .

We assume that working in the second period z_{t+1} implies a disutility defined in monetary terms $v(z_{t+1}, \ell)$, where $\frac{\partial v}{\partial z} > 0$, $\frac{\partial^2 v}{\partial z^2} > 0$ are imposed for the existence of a unique solution. In addition, disutility from working in the second period of life is a decreasing function of longevity, i.e. $\frac{\partial v}{\partial \ell} < 0$, which reflects the idea that an increase in longevity fosters later retirement. Note that, for simplicity, earnings in the second period of life is

not taxed. Intuitively, the end of the first period can be interpreted as the statutory age of retirement, unless otherwise indicated by an explicit mandatory retirement age. Any savings in funded social security system is not modeled explicitly, and it is assumed to be identical to other savings. Thus, the pension contribution parameter τ measures the relative size of the unfunded pensions. In other words, $\tau = 0$ implies that the whole pension system is funded.

Denoting by $u(\cdot)$ the utility function for consumption c or d , and U the lifetime utility, the problem of an individual of generation t is:

$$\max U_t = u(w_t - \tau - s_t) + \beta \ell u\left(\frac{w_{t+1}z_{t+1} + R_{t+1}s_t + p - v(z_{t+1}, \ell)}{\ell}\right), \quad (1)$$

where $p = \tau(1 + n)$ is the pension benefit in period $t + 1$ and β is the time discount factor. The gross rate of population growth $(1 + n)$ is equivalent to the number of children per individual in this set up. The argument of second period utility is net amount of resources then available divided by the length of the second period.¹

The first order conditions for life time utility maximization are simply given by:

$$v'_{z_{t+1}}(z_{t+1}, \ell) = w_{t+1}, \quad (2)$$

$$\beta R_{t+1} u'(d_{t+1}) - u'(c_t) = 0, \quad (3)$$

where c_t and d_{t+1} denote the first and second period consumption. The first condition (2) shows that the marginal disutility from second period work needs to be equal to the wage rate at the optimum. The second condition is the consumption Euler equation, and it shows that the individual cannot gain further utility by reallocating consumption between periods. In order to be able to show some of our results analytically, we will use simple functional forms for $u(\cdot)$ and $v(\cdot)$. Accordingly, we assume that the period utility function is logarithmic $u(x) = \ln x$, and the monetary disutility function is quadratic in its main argument $v(x) = x^2/2\gamma\ell$. One clearly sees from the latter functional form that the disutility of working longer can be mitigated by an increase in longevity. We can now rewrite the problem of the individual as the following:

$$U_t = \ln(w_t - \tau - s_t) + \beta \ell \ln\left(\frac{w_{t+1}z_{t+1} + R_{t+1}s_t - z_{t+1}^2/2\gamma\ell + p}{\ell}\right). \quad (4)$$

¹Suppose T_1 (T_2) is the number of years of the first (second) period, where $T_1 > T_2$. The life time utility would then be: $U = T_1 u[(w - \tau - s)/T_1] + \beta T_2 u((wz + Rs + p - v(z, T_2))/T_2)$, where we normalize T_1 and T_2 so that $T_1 = 1$ and $T_2 = l$.

The first order condition with respect to z_{t+1} yields :

$$z_{t+1} = z_{t+1}^* = \gamma \ell w_{t+1}, \quad (5)$$

where, an asterisk (*) denotes an optimal solution.² Using this optimality condition, and incorporating $p = \tau(1 + n)$, we can then get an explicit solution for the optimal saving rates:

$$s_t = \frac{\beta \ell}{1 + \beta \ell} w_t - \frac{\gamma \ell w_{t+1}^2}{2R_{t+1}(1 + \beta \ell)} - \tau \left(\frac{\beta \ell}{1 + \beta \ell} + \frac{1 + n}{(1 + \beta \ell) R_{t+1}} \right). \quad (6)$$

In many countries, z is not the outcome of a choice without a distortion. Through an array of programs, workers are induced to retire at ages different from what they would choose in the absence of these programs. We consider a case where the workers are induced to retire earlier than they wish to do so, and denote this induced early retirement by \bar{z} .³ In the case of this mandatory early retirement, we rewrite equations (5) and (6) as follows :

$$z_{t+1} = \bar{z},$$

$$s_t = \frac{\beta \ell}{1 + \beta \ell} w_t - \frac{\bar{z}}{R_{t+1}(1 + \beta \ell)} (w_{t+1} - \bar{z}/2\gamma \ell) - \tau \left(\frac{\beta \ell}{1 + \beta \ell} + \frac{1 + n}{(1 + \beta \ell) R_{t+1}} \right). \quad (7)$$

We now turn to the production side of the economy. The technology is characterized by a Cobb-Douglas production function:

$$Y_t = F(K_t, N_t) = AK_t^\alpha N_t^{1-\alpha}, \quad (8)$$

where K is the stock of capital, A is a productivity parameter, and N is the labor force. We

²Note that we assume that lifetime utility is increasing in longevity: $\frac{\partial U}{\partial l} = \beta \left[u(d) - u'(d)d - u'(d) \frac{\partial z(z,l)}{\partial l} \right] > 0$, which is satisfied if $\frac{u'(d)d}{u(d)} < 1$. Intuitively speaking one more year of life is worth living. With the functional forms we use, this condition is reduced to $\frac{\partial U}{\partial l} = \beta \left[\log(d) - 1 + \frac{\gamma w^2}{d} \right] > 0$.

³An alternative specification could be that second period labor is subject to a proportional tax θ whose proceeds are returned to the old workers. Their problem would be to choose z such as to maximize : $wz(1 - \theta) + T - v(z, \ell)$. With $T = \theta w z$ and $v = z^2/2\gamma \ell$, this yields $z = \gamma \ell w(1 - \theta)$. In the case of optimal retirement with no distortions, $z = z^* = \gamma \ell w$. In the case of induced early retirement, $z = \bar{z} = \gamma \ell w(1 - \theta)$, where θ is chosen such as to generate $\bar{z} < z^*$.

distinguish the labor force N_t from the size of generation t , L_t . The labor force comprises the young population of generation t and the labor force participation from the old generation $t - 1$. Incorporating the population growth, $L_t = L_{t-1} (1 + n)$, the labor force can then be written as $N_t = L_t + L_{t-1}z_t = L_{t-1} (1 + n + z_t)$. In comparison, total population at time t is :

$$L_t + \ell L_{t-1} = L_{t-1} (1 + \ell + n).$$

Denoting $K_t/N_t \equiv k_t$ and $Y_t/N_t \equiv y_t$, we obtain the income per worker (and not per capita):

$$y_t = f(k_t) = Ak_t^\alpha.$$

Factors of production are paid according to their marginal contributions :

$$R_t = f'(k_t) = A\alpha k_t^{\alpha-1}, \quad (9)$$

$$w_t = f(k_t) - f'(k_t)k_t = (1 - \alpha) Ak_t^\alpha. \quad (10)$$

equilibrium conditions in the labor and capital markets are as follows:

$$N_t = L_{t-1} (1 + n + z_t), \quad (11)$$

$$K_{t+1} = L_t s_t, \quad (12)$$

where the latter expression reflects the fact that capital is assumed to depreciate completely after each period. Although this assumption arises from convenience, it is not unrealistic considering the fact that a period denotes several decades in calendar. Using the optimality condition for savings derived before, the latter expression can be rewritten as follows :

$$\begin{aligned} G_t \equiv & (1 + n + z_{t+1}) k_{t+1} - \frac{\beta\ell}{1 + \beta\ell} A(1 - \alpha) k_t^\alpha + \tau \left(\frac{\beta\ell}{1 + \beta\ell} + \frac{(1 + n) k_{t+1}^{1-\alpha}}{A\alpha (1 + \beta\ell)} \right), \\ & + \frac{z_{t+1} k_{t+1}^{1-\alpha}}{(1 + \beta\ell) A\alpha} \left(A(1 - \alpha) k_{t+1}^\alpha - \frac{z_{t+1}}{2\gamma\ell} \right) = 0, \end{aligned} \quad (13)$$

which explicitly defines the dynamic behavior of capital stock. Note that the standard (Diamond) case with no social security and work in second period of life can be deduced

by shutting down these two sections, $z = \tau = 0$, which generates the following:

$$G_t \equiv (1 + n)k_{t+1} - \frac{\beta\ell}{1 + \beta\ell}A(1 - \alpha)k_t^\alpha. \quad (14)$$

Comparing (13) and (14), we observe two main differences. First, the third term on the right hand side of (13) denotes the double burden that the PAYG imposes to saving. Second, the fourth term reflects the double effect of working in the second period: a distortionary effect if z is not optimal and a saving inducement if $z < z^*$.

In equations (6) and (7) we assumed a pension system that relies on a defined contribution (DC) formula in which the tax $\bar{\tau}$ is given and thus the benefits p has to follow through based on demographic shifts. A alternative system can also be considered, which offers constant annuity benefits \bar{a} (DB) during retirement.⁴ The two revenue constraints that these systems imply are as follows:

$$\begin{aligned} DC : \bar{\tau}(1 + n) &= p, \\ DB : \bar{a}(\ell - z) &= \tau(1 + n), \end{aligned}$$

where an upper bar denotes the defined variable. For example, \bar{a} is the defined annuity and τ has to adjust to variations in z , ℓ and n in this case. Note that for each type of pension system, the individual utility has to adjust accordingly.⁵

We have so far identified two major dimensions which characterize a social security system : is it DC or DB, and is there mandatory early retirement or not? Altogether, these two dimensions provide four different ways to describe the equation (13). These four cases are presented on Table 1, and the corresponding equations for G_{it} are provided in the Appendix 1.

Before proceeding with comparative statics, it is important to further discuss some of the assumptions made in this paper. First, we adopt simple functional forms for utility and production. This allows us to obtain analytical results to the extent possible and lets us assume the existence of a unique and stable equilibrium. This equilibrium is defined by the dynamics of capital accumulation (13), where $0 < \frac{\partial k_{t+1}}{\partial k_t} < 1$. This condition implies that $G_k = \frac{\partial G}{\partial k} > 0$ holds in the steady state.⁶ Second, by we assume a quasi-linear

⁴With defined benefits, there are two ways of exiting from the pension system at the time of retirement: either through annuities or by cashing some capital. Most public DB systems comprise an exit in annuities. Some DB private systems, on the other hand, provide the option to choose between the two types of exits.

⁵Note that we could include a defined contribution system with annuities, where $\bar{\tau}(1 + n) = a(1 - z)$, or a defined benefit system with lump sum benefits, where $\tau(1 + n) = \bar{p}$. However, we restrict attention to the more commonly known cases.

⁶In this paper we refrain from discussing the issue of existence, unicity and stability of the equilibria by

Table 1: Different Social Security and Retirement Regimes

Type of Social Security System		Retirement Age Regulation Scheme
Case-1	Defined Contribution	Mandatory Early Retirement
Case-2	Defined Benefit	Mandatory Early Retirement
Case-3	Defined Contribution	Optimal Retirement
Case-4	Defined Benefit	Optimal Retirement

Notes: The optimal retirement is given by $z^* = \gamma \ell w$, and the mandatory early retirement is given by $z = \bar{z} < z^*$.

disutility from working in old ages. On the one hand, this means that we neglect income effects that could clearly play some role in the decision of retirement. On the other hand, this assumption allows us to better contrast mandatory early retirement and freely chosen retirement. Finally, we assume away the intensive margin in the first period, just focus on the intensive margin in the second period. This simplifies the analysis, and is in agreement with the literature on age taxation that focuses on the retirement decision; see for example Lozachmeur (1996).

3 Comparative Statics

In this section, we investigate the comparative statics for the four alternative cases of social security systems identified in the previous section. Our main aim is to elaborate on the behavior of capital accumulation when the economy experiences aging due to lower fertility or higher longevity.

3.1 Mandatory Early Retirement

We begin by showing the impact of a decrease in fertility in a mandatory early retirement system:

$$DC : \Psi \frac{\partial k^1}{\partial n} = -k - \frac{\bar{\tau} k^{1-\alpha}}{A\alpha(1+\beta\ell)} < 0, \quad (15)$$

$$DB : \Psi \frac{\partial k^2}{\partial n} = -k + \frac{\bar{a}(\ell - \bar{z})\beta\ell}{(1+\beta\ell)(1+n)^2} \geq 0, \quad (16)$$

choosing commonly known and well behaving functional forms. For a general technical discussion on this, see de la Croix and Michel (2002), and for a discussion in a framework similar to our model, see Ludwig and Vogel (2010). Finally, Rogerson and Wallenius (2009) provide a nice discussion on preference specifications that are consistent with balanced growth.

where $\Psi = \left(\frac{\partial G}{\partial k}\right)^{-1} > 0$, and superscripts denote the type of social security as defined in Table 1. In a standard case (Diamond), an increase in fertility has a depressive effect on capital accumulation in the absence of a PAYG pension system. This is shown by the first term on the right hand side of each equation above, which is often called the “capital dilution” effect. This depressive effect is reinforced with a DC pension system as shown by the negative second term in (15), but it is weakened or possibly reversed with DB pensions as shown by positive second terms in (16). The explanation is quite intuitive. With a DC system, an increase in fertility implies an increase in pension, which discourages saving. With a DB system the pension level is kept constant and thus the contribution rate decreases, which fosters saving. We call this effect the “saving displacement” effect. Note that, in DB, the lower the \bar{z} , the larger the negative saving displacement effect. With a too generous pension, an increase in fertility could even lead to an increase in capital.

Next, we turn to the impact of an increase in longevity on equilibrium capital per worker in an induced early retirement system:

$$DC : \quad \Psi \frac{\partial k^1}{\partial \ell} = \frac{1}{(1 + \beta\ell)^2} [\beta w - \bar{\tau}\Pi - \bar{z}\Pi] \geq 0, \quad (17)$$

$$DB : \quad \Psi \frac{\partial k^2}{\partial \ell} = \frac{1}{(1 + \beta\ell)^2} [\beta w - \bar{a}\Theta - \bar{z}\Omega] \geq 0, \quad (18)$$

where $\Pi = \beta\left(\frac{1+n}{R}\right) [R - (1+n)]$, $\Theta = \frac{\beta(\ell-\bar{z})}{(1+n)R} [R - (1+n)] + \frac{(1+\beta\ell)}{(1+n)R} [R + (1+n)]$, and $\Omega = \frac{1}{2R\gamma\ell^2} [\bar{z} + 2\beta\ell(\bar{z} - z^*)]$.

In both (17) and (18) we have three terms in brackets. The first term, which is common to both cases, represents the capital dilution effect that is always positive. The third term is also common to the two expressions. It represents the saving displacement effect due to the mandatory retirement. When $\bar{z} = 0$, this term vanishes. The need for saving is then at its highest as there are no earnings in the second period. Thus, as \bar{z} increases, the effect of longevity on capital decreases. The second term varies from one expression to the other. The second term varies between the two expressions. It represents the saving displacement effect due to social security. They vanish when $\bar{\tau} = \bar{a} = 0$. Remember that we assume dynamic efficiency, $R > (1+n)$, and as shown by Aaron (1966), when the marginal productivity of capital is higher than the rate of population growth, social security depresses both capital and welfare. With DB, the saving displacement effect due to the defined annuity \bar{a} comprises not only the gap between R and $(1+n)$ but also the sum of the two rate.

Next, we investigate the effects on capital accumulation in the absence of a mandatory

retirement age.

3.2 Optimal Retirement

We now relax the early retirement assumption, and analyze the impact of aging on capital accumulation when retirement is chosen optimally. The analysis here shows that, compared to the case with early retirement, the ability to adjust the retirement age optimally leads to less distortion in equilibrium, but it also diminishes the incentives for saving.

$$DC : \quad \Psi \frac{\partial k^3}{\partial n} = -k - \frac{\bar{\tau} k^{1-\alpha}}{A\alpha(1+\beta\ell)} < 0, \quad (19)$$

$$DB : \quad \Psi \frac{\partial k^4}{\partial n} = -k + \frac{\bar{a}\beta\ell^2(1-\gamma(w-\bar{a}))}{(1+\beta\ell)(1+n)^2} \geq 0. \quad (20)$$

Similar to the case with early retirement, the PAYG pension system reinforces the depressive effect of an increase in fertility on capital accumulation in the DC case and weakens or possibly reverses it in the DB case. Turning to the effect of longevity when z is endogenous, we have:

$$DC : \quad \Psi \frac{\partial k^3}{\partial \ell} = \frac{1}{(1+\beta\ell)^2} [\beta w - \bar{\tau}\Pi - \Phi] \geq 0, \quad (21)$$

$$DB : \quad \Psi \frac{\partial k^4}{\partial \ell} = \frac{1}{(1+\beta\ell)^2} [\beta w - \bar{a}\chi - \Psi] \geq 0, \quad (22)$$

where the expressions within brackets are defined as $\Phi = \frac{k\gamma w}{2\alpha} [2\alpha(1+\beta\ell)^2 + (1-\alpha)]$, $\chi = \left(\frac{1-\gamma(w-\bar{a})}{(1+n)R}\right) [(1+n)R + \beta l R(1+\beta l)]$, and $\Psi = \frac{k\alpha(w-\bar{a})}{2\alpha} [2\alpha(1+\beta\ell)^\alpha + \frac{\bar{a}}{Ak^\alpha} + (1-\alpha)]$.

In both expressions above, we observe a positive capital dilution effect. We also have two saving displacement effects that are both negative and different for the DC and the DB regimes. As in the mandatory retirement case, with $\bar{\tau} = \bar{a} = 0$, the saving displacement effect of social security vanishes. However, this effect increases with the two social security parameters, $\bar{\tau}$ in DC and \bar{a} in DB. The terms ϕ and Ψ represent the saving displacement effect due to second period activity. They vanish when $\gamma = 0$, which implies no activity in the second period. Both of these terms increase with longevity as z^* increases with l . There is an additional depressing term with DA, which increases with \bar{a} .

Table 2: The Effects of Aging on Equilibrium Capital Per Worker

	Standard Case	Defined Contribution	Defined Benefit
		<i>Mandatory Early Retirement</i>	
Decrease in Fertility	> 0	> 0	≤ 0
Increase in Longevity	> 0	≤ 0	≤ 0
		<i>Optimal Retirement</i>	
Decrease in Fertility	> 0	> 0	≤ 0
Increase in Longevity	> 0	≤ 0	≤ 0

We summarize the above findings in Table 2. The only unambiguous case is the positive effect of declining fertility with DC. Increasing longevity is ambiguous but has a higher effect with $\bar{z} = 0$ than with z^* . With DB, both sources of aging have an ambiguous effect; but, again, the effect is higher with $\bar{z} = 0$ than with z^* . These results are intuitive: with $\bar{z} = 0$ or at least $\bar{z} < z^*$, individuals anticipate a lower level of earnings in the second period, and compensates this by additional saving. As we consistently assume dynamic efficiency, all things being, utility under optimal retirement is not necessarily higher than under mandatory retirement. This is mainly because mandatory early retirement induce higher saving and capital accumulation as desired consumption in the second period of life time cannot be financed by extending the work hours. As a result, mandatory early retirement presents a case that is closer to the golden rule than the optimal retirement. This is a standard second-best problem where a distortion makes a second distortion desirable.

3.3 Aging and Utility

We next study the effects of aging on welfare. To do this, we need to distinguish between two main drivers of welfare: the capital stock that determines the aggregate production and the PAYG system that has differential impact under DC and DB. In order to provide some analytical results for the PAYG effect we look at the effect of aging in a fixed factor prices setting, which corresponds to the case of a small open economy. We prove the following proposition in Appendix 2.

Proposition 1. *In a fixed factor price setting with PAYG social security and endogenous retirement age,*

1. *A decrease in the fertility rate always increases the individual welfare. In the case of DC the effect is the same whether retirement is freely chosen or mandatory. In the case of DB, the effect is smaller when retirement age is not regulated.*

2. *Assuming that surviving one more year in the second period is desirable, the welfare incidence of an increase in longevity is consistently positive. It is higher with optimal retirement than with mandatory retirement.*

To see the effect of aging on welfare we have to combine the capital effect that was studied in the previous subsections and the PAYG effect. We resort to numerical simulations to be able to show these combined effects.

4 Simulations

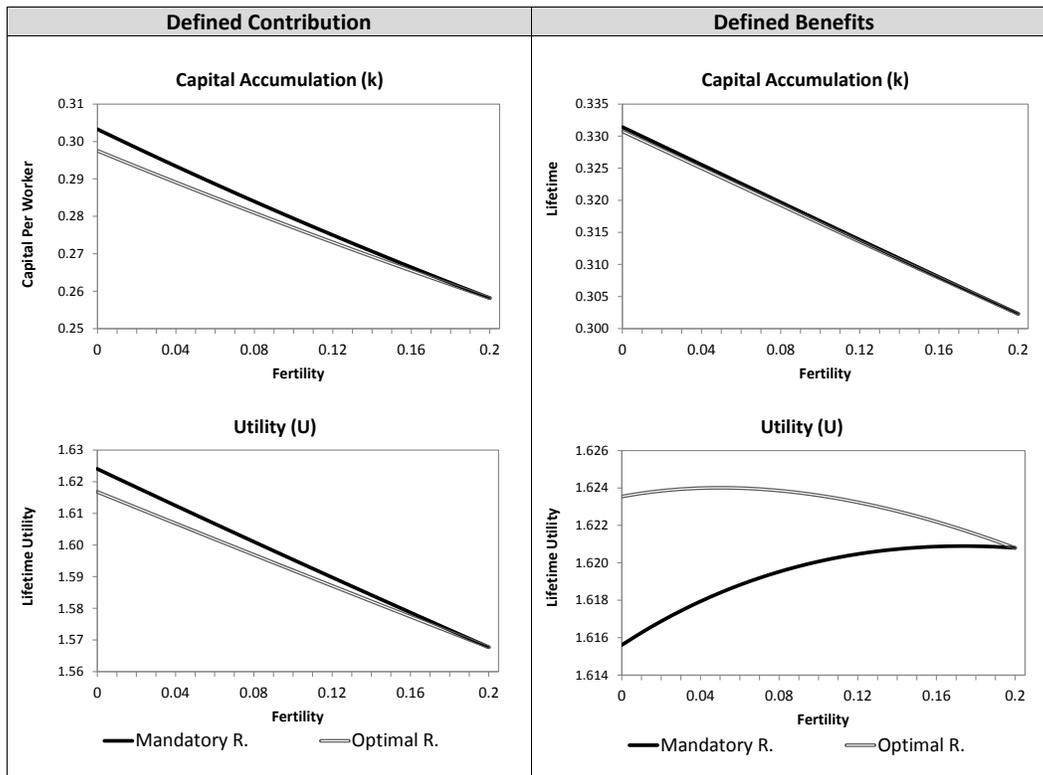
The analysis so far has shown that demographic change has different implications for capital accumulation under alternative PAYG systems. Analytically, these results are sufficient to show that the impacts are different quantitatively. In order to better grasp these effects, and to study the welfare effects, we next employ a numerical example. To this effect, we use common values of parameters from the literature to simulate the equilibrium profiles for the agent's lifetime utility U and the capital per worker k with different values of fertility n and longevity ℓ .⁷ Figures 1 and 2 show the results with respect to variations in fertility and in longevity cases, respectively.

We start with the interpretation of the declining fertility case. Note that we impose a condition that the mandatory age of retirement is not distortionary at the highest level of fertility level in the graph, *i.e.* it is set at the optimal retirement age. This explains why both the level of capital and the generational utility are identical between mandatory and optimal retirement systems at that point.

In both pension regimes considered, DC and DB, the capital stock increases regardless of the retirement regulations when fertility decreases. However, the increase is more pronounced with a mandatory retirement system in the DC case. To see this, note that with a monotonic increase in capital, we expect that wage and thus the optimal age of retirement to increase. As mandatory retirement age is fixed, the labor distortion implied by mandatory retirement increases when n decreases. This explains why the capital stock with mandatory retirement (solid black line) dominates the capital stock under optimal

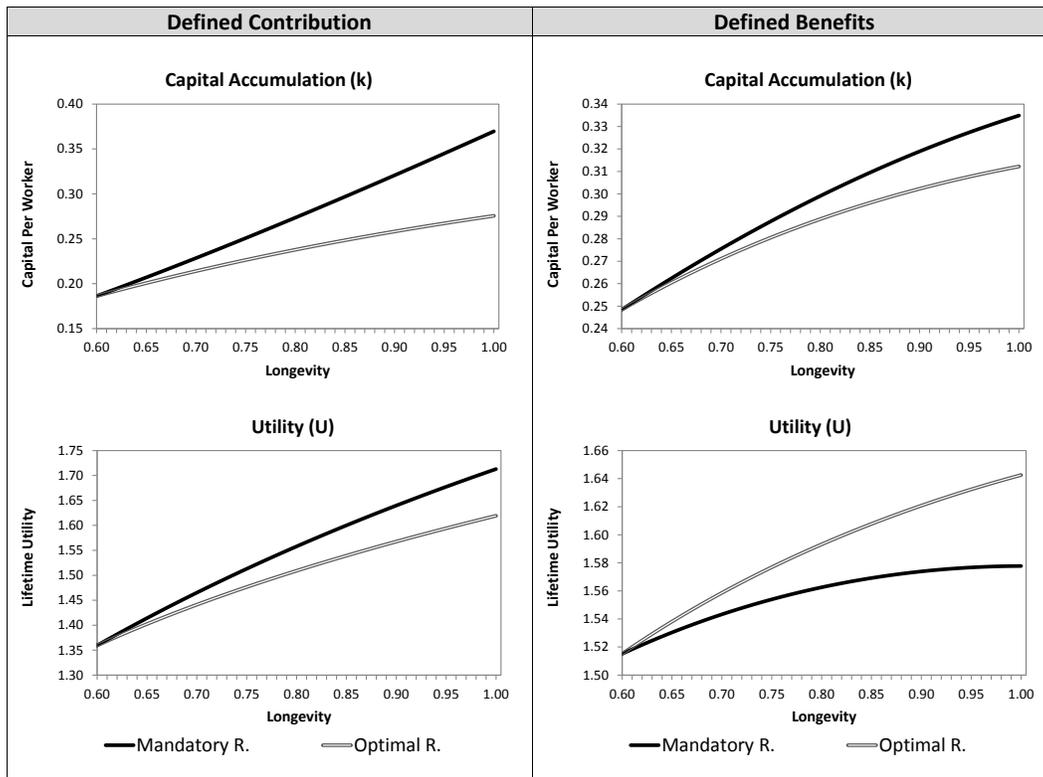
⁷Note that we do not attempt to calibrate the simulations to any country cases. As the model is built to demonstrate our results (preferably analytically) by using the simplest possible case, it does not lend itself to replicating the complex demographic and institutional aspects of actual country examples. Nevertheless, we choose commonly used parameter values: $A = 10$, $\alpha = 0.33$, $\beta = 0.25$, and $\gamma = 0.15$, where A is a scaling parameter and plays no important role, and capital's share in income in the economy (α) reflects the averages for the OECD countries, see Jones (2003). Considering the fact that each period represents approximately 35 years in our model, β is equivalent to 0.96 in annual terms. In addition, the following assumptions were made when needed: $a = 1.6$, $\bar{z} = 0.394$ in defined annuity case with mandatory retirement, and $\bar{z} = 0.579$ in defined benefit and defined contribution cases. These values are chosen to make the initial conditions and changes comparable with alternative social security systems.

Figure 1: Equilibrium Effects of Fertility on Lifetime Utility and Capital Per Worker



Note. Simulations reflect the parameter values: $A = 10$, $\alpha = 0.33$, $\beta = 0.25$, and $\gamma = 0.15$. In addition, the following assumptions were made when needed: $a = 1.6$, $\bar{z} = 0.394$ in defined benefit case with mandatory retirement, and $\bar{z} = 0.579$ in defined benefit and defined contribution cases. These values equalize the starting points of mandatory and optimal retirement systems at $n = 0.2$.

Figure 2: Equilibrium Effects of Longevity on Lifetime Utility and Capital Per Worker



Note. Simulations reflect the parameter values: $A = 10$, $\alpha = 0.33$, $\beta = 0.25$, and $\gamma = 0.15$. In addition, the following assumptions were made when needed: $a = 1.6$, $\bar{z} = 0.394$ in defined benefit case with mandatory retirement, and $\bar{z} = 0.579$ in defined benefit and defined contribution cases. These values equalize the starting points of mandatory and optimal retirement systems at $l = 0.6$.

retirement (hollow line) in the figure. The same relationship is also true for the utility profiles under the mandatory and optimal retirement systems with DC.

The utility profiles under the DB regime provide curious cases. A decrease in fertility leads to an initially increasing and then decreasing (hump-shaped) lifetime utility with optimal retirement. In comparison, utility always decreases with mandatory retirement. This is consistent with our theoretical findings above. The capital stock increases about at the same pace with \bar{z} and z^* , which implies the same positive impact on utility from the capital dilution channel. But, given Proposition 1, we know that the negative PAYG effect is larger with $\bar{z} = 0$ than with z^* .

We next turn to aging due to an increase in longevity. Note that, in all cases, the capital per worker increases with longevity. However, the effect is more pronounced in the mandatory retirement scheme under both DC and DB. This is also consistent with our theoretical results. With a longer life span, savings are increased to smooth the consumption between two periods. The increase in savings is more dramatic when labor cannot adjust in the second period of life. Similarly, lifetime utility increases under all social security and retirement regulation schemes. However, in the case of DB optimal retirement dominates mandatory retirement, whereas the opposite occurs with DC. The reason for this contrasting result is straightforward: with DC, the increase in capital is relatively more important when there is mandatory early retirement. In other words, the gain in dynamic optimality (getting closer to the golden rule) dominates the loss in static efficiency (on the labor market). The opposite holds in the case of DB.

Overall, the static simulations in this section show that the DC pension system seems to outperform the DB in aging societies regardless of the retirement age regulation. This is, however, particularly the case when the mandatory early retirement rule prevails. Note, however that, these observations compare the long-term performances. Next we investigate the transition dynamics between two static equilibria brought about by a change in fertility or longevity.

5 Dynamics

In this section, we investigate if the short term impact of demographic shift is different than the long term implications in different settings of pension systems.

The simulations in this section use the same parameter values as in the static simulations in the previous section. However, in this case, we need to specify the magnitude of demographic transition between two steady states as a single value. In order to make

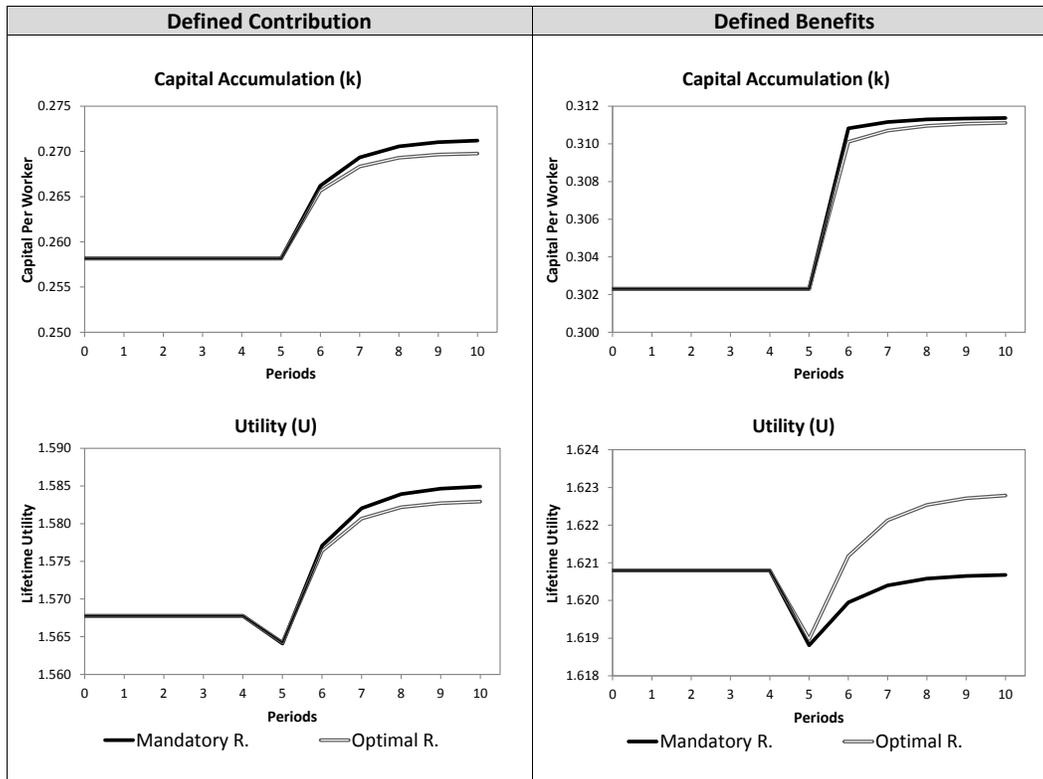
comparable the changes in n and ℓ , we characterize the fertility driven aging by decreasing the n from 0.2 to 0.137, and the longevity driven aging by increasing the ℓ from 0.9 to 0.95. These changes, in turn, increase the rate of dependency $\frac{\ell}{1+n}$ by about 5.6 percent in comparison to the initial steady state. We assume there's perfect foresight, *i.e.* both changes are anticipated by the agents in advance. We introduce both shocks in period-6 in order to observe the anticipatory adjustments as well as the after-shock propagations. These demographic shocks lead to new equilibria over time, however, not necessarily in a monotonic manner as we show.

Figures 3 and 4 show the dynamic adjustment paths in different social security systems and demographic shocks. In the case of capital accumulation, transitions are generally monotonic with the expected signs with the exception of an increase in longevity in DB, which we explain below. In all cases, capital per worker increases starting from period-6. Note that the capital per worker in period-6 is determined by a combination of two factors: savings of the generation that is born in period-5, and the size of the workforce in period 6 (which is a combination of the size of generation period-6 and the old age work hours of generation-5). In the simplest case, where retirement age is regulated by a mandatory retirement system and aging is driven by longevity, capital by worker increases mainly because the generation born in period-5 increase their savings to mooth consumption over a longer life. When longevity increases in period-6, the generation-5 again increase their savings to finance consumption over a longer life. However, they do not bear the burden of financing a larger pension bill yet since they contribute in period-5, before the demographic shock hits. In comparison, the generation-6 and following generations pay higher social security contributions, which reduces their first period income. Thus, they cannot save as much as the generation-5, which translates into a temporary overshooting in capital accumulation in period-6 as shown in figure 4.

The other cases present more complex mechanisms. For instance, in an optimal retirement system with fertility driven aging, capital per worker increases due the capital dilution effect, *i.e.* holding the generation-5 savings constant, each worker gets to use more capital as there are fewer workers in generation-6. At the same time, the old from generation-5 increase their labor force participation, which compensates the decrease n partially. Moreover, if the pension system is DC, then generation-5 may also adjust their savings in anticipation of a change in income in period-6, which comes as a result of a decrease in pensions as well as changes in interest earnings and wages as a result of the capital deepening that occurs in period-6.

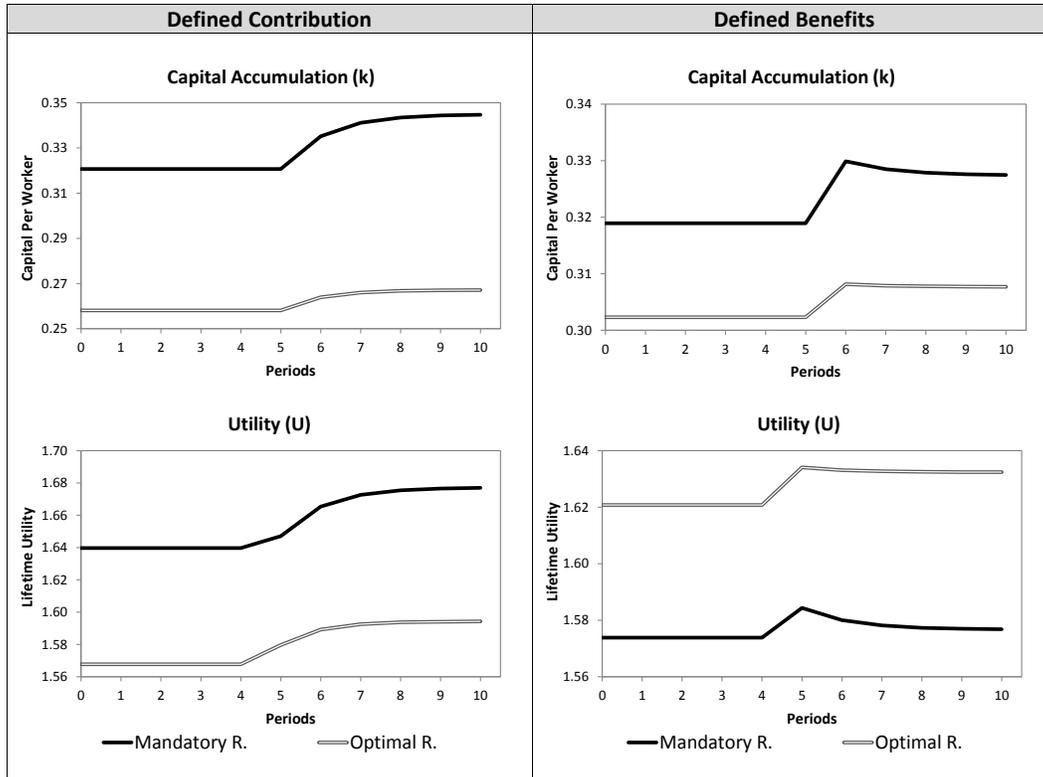
The most striking result, however, is a transitory loss in lifetime utility that comes with a fertility shock. Consider the case of fertility driven aging in a defined contribution

Figure 3: Dynamics of Fertility Driven Aging



Note. Simulations reflect the parameter values: $A = 10$, $\alpha = 0.33$, $\beta = 0.25$, and $\gamma = 0.15$. In addition, the following assumptions were made when needed: $a = 1.6$, $\bar{z} = 0.394$ in defined benefit case with mandatory retirement, and $\bar{z} = 0.579$ in defined benefit and defined contribution cases. These values equalize the mandatory and optimal retirement systems at $l = 0.6$. The fertility shock denotes a decrease in n from 0.2 to 0.137 in period 6. Utility level at period t shows the lifetime utility of the generation who are born in period t , and live in period t and $t + 1$.

Figure 4: Dynamics of Longevity Driven Aging



Note. Simulations reflect the parameter values: $A = 10$, $\alpha = 0.33$, $\beta = 0.25$, and $\gamma = 0.15$. In addition, the following assumptions were made when needed: $a = 1.6$, $\bar{z} = 0.394$ in defined benefit case with mandatory retirement, and $\bar{z} = 0.579$ in defined benefit and defined contribution cases. These values equalize the mandatory and optimal retirement systems at $l = 0.6$. The longevity shock is an increase in l from 0.9 to 0.95 in period 6. Utility level at period t shows the lifetime utility of the generation who are born in period t , and live in period t and $t + 1$.

system for instance. Although the long term impact of a fertility shock in period-6 on welfare is positive, the generation that is born in period-5 experiences a decline in their lifetime utility. In order to see this, note that the fertility shock reduces the pensions of generation-5 and, thus, the old age consumption. At the same time, because this particular generation already worked and contributed to social security system in period-5, it does not benefit from efficiency gains available to future generations to the same extent. As the decrease in fertility brings the economy closer to golden rule savings, the lifetime utility increases for the following generations.

The same principle is also operative in the fertility driven aging with defined benefits case. Although pensions do not change in this case, there are other factors that affect the income of generation-5 in period-6. Note that when fertility decreases in period-6, it leads to capital deepening through adjustments in capital dilution effect as well as saving displacement effect. As a result, interest earnings on period-5 savings decreases and the wages increases in period-6. However, the latter benefits the generation-5 only partially as the work hours in period-6 is limited. Thus, the decrease in interest earnings dominates the gain in wages, leading to a loss in lifetime utility.⁸

Overall, the dynamic simulations highlight an interesting tradeoff between alternative social security systems. Regardless of the source of aging, the long-term welfare gain under DC is larger than the DB. This can be seen by comparing the changes in equilibrium capital per worker and lifetime utility as a share of the initial equilibrium values displayed in figures 3 and 4. However, this long term performance comes at a cost of utility losses in the transition generations.

6 Conclusions

In this paper, we evaluate the implications aging on capital accumulation and lifetime utility. We show that these implications differ both quantitatively and qualitatively depending on the type of social security system, type of retirement regulations and the time frame of the analysis. The effects of an increase in longevity or a decrease in fertility, two phenomena that contribute to aging, change depending on a number of features in the pension system. In a standard overlapping generations model with no social security or second period work, a decrease in fertility and increase in longevity should lead to an increase in capital accumulation. However, this is not necessarily the case when we in-

⁸Note that since the underlying model is a simple two period OLG framework, changes in key variables are not expected to reflect smooth adjustments that could happen in large calibration models where one period is approximately one calendar year.

roduce these elements. The effect of a decrease in fertility on capital accumulation is still positive with a defined contribution system regardless of the retirement age regulation; however, the results are ambiguous with other unfunded social security systems and with an increase in longevity.

Our findings have important implications, as we know that countries do not age the same way and social security systems tend to shift progressively from a regime of defined benefits towards one of defined contributions. Our dynamic simulations show that this shift could bring long term gains; however, the transition could impose welfare costs for current generations. On our further research agenda, we intend to look at the joint effect of aging and changes in the social security regimes explicitly: from DC to DB and from early retirement to flexible retirement. A limitation of the current analysis is the assumption of identical individuals. With heterogeneity in wages, we could find more merits in the defined benefit formula, which is also left for future research.

References

- [1] Aaron, H. (1966), The social security paradox, *Canadian Journal of Economics and Political Science* 32, 371-4
- [2] Ben Porath, Y., (1967), The Production of Human Capital and the Life-cycle of Earnings, *Journal of Political Economy*, 75(3), 352-365.
- [3] Bloom, D., Canning, D. and Graham, B., (2003), Longevity and Life-cycle Savings, *Scandinavian Journal of Economics* 105(3), 319-338.
- [4] Bloom, D.E., Canning, D., Mansfield, R.K., and Moore, M. (2007), Demographic Change, Social Security Systems, and Savings, *Journal of Monetary Economics* 54, 92-114.
- [5] de la Croix, D. and Licandro, O., (1999), Life expectancy and endogenous growth, *Economics Letters* 65, 255-263.
- [6] De la Croix D. and P. Michel, (2002), A Theory of Economic Growth: Dynamics and Policy in Overlapping Generations, Cambridge University Press, 2002
- [7] Echevarria, C., (2004), Life expectancy, retirement and endogenous growth, *Economic Modelling* 21(1), 147-174.

- [8] Gruber, J. and D. Wise, eds., (1999), *Social Security and Retirement Around the World*, Chicago: University of Chicago Press.
- [9] Heijdra, B., and Mierau, J. (2011), *The Individual Life Cycle and Economic Growth: An Essay on Demographic Macroeconomics*, *De Economist* 159(1), 63-87
- [10] Ito, H., and Tabata, K. (2008), *Demographic Structure and Growth: The Effects of Unfunded Social Security*, *Economic Letters* 100, 288-291
- [11] Jones, C.I. (2003), *Growth, Capital Shares, and a New Perspective on Production Functions*, Berkeley University manuscript.
- [12] Kalemli-Ozcan, S., Ryder, H. and Weil, D., (2000), *Mortality decline, human capital investment, and economic growth*, *Journal of Development Economics* 62, 1-23.
- [13] Kinugasa, T. and Mason, A., (2006), *Why countries become wealthy: the effects of adult longevity on saving*, *World Development* 35, 1-23.
- [14] Lozachmeur, J.-M. (2006), *Optimal Age-Specific Income Taxation*, *Journal of Public Economic Theory*, 8, 697-711.
- [15] Ludwig, A. and Vogel, E., (2010), *Mortality, fertility, education and capital accumulation in a simple OLG economy*, *Journal of Population Economics*, Springer, vol. 23(2), 703-735.
- [16] Miyazawa, K., (2006), *Growth and Inequality: A Demographic Explanation*, *Journal of Population Economics* 19, 559-578.
- [17] Nishimura, N., Pestieau, P., and Ponthiere, G. (2015), *Education Choices, Longevity and Optimal Policy in a Ben-Porath Economy*, CORE Discussion Paper.
- [18] Rogerson, R. and Wallenius, J., (2007), *Micro and macro elasticities in a life cycle model with taxes*, NBER Working Paper 13017.
- [19] Tabata, K. (2014), *Population Aging and Growth: The Effect of PAYG Pension Reform*, unpublished.
- [20] Zhang, J., Zhang, J., Lee, R., (2001), *Mortality decline and long-run economic growth*, *Journal of Public Economics* 80, 485-507
- [21] Zhang, J., Zhang, J., and Lee, R., (2003), *Rising longevity, education, savings, and growth*, *Journal of Development Economics* 70, 83-101

Appendix 1: Motions of Capital with Alternative Designs of Social Security

$$\begin{aligned} \text{Case 1 : } G_{1t} &\equiv (1+n+z_{t+1})k_{t+1} - \frac{\beta\ell}{1+\beta\ell}A(1-\alpha)k_t^\alpha + \frac{\bar{\tau}}{1+\beta\ell} \left(\beta\ell + \frac{(1+n)k_{t+1}^{1-\alpha}}{A\alpha} \right) \\ &+ \frac{z_{t+1}k_{t+1}^{1-\alpha}}{(1+\beta\ell)A\alpha} \left(A(1-\alpha)k_{t+1}^\alpha - \frac{z_{t+1}}{2\gamma\ell} \right) = 0 \end{aligned}$$

$$\begin{aligned} \text{Case 2 : } G_{3t} &\equiv (1+n+z_{t+1})k_{t+1} - \frac{\beta\ell}{1+\beta\ell}A(1-\alpha)k_t^\alpha + \frac{\bar{a}(\ell-\bar{z})}{1+\beta\ell} \left(\frac{\beta\ell}{(1+n)} + \frac{k_{t+1}^{1-\alpha}}{A\alpha} \right) \\ &+ \frac{z_{t+1}k_{t+1}^{1-\alpha}}{(1+\beta\ell)A\alpha} \left(A(1-\alpha)k_{t+1}^\alpha - \frac{z_{t+1}}{2\gamma\ell} \right) = 0 \end{aligned}$$

$$\begin{aligned} \text{Case 2 : } G_{3t} &\equiv (1+n+A(1-\alpha)k^\alpha\gamma\ell)k_{t+1} - \frac{\beta\ell}{1+\beta\ell}A(1-\alpha)k_t^\alpha \\ &+ \frac{\bar{\tau}}{1+\beta\ell} \left(\beta\ell + \frac{(1+n)k_{t+1}^{1-\alpha}}{A\alpha} \right) + \frac{A^2(1-\alpha)^2\gamma\ell k_{t+1}^{1+\alpha}}{2(1+\beta\ell)A\alpha} = 0 \end{aligned}$$

$$\begin{aligned} \text{Case 2 : } G_{4t} &\equiv (1+n+D_t\gamma\ell)k_{t+1} - \frac{\beta\ell}{1+\beta\ell}A(1-\alpha)k_t^\alpha + \frac{\bar{a}(\ell-D_t\gamma\ell)}{1+\beta\ell} \left(\frac{\beta\ell}{(1+n)} + \frac{k_{t+1}^{1-\alpha}}{A\alpha} \right) \\ &+ \frac{D_t\gamma\ell k_{t+1}^{1-\alpha}}{(1+\beta\ell)A\alpha} \left(A(1-\alpha)k_{t+1}^\alpha - \frac{D_t}{2} \right) = 0 \end{aligned}$$

where $D_t = (A(1-\alpha)k_t^\alpha - \bar{a})$

Appendix 2: Open Economy Case

In this section, we elaborate on the effect of aging on welfare in a small open economy case. Note that, in this case, returns to capital (R) and labor (w) are fixed at international levels. We also assume that $\bar{z} = 0$ when mandatory retirement is implemented, and that $v(z, l) = \frac{z^2}{2\gamma l}$ and $R = 1$ for the sake of simplicity. Given these assumptions, we can write down the lifetime utilities for DC under optimal and mandatory retirement regulations as follows:

$$U_1 = u(w - \bar{\tau} - s) + lu \left(\frac{Rs + \bar{\tau}(1+n) + w^2\gamma l}{l} \right)$$

$$U_2 = u(w - \bar{\tau} - s) + lu \left(\frac{Rs + \bar{\tau}(1+n)}{l} \right)$$

Similarly, lifetime utilities with DB when retirement is chosen optimally by agents or regulated by a mandatory retirement rule is given by:

$$U_3 = u \left(w - s - \frac{\bar{a}}{1+n} l(1-wl) \right) + lu \left(\frac{Rs + \frac{w^2\gamma l}{2} + \bar{a}l(1-wl)}{l} \right)$$

$$U_4 = u \left(w - s - \frac{\bar{a}}{1+n} l \right) + lu \left(\frac{Rs + \bar{a}l}{l} \right)$$

Next, we evaluate the effect of a marginal change in longevity and fertility by differentiating the definitions above. In all cases, a small increase in fertility is found to increase the welfare when wages and interest rates are unaffected.

$$\frac{\partial U_1}{\partial n} = lu'(d)\bar{\tau} > 0$$

$$\frac{\partial U_2}{\partial n} = lu'(d)\bar{\tau} > 0$$

$$\frac{\partial U_3}{\partial n} = u'(c)\bar{a} \frac{l(1-\gamma w)}{(1+n)^2} > 0$$

$$\frac{\partial U_4}{\partial n} = u'(c) \frac{l\bar{\tau}}{(1+n)^2} > 0$$

Similarly, the effects of longevity are given by:

$$\frac{\partial U_1}{\partial l} = u(d) - u'(d)d + u'(d) \frac{w^2\gamma}{2} > 0$$

$$\frac{\partial U_2}{\partial l} = u(d) - u'(d)d > 0$$

$$\frac{\partial U_3}{\partial l} = u'(d)\bar{a} \left(1 - \frac{R}{1+n} \right) (1-\gamma w) + u(d) + u'(d) \frac{w^2\gamma}{2}$$

$$\frac{\partial U_4}{\partial l} = u'(d)\bar{a} \left(1 - \frac{R}{1+n} \right) + u(d)$$

where we assume that $u(d) - u'(d)d > 0$ and that $R > 1+n$.