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Research Report

Solving Hayashi and Prescott (2007) "The 1990s: Japan's Lost Decade"(extended with an exogenous population growth and labour-augmenting technical progress) using a Linear-Quadratic Approximation as in Ljungqvist and Sargent (2004)

Solving Hayashi and Prescott (2007) "The 1990s: Japan's Lost Decade"(extended with an exogenous population growth and labour-augmenting technical progress) using a Linear-Quadratic Approximation as in Ljungqvist and Sargent (2004)

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January 25, 2016

1 Linear Quadratic Approximation as in Ljungqvist and Sargent (2004), pp. 129-31

This note describes an important use of the optimal linear regulator: to approximate the solution of more complicated dynamic programs. Optimal linear regulator problems are often used to approximate problems of the following form: maximize over $\{u_t\}_{t=0}^{\infty}$

$$\begin{aligned} &E_0 \sum_{t=0}^{\infty} \beta^t r(z_t) \\ &x_{t+1} = Ax_t + Bu_t + Cw_{t+1} \end{aligned} \tag{1}$$

where $\{u_t\}_{t=0}^{\infty}$ is a vector i.i.d. random disturbances with mean zero and finite variance, and $r(z_t)$ is a concave and twice continuously differentiable function of $z_t = \begin{pmatrix} x_t \\ u_t \end{pmatrix}$. All non-linearities in the original problem are absorbed into the composite function $r(z_t)$.

2 Hayashi-Prescott's Model

The problem is to choose $\{c_t, h_t, e_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize

$$
\begin{aligned}
&E_0 \sum_{t=0}^{\infty} \beta^t \{ln(c_t) - g(h_t)e_t\} \\
&c_t + i_t = (1-\theta\tau_k)A_t k_t^{\theta}(h_t e_t)^{1-\theta} \\
&\gamma\eta k_{t+1} = i_t + (1-\delta)k_t \\
&\ln A_{t+1} = \rho \ln A_t + w_{t+1}
\end{aligned}
\tag{2}
$$

where $\{w_t\}_{t=0}^{\infty}$ is an i.i.d. stochastic process with mean zero and finite variance, A_t is a technology shock, and $\tilde{A}_t \equiv \ln A_t$. To get this problem into the form above,

take $x_t = \begin{pmatrix} k_t \\ \tilde{A}_t \end{pmatrix}$,

$u_t = \begin{pmatrix} i_t \\ h_t \\ e_t \end{pmatrix}$,

$r(z_t) = \ln((1-\theta\tau_k)\exp(\tilde{A}_t)k_t^{\theta}(h_t e_t)^{1-\theta} - i_t) - g(h_t)e_t$

and we write the laws of motion as

$$
\begin{pmatrix} 1 \\ k_{t+1} \\ \tilde{A}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-\delta)/\eta\gamma & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} 1 \\ k_t \\ \tilde{A}_t \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1/\gamma\eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i_t \\ h_t \\ e_t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} w_{t+1}
$$

where it is convenient to add the constant 1 as the first component of the state vector.

Therefore the matrices we need as input in **olrp.m** are

$$
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-\delta)/\eta\gamma & 0 \\ 0 & 0 & \rho \end{pmatrix}
$$

$$
B = \begin{pmatrix} 0 & 0 & 0 \\ 1/\gamma\eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
$$

3 Kydland and Prescott's method

We want to replace $r(z_t)$ by a quadratic $z_t' M z_t$.
In our case

$$r(z_t) = \ln((1-\theta\tau_k)\exp(\tilde{A}_t) + (1-\delta)k_t - \gamma\eta k_{t+1}) - g(h_t)e_t \tag{3}$$

We choose a point $\bar{z}$ and approximate with the first two terms of a Taylor series:

$$\hat{r}(z) = r(\bar{z}) + (z-\bar{z})'\frac{\partial r}{\partial z} + \frac{1}{2}(z-\bar{z})'\frac{\partial^2 r}{\partial z \partial z'}(z-\bar{z}) \tag{4}$$

If the state x_t is $n \times 1$ and the control u_t is $k \times 1$, then the vector z_t is $(n+k)\times 1$. Let e be the $(n+k)\times 1$ vector with 0's everywhere except for a 1 in the row corresponding to the location of the constant utility in the state vector, so that $1 \equiv e' z_t \ \forall t$.

Repeatedly using $z'e = e'z = 1$, we can express the equation above as

$$\hat{r}(z) = z' M z \tag{5}$$

where

$$M = e[r(\bar{z}) - (\frac{\partial r}{\partial z})'\bar{z} + \frac{1}{2}\bar{z}'\frac{\partial^2 r}{\partial z \partial z'}\bar{z}]e'$$
$$+\frac{1}{2}(\frac{\partial r}{\partial z}e' - e\bar{z}'\frac{\partial^2 r}{\partial z \partial z'} - \frac{\partial^2 r}{\partial z \partial z'}\bar{z}e' + e\frac{\partial r'}{\partial z})$$
$$+\frac{1}{2}(\frac{\partial^2 r}{\partial z \partial z'})$$

where the partial derivatives are evaluated at $\bar{z}$.

In Hayashi-Prescott model,

$$\bar{r} = r(\bar{z}) = \ln((1-\theta\tau_k)\exp(\bar{\tilde{A}})\bar{k}^{\theta}(\bar{h}\bar{e})^{1-\theta} - (\gamma\eta - 1 + \delta)\bar{k}) - g(\bar{h})\bar{e}$$

$$\frac{\partial r}{\partial z} = \begin{bmatrix} \bar{r}_A \\ \bar{r}_k \\ \bar{r}_i \\ \bar{r}_h \\ \bar{r}_e \end{bmatrix} \tag{6}$$

$$\frac{\partial^2 r}{\partial z \partial z'} = \begin{bmatrix} \bar{r}_{AA} & \bar{r}_{Ak} & \bar{r}_{Ai} & \bar{r}_{Ah} & \bar{r}_{Ae} \\ \bar{r}_{kA} & \bar{r}_{kk} & \bar{r}_{ki} & \bar{r}_{kh} & \bar{r}_{ke} \\ \bar{r}_{iA} & \bar{r}_{ik} & \bar{r}_{ii} & \bar{r}_{ih} & \bar{r}_{ie} \\ \bar{r}_{iA} & \bar{r}_{ik} & \bar{r}_{ii} & \bar{r}_{ih} & \bar{r}_{ie} \\ \bar{r}_{hA} & \bar{r}_{hk} & \bar{r}_{hi} & \bar{r}_{hh} & \bar{r}_{he} \\ \bar{r}_{eA} & \bar{r}_{ek} & \bar{r}_{ei} & \bar{r}_{eh} & \bar{r}_{ee} \end{bmatrix} \tag{7}$$

Both objects are to be evaluated at $z = \bar{z}$.

4 Filling the elements

$$\begin{aligned}
\bar{r}_A &= ((1-(\theta\tau_k))\bar{y})/((1-(\theta\tau_k))\bar{y}-\bar{i}) \\
\bar{r}_k &= ((1-(\theta\tau_k))\alpha\bar{y}/\bar{k})/((1-(\theta\tau_k))\bar{y}-\bar{i}) \\
\bar{r}_i &= (-1)/((1-(\theta\tau_k))\bar{y}-\bar{i}) \\
\bar{r}_h &= (((1-(\theta\tau_k))\bar{y}/\bar{h})/((1-(\theta\tau_k))\bar{y}-\bar{i})-\alpha\bar{e} \\
\bar{r}_e &= (((1-(\theta\tau_k))\bar{y}/\bar{e})/((1-(\theta\tau_k))\bar{y}-\bar{i})-\alpha\bar{h} \\
\bar{r}_{AA} &= -(((1-(\theta\tau_k))\bar{y})/(((1-(\theta\tau_k))\bar{y}-\bar{i}))^2 \\
\bar{r}_{kk} &= -(((1-(\theta\tau_k))\theta\bar{y}/\bar{k})/(((1-(\theta\tau_k))\bar{y}-\bar{i}))^2+((1-(\theta\tau_k))\alpha(\alpha-1)(\bar{y}/\bar{k}^2))/((1-(\theta\tau_k))\bar{y}-\bar{i} \\
\bar{r}_{ii} &= (-1)/(((1-(\theta\tau_k))\bar{y}-\bar{i})^2) \\
\bar{r}_{ee} &= -(1/(\bar{c}^2))((1-\theta\tau_k)^2)((1-\theta)^2)((\bar{y}/\bar{c})^2)+(1/\bar{c})*(1-(\theta\tau_k))(1-\theta)(-\theta)(\bar{y}/\bar{e}^2) \\
\bar{r}_{hh} &= -(1/\bar{c})((1-(\theta\tau_k))(1-\theta)(\bar{y}/\bar{h}))^2+(1/\bar{c})(1-(\theta\tau_k))(1-\theta)(\bar{y}/\bar{h}) \\
\bar{r}_{he} &= -(1/\bar{c})((1-(\theta\tau_k))(1-\theta)(\bar{y}/\bar{h}\bar{e}))^2+(1/\bar{c})(1-(\theta\tau_k))((1-\theta)^2)(\bar{y}/\bar{h}\bar{e}) \\
\bar{r}_{Ak} &= ((\theta(\bar{y}/\bar{k})\bar{c})-(\bar{y}\theta(\bar{y}/\bar{k})))/((\bar{c})^2) \\
\bar{r}_{Ai} &= ((1-(\theta\tau_k))\bar{y})/(((1-(\theta\tau_k))\bar{y}-\bar{i})^2) \\
\bar{r}_{Ah} &= (1-(\tau_k\theta))(\bar{y}/(\bar{c}\bar{h}))-(1/(\bar{c}^2))((1-(\theta\tau_k))(1-\theta)(\bar{y}^2)/\bar{h}) \\
\bar{r}_{Ae} &= (1-(\tau_k\theta))(\bar{y}/(\bar{c}\bar{e}))-(1/(\bar{c}^2))((1-(\theta\tau_k))(1-\theta)(\bar{y}^2)/\bar{e}) \\
\bar{r}_{ki} &= ((1-(\theta\tau_k))\theta\bar{y}/\bar{k})/(((1-(\theta\tau_k))\bar{y}-\bar{i})^2) \\
\bar{r}_{kh} &= -(1/(\bar{c}^2))((1-(\theta\tau_k))\theta\bar{y}/\bar{k})(1-(\theta\tau_k))(\theta-1)(\bar{y}/\bar{h})+(1/\bar{c})((1-(\theta\tau_k))\theta(\theta-1)(\bar{y}/(\bar{k}\bar{h}))) \\
\bar{r}_{ke} &= -(1/(\bar{c}^2))((1-(\theta\tau_k))\theta\bar{y}/\bar{k})(1-(\theta\tau_k))(\theta-1)(\bar{y}/\bar{e})+(1/\bar{c})((1-(\theta\tau_k))\theta(\theta-1)(\bar{y}/(\bar{k}\bar{e}))) \\
\bar{r}_{ih} &= (((1-(\theta\tau_k))\bar{y}/\bar{h})/(((1-(\theta\tau_k))\bar{y}-\bar{i}))^2) \\
\bar{r}_{ie} &= (((1-(\theta\tau_k))\bar{y}/\bar{e})/(((1-(\theta\tau_k))\bar{y}-\bar{i}))^2)
\end{aligned} \tag{8}$$

5 Partitioning M matrix

Partition M, so that

$$z'Mz = \begin{pmatrix} x \\ u \end{pmatrix}' \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}' \begin{pmatrix} R & W \\ W' & Q \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{9}$$

We need the partitioned matrices R, W, Q as an input in the **olrp.m**, and we are done.

6 Determination of $\bar{z}$

Usually, the point $\bar{z}$ is chosen as the (optimal) stationary state of the *non-stochastic* version of the original nonlinear model:

$$\begin{aligned} &E_0 \sum_{t=0}^{\infty} \beta^t r(z_t) \\ &x_{t+1} = Ax_t + Bu_t \end{aligned} \tag{10}$$

The stationary point is obtained in these steps:
1. Find the Euler equations.
2. Substitute

$$z_{t+1} = \begin{pmatrix} x_{t+1} \\ u_{t+1} \end{pmatrix} = z_t = \begin{pmatrix} x_t \\ u_t \end{pmatrix} \equiv \bar{z} = \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} \tag{11}$$

into the Euler equations and transition laws, and solve the resulting system of non-linear equations for $\bar{z}$. This purpose can be accomplished, by using the non-linear equation solver **fsolve.m** in MATLAB. We can also solve for the deterministic steady-state analytically.

6.1 Algorithm: How to solve for the deterministic steady-state ($A = 1$)

We use the steady-state condition that equates after-tax gross real return on capital and gross discount rate $\frac{1}{\beta} = 1 - \delta + (1 - \tau_k)\theta A(\frac{k}{he})^{\theta-1}$, whith gives us $\frac{k}{he}$. Next, impose $he = 1/3$ as done in the literature. This gives us k. Now we have enough information to calculate steady-state consumption $c = y - \delta k$. Next, $i = (\eta - 1 + \delta)k$, thus we know i. Finally, from marginal rate of substitution between consumption and leisure in the steady-state we obtain α: $\alpha(1 + 40(h - 40))c = w = \theta \frac{y}{e}$.