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## Asymptotics for parametric GARCH-in-Mean Models

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GARCH-in-Mean Models**

**Christian Conrad  
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January 2015

# ASYMPTOTICS FOR PARAMETRIC GARCH-IN-MEAN MODELS\*

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## Abstract

In this paper we develop an asymptotic theory for the parametric GARCH-in-Mean model. The asymptotics is based on a study of the volatility as a process of the model parameters. The proof makes use of stochastic recurrence equations for this random function and uses exponential inequalities to localize the problem. Our results show why the asymptotics for this specification is quite complex although it is a rather standard parametric model. Nevertheless, our theory does not yet treat all standard specifications of the mean function.

**Keywords:** GARCH-in-Mean, stochastic recurrence equations, risk-return relationship

**JEL Classification:** C13, C22, C51, G12

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# 1 Introduction

The aim of this paper is to develop an asymptotic theory for the Quasi-Maximum Likelihood Estimator (QMLE) in GARCH-in-Mean (GARCH-M) models for the special case of GARCH(1,1)-innovations. This model was suggested in Engle et al. (1987) and has been frequently used in empirical finance for investigating the risk-return trade-off implied by Merton's (1973) intertemporal CAPM (see, among many others, French et al., 1987 or Lundblad, 2007). Despite of its popularity in empirical applications, up to now there is no asymptotic theory for the QMLE of the GARCH-M. We will explain why the proof of the asymptotic normality of the QMLE is so difficult in this simple classical parametric model. There is also a mathematical motivation for investigating this model, because difficulties in the study of the model arise from nonstationarities of derivatives of the likelihood function that create some nonstandard mathematical problems.

The GARCH(1,1)-M model is given by

$$Y_t = m_{\gamma_0}(\bar{h}_t(\theta_0)) + \bar{h}_t(\theta_0)^{1/2} Z_t, \quad (1)$$

$$\bar{h}_t(\theta) = \omega + \alpha(Y_{t-1} - m_{\gamma}(\bar{h}_{t-1}(\theta)))^2 + \beta\bar{h}_{t-1}(\theta) \quad (2)$$

with i.i.d. mean zero variables  $Z_t$  with variance equal to one. Here,  $\theta = (\psi, \gamma)$  is the unknown parameter, consisting of the regression parameter  $\gamma$  and the GARCH parameter  $\psi = (\omega, \alpha, \beta)$ . The true parameters are denoted by  $\theta_0, \psi_0, \gamma_0, \omega_0, \alpha_0$  and  $\beta_0$ . We also write  $h_t$  for  $\bar{h}_t(\theta_0)$  and  $m_0$  for  $m_{\gamma_0}$ . The function  $\bar{h}_t(\theta)$  is defined as the strictly stationary and ergodic solution of (2). Below we will state conditions under which such a solution exists and is unique (see Lemma 1). The existence and uniqueness of such a solution at the value  $\theta = \theta_0$  is guaranteed if  $E[\ln(\alpha_0 Z_t^2 + \beta_0)] < 0$ , see Nelson (1990). We also write  $\hat{h}_t(\theta)$  for a solution of (2) with fixed starting value  $\zeta_0$ , that is

$$\hat{h}_t(\theta) = \omega + \alpha(Y_{t-1} - m_{\gamma}(\hat{h}_{t-1}(\theta)))^2 + \beta\hat{h}_{t-1}(\theta) \text{ with } \hat{h}_0(\theta) \equiv \zeta_0. \quad (3)$$

In the following, the quasi-likelihood function will be based on  $\hat{h}_t(\theta)$ . Lee and Hansen (1994) and Lumsdaine (1996) were the first to derive the distribution theory for the QMLE of the GARCH(1,1) model. The theory has been extended to the general GARCH( $p, q$ ) case by Berkes et al. (2003) and Francq and Zakoïan (2004), among others. The result for the GARCH(1,1) can be easily extended to a GARCH(1,1) model with a constant function  $m_{\gamma}$ . Also, one can use results from the GARCH(1,1) literature to study the properties of the GARCH(1,1)-M model. For example, in Carrasco and Chen (2002) it has been shown that  $h_t$  in the GARCH(1,1) model is  $\beta$ -mixing with exponentially decaying mixing coefficients. A detailed discussion of the dependence structure of  $Y_t$  and  $h_t$  is provided in Conrad and Karanasos (2014). Some properties of the volatility process  $h_t$  follow directly from the ARCH( $\infty$ ) representation of  $h_t$ . Christensen et al. (2012) give a complete asymptotic analysis for a GARCH(1,1)-M model with

modified recurrence equation (2). In their model it is assumed that  $h_t(\theta) = w + \alpha Y_{t-1}^2 + \beta h_{t-1}(\theta)$ . Then, by construction, the ARCH( $\infty$ ) representation of  $h_t$  does no longer depend on  $m$ . This allows them to develop a detailed theory, also for nonparametric  $m$ . Alternative estimators for a nonparametric  $m$  have been studied in Linton and Perron (2003) and Conrad and Mammen (2008).

For a parametric  $m$  function, it is standard to assume that the conditional mean can be written as  $m_\gamma(x) = \gamma_1 + \gamma_2 g(x)$  with a fixed function  $g$ . The original specification of Engle et al. (1987) assumes either  $g(h_t(\theta_0)) = h_t(\theta_0)$  or  $g(h_t(\theta_0)) = \sqrt{h_t(\theta_0)}$ , while some authors also use  $g(h_t(\theta_0)) = \ln(h_t(\theta_0))$ . The linear specification is directly motivated by Merton's (1973) intertemporal CAPM, which suggests that the expected excess return on the market should be proportional to the conditional variance of the market return. As noted by Pagan and Hong (1990), the log specification may be unsatisfactory, since as  $h_t(\theta_0) \rightarrow 0$  the conditional variance in logs takes very large negative values and the relationship between the conditional variance and  $Y_t$  may be overstated.

In this paper we will develop an asymptotic theory for GARCH-M models. For doing so, we will assume that (2) behaves like a "stochastic contraction". Our approach will cover the specifications  $g(x) = \sqrt{x}$  and  $g(x) = \ln(x)$  but will not apply when  $g(x) = x$ .

## 2 Asymptotics for GARCH(1,1)-M models

Our main result is a theorem on the asymptotic normality of the QMLE  $\hat{\theta}$ . The proof of this result proceeds in several steps where in the first step consistency is shown. In the second step, we derive rates of convergence for  $\hat{\theta}$ . In the final step, this result is used to get local expansions of the quasi-likelihood function that allow to establish asymptotic normality.

In the first step, our treatment of the quasi-maximum likelihood estimator is based on a stochastic recurrence equations approach as developed in Straumann (2005) and Straumann and Mikosch (2006). In those papers, stochastic recurrence equations of the quasi-likelihood function and of its derivatives have been used to show that they have a stationary ergodic functional solution. In the GARCH(1,1)-M model, we can use these arguments to show that the quasi-likelihood function has a stationary ergodic functional solution. But this argument does not apply for the derivatives of the quasi-likelihood function, at least under reasonable assumptions. We argue that the derivatives of the quasi-likelihood functions show exploding behavior in a neighborhood of the true parameter and that they only have a stable behavior in a shrinking neighborhood. For this reason, in a second step we have to show convergence rates for  $\hat{\theta}$ . Having these rates, we only have to consider the derivatives of the quasi-likelihood functions in shrinking neighborhoods.

We make the following assumptions.

**Assumption 1.** *The parameter set  $\Theta$  is compact and equal to the closure of its interior. The true parameter  $\theta_0$  lies in the interior of  $\Theta$ . The function  $(\gamma, u) \rightsquigarrow m_\gamma(u)$  is continuous with respect to  $\gamma$  and differentiable with respect to  $u$ . It holds that  $\omega \geq \omega_* > 0$ ,  $\alpha \geq \alpha_* > 0$ ,  $\beta \geq \beta_* > 0$  for all  $\theta \in \Theta$ . The innovations  $Z_t$  are i.i.d. with  $E[Z_t] = 0$  and  $E[Z_t^2] = 1$ .*

**Assumption 2.** *It holds that  $\alpha_0 + \beta_0 < 1$ .*

Assumptions 1 and 2 imply that  $\sqrt{h_t}Z_t$  is a covariance-stationary process with unconditional variance equal to  $\omega_0/(1 - \alpha_0 - \beta_0)$  (see Bollerslev, 1986). Further, they imply that  $E[\ln(\alpha_0 Z_t^2 + \beta_0)] < 0$ . As mentioned above, this guarantees that (2) has a strictly stationary and ergodic solution  $h_t$  for  $\theta = \theta_0$ . In the proof of consistency of the quasi-maximum likelihood estimator, we make use of the theory on stochastic recurrence equations. The essential assumption needed in this approach is stated below. In the following, we denote derivatives of functions  $m_\gamma(u)$  w.r.t. the argument  $u$  by  $m'_\gamma(u)$ ,  $m''_\gamma(u)$ , ... Derivatives w.r.t. the parameter  $\gamma$  are denoted by  $\dot{m}_\gamma(u)$ ,  $\ddot{m}_\gamma(u)$ , ...

**Assumption 3.** *It holds that*

$$E[\ln(U_t)] < 0, \quad D_1 < +\infty, \quad D_2 < +\infty,$$

where

$$\begin{aligned} U_t &= \sup_{\alpha, \beta} 2\alpha D_1 |Z_t \sqrt{h_t} + m_0(h_t)| + D_2 + \beta, \\ D_1 &= \sup_{\gamma, u} |m'_\gamma(u)|, \\ D_2 &= \sup_{\gamma, u} |m_\gamma(u) m'_\gamma(u)|. \end{aligned}$$

Next, we explain why this assumption naturally arises here. For this purpose, we shortly come back to the classical assumption that  $E[\ln(\alpha_0 Z_t^2 + \beta_0)] < 0$ . We recall why it implies that there exists a stationary ergodic solution  $h_t$  of the GARCH equation. We will later explain why Assumption 3 will be useful for similar reasons. Afterwards, we will discuss how restrictive the assumption is. We start with a brief discussion of stochastic recurrence equations.

Consider first two sequences  $h_t^*$  and  $h_t^{**}$  with different starting values  $\zeta_0^* > 0$  and  $\zeta_0^{**} > 0$  that fulfill the recurrence equation of  $h_t = \bar{h}_t(\theta_0)$ :

$$\begin{aligned} h_t^* &= \omega_0 + h_{t-1}^* (\alpha_0 Z_{t-1}^2 + \beta_0), \\ h_t^{**} &= \omega_0 + h_{t-1}^{**} (\alpha_0 Z_{t-1}^2 + \beta_0). \end{aligned}$$

Then  $h_t^{**} - h_t^* = (h_{t-1}^{**} - h_{t-1}^*) (\alpha_0 Z_{t-1}^2 + \beta_0)$  and the condition  $E[\ln(\alpha_0 Z_t^2 + \beta_0)] < 0$  implies that  $h_t^{**} - h_t^* \rightarrow 0$  a.s.. It can be shown that this result implies that there exists a unique stationary ergodic solution of the recurrence equation of  $h_t = \bar{h}_t(\theta_0)$ . The approach of stochastic recurrence equations has

been generalized w.r.t. two aspects: First, one can consider nonlinear recurrence equations. Then one needs conditions of the type  $E[\ln(\Lambda)] < 0$  where  $\Lambda$  is the (random) Lipschitz constant of the recurrence equation. Second, instead of real valued random variables one can consider random elements of function spaces.

We use the second approach with the random functions

$$\bar{h}_t(\cdot) = \omega + \alpha(Y_{t-1} - m_\gamma(\bar{h}_{t-1}(\cdot)))^2 + \beta\bar{h}_{t-1}(\cdot).$$

Consider two sequences  $\bar{h}_t(\cdot)^{**}$  and  $\bar{h}_t^*(\cdot)$  again with different starting values  $\zeta_0^{**} > 0$  and  $\zeta_0^* > 0$ :

$$\begin{aligned}\bar{h}_t^{**}(\cdot) &= \omega + \alpha(Y_{t-1} - m_\gamma(\bar{h}_{t-1}^{**}(\cdot)))^2 + \beta\bar{h}_{t-1}^{**}(\cdot), \\ \bar{h}_t^*(\cdot) &= \omega + \alpha(Y_{t-1} - m_\gamma(\bar{h}_{t-1}^*(\cdot)))^2 + \beta\bar{h}_{t-1}^*(\cdot).\end{aligned}$$

One can show the following Lipschitz inequality:

$$\|\bar{h}_t^{**}(\cdot) - \bar{h}_t^*(\cdot)\|_\infty \leq U_t \|\bar{h}_{t-1}^{**} - \bar{h}_{t-1}^*\|_\infty$$

with  $U_t$  defined above and  $\|\dots\|_\infty$  equal to the sup-norm. In our Assumption 3, we had assumed that  $E[\ln(U_t)] < 0$ . This assumption implies that the recurrence equation

$$\bar{h}_t(\theta) = \omega + \alpha(Y_{t-1} - m_\gamma(\bar{h}_{t-1}(\theta)))^2 + \beta\bar{h}_{t-1}(\theta)$$

has a stationary ergodic solution  $\bar{h}_t(\theta)$ . This is stated in the following lemma.

**Lemma 1.** *Let Assumptions 1–3 be satisfied. Then (2) has a solution  $\bar{h}_t(\cdot)$  that is unique, stationary and ergodic. Furthermore, it holds that there exists a  $\rho > 1$  such that*

$$\rho^t \sup_{\theta \in \Theta} |\hat{h}_t(\theta) - \bar{h}_t(\theta)| \rightarrow 0, \quad a.s. \quad (4)$$

for the random function  $\hat{h}_t$  that solves (3) for  $t \geq 1$  with fixed starting value  $\zeta_0 > 0$ .

For the convergence statement in (4), one also says that  $\hat{h}_t(\cdot)$  converges exponentially fast almost surely to  $\bar{h}_t(\cdot)$ .

Next, we discuss that Assumption 3 is rather restrictive. It is always fulfilled if  $\beta < 1$  and  $D_1$  and  $D_2$  are small enough. The assumption  $D_2 < +\infty$  states that our function  $m$  does not grow faster than  $x \rightarrow a\sqrt{x}$ . The treatment of functions with faster growth would require a different approach. Consider e.g. the recurrence equation for the linear function  $m_\gamma(x) = \gamma_1 + \gamma_2 x$ . Here, we get that

$$\bar{h}_t(\theta) - \bar{h}_t(\theta_0) = \omega - \omega_0 + \dots + \alpha\gamma_2^2[\bar{h}_{t-1}(\theta) - \bar{h}_{t-1}(\theta_0)]^2 + \dots$$

It needs a very careful check why the quadratic term in the recurrence equation does not lead to an explosive behavior during  $0 \leq t \leq T$ . The process is not stationary and it is to be expected that the

process explodes for  $t \rightarrow \infty$ . In order to illustrate this behavior we simulate the GARCH(1,1)-M model with the following parameters:  $\alpha_0 = 0.1$ ,  $\beta_0 = 0.85$ ,  $\gamma_{01} = 0$  and  $\gamma_{02} = 0.5$ . For the process  $\bar{h}_t(\theta)$ , all parameters but  $\gamma_2$  are chosen as in  $\bar{h}_t(\theta_0)$ , while  $\gamma_2 \in \{0.6, 0.7, 0.8, 0.9, 1\}$ . We choose  $T = 3000$  and consider  $M = 1000$  replications. For different values of  $\Delta\gamma_2 = \gamma_2 - \gamma_{02}$ , the following tables show the fraction of cases in which  $\bar{h}_t(\theta) - \bar{h}_t(\theta_0)$  is diverging (defined as  $\bar{h}_t(\theta) - \bar{h}_t(\theta_0) > 100$ ) and the average point in time  $t$  when this is happening (explosion time). As Tables 1-3 clearly show, the fraction of cases in which  $\bar{h}_t(\theta) - \bar{h}_t(\theta_0)$  is diverging (# divergence) is increasing in  $\Delta\gamma_2$ . Also, the larger  $\Delta\gamma_2$  the earlier the difference  $\bar{h}_t(\theta) - \bar{h}_t(\theta_0)$  explodes. Further, by considering different values of  $\omega_0$  it becomes evident that divergence occurs more often and earlier the larger is the expected value of  $h_t = \bar{h}_t(\theta_0)$ .

Table 1: Comparison of  $\bar{h}_t(\theta) - \bar{h}_t(\theta_0)$  when  $\omega_0 = 0.05$ .

$\Delta\gamma_2$	0.1	0.2	0.3	0.4	0.5
# divergence	0.02	0.50	0.99	1	1
explosion time	1403.8	1316.5	645.96	187.67	82.73

Notes: The table reports the fraction of cases in which  $\bar{h}_t(\theta) - \bar{h}_t(\theta_0)$  is diverging (# divergence). For those simulations for which we observe divergence the average explosion time  $t$  is reported.

Table 2: Comparison of  $\bar{h}_t(\theta) - \bar{h}_t(\theta_0)$  when  $\omega_0 = 0.1$ .

$\Delta\gamma_2$	0.1	0.2	0.3	0.4	0.5
# divergence	0.41	1	1	1	1
explosion time	1327.5	363.15	90.85	40.91	25.09

Notes: see Table 1.

Table 3: Comparison of  $\bar{h}_t(\theta) - \bar{h}_t(\theta_0)$  when  $\omega_0 = 0.2$ .

$\Delta\gamma_2$	0.1	0.2	0.3	0.4	0.5
# divergence	0.99	1	1	1	1
explosion time	513.39	69.49	27.84	16.78	11.67

Notes: see Table 1.



Next, the quasi-maximum likelihood estimator  $\hat{\theta}$  is defined as

$$\hat{\theta} = \arg \max_{\theta \in \Theta} \hat{L}_T(\theta),$$

where  $\hat{L}_T(\theta)$  is the quasi-likelihood function:

$$\hat{L}_T(\theta) = -\frac{1}{2} \sum_{t=1}^T [\ln(\hat{h}_t(\theta)) + \hat{h}_t(\theta)^{-1} (Y_t - m_\gamma(\hat{h}_t(\theta)))^2]. \quad (5)$$

For the consistency of  $\hat{\theta}$ , we need one further assumption.

**Assumption 4.** *The distribution of the random variable  $Z_t$  allows for a strictly positive density on an interval  $[z^*, z^{**}]$  with  $z^* < z^{**}$ . The following identifiability condition holds:*

$$m_\gamma(h_0) = m_0(h_0) \quad a.s. \quad \text{if and only if } \gamma = \gamma_0.$$

The next theorem states the asymptotic consistency of the QMLE. The proof makes essential use of the ergodicity of the process  $\bar{h}(\theta)$ . In particular, this also implies that the quasi-likelihood function converges to its expectation.

**Theorem 1.** *For the model given by (1) - (2), let Assumptions 1 - 4 be satisfied. Then it holds that  $\hat{\theta} \xrightarrow{P} \theta_0$ .*

In a next step, we will show the asymptotic normality of the QMLE  $\hat{\theta}$ . Unfortunately, as mentioned above, the derivatives of the quasi-likelihood do not behave well in fixed neighborhoods of the true parameter  $\theta_0$ . The basic reason is that under reasonable conditions the derivatives of  $\hat{h}_t$  do not behave well at points  $\theta \neq \theta_0$ . Only for  $\theta$  in a shrinking neighborhood of  $\theta_0$ , one can control the asymptotic behavior of the derivatives. For this reason, we need a stronger result than Theorem 1. In our next theorem, we will show that  $\hat{\theta}$  converges to  $\theta_0$  with nearly parametric rate  $O_P(\ln(T)T^{-1/2})$ . In a next step, we will show that the first two derivatives  $\hat{h}_t^{(l)}(\theta)$  ( $l \in \{1, 2\}$ ) of  $\hat{h}_t(\theta)$  converge to a stationary ergodic processes, uniformly over  $\theta$  with  $\|\theta - \theta_0\| \leq \ln(T)T^{-1/2}$ . The limiting processes do not depend on  $\theta$  in this shrinking neighborhood. This can be used to show asymptotic normality of the QMLE. For our next theorem, we need the following additional assumptions:

**Assumption 5.** *For some  $D > 0$  it holds for  $\|\theta - \theta_0\| \leq \delta$  that for the functions  $g_1(s) = m_\gamma(s)\dot{m}_\gamma(s)$ ,  $g_1(s) = m_\gamma(s)m'_\gamma(s)$ ,  $g_2(s) = m'_\gamma(s)$ ,  $g_2(s) = m_\gamma(s)$ ,  $g_2(s) = \dot{m}_\gamma(s)$ ,  $g_2(s) = \dot{m}'_\gamma(s)$ ,  $g_2(s) = m''_\gamma(s)$ ,  $g_2(s) = \ddot{m}_\gamma(s)$ ,  $g_3(s) = \dot{m}_\gamma(s)$ ,  $g_4(s) = m'_\gamma(s)$ ,  $g_4(s) = \dot{m}'_\gamma(s)$  with some constant  $D > 0$*

$$\begin{aligned} \|g_1(s) - g_1(s')\| &\leq D|s - s'|, \\ \|g_2(s) - g_2(s')\| &\leq D|s - s'|s^{-1/2}, \\ \|g_3(s)\| &\leq D\sqrt{s}, \\ \|g_4(s)\| &\leq Ds^{-1/2} \end{aligned}$$

for  $s > s' \geq \omega_*$ . Here,  $\dot{f}_\gamma(s)$  and  $\ddot{f}_\gamma(s)$  denote the first or second order partial derivative of a function  $f_\gamma(s)$  with respect to  $\gamma$ .

**Assumption 6.** *There exists  $\delta > 0, D_3 > 0$  such that for*

$$V_t = \sup_{\|\theta - \theta_0\| \leq \delta} \frac{4D_3\alpha|Z_t| + \alpha D_3^2 + 4DD_3\delta + 4\beta}{4(\alpha_0 Z_t^2 + \beta_0)}$$

it holds that

$$E \ln(V_t) < 0.$$

Here,  $D_3$  is chosen such that

$$|m_\gamma(s) - m_\gamma(s^*)| \leq \frac{D_3 |s - s^*|}{2\sqrt{s}}$$

for  $s, s^* \geq \omega_*$  and  $\|\theta - \theta_0\| \leq \delta$ . If  $V_t$  fulfills that  $P[V_t > 1] > 0$ , then we define  $\kappa_1 > 0$  as the solution of the equation  $EV_t^{\kappa_1} = 1$ . If  $P[V_t > 1] = 0$  we set  $\kappa_1 = \infty$  and  $\kappa_1^{-1} = 0$ .

If  $V_t$  fulfills that  $P[V_t > 1] > 0$ , then there exists a unique solution  $0 < \kappa_1 < \infty$  of the equation  $EV_t^{\kappa_1} = 1$ , see Theorem 2.1 in Mikosch and Stărică (2000).

We make the following assumption on the moments of  $Z_t$ .

**Assumption 7.** *The random variables  $Z_t$  fulfill the following moment condition:*

$$EZ_t^{\kappa_2} < \infty$$

for some  $\kappa_2 > 4$ .

**Assumption 8.** *The random variables  $Z_t$  fulfill the following moment condition:*

$$E(\alpha_0 Z_t^2 + \beta_0)^{\kappa_3/2} = 1$$

for some  $\kappa_3 > 0$  with  $2\kappa_3^{-1} + 2\kappa_1^{-1} < 1$ .

**Assumption 9.** *The matrix  $S = S_1 + S_2$  with*

$$S_1 = E \left[ \frac{\bar{h}'_t(\theta_0) \bar{h}'_t(\theta_0)^\top}{h_t^2} \right],$$

$$S_2 = E \left[ \frac{(\dot{m}_{\gamma_0}(h_t) + m'_{\gamma_0}(h_t) \bar{h}'_t(\theta_0)) (\dot{m}_{\gamma_0}(h_t) + m'_{\gamma_0}(h_t) \bar{h}'_t(\theta_0))^\top}{h_t} \right]$$

is non-singular.

We now state the following result on the convergence rate of the quasi-maximum likelihood estimator.

**Proposition 1.** *For the model given by (1) - (2), let Assumptions 1 - 9 with  $\delta > 0$  small enough be satisfied. Then it holds that*

$$\hat{\theta} - \theta_0 = O_P \left( \ln(T) T^{-1/2} \right).$$

Proposition 1 allows to restrict attention to local expansions of the quasi-likelihood and this is the essential step to derive asymptotic normality of the maximum quasi-likelihood estimator as stated in our main theorem.

**Theorem 2.** *For the model (1) - (2), make the Assumptions 1 - 9 with  $\delta > 0$  small enough. Then it holds that*

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(\Sigma_2^{-1}\Sigma_1\Sigma_2^{-1}),$$

where

$$\begin{aligned} \Sigma_1 &= E \left[ \left\{ \frac{1}{2} \frac{\bar{h}'_t}{\bar{h}_t} (Z_t^2 - 1) + \bar{h}_t^{-1/2} (\dot{m}_{\gamma_0}(\bar{h}_t) + m'_{\gamma_0}(\bar{h}_t)\bar{h}'_t) Z_t \right\} \left\{ \frac{1}{2} \frac{\bar{h}'_t}{\bar{h}_t} (Z_t^2 - 1) + \bar{h}_t^{-1/2} (\dot{m}_{\gamma_0}(\bar{h}_t) \right. \right. \\ &\quad \left. \left. + m'_{\gamma_0}(\bar{h}_t)\bar{h}'_t) Z_t \right\}^\top \right], \\ \Sigma_2 &= E \left[ \frac{1}{2} \frac{\bar{h}'_t(\bar{h}'_t)^\top}{\bar{h}_t^2} + \frac{1}{\bar{h}_t} (\dot{m}_{\gamma_0}(\bar{h}_t) + m'_{\gamma_0}(\bar{h}_t)\bar{h}'_t) (\dot{m}_{\gamma_0}(\bar{h}_t) + m'_{\gamma_0}(\bar{h}_t)\bar{h}'_t)^\top \right]. \end{aligned}$$

Note that for Gaussian  $Z_t$ , we have  $\Sigma_1 = \Sigma_2$  and the asymptotic covariance is equal to  $\Sigma_1^{-1}$ . On the other hand, if there is no mean function, i.e.  $m_{\gamma_0}(h_t) = 0$ , then  $\Sigma_1$  and  $\Sigma_2$  reduce to

$$\Sigma_1 = \frac{1}{2} E \left[ \frac{\bar{h}'_t(\bar{h}'_t)^\top}{\bar{h}_t^2} \right] \quad \text{and} \quad \Sigma_2 = \frac{1}{4} (E[Z_t^4] - 1) E \left[ \frac{\bar{h}'_t(\bar{h}'_t)^\top}{\bar{h}_t^2} \right]$$

and the covariance reduces to the one of the standard GARCH(1,1) (see Theorem 2.2 in Francq and Zakoian, 2004).

### 3 Conclusions

Finding sufficient regularity conditions that ensure consistency and asymptotic normality of the QMLE for the GARCH-M model has been a long-standing problem in financial econometrics. We consider the special case of a parametric GARCH-M model with innovations that follow a GARCH(1,1) process, which is the specification most often used in empirical applications. Following Straumann (2005) and Straumann and Mikosch (2006), we make use of stochastic recurrence equations and employ exponential inequalities to show the consistency and asymptotic normality of the QMLE for certain specifications of the mean function that do not grow too fast.

## Appendix

**Proof of Lemma 1.** Put

$$g_\theta(y, s) = \omega + \alpha(y - m_\gamma(s))^2 + \beta s \quad (6)$$

and consider the sequence of random functions  $\phi_t$  with:

$$[\phi_t(s)](\theta) = g_\theta(Y_t, s(\theta)).$$

The functions  $\phi_t$  map continuous functions  $s : \Theta \rightarrow [0, \infty)$  onto the class of such functions. Note that

$$h_{t+1}(\theta) = [\phi_t(h_t)](\theta).$$

Because  $Y_t$  is a stationary and ergodic sequence, the same holds for  $\phi_t$ .

Consider functions  $s, s^*$  with  $\|s - s^*\|_\infty \leq \delta$ . It holds that

$$\begin{aligned} |\phi_t(s) - \phi_t(s^*)|(\theta) &\leq 2\alpha|\sqrt{\bar{h}_t}Z_t + m_{\gamma_0}(h_t)||m_\gamma(s) - m_\gamma(s^*)| + \alpha|m_\gamma(s)^2 - m_\gamma(s^*)^2| \\ &\quad + \beta|s - s^*|(\theta) \\ &\leq U_t|s - s^*|(\theta). \end{aligned}$$

The lemma follows from  $E[\ln(U_t)] < 0$  by application of Theorem 3.1 in Bougerol (1993), see also Proposition 5.2.12 in Straumann (2005). See also the discussion before the statement of Lemma 1.  $\square$

**Proof of Theorem 1.** The theorem can be shown by similar arguments as in Theorem 5.3.1. in Straumann (2005). There, the proof is based on the comparison of  $\hat{L}_T(\theta)$ ,  $\bar{L}_T(\theta)$  and  $L(\theta)$ , where  $\hat{L}_T(\theta)$  is defined in (5) and

$$\begin{aligned} \bar{L}_T(\theta) &= -\frac{1}{2} \sum_{t=1}^T \ln(\bar{h}_t(\theta)) + \bar{h}_t(\theta)^{-1} (Y_t - m_\gamma(\bar{h}_t(\theta)))^2, \\ L(\theta) &= -\frac{1}{2} E [\ln(\bar{h}_t(\theta)) + \bar{h}_t(\theta)^{-1} (Y_t - m_\gamma(\bar{h}_t(\theta)))^2]. \end{aligned}$$

The proof in Straumann (2005) is based on showing:

$$\frac{1}{T} \|\hat{L}_T - \bar{L}_T\|_\infty \rightarrow 0 \text{ (in probability), } \quad \left\| \frac{1}{T} \bar{L}_T - L \right\|_\infty \rightarrow 0 \text{ (in probability), } \quad L(\theta) < L(\theta_0) \text{ for } \theta \neq \theta_0.$$

The second claim follows from the fact that  $\bar{h}_t$  is a stationary ergodic process (see Straumann, 2005).

For the first claim, one uses the bound

$$\begin{aligned} \frac{1}{T} |\hat{L}_T(\theta) - \bar{L}_T(\theta)| &\leq \frac{c}{T} \sum_{t=1}^T \Delta_t(\theta) \left\{ \frac{(Y_t - m_\gamma(\hat{h}_t(\theta)))^2}{s_t(\theta)} + \frac{(Y_t - m_\gamma(\bar{h}_t(\theta)))^2}{s_t(\theta)} + 1 \right\} \\ &\quad + \frac{c}{T} \sum_{t=1}^T \frac{|m_\gamma(\hat{h}_t(\theta)) - m_\gamma(\bar{h}_t(\theta))| |Y_t|}{s_t(\theta)} \\ &\quad + \frac{c}{T} \sum_{t=1}^T \frac{|m_\gamma(\hat{h}_t(\theta)) - m_\gamma(\bar{h}_t(\theta))|^2}{s_t(\theta)}, \end{aligned}$$

where  $s_t(\theta) = \hat{h}_t(\theta) + \bar{h}_t(\theta)$ ,  $\Delta_t(\theta) = |\hat{h}_t(\theta) - \bar{h}_t(\theta)|$  and  $c > 0$  is a constant, not depending on  $\theta$ . Using Assumption 3, we have  $m_\gamma^2(x) \leq c' x$ ,  $|m_\gamma(x) - m_\gamma(y)| \leq c' |x - y|$ ,  $|m_\gamma(x)^2 - m_\gamma(y)^2| \leq c' |x - y|$  with a constant  $c' > 0$ . Using  $s_t(\theta) \geq 2\omega_*$  (see Assumption 1) we get with a constant  $c''$ :

$$\frac{1}{T}[\hat{L}_T(\theta) - \bar{L}_T(\theta)] \leq \frac{c''}{T} \sum_{t=1}^T \Delta_t(\theta) \{1 + h_t + h_t Z_t^2 + Z_t^2\}. \quad (7)$$

Because of Assumptions 1 and 2, we have that  $E[h_t Z_t^2] = E[h_t] < \infty$ . Further, because of the ergodicity of  $h_t$ ,  $h_t Z_t^2$  and  $Z_t^2$ , this implies that  $\frac{1}{s} \sum_{t=1}^s [1 + h_t + h_t Z_t^2 + Z_t^2] = O_P(1)$  for  $s \rightarrow \infty$ . We apply this result with  $s = T$  and  $s = C \ln T$  with  $C > 0$  large enough. With this bound, (7) and (4), we get if  $C$  is chosen large enough that

$$\begin{aligned} \frac{1}{T}[\hat{L}_T(\theta) - \bar{L}_T(\theta)] &\leq O_P\left(\frac{C \ln T}{T}\right) \frac{1}{C \ln T} \sum_{t=1}^{C \ln T} \{1 + h_t + h_t Z_t^2 + Z_t^2\} \\ &\quad + O_P(\rho^{-C \ln T}) \frac{1}{T} \sum_{t=1}^T \{1 + h_t + h_t Z_t^2 + Z_t^2\} \\ &= O_P\left(\frac{\ln T}{T}\right), \end{aligned} \quad (8)$$

uniformly over  $\theta \in \Theta$ . In particular, we have that  $\frac{1}{T}[\hat{L}_t(\theta) - \bar{L}_t(\theta)]$  converges almost surely to 0, uniformly over  $\theta \in \Theta$ .

It remains to check the last claim:  $L(\theta) < L(\theta_0)$  for  $\theta \neq \theta_0$ . For the proof of this claim, it suffices to check that

$$\bar{h}_0(\theta) = h_0 \text{ a.s. implies } \theta = \theta_0.$$

From  $\bar{h}_0(\theta) = h_0$  a.s. and stationarity of  $(\bar{h}_t(\theta), h_t)$ , we get that  $\bar{h}_1(\theta) = h_1$  a.s.. Thus, we have that

$$0 = \omega - \omega_0 + (\alpha - \alpha_0)Y_0^2 - 2Y_0(\alpha m_\gamma(h_0) - \alpha_0 m_0(h_0)) + \alpha m_\gamma^2(h_0) - \alpha_0 m_0^2(h_0) + (\beta - \beta_0)h_0 \text{ a.s.}$$

Using Assumption 4 and considering the first two derivatives of the right hand side with respect to the  $y$  value, we get that  $\alpha = \alpha_0$ ,  $m_\gamma = m_0$  on the support of  $h_0$  (a.s.), and  $\omega = \omega_0$ .  $\square$

For the proof of Proposition 1, we will make use of the following lemmas.

**Lemma 2.** *Make the assumptions of Proposition 1. Then with  $\hat{h}_t = \hat{h}_t(\hat{\theta})$  and  $\hat{m} = m_{\hat{\gamma}}$  it holds that*

$$\begin{aligned} &\frac{1}{2} \sum_{t=1}^T \frac{(\hat{h}_t - h_t)^2}{\hat{h}_t^2 \vee h_t^2} + \sum_{t=1}^T \frac{(\hat{m}(\hat{h}_t) - m(h_t))^2}{\hat{h}_t} \\ &\leq - \sum_{t=1}^T \frac{(\hat{h}_t - h_t)}{\hat{h}_t} (Z_t^2 - 1) + 2 \sum_{t=1}^T (\hat{m}(\hat{h}_t) - m(h_t)) \frac{\sqrt{\hat{h}_t}}{\hat{h}_t} Z_t + O_P(\ln T), \end{aligned} \quad (9)$$

where  $a \vee b$  denotes the maximum of the real numbers  $a$  and  $b$ .

**Proof of Lemma 2.** Note that by definition of the quasi-likelihood estimator  $\hat{\theta}$ , we have that  $\hat{L}_T(\hat{\theta}) \geq \hat{L}_T(\theta_0)$ . Because of (8), this implies that  $\bar{L}_T(\hat{\theta}) \geq \bar{L}_T(\theta_0) + O_P(\ln T)$ . We make use of the inequality

$\ln(1+x) \leq x - x^2(1+(x)_+)^{-1}$ , where  $(x)_+$  is the positive part of  $x$ . This inequality follows easily from a Taylor expansion around  $x=0$ . From these two inequalities, we get that

$$\begin{aligned} O_P(\ln T) &\leq \frac{1}{2} \sum_{t=1}^T \ln \left( \frac{h_t}{\hat{h}_t} \right) - \frac{1}{2} \sum_{t=1}^T \left[ \hat{h}_t^{-1} (h_t^{1/2} Z_t + m_0(h_t) - \hat{m}(\hat{h}_t))^2 - Z_t^2 \right] \\ &\leq \frac{1}{2} \sum_{t=1}^T \frac{h_t - \hat{h}_t}{\hat{h}_t} - \frac{1}{4} \sum_{t=1}^T \frac{(h_t - \hat{h}_t)^2}{\hat{h}_t^2 \vee h_t^2} - \frac{1}{2} \sum_{t=1}^T \frac{h_t - \hat{h}_t}{\hat{h}_t} Z_t^2 \\ &\quad - \sum_{t=1}^T \frac{h_t^{1/2}}{\hat{h}_t} (m_0(h_t) - \hat{m}(\hat{h}_t)) Z_t - \frac{1}{2} \sum_{t=1}^T \hat{h}_t^{-1} (m_0(h_t) - \hat{m}(\hat{h}_t))^2. \end{aligned}$$

The claim of the lemma follows by rearrangement of the terms.  $\square$

**Lemma 3.** *Make the assumptions of Proposition 1. There exist random variables  $W_t$  with  $\sup_{1 \leq t \leq T} |W_t| = O_P(T^{1/\kappa_1})$  such that for  $\|\theta - \theta_0\| \leq \delta$*

$$\left| \frac{\bar{h}_t(\theta) - h_t}{h_t} \right| \leq \|\theta - \theta_0\| W_t, \quad (10)$$

$$\left| \frac{\hat{h}_t(\theta) - \hat{h}_t}{h_t} \right| \leq \|\theta - \theta_0\| W_t. \quad (11)$$

**Proof of Lemma 3.** We only show claim (10). Claim (11) follows by similar arguments. For the proof of claim (10), we show that for some constant  $C > 0$  for  $\|\theta - \theta_0\| \leq \delta$

$$\left| \frac{\bar{h}_{t+1}(\theta) - h_{t+1}}{h_{t+1}} \right| \leq C \|\theta - \theta_0\| + V_t \left| \frac{\bar{h}_t(\theta) - h_t}{h_t} \right|. \quad (12)$$

For a proof of this claim, write  $\bar{h}_t = \bar{h}_t(\theta)$  and  $\Delta\omega = |\omega - \omega_0|$ ,  $\Delta\alpha = |\alpha - \alpha_0|$  and  $\Delta\beta = |\beta - \beta_0|$ . We get that for some constants  $C_1 > 0$  for  $\|\theta - \theta_0\| \leq \delta$

$$\begin{aligned} |\bar{h}_{t+1} - h_{t+1}| &\leq \Delta\omega + \Delta\alpha h_t Z_t^2 + \alpha \left| (h_t^{1/2} Z_t + m_0(h_t) - m_\gamma(\bar{h}_t))^2 - h_t Z_t^2 \right| + \Delta\beta h_t + \beta |\bar{h}_t - h_t| \\ &\leq C_1 \|\theta - \theta_0\| (1 + h_t + h_t Z_t^2) + 2\alpha |m_\gamma(h_t) - m_\gamma(\bar{h}_t)| h_t^{1/2} Z_t \\ &\quad + 2\alpha |m_\gamma(h_t) - m_\gamma(\bar{h}_t)| |m_\gamma(h_t) - m_0(h_t)| + \alpha |m_\gamma(h_t) - m_\gamma(\bar{h}_t)|^2 + \beta |\bar{h}_t - h_t| \\ &\leq C_1 \|\theta - \theta_0\| (1 + h_t + h_t Z_t^2) + |\bar{h}_t - h_t| \left[ \alpha D_3 Z_t + \alpha D D_3 \delta + \frac{\alpha D_3^2}{4} + \beta \right]. \end{aligned}$$

If we divide both sides of this inequality by  $h_{t+1}$ , we get equation (12), because of  $h_{t+1} \geq (\alpha_0 Z_t^2 + \beta_0) h_t$ .

For the proof of the lemma, it remains to show that (12) implies (10). Put  $\Delta_t = |\bar{h}_t - h_t|/h_t$  and define  $\Delta_t^*$  as the stationary solution of the recurrence equation  $\Delta_{t+1}^* = 1 + V_t \Delta_t^*$ . We have that

$$\Delta_{t+1} - C \|\theta - \theta_0\| \Delta_{t+1}^* \leq V_t (\Delta_t - C \|\theta - \theta_0\| \Delta_t^*).$$

This implies that  $(\Delta_{t+1} - C \|\theta - \theta_0\| \Delta_{t+1}^*)_+ \leq V_t (\Delta_t - C \|\theta - \theta_0\| \Delta_t^*)_+$ , where  $(x)_+$  denotes the positive part of  $x$ . Because  $(\Delta_t - C \|\theta - \theta_0\| \Delta_t^*)_+$  is stationary and  $E \ln(V_t) < 0$ , we get that  $(\Delta_t - C \|\theta - \theta_0\| \Delta_t^*)_+ = 0$  a.s.. Thus for  $W_t = C \Delta_t^*$  we have that  $\Delta_t \leq \|\theta - \theta_0\| W_t$  a.s.. For the proof of the lemma, it remains

to be shown that  $\sup_{1 \leq t \leq T} |W_t| = O_P(T^{1/\kappa_1})$ . If  $P[V_t > 1] = 0$ , we can bound  $V_t$  by a random variable  $V_t^*$  with  $P[V_t^* > 1] = 0$  and  $E(V_t^*)^{\kappa_1^*} = 1$  with  $\kappa_1^*$  as large as we like. Thus, w.l.o.g. we can assume that  $P[V_t > 1] > 0$ . For this case we get from Theorem 4.1 in Goldie (1991) that  $P(W_t \geq x) \sim cx^{-\kappa_1}$  for  $x \rightarrow \infty$  for some constant  $c > 0$ . This implies  $\sup_{1 \leq t \leq T} |W_t| = O_P(T^{1/\kappa_1})$ . Note also that  $V_t$  is bounded by definition.  $\square$

Denote by  $\hat{h}'_t(\theta)$  the solution of

$$\hat{h}'_{t+1}(\theta) = \partial_\theta g_\theta(Y_t, \hat{h}_t(\theta)) + \partial_s g_\theta(Y_t, \hat{h}_t(\theta)) \hat{h}'_t(\theta) \quad (13)$$

with deterministic starting value  $\hat{h}'_0(\theta) = \zeta_1$ . The function  $g_\theta$  was defined in (6). Furthermore,  $\partial_\theta g_\theta$  and  $\partial_s g_\theta$  are the partial derivatives of  $g_\theta$  with respect to  $\theta$  or  $s$ , respectively. We also define  $\hat{h}''_t(\theta)$  as the solution of

$$\hat{h}''_{t+1}(\theta) = \partial_{\theta\theta} g_\theta(Y_t, \hat{h}_t(\theta)) + 2\partial_{\theta s} g_\theta(Y_t, \hat{h}_t(\theta)) \hat{h}'_t(\theta) + \partial_{ss} g_\theta(Y_t, \hat{h}_t(\theta)) \hat{h}''_t(\theta) \quad (14)$$

with deterministic starting value  $\hat{h}''_0(\theta) = \zeta_2$ . Here  $\partial_{\theta\theta} g_\theta$ ,  $\partial_{\theta s} g_\theta$  and  $\partial_{ss} g_\theta$  denote second order partial derivatives of  $g_\theta$ .

The next lemma states that

$$d_{t+1}^*(\theta) = \partial_\theta g_\theta(Y_t, \bar{h}_t(\theta)) + \partial_s g_\theta(Y_t, \bar{h}_t(\theta)) d_t^*(\theta) \quad (15)$$

has a unique stationary solution  $d_t^*(\cdot)$ . We denote this solution by  $\bar{h}'_t(\cdot) = d_t^*(\cdot)$ .

**Lemma 4.** *Make the assumptions of Proposition 1. Equation (15) has a unique stationary solution  $\bar{h}'_t(\cdot) = d_t^*(\cdot)$  that is ergodic. For  $\delta > 0, \rho > 1$  small enough it holds that*

$$\rho^t \sup_{\|\theta - \theta_0\| \leq \delta} \|\bar{h}'_t(\theta) - \hat{h}'_t(\theta)\| \rightarrow 0, \quad a.s.$$

Furthermore, it holds that  $\bar{h}'_t$  is identical to the derivative of  $\bar{h}_t$ , a.s., and that it is continuous.

**Proof of Lemma 4.** According to Proposition 5.5.1 of Straumann (2005), it suffices for the statement of the lemma to verify that: (i)  $g_\theta(y, s)$  is continuously differentiable with respect to  $\theta$  and  $s$  for  $y$  fixed. (ii) For some  $\kappa > 0$  and a stationary process  $C_t$  with  $E[\ln^+(C_t)] < \infty$  it holds that  $\|\Delta g_\theta(Y_t, s) - \Delta g_\theta(Y_t, s^*)\| \leq C_t |s - s^*|^\kappa$  for  $s, s^* \geq \omega_*$ ,  $\|\theta - \theta_0\| \leq \delta$ . (iii)  $E[\ln^+(\sup_{\|\theta - \theta_0\| \leq \delta} \Delta g_\theta(Y_0, \bar{h}_0(\theta)))] < \infty$ .

Here,  $\Delta g_\theta(y, s)$  denotes the vector of the first order derivatives of  $g_\theta(y, s)$  with respect to  $\theta$  and  $s$  for  $y$  fixed. We now check (i)–(iii). Condition (i) directly follows from our Assumption 1. For the check of (ii) we note that from Assumption 1 we get by direct calculations that (ii) holds with  $C_t = C(1 + \sqrt{h_t} + \sqrt{h_t}|Z_t|)$  if the deterministic constant  $C$  is chosen large enough. The condition  $E[\ln^+(C_t)] < \infty$  follows from  $EZ_t^2 < \infty$  and  $Eh_t < \infty$ .

For the proof of (iii), one shows the following bound for  $\|\theta - \theta_0\|$  small enough and  $C > 0$  large enough

$$\begin{aligned} \|\Delta g_\theta(Y_0, \bar{h}_0(\theta)) - \Delta g_{\theta_0}(Y_0, \bar{h}_0(\theta_0))\| &\leq C[1 + h_0 + h_0|Z_0| + (1 + |Z_0|)|\bar{h}_0(\theta) - h_0|] \\ &\leq C[1 + h_0 + h_0|Z_0| + (1 + |Z_0|)h_0W_0], \end{aligned}$$

where in the last inequality Lemma 3 has been used. Now, by direct calculations with  $C^* > 0$  large enough

$$\|\Delta g_{\theta_0}(Y_0, \bar{h}_0(\theta_0))\| \leq C^*[1 + h_0 + h_0|Z_0|].$$

This gives for  $\|\theta - \theta_0\|$  small enough and  $C^{**} > 0$  large enough

$$\|\Delta g_\theta(Y_0, \bar{h}_0(\theta))\| \leq C^{**}h_0(1 + Z_0^2)W_0.$$

Claim (iii) follows from  $E[\ln^+(h_0)] < \infty$ ,  $E[\ln^+(1 + Z_0^2)] < \infty$  and  $E[\ln^+(W_0)] < \infty$ . This concludes the proof of Lemma 4.  $\square$

**Lemma 5.** *Make the assumptions of Proposition 1. It holds for  $\rho \rightarrow 0$  that*

$$\sup_{\|\theta - \theta_0\| \leq \rho} \|\theta - \theta_0\|^{-2} \left| E \left[ \frac{(\bar{h}_t(\theta) - h_t)^2}{\bar{h}_t(\theta)^2 \vee h_t^2} \right] - (\theta - \theta_0)^\top S_1(\theta - \theta_0) \right| \rightarrow 0, \quad (16)$$

$$\sup_{\|\theta - \theta_0\| \leq \rho} \|\theta - \theta_0\|^{-2} \left| E \left[ \frac{(\bar{h}_t(\theta) - h_t)^2}{h_t^2} \right] - (\theta - \theta_0)^\top S_1(\theta - \theta_0) \right| \rightarrow 0, \quad (17)$$

$$\sup_{\|\theta - \theta_0\| \leq \rho} \|\theta - \theta_0\|^{-2} \left| E \left[ \frac{(m_\gamma(\bar{h}_t(\theta)) - m_0(h_t))^2}{\bar{h}_t(\theta)} \right] - (\theta - \theta_0)^\top S_2(\theta - \theta_0) \right| \rightarrow 0, \quad (18)$$

$$\sup_{\|\theta - \theta_0\| \leq \rho} \|\theta - \theta_0\|^{-2} \left| E \left[ \frac{(m_\gamma(\bar{h}_t(\theta)) - m_0(h_t))^2}{h_t} \right] - (\theta - \theta_0)^\top S_2(\theta - \theta_0) \right| \rightarrow 0. \quad (19)$$

**Proof of Lemma 5.** Choose  $\theta_n$  with  $\theta_n \rightarrow \theta_0$  and  $\|\theta_n - \theta_0\|^{-1}(\theta_n - \theta_0) \rightarrow e$  for a unit vector  $e$ . For claim (16), we have to show that

$$\left| \|\theta_n - \theta_0\|^{-2} E \left[ \frac{(\bar{h}_t(\theta_n) - h_t)^2}{\bar{h}_t^2(\theta_n) \vee h_t^2} \right] - e^\top E \left[ \frac{\bar{h}'_t(\theta_0)\bar{h}'_t(\theta_0)^\top}{h_t^2} \right] e \right| \rightarrow 0.$$

For a proof of this claim first note that because of Lemma 4 we have that

$$\|\theta_n - \theta_0\|^{-2} [(\bar{h}_t(\theta_n) - h_t)^2(\bar{h}_t^2(\theta_n) \vee h_t^2)^{-1}] \rightarrow e^\top [\bar{h}'_t(\theta_0)\bar{h}'_t(\theta_0)^\top h_t^{-2}] e \text{ a.s.}$$

Thus, the claim follows by dominated convergence since

$$\|\theta_n - \theta_0\|^{-2} [(\bar{h}_t(\theta_n) - h_t)^2(\bar{h}_t^2(\theta_n) \vee h_t^2)^{-1}] \leq W_t^2 h_t \bar{h}_t(\theta_n)^{-1} \leq W_t^2 h_t \omega_*^{-1} \leq W_t^4 \omega_*^{-1} + h_t^2 \omega_*^{-1}, \quad (20)$$

$EW_t^4 < \infty$  and  $Wh_t^2 < \infty$ , see also Lemma 3.

Claims (17)–(19) can be shown by similar arguments.  $\square$



**Lemma 6.** *Make the assumptions of Proposition 1. With some constants  $c^+ > c^- > 0$ , it holds that*

$$c^- \|\theta - \theta_0\|^2 \leq \frac{1}{T} \sum_{t=1}^T \frac{(\bar{h}_t(\theta) - h_t)^2}{\bar{h}_t^2(\theta) \vee h_t^2} + \frac{1}{T} \sum_{t=1}^T \frac{(m_\gamma(\bar{h}_t(\theta)) - m(h_t))^2}{\bar{h}_t(\theta)} \leq c^+ \|\theta - \theta_0\|^2$$

for all  $\|\theta - \theta_0\| \leq \delta$  with probability tending to one.

**Proof of Lemma 6.** Put  $\varphi_t(\theta_0) = 0$  and define for  $\theta \neq \theta_0$

$$\begin{aligned} \varphi_t(\theta) &= \|\theta - \theta_0\|^{-2} \left[ \frac{(\bar{h}_t(\theta) - h_t)^2}{\bar{h}_t^2(\theta) \vee h_t^2} + \frac{(m_\gamma(\bar{h}_t(\theta)) - m(h_t))^2}{\bar{h}_t(\theta)} - (\theta - \theta_0)^\top R_t (\theta - \theta_0) \right], \\ R_t &= \frac{\bar{h}'_t(\theta_0) \bar{h}'_t(\theta_0)^\top}{h_t^2} + \frac{(\dot{m}_{\gamma_0}(h_t) + m'_{\gamma_0}(h_t) \bar{h}'_t(\theta_0)) (\dot{m}_{\gamma_0}(h_t) + m'_{\gamma_0}(h_t) \bar{h}'_t(\theta_0))^\top}{h_t}. \end{aligned}$$

Because of Lemmas 1 and 4, we have that  $\varphi_t$  is ergodic and stationary. Using the bound (20) for the first term of  $\varphi_t$  and a similar bound for the second term we get that  $E[\sup_{\|\theta - \theta_0\| \leq \delta} |\varphi_t(\theta)|] < \infty$ . Thus, we have that

$$\sup_{\|\theta - \theta_0\| \leq \delta} \left| \frac{1}{T} \sum_{t=1}^T \varphi_t(\theta) - E[\varphi_t(\theta)] \right| = o_P(1).$$

From Lemma 5, we know that  $\sup_{\|\theta - \theta_0\| \leq \rho} \|E[\varphi_t(\theta)]\| \rightarrow 0$  for  $\rho \rightarrow 0$ . Here,  $\|\cdot\|$  denotes the spectral norm of a matrix, i.e.  $\theta \rightarrow E[\varphi_t(\theta)]$  is continuous in  $\theta = \theta_0$ . The statement of the lemma now follows from  $\frac{1}{T} \sum_{t=1}^T R_t = S + o_P(1)$ , see Lemma 4, and our assumption that  $S$  is non-singular. Here, we make the assumption that  $\delta$  is chosen small enough.  $\square$

Our next lemma contains an exponential inequality for martingales. This inequality is a modification of e.g. Lemma 8.9 in van de Geer (2000).

**Lemma 7.** *For random variables  $\dots, e_{-1}, e_0, e_1, \dots, e_T$  suppose that  $e_t$  is  $\mathcal{F}_t$ -measurable for an increasing  $\sigma$ -field  $\mathcal{F}_t$ , that  $\mathbf{E}[e_t | \mathcal{F}_{t-1}] = 0$  and that  $\sup_t \mathbf{E}[\exp(c|e_t|) | \mathcal{F}_{t-1}] < \infty$  (a.s.) for a constant  $c > 0$  small enough. Consider a sequence of random variables  $r_1, r_2, \dots$  where  $r_t$  is measurable with respect to the  $\sigma$ -field generated by  $\mathcal{F}_{t-1}$ . Assume that  $\max_{1 \leq t \leq T} |r_t| \leq c/2$  (a.s.). Then it holds that*

$$\mathbf{E} \left[ \exp \left( \sum_{t=1}^T r_t e_t \right) \right] \leq \left\{ \mathbf{E} \left[ \exp \left( C \sum_{t=1}^T r_t^2 \right) \right] \right\}^{1/2},$$

where  $C$  is a deterministic a.s. bound of  $\mathbf{E} [2e_t^2 \exp(c|e_t|) | \mathcal{F}_{t-1}]$ .

We will make use of this lemma in the proof of the following lemma. For completeness we will give a proof of Lemma 7, although proofs of related versions of the result must be available elsewhere.

**Proof of Lemma 7.** We will show that for  $0 \leq s \leq T$

$$\mathbf{E} \left[ \exp \left( \sum_{t=1}^T r_t e_t \right) \right] \leq \left\{ \mathbf{E} \left[ \exp \left( \sum_{t=1}^s r_t e_t \right) \sqrt{\mathbf{E}_{s+1} \left[ \exp \left( C \sum_{t=s+1}^T r_t^2 \right) \right]} \right] \right\}, \quad (21)$$

where  $\mathbf{E}_{s+1}[\dots]$  denotes the conditional expectation  $\mathbf{E}[\dots|\mathcal{F}_s]$ . Note that claim (21) with  $s = 0$  implies the statement of the lemma because of

$$\mathbf{E} \left[ \sqrt{\mathbf{E}_1 \left[ \exp\left(C \sum_{t=1}^T r_t^2\right) \right]} \right] \leq \left\{ \mathbf{E} \left[ \mathbf{E}_1 \left[ \exp\left(C \sum_{t=1}^T r_t^2\right) \right] \right] \right\}^{1/2} = \left\{ \mathbf{E} \left[ \exp\left(C \sum_{t=1}^T r_t^2\right) \right] \right\}^{1/2}.$$

Furthermore, (21) with  $s = T$  holds trivially. We will show that (21) for  $s = u + 1$  implies that (21) holds for  $s = u$ , where  $u = 1, \dots, T - 1$ . Thus by an induction argument we get (21) with  $s = 0$  and this implies the statement of the lemma.

Suppose that (21) with  $s = u + 1$  for some  $u = 1, \dots, T - 1$ . then we get by application of the Cauchy-Schwartz inequality that

$$\begin{aligned} \mathbf{E} \left[ \exp \left( \sum_{t=1}^T r_t e_t \right) \right] &\leq \mathbf{E} \left[ \exp \left( \sum_{t=1}^{u+1} r_t e_t \right) \sqrt{\mathbf{E}_{u+2} \left[ \exp\left(C \sum_{t=u+2}^T r_t^2\right) \right]} \right] \\ &= \mathbf{E} \left[ \mathbf{E}_{u+1} \left[ \exp \left( \sum_{t=1}^{u+1} r_t e_t \right) \sqrt{\mathbf{E}_{u+2} \left[ \exp\left(C \sum_{t=u+2}^T r_t^2\right) \right]} \right] \right] \\ &\leq \mathbf{E} \left[ \left\{ \mathbf{E}_{u+1} \left[ \exp \left( \sum_{t=1}^{u+1} 2r_t e_t \right) \right] \right\}^{1/2} \left\{ \mathbf{E}_{u+1} \left[ \mathbf{E}_{u+2} \left[ \exp\left(C \sum_{t=u+2}^T r_t^2\right) \right] \right] \right\}^{1/2} \right] \\ &= \mathbf{E} \left[ \exp \left( \sum_{t=1}^u r_t e_t \right) \left\{ \mathbf{E}_{u+1} [\exp(2r_{u+1}e_{u+1})] \right\}^{1/2} \left\{ \mathbf{E}_{u+1} \left[ \exp\left(C \sum_{t=u+2}^T r_t^2\right) \right] \right\}^{1/2} \right]. \end{aligned}$$

We now argue that

$$\mathbf{E}_{u+1} [\exp(2r_{u+1}e_{u+1})] \leq \exp(Cr_{u+1}^2). \quad (22)$$

If one plugs this into the last inequality one gets (21) with  $s = u$ . This shows the statement of the lemma. Thus it remains to show (22). This claim follows by a simple Taylor expansion. One gets with  $|\eta_{u+1}| \leq |r_{u+1}| |e_{u+1}| \leq c/2 |e_{u+1}|$  that

$$\begin{aligned} \mathbf{E}_{u+1} [\exp(2r_{u+1}e_{u+1})] &= \mathbf{E}_{u+1} [1 + 2r_{u+1}e_{u+1} + 2r_{u+1}^2e_{u+1}^2 \exp(2\eta_{u+1})] \\ &= \mathbf{E}_{u+1} [1 + 2r_{u+1}^2e_{u+1}^2 \exp(2\eta_{u+1})] \\ &\leq \mathbf{E}_{u+1} [1 + Cr_{u+1}^2] \\ &= 1 + Cr_{u+1}^2 \\ &\leq \exp(Cr_{u+1}^2). \end{aligned}$$

□

**Lemma 8.** *Make the assumptions of Proposition 1. It holds that*

$$\sum_{t=1}^T \frac{(\hat{h}_t - h_t)}{\hat{h}_t} (Z_t^2 - 1) + \sum_{t=1}^T (\hat{m}(\hat{h}_t) - m(h_t)) \frac{\sqrt{\hat{h}_t}}{\hat{h}_t} Z_t \quad (23)$$

$$= O_P(\ln(T)) \left[ \sum_{t=1}^T \frac{(\hat{h}_t - h_t)^2}{\hat{h}_t^2} + \sum_{t=1}^T \frac{h_t (\hat{m}(\hat{h}_t) - m(h_t))^2}{\hat{h}_t^2} \right]^{1/2},$$

$$\sum_{t=1}^T \frac{(\hat{h}_t - h_t)}{\hat{h}_t} (Z_t^2 - 1) + \sum_{t=1}^T (\hat{m}(\hat{h}_t) - m(h_t)) \frac{\sqrt{\hat{h}_t}}{\hat{h}_t} Z_t \quad (24)$$

$$= O_P(\ln(T) T^{1/\kappa_3}) \left[ \sum_{t=1}^T \frac{(\hat{h}_t - h_t)^2}{\hat{h}_t^2 \vee h_t^2} + \sum_{t=1}^T \frac{(\hat{m}(\hat{h}_t) - m(h_t))^2}{\hat{h}_t} \right]^{1/2}.$$

**Proof of Lemma 8.** We will show that for  $\delta > 0$  small enough

$$\begin{aligned} & \sup_{\|\theta - \theta_0\| \leq \delta} \left[ \sum_{t=1}^T \frac{(\bar{h}_t(\theta) - h_t)^2}{\bar{h}_t(\theta)^2} + \sum_{t=1}^T \frac{h_t (m_\gamma(\bar{h}_t(\theta)) - m(h_t))^2}{\bar{h}_t(\theta)^2} \right]^{-1/2} \\ & \times \left[ \sum_{t=1}^T \frac{(\bar{h}_t(\theta) - h_t)}{\bar{h}_t(\theta)} (Z_t^2 - 1) + \sum_{t=1}^T (m_\gamma(\bar{h}_t(\theta)) - m(h_t)) \frac{\sqrt{\bar{h}_t(\theta)}}{\bar{h}_t(\theta)} Z_t \right] = O_P(\ln(T)). \end{aligned} \quad (25)$$

Because of Lemma 1 and consistency of  $\hat{\theta}$  this implies (23).

For the proof of (25) we will apply Lemma 7 with  $e_t = e_t^* - E[e_t^*]$ ,  $e_t^* = (Z_t^2 - 1)I[|Z_t| \leq T^{1/\kappa_2}]$  and  $\bar{r}_t(\theta) = r_t^*(\theta)I[|r_t^*(\theta)| \leq cT^{2/\kappa_1}]$ ,  $r_t^*(\theta) = \|\theta - \theta_0\|^{-1}(\bar{h}_t(\theta) - h_t)/h_t$  for  $\theta$  in a  $\delta$ -neighborhood of  $\theta_0$ . Note that with probability tending to one  $e_t = e_t^*$  for  $t = 1, \dots, T$ . Furthermore, we have that for all  $\varepsilon > 0$  the constant  $c$  can be chosen such that with probability  $\geq 1 - \varepsilon$  it holds that  $r_t(\theta) = r_t^*(\theta)$  for  $t = 1, \dots, T$  and for all  $\theta$  in a  $\delta$ -neighborhood of  $\theta_0$ . We now show that for  $\rho > 0$ , one can choose a constant  $c_\rho > 0$  such that

$$P \left( \sum_{t=1}^T \bar{r}_t(\theta) e_t > c_\rho \ln(T) \left[ T + \sum_{t=1}^T \bar{r}_t(\theta)^2 \right]^{1/2} \right) \leq CT^{-\rho} \quad (26)$$

for  $T \geq T_0$  with constant  $C$  not depending on  $\theta$  and  $\varepsilon$  and  $T_0$  not depending on  $\theta$ . For a proof of (26), we use the inequality

$$\begin{aligned} & P \left( \sum_{t=1}^T \bar{r}_t(\theta) e_t > c_\rho \ln(T) \left[ T + \sum_{t=1}^T \bar{r}_t(\theta)^2 \right]^{1/2} \right) \\ & \leq \mathbf{E} \left[ \exp \left( \sum_{t=1}^T r_t(\theta) e_t \right) \right] \exp(-c_\rho \ln(T)) \end{aligned}$$

with  $r_t(\theta) = \left[ T + \sum_{t=1}^T \bar{r}_t(\theta)^2 \right]^{-1/2} \bar{r}_t(\theta)$ . Application of the last lemma with  $r_t = r_t(\theta)$  gives (26).

In a second step, we apply Lemma 7 with  $e_t = e_t^* - E[e_t^*]$ ,  $e_t^* = Z_t I[|Z_t| \leq T^{1/\kappa_2}]$  and  $\bar{r}_t(\theta) = r_t^*(\theta)I[|r_t^*(\theta)| \leq cT^{2/\kappa_1}]$ ,  $r_t^*(\theta) = \|\theta - \theta_0\|^{-1}(m_\gamma(\bar{h}_t(\theta)) - m(h_t)) \frac{\sqrt{\bar{h}_t(\theta)}}{\bar{h}_t(\theta)} Z_t$  for  $\theta$  in a  $\delta$ -neighborhood of  $\theta_0$ .

Also, with these choices (26) holds.

We now note that it suffices to show (25) with the supremum running only over a grid of polynomially many  $\theta$ -values. This follows by using rough estimates for neighbored values of  $\theta$ . Thus, (25) follows from equation (26) with the two choices of  $e_t$  and  $\bar{r}_t(\theta)$ . At this stage, also Lemma 6 is used. This concludes the proof of (23). Claim (24) follows from (23) by using the bound  $\sup_{1 \leq t \leq T} h_t / \hat{h}_t \leq \omega_*^{-2} \sup_{1 \leq t \leq T} h_t = O_P(T^{1/\kappa_3})$ . Here, the last equality follows from Assumption 8, see Theorem 2.1 in Mikosch and Stărică (2000) and the arguments at the end of the proof of Lemma 3.  $\square$

**Proof of Proposition 1.** From Lemmas 2 and 8, we get that

$$\begin{aligned} & \sum_{t=1}^T \frac{(\hat{h}_t - h_t)^2}{\hat{h}_t^2 \vee h_t^2} + \sum_{t=1}^T \frac{(\hat{m}(\hat{h}_t) - m(h_t))^2}{\hat{h}_t} \\ & \leq - \sum_{t=1}^T \frac{(\hat{h}_t - h_t)}{\hat{h}_t} (Z_t^2 - 1) + 2 \sum_{t=1}^T (\hat{m}(\hat{h}_t) - m(h_t)) \frac{\sqrt{\hat{h}_t}}{\hat{h}_t} Z_t + O_P(\ln T) \\ & \leq O_P(\ln(T) T^{1/\kappa_3}) \left\{ \left[ \sum_{t=1}^T \frac{(\hat{h}_t - h_t)^2}{\hat{h}_t^2 \vee h_t^2} \right]^{1/2} + \left[ \sum_{t=1}^T \frac{(\hat{m}(\hat{h}_t) - m(h_t))^2}{\hat{h}_t} \right]^{1/2} \right\}. \end{aligned}$$

This implies that

$$\sum_{t=1}^T \frac{(\hat{h}_t - h_t)^2}{\hat{h}_t^2 \vee h_t^2} + \sum_{t=1}^T \frac{(\hat{m}(\hat{h}_t) - m(h_t))^2}{\hat{h}_t} = O_P(\ln(T)^2 T^{2/\kappa_3}).$$

Because of Lemma 6, this shows that

$$\|\hat{\theta} - \theta_0\|^2 = O_P(\ln(T)^2 T^{-1+(2/\kappa_3)}).$$

With Lemma 3, we get from this bound that

$$\left| \frac{\hat{h}_t - h_t}{h_t} \right| = O_P(\ln(T)^2 T^{-(1/2)+(1/\kappa_1)+(1/\kappa_3)}) = o_P(1).$$

Thus,

$$\left[ \sum_{t=1}^T \frac{(\hat{h}_t - h_t)^2}{\hat{h}_t^2 \vee h_t^2} + \sum_{t=1}^T \frac{(\hat{m}(\hat{h}_t) - m(h_t))^2}{\hat{h}_t} \right] \left[ \sum_{t=1}^T \frac{(\hat{h}_t - h_t)^2}{h_t^2} + \sum_{t=1}^T h_t \frac{(\hat{m}(\hat{h}_t) - m(h_t))^2}{\hat{h}_t^2} \right]^{-1} \rightarrow 1,$$

in probability and we get by using the above arguments with (23) instead of (24) that

$$\sum_{t=1}^T \frac{(\hat{h}_t - h_t)^2}{\hat{h}_t^2 \vee h_t^2} + \sum_{t=1}^T \frac{(\hat{m}(\hat{h}_t) - m(h_t))^2}{\hat{h}_t} = O_P(\ln(T)^2).$$

Because of Lemma 6, this proves the statement of the proposition.  $\square$

For the proof of Theorem 2 we will make use of the following expansions for functions  $\theta \rightarrow s(\theta)$ :

$$\partial_{\theta} g_{\theta}(y, s(\theta)) = \begin{pmatrix} 1 \\ v^2 \\ s \\ -2\alpha v \dot{m} \end{pmatrix} + (\beta - 2\alpha v m') s', \quad (27)$$

$$\begin{aligned} \partial_{\theta\theta} g_{\theta}(y, s(\theta)) = & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2v \dot{m} \\ 0 & 0 & 0 & 0 \\ 0 & -2v \dot{m} & 0 & 2\alpha \dot{m} \dot{m}^{\top} - 2\alpha v \ddot{m} \end{pmatrix} + \begin{pmatrix} 0 \\ -4v m' \\ 2 \\ 4\alpha m' \dot{m} - 4\alpha v \dot{m}' \end{pmatrix} s' \\ & + (\beta - 2\alpha v m') s''. \end{aligned} \quad (28)$$

Here, we denote by  $\partial_{\theta}$  and  $\partial_{\theta\theta}$  the first and second order partial derivatives with respect to  $\theta$ . Furthermore, we define  $v = y - m_{\gamma}(s)$  and we write  $s$  instead of  $s(\theta)$ . These equations can be used to show for  $h_t(\theta) = \hat{h}_t(\theta)$  or  $h_t(\theta) = \bar{h}_t(\theta)$  that

$$\begin{aligned} \|h'_{t+1}(\theta) - h'_{t+1}(\theta_0)\| \leq & C [\|\theta - \theta_0\| (1 + h_t(\theta_0)|Z_t| + h_t + h'_t(\theta_0)|Z_t|) \\ & + |h_t(\theta) - h_t(\theta_0)| (1 + |Z_t| + h'_t(\theta_0)|Z_t|) \\ & + \|h'_t(\theta) - h'_t(\theta_0)\| V_t(\alpha_0 Z_t^2 + \beta_0)] \end{aligned} \quad (29)$$

for some positive constant  $C$ .

Using the last inequality and Lemma 3, we get the statement of the following lemma.

**Lemma 9.** *Make the assumptions of Proposition 1. For a constant  $C > 0$  it holds that*

$$\begin{aligned} \sup_{\|\theta - \theta_0\| \leq C \ln(T) T^{-1/2}} \sup_{1 \leq t \leq T} h_t^{-1}(\theta_0) \|h'_t(\theta) - h'_t(\theta_0)\| &= o_P(1), \\ \sup_{1 \leq t \leq T} h_t^{-1}(\theta_0) \|h'_t(\theta_0)\| &= O_P(T^{1/\kappa_1}). \end{aligned}$$

The next lemma states that

$$d_{t+1}^{**} = \partial_{\theta\theta} g_{\theta_0}(Y_t, \hat{h}_t(\theta_0)) + 2\partial_{\theta s} g_{\theta_0}(Y_t, \hat{h}_t(\theta_0)) \hat{h}'_t(\theta_0) + \partial_{ss} g_{\theta_0}(Y_t, \hat{h}_t(\theta_0)) d_{t+1}^{**}(\theta) \quad (30)$$

has a unique stationary solution  $d_t^{**}$ . We denote this solution by  $\bar{h}_t'' = d_t^{**}$ . Note that this is a random value and not a random function.

**Lemma 10.** *Make the assumptions of Proposition 1. Equation (30) has a unique stationary solution  $\bar{h}_t'' = d_t^{**}$  that is ergodic. For  $\rho > 1$  small enough it holds that*

$$\rho^t \|\bar{h}_t'' - \hat{h}_t''(\theta_0)\| \rightarrow 0, \quad a.s.$$

Equation (28) can be used to show that

$$\begin{aligned} \left\| \hat{h}_{t+1}''(\theta) - \hat{h}_{t+1}''(\theta_0) \right\| &\leq C \left[ \|\theta - \theta_0\| (1 + |Z_t| (1 + h_t + \|h_t'\| + \|h_t'\|^2 + \|h_t''\|)) \right. \\ &\quad + |\hat{h}_t(\theta) - \hat{h}_t(\theta_0)| (1 + |Z_t|) (1 + \|h_t'\| + \|h_t'\|^2 + \|h_t''\|) \\ &\quad + \|\hat{h}_t'(\theta) - \hat{h}_t'(\theta_0)\| (1 + |Z_t|) (1 + \|h_t'\|) \\ &\quad \left. + \left\| \hat{h}_t''(\theta) - \hat{h}_t''(\theta_0) \right\| V_t(\alpha_0 Z_t^2 + \beta_0) \right] \end{aligned}$$

for some positive constant  $C$ . Using the last inequality, we get the statement of the following lemma.

**Lemma 11.** *Make the assumptions of Proposition 1. For constants  $C > 0$  it holds that*

$$\sup_{\|\theta - \theta_0\| \leq C \ln(T) T^{-1/2}} \sup_{1 \leq t \leq T} h_t^{-1}(\theta_0) \left\| \hat{h}_t''(\theta) - \hat{h}_t''(\theta_0) \right\| = o_P(1).$$

By making use of the derived results, we now get the statement of Theorem 2.

**Proof of Theorem 2.** We make use of  $0 = \hat{L}'_T(\hat{\theta}) = \hat{L}'_T(\theta_0) + \hat{L}''_T(\hat{\theta}^*)(\hat{\theta} - \theta_0)$  for some random  $\hat{\theta}^*$  with  $\hat{\theta}^* = O_P(\ln(T) T^{-1/2})$ . This gives:

$$\sqrt{T}(\hat{\theta} - \theta_0) = (T^{-1} \hat{L}''_T(\hat{\theta}^*))^{-1} T^{-1/2} \hat{L}'_T(\theta_0).$$

Using the above discussions, we get that

$$\begin{aligned} T^{-1} \hat{L}''_T(\hat{\theta}^*) &\rightarrow \Sigma_2, \text{ in probability,} \\ T^{-1/2} \hat{L}'_T(\theta_0) &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{1}{2} \frac{\bar{h}'_t}{\bar{h}_t} (Z_t^2 - 1) + \bar{h}_t^{-1/2} (\dot{m}_{\gamma_0}(\bar{h}_t) + m'_{\gamma_0}(\bar{h}_t) \bar{h}'_t) Z_t + o_P(1). \end{aligned}$$

The theorem follows by application of a martingale central limit theorem, see Hall and Heyde (1980).  $\square$

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