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**Research Report**

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# Solving Hayashi and Prescott (2007) by Log-linearization

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## 1 Growth Accounting

The aggregate production function is Cobb-Douglas

$$Y_t = A_t K_t^\theta (\gamma^t h_t E_t)^{1-\theta} \quad (1)$$

where

$Y_t$  is aggregate output,  $A_t$  is total factor productivity,  $K_t$  is aggregate capital,  $h_t$  is hours per employee,  $E_t$  is aggregate employment.

The Cobb-Douglas production function is increasing and concave in all arguments, CRS, differentiable, satisfies Inada conditions (ensure interior solutions)

$\gamma \geq 1$  is the gross growth rate of the labour-augmenting technological progress.

In addition, there is also exogenous population growth at the rate of  $\eta \geq 1$ , i.e.  $N_t = N_0 \eta^t$ , where  $N_t$  denotes the working-age population. Without loss of generality (WLOG) we can normalize  $N_0 = 1$ .

### Growth Accounting

First, we will define some useful ratios: Let  $y_t = \frac{Y_t}{N_t}$ ,  $e_t = \frac{E_t}{N_t}$  (those two are non-stationary), and  $x_t = \frac{K_t}{Y_t}$ . Non-stationarity is an issue when we want to solve for the steady-state, so we postpone detrending the model variables until we finish with the growth accounting exercise.

Observe that

$$x_t = \frac{K_t}{Y_t} = \frac{K_t/N_t}{Y_t/N_t} = \frac{k_t}{y_t} \quad (2)$$

$$\frac{Y_t}{N_t} = \frac{A_t K_t^\theta (\gamma^t h_t E_t)^{1-\theta}}{N_t} \quad (3)$$

$$y_t = A_t \left( \frac{K_t}{N_t} \right)^\theta \left( \frac{\gamma^t h_t E_t}{N_t} \right)^{1-\theta} \quad (4)$$

$$y_t = A_t k_t^\theta \gamma^{(1-\theta)t} h_t^{1-\theta} e_t^{1-\theta} \quad (5)$$

Divide both sides by  $y_t^\theta$

$$\frac{y_t}{y_t^\theta} = \frac{A_t k_t^\theta \gamma^{(1-\theta)t} h_t^{1-\theta} e_t^{1-\theta}}{y_t^\theta} \quad (6)$$

$$y_t^{1-\theta} = A_t \left( \frac{k_t}{y_t} \right)^\theta \gamma^{(1-\theta)t} h_t^{1-\theta} e_t^{1-\theta} \quad (7)$$

$$y_t^{1-\theta} = A_t x_t^\theta \gamma^{(1-\theta)t} h_t^{1-\theta} e_t^{1-\theta} \quad (8)$$

Raise to  $\frac{1}{1-\theta}$  power to obtain

$$y_t = A_t^{\frac{1}{1-\theta}} x_t^{\frac{\theta}{1-\theta}} \gamma^t h_t e_t \quad (9)$$

Take logs from both sides

$$\log y_t = \frac{1}{1-\theta} \log A_t + \frac{\theta}{1-\theta} \log x_t + t \log \gamma + \log h_t + \log e_t \quad (10)$$

where

$\log y_t$  is output per adult growth rate

$\log A_t$  is TFP factor growth rate

$\log x_t$  is  $K/Y$  growth rate

$\log \gamma$  is the trend growth rate

$\log h_t$  is growth of the workweek (hours per employee)

$\log e_t$  is employment growth rate.

Note that for "small"  $\gamma$ ,  $\log \gamma = \gamma - 1$  at first approximation, thus

$$\log y_t = \frac{1}{1-\theta} \log A_t + \frac{\theta}{1-\theta} \log x_t + t(\gamma - 1) + \log h_t + \log e_t \quad (11)$$

## 2 Model Description

The theoretical setup is a Dynamic Stochastic General Equilibrium (DSGE) model used in Real Business Cycle (RBC) studies. There is an exogenous and deterministic labor-augmenting technological progress. In addition, there are also shocks to total factor productivity (TFP).

## 2.1 Households

There is a unit measure of identical households. The population size,  $N_t$ , grows at a constant gross rate  $\eta \geq 1$ , i.e.  $N_{t+1} = \eta N_t$ , and without loss of generality (WLOG), we normalize  $N_0 = 1$ . Households maximize total utility

$$E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, h_t, E_t) \quad (12)$$

where  $E_0$  is the expectation operator as of period 0;  $C_t$  is aggregate consumption at time  $t$ ; and  $0 < \beta < 1$  is the discount factor. The instantaneous utility function is increasing, concave and satisfies the Inada conditions. We use the following form for utility:

$$U(C_t, h_t, E_t) = \ln(C_t) - \alpha(1 + 40(h_t - 40))E_t \quad (13)$$

Households save by investing in capital  $I_t$ , and as owners of capital, receive income  $r_t K_t$  from renting the capital to the firms and  $w_t h_t E_t$  from selling their labor services to firms;  $r_t$  is the return to (private) capital and  $K_t$  denotes the aggregate capital stock in the beginning of period  $t$ . Households are also endowed with time that can be allocated to work or leisure. Aggregate labor income is  $w_t h_t E_t$ , where  $w_t$  is the hourly wage rate,  $h_t$  is hours worked per household, and  $E_t$  is the employment level. Finally, households are owners of the firms in the economy, and receive all profit in the form of dividends. Households' budget constraint is

$$C_t + I_t \leq w_t h_t E_t + (1 - \tau)r_t K_t - \pi_t^* \quad (14)$$

where  $0 < \tau < 1$  is the distortionary capital income tax rate, and  $\pi_t^*$  is the lump-sum tax/subsidy.

Aggregate physical capital evolves according to the following law of motion

$$K_{t+1} = I_t + (1 - \delta)K_t \quad (15)$$

where  $\delta$  is the constant linear depreciation rate of capital.

Households act competitively by taking prices and policy variables as given. They choose the aggregate paths  $\{C_t^*, h_t^*, E_t^*\}_{t=0}^{\infty}$  to maximize Equation(12) subject to Equations (13)-(15), and initial conditions for  $K_0$ .

## 2.2 Consumer Optimization Problem

Set up the Lagrangean

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \{ \ln(C_t) - \alpha(1 + 40(h_t - 40))E_t - \Lambda_t (C_t + K_{t+1} - (1 - \delta)K_t - w_t h_t E_t - (1 - \tau)r_t K_t + \pi_t^*) \} \quad (16)$$

This is a standard concave programming problem, so the FOCs are both necessary and sufficient for an optimum.

$$C_t : \beta^t \left\{ \frac{1}{C_t} - \Lambda_t \right\} = 0 \rightarrow \frac{1}{C_t} = \Lambda_t \quad (17)$$

In economic terms, this means that an additional unit of consumption brings additional utility equal to marginal utility of wealth, as measured by the shadow price  $\Lambda$  of relaxing households' budget constraint.

$$h_t : -40\alpha E_t + \Lambda_t w_t^* E_t = 0 \quad (18)$$

$$\rightarrow 40\alpha = \Lambda_t w_t^* \quad (19)$$

The above optimality condition means that households equate the cost of working additional hour during the work week, measured in disutility from supplying labor services, to the benefit, which is the marginal product of working an additional hour.

$$E_t : -\alpha(1 + 40(h_t - 40)) + \Lambda_t w_t^* h_t = 0 \quad (20)$$

$$\rightarrow \alpha(1 + 40(h_t - 40)) = \Lambda_t w_t^* h_t \quad (21)$$

This optimality condition means that households equate the cost of an additional person working, measured in disutility from working, to the benefit, which is the marginal product of an additional person working.

$$K_{t+1} : -\Lambda_t + \beta E_t \Lambda_{t+1} (1 - \delta + (1 - \tau)r_{t+1}^*) = 0 \quad (22)$$

$$\rightarrow \Lambda_t = \beta E_t \Lambda_{t+1} (1 - \delta + (1 - \tau)r_{t+1}^*) \quad (23)$$

From the firm's problem we obtain below that  $r_{t+1} = \theta \frac{Y_{t+1}}{K_{t+1}}$ . Plugging (17) in (22), we obtain

$$\frac{1}{C_t} = \beta E_t \left[ \frac{1}{C_{t+1}} (1 - \delta + (1 - \tau)\theta \frac{Y_{t+1}}{K_{t+1}}) \right] \quad (24)$$

The equation above says that if we cut consumption in the current period, we increase saving by 1 unit. Saving is channelled to extra investment in capital, which is the only asset in the economy. There is no direct benefit from owning capital (it does not enter utility), only indirect one (providing capital income). Therefore, can increase consumption in the next period by the product of this additional unit of capital, which is by definition the marginal product of capital. In addition, we can eat the undepreciated capital (we are in a "putty-putty" economy). All this additional consumption is discounted one period at the rate  $\beta$ . This is a cost-benefit equalization: Cost is 1, while benefit is  $r_{t+1} + (1 - \delta)$ . If cost  $>$  ( $<$ ) benefit, agent can short sell (buy) infinite amount of capital and buy it back (sell) next period, realizing profit from that.

Aside from intertemporal optimality (choosing optimal allocations over time),

agents in the economy have also some intra-temporal (within period) optimality conditions to follow. The latter have to do with relative prices of consumption to hours and employment, respectively. The two conditions are in the form of marginal rate of substitution = relative price.

$$40\alpha = \frac{1}{C_t} w_t^* \quad (25)$$

$$\alpha(1 + 40(h_t - 40)) = \frac{1}{C_t} w_t^* h_t \quad (26)$$

Note: it seems there is a mistake in Hayashi-Prescott. When we divide side by side (19) and (22), we obtain

$$\frac{40\alpha}{\alpha(1 + 40(h_t - 40))} = \frac{\Lambda_t w_t^*}{\Lambda_t w_t^* h_t} \quad (27)$$

Then

$$\frac{40}{(1 + 40(h_t - 40))} = \frac{1}{h_t} \quad (28)$$

or,  $40h_t = 1 + 40h_t - 160$ , which does not have a solution.

Therefore, I use  $g(h_t) = \alpha h_t$ , which leads to a singular problem.

The FOCs that change are

$$\alpha = \Lambda_t w_t^* \quad (29)$$

$$\alpha h_t = \Lambda_t w_t^* h_t, \quad (30)$$

hence the singularity follows.

Transversality condition:  $\lim_{t \rightarrow \infty} \beta^t \frac{1}{C_t} K_{t+1} = 0$

The way of deriving the transversality condition is a bit "esoteric" in the literature, as many economists do not really understand the full mathematical machinery required to do so. For our purposes, we regard the transversality condition as a kind of "last" first-order condition. This is easy if we look at the finite-horizon version of the problem. In the finite case, the Lagrangean function is finite dimensional. The infinite horizon version can be thought of as  $\lim_{t \rightarrow \infty} \mathcal{L}_t = \mathcal{L}_\infty$ . In this sense, transversality condition is like taking the limit to the boundary condition in the finite case. The name "Transversality" comes to remind you the condition tells what happens once we cross the boundary (infinity and beyond).

Aside from the mathematical technicalities, the economic interpretation of the

transversality condition is straightforward. It requires that the present discounted value of the capital stock when we approach infinity is zero. If it were not so, then the original solution (plan) was not optimal, as we can re-optimize and increase the sum of discounted utilities.

Also, as the price sequence has all entries being strictly positive, it does not pay to leave any capital unused at the end of time.

### 2.3 Firms

There is a unit mass of firms as well. They all produce a homogeneous final product using the same production function that requires physical capital and labor hours. This allows us to aggregate total output using the same production function

$$Y_t = A_t K_t^\theta (\gamma^t h_t E_t)^{1-\theta} \quad (31)$$

where  $A_t$  measures the level of Hicks neutral technology available to the economy in period  $t$ ,  $0 < \theta, (1 - \theta) < 1$  are the productivity of capital and labor, respectively.

Firms act competitively by taking prices and policy variables as given. Accordingly, subject to Equation (29),  $K_t, h_t, E_t$  are chosen every period to maximize static aggregate profit,

$$\Pi_t = A_t K_t^\theta (\gamma^t h_t E_t)^{1-\theta} - r_t^* K_t - w_t^* h_t E_t \quad (32)$$

### 2.4 Firm's FOCs

$$K_t : \theta A_t K_t^{\theta-1} (\gamma^t h_t E_t)^{1-\theta} = r_t^* \quad (33)$$

$$E_t : (1 - \theta) A_t K_t^\theta (\gamma^t h_t)^{1-\theta} E_t^{-\theta} = w_t^* \quad (34)$$

$$h_t : (1 - \theta) A_t K_t^\theta (\gamma^t E_t)^{1-\theta} h_t^{-\theta} = w_t^* \quad (35)$$

i.e. price of labor is equal to its marginal product, and the rental rate of capital is equal to the marginal product of capital. Then

$$\Pi_t = 0 \quad (36)$$

### 2.5 Government budget constraint

Total government expenditure,  $G$ , and lump-sum transfers/taxes, are financed by levying proportional taxes on capital income. It is assumed that government conducts spending in a wasteful manner. Thus,

$$G_t = \tau r_t^* K_t^* + \pi_t \quad (37)$$

where only two of the three  $\{G_t, \pi_t, \tau\}$  policy instruments can be exogenously set. We will choose the tax rate  $\tau$  on capital income to be deterministic, and the

path for  $\{G_t\}$  to be exogenously set. Then the path for  $\{\pi_t\}$  will be endogenously determined as a residual from the per-period budget balance constraint.

### 3 Decentralized Competitive Equilibrium

Given the paths of the policy instruments  $\{G_t, \pi_t\}_{t=0}^{\infty}$ , and initial conditions for the state variable  $K_0$ , a decentralized competitive equilibrium (DCE) is defined to be a sequence of allocations  $\{C_t, h_t, E_t, I_t, K_{t+1}\}_{t=0}^{\infty}$ , prices  $\{r_t, w_t\}_{t=0}^{\infty}$  and the tax rate  $\{\tau\}$  such that (i) households maximize utility; (ii) firms maximize profits; (iii) all markets clear and (iv) the government budget constraint is satisfied in each time period.

### 4 Transformed Optimality Conditions

Since labor-augmenting progress is the engine of the LR exogenous growth, we transform variables to make them stationary.

We will define per capita units  $\bar{X}_t = \frac{X_t}{N_t}$  and  $x_t = \frac{\bar{X}_t}{\gamma^t}$  in effective per capita units, where  $X_t = \{Y_t, C_t, I_t, K_t, E_t\}$

#### 4.1 Transformed budget and physical capital constraints

#### 4.2 Aggregate

$$C_t + K_{t+1} - (1 - \delta)K_t + G_t = Y_t \quad (38)$$

#### 4.3 Per capita

$$\frac{C_t}{N_t} + \frac{K_{t+1}}{N_t} - (1 - \delta)\frac{K_t}{N_t} + \frac{G_t}{N_t} = \frac{Y_t}{N_t} \quad (39)$$

$$\bar{C}_t + \eta\bar{K}_{t+1} - (1 - \delta)\bar{K}_t + \bar{G}_t = \bar{Y}_t \quad (40)$$

#### 4.4 Stationary per capita

$$\frac{\bar{C}_t}{\gamma^t} + \eta\frac{\bar{K}_{t+1}}{\gamma^t} - (1 - \delta)\frac{\bar{K}_t}{\gamma^t} + \frac{\bar{G}_t}{\gamma^t} = \frac{\bar{Y}_t}{\gamma^t} \quad (41)$$

$$c_t + \eta k_{t+1} - (1 - \delta)k_t + g_t = y_t \quad (42)$$

#### 4.5 Transformed Production Function

#### 4.6 Aggregate

$$Y_t = A_t K_t^\theta (\gamma^t h_t E_t)^{1-\theta} \quad (43)$$



#### 4.7 Per capita

$$\frac{Y_t}{N_t} = A_t \frac{K_t^\theta}{N_t} (\gamma^t h_t \frac{E_t}{N_t})^{1-\theta} \quad (44)$$

$$\bar{Y}_t = A_t \bar{K}_t^\theta (\gamma^t h_t \bar{E}_t)^{1-\theta} \quad (45)$$

#### 4.8 Stationary per capita

$$\frac{\bar{Y}_t}{\gamma^t} = A_t \left(\frac{\bar{K}_t}{\gamma^t}\right)^\theta \left(h_t \frac{\bar{E}_t}{\gamma^t}\right)^{1-\theta} \quad (46)$$

$$y_t = A_t h_t^\theta (h_t e_t)^{1-\theta} \quad (47)$$

#### 4.9 Transformed Optimality Condition (FOC Consumption)

#### 4.10 Aggregate

$$\frac{1}{C_t} = \Lambda_t N_t \quad (48)$$

#### 4.11 Per-capita

$$\frac{1}{\frac{C_t}{N_t}} = \Lambda_t \quad (49)$$

$$\frac{1}{C_t} = \Lambda_t \quad (50)$$

#### 4.12 Stationary per capita

$$\frac{\gamma^t}{C} = \Lambda_t \quad (51)$$

$$\frac{1}{c_t} = \Lambda_t / \gamma^t = \lambda_t \quad (52)$$

#### 4.13 Transformed Optimality Condition (physical capital)

#### 4.14 Aggregate

$$\Lambda_t = \beta E_t \Lambda_{t+1} (1 - \delta + (1 - \tau) \theta A K_t^{\theta-1} (\gamma^t h_t E_t)^{1-\theta}) \quad (53)$$

#### 4.15 Per capita

$$\Lambda_t = \beta E_t \Lambda_{t+1} (1 - \delta + (1 - \tau) \theta A \bar{K}_t^{\theta-1} (\gamma^t h_t \bar{E}_t)^{1-\theta}) \quad (54)$$

#### 4.16 Stationary per capita

$$\lambda_t = (\beta/\gamma)E_t\lambda_{t+1}(1 - \delta + (1 - \tau)\theta Ak_t^{\theta-1}(\gamma^t h_t e_t)^{1-\theta}) \quad (55)$$

where

$$\lambda_t = \Lambda_t/\gamma^t \quad (56)$$

### 5 Per capita stationary DCE

Using this notation, we obtain the following per capita stationary DCE

$$y_t = c_t + \eta\gamma k_{t+1} - (1 - \delta)k_t + g_t \quad (57)$$

$$y_t = A_t k_t^\theta (h_t e_t)^{1-\theta} \quad (58)$$

$$\lambda_t = c_t^{-1} \quad (59)$$

$$\lambda_t = E_t\lambda_{t+1} \frac{\beta}{\gamma} [(1 - \tau)\theta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta] \quad (60)$$

$$\alpha c_t = (1 - \theta) \frac{y_t}{h_t e_t} \quad (61)$$

where  $\lambda_t$  is the transformed shadow price associated with Equation (22) in the households' problem.

Therefore, the stationary DCE is summarized by the above system of six equations in the paths of the following six variables  $(y_t, c_t, h_t, K_{t+1}, \lambda_t)$  given the path of the exogenous technology process  $\{A_t\}$ , whose motion is specified in the next subsection.

#### 5.1 Process for technology

To complete the model, we need to specify the process governing the exogenous technology process. We assume it to follow an AR(1) process:

$$A_t = A^{1-\rho} A_{t-1}^\rho e^{\epsilon_t} \quad (62)$$

where  $A > 0$  is a constant,  $0 < \rho < 1$  is the first-order autoregressive persistence parameter and  $\epsilon_t \sim iidN(0, \sigma^2)$  are stochastic shocks to productivity.

### 6 How do we work

We will log-linearize around the deterministic steady-state of the system. In order to study fluctuations in a small neighborhood of the steady-state, we first calculate it and calibrate the model in such a way that the parameters of the model are able to match long-run behavior of the aggregate macroeconomic variables.

## 6.1 Data and Calibration

### 6.2 Data

The annual data required for the calibration can be obtained from Bureau of Economic Analysis (NIPA accounts), OECD(Economic Outlook Database), US Department of Labor, Bureau of Labor Statistics (BLS) and ECFIN Effective Average Tax Base (Martinez-Mongray, 2000)

### 6.3 Calibration

The values of the model's parameters are summarized in Table 1. In order to calibrate the model, we proceed as follows. We set the value of  $(1 - \theta)$  equal to labor's share of income (i.e. 0.578) using compensation of employees data from the OECD Economic Outlook. This figure is similar to the one obtained in Gollin (2000). Capital share  $\theta$  is determined residually.

The discount rate  $1/\beta$  equals 1 plus the *ex post* real interest rate, which can be obtained from OECD Economic Outlook. This implies  $\beta = 0.964$ , which is consistent with previous studies. The population gross growth rate  $\eta$  is set equal to the post war labor force gross growth rate 1.016, obtained from Bureau of Labor Statistics. The depreciation rate for physical capital  $\delta$  is calculated to be 0.049 on average. The scale parameter of the technology process  $A$  is set to 1 WLOG. The parameters for the stationaty TFP process are as in Lansing (1998)  $\rho = 0.933$  and  $\sigma = 0.1$ .

Table 1: Parameter Values (base calibration)

Parameter	Value	Definition
$A$	1.000	Technological progress in goods production
$\theta$	0.422	Productivity of private capital
$1 - \theta$	0.578	Productivity of effective labor
$\beta$	0.964	Discount factor
$\delta$	0.049	Depreciation rate on physical capital
$\alpha$	1.373	Weight on disutility from work
$\tau$	0.48	Effective tax rate on capital income
$\rho$	0.933	AR(1) parameter technology
$\sigma$	0.01	SD of technology innovation

## 7 Steady-State

Suppressing time indices and shutting down stochasticity

$$y = c + \eta\gamma k - (1 - \delta)k + g \tag{63}$$

$$y = c + (\eta\gamma - 1 + \delta)k + g \quad (64)$$

$$y = Ak^\theta(he)^{1-\theta} \quad (65)$$

$$\lambda = c^{-1} \quad (66)$$

$$1 = \frac{\beta}{\gamma}[(1 - \tau)\theta\frac{y}{k} + 1 - \delta] \quad (67)$$

$$\alpha c = (1 - \theta)y/he \quad (68)$$

## 7.1 Solving Analytically for the Steady-State

Algorithm: How to solve for the deterministic steady-state ( $A = 1$ ):

$$\frac{1}{\beta} = 1 - \delta + (1 - \tau_k)\theta A\left(\frac{k}{he}\right)^{\theta-1} \quad (69)$$

This gives us  $\frac{k}{he}$ . Then we know

$$w = (1 - \theta)A\left(\frac{k}{he}\right)^\theta \quad (70)$$

$$c = y - \delta k \quad (71)$$

$$c/k = y/k - \delta = A\left(\frac{k}{he}\right)^{\theta-1} - \delta \quad (72)$$

$$i/k = (\eta - 1 + \delta) \quad (73)$$

Hence we know  $i/k$ . Then from MRS:

$$g(h)c = w = (1 - \theta)A\left(\frac{k}{he}\right)^\theta \quad (74)$$

$$(g(h)/k)(c/k) = w = (1 - \theta)A\left(\frac{k}{he}\right)^\theta \quad (75)$$

determines  $he$ .

## 8 Linearization

### 8.1 Linearized Market Clearing

$$c_t + \eta\gamma k_{t+1} + g_t - (1 - \delta)k_t = y_t \quad (76)$$

$$\ln(c_t + \eta\gamma k_{t+1} + g_t - (1 - \delta)k_t) = \ln(y_t) \quad (77)$$

$$\frac{d \ln(c_t + \eta\gamma k_{t+1} + g_t - (1 - \delta)k_t)}{dt} = d \ln(y_t) \quad (78)$$

$$\left(\frac{1}{c_t + \eta\gamma k_{t+1} + g_t - (1 - \delta)k_t}\right) \left[\frac{dc_t}{dt} \frac{c}{c} + \gamma\eta \frac{dk_{t+1}}{dt} \frac{k}{k} - (1 - \delta) \frac{dk_t}{dt} \frac{k}{k}\right] = \frac{dy_t}{dt} \frac{1}{y} \quad (79)$$

$$\frac{1}{y} [\hat{c}_t c + \gamma\eta \hat{k}_{t+1} k - (1 - \delta) \hat{k}_t k] = \hat{y}_t \quad (80)$$

$$-\left[\frac{\eta\gamma}{y}\right]\hat{k}_{t+1} = \left[\frac{c}{y}\right]\hat{c}_t - \hat{y}_t - \left[\frac{(1-\delta)k}{y}\right]\hat{k}_t \quad (81)$$

$$-\omega_1\hat{k}_{t+1} = \omega_2\hat{c}_t - \hat{y}_t - \omega_3\hat{k}_t \quad (82)$$

$$\text{where } \omega_1 \equiv \frac{\eta\gamma}{y}, \omega_2 \equiv \frac{c}{y}, \omega_3 \equiv \frac{(1-\delta)k}{y}$$

## 8.2 Linearized Production Function

$$y_t = A_t k_t^\theta (h_t e_t)^{1-\theta} \quad (83)$$

$$\ln y_t = \ln A_t + \theta \ln k_t + (1-\theta) \ln h_t + (1-\theta) \ln e_t \quad (84)$$

$$\frac{d \ln y_t}{dt} = \frac{d \ln A_t}{dt} + \theta \frac{d \ln k_t}{dt} + (1-\theta) \frac{d \ln h_t}{dt} + (1-\theta) \frac{d \ln e_t}{dt} \quad (85)$$

$$\frac{1}{y} \frac{dy_t}{dt} = \frac{1}{A} \frac{dA_t}{dt} + \frac{\theta}{k} \frac{dk_t}{dt} + \frac{(1-\theta)}{h} \frac{dh_t}{dt} + \frac{(1-\theta)}{e} \frac{de_t}{dt} \quad (86)$$

$$0 = -\hat{y}_t + \hat{A}_t + \theta \hat{k}_t + (1-\theta) \hat{h}_t + (1-\theta) \hat{e}_t \quad (87)$$

$$0 = -\hat{y}_t + \hat{A}_t + \omega_4 \hat{k}_t + \omega_5 \hat{h}_t + \omega_5 \hat{e}_t \quad (88)$$

$$\text{where } \omega_4 \equiv \theta, \omega_5 \equiv 1-\theta$$

## 8.3 Linearized FOC for consumption

$$\lambda_t = c_t^{-1} \quad (89)$$

$$\ln \lambda_t = -\ln c_t \quad (90)$$

$$\frac{d \ln \lambda_t}{dt} = -\frac{d \ln c_t}{dt} \quad (91)$$

$$\frac{1}{\lambda} \frac{d\lambda_t}{dt} = -\frac{1}{c} \frac{dc_t}{dt} \quad (92)$$

$$0 = -\hat{c}_t - \hat{\lambda}_t \quad (93)$$

## 8.4 Linearized No-arbitrage condition

$$\lambda_t = \beta/\gamma E_t[\lambda_{t+1}((1-\tau)\theta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta)] \quad (94)$$

$$\ln \lambda_t = \ln E_t[\lambda_{t+1}((1-\tau)\theta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta)] \quad (95)$$

$$\frac{d \ln \lambda_t}{dt} = \frac{d \ln E_t[\lambda_{t+1}((1-\tau)\theta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta)]}{dt} \quad (96)$$

$$\hat{\lambda}_t = E_t[\hat{\lambda}_{t+1} + \left[ \frac{(1-\tau)\theta y}{((1-\tau)\theta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta)k} \hat{y}_{t+1} - \frac{(1-\tau)\theta y}{((1-\tau)\theta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta)k} \hat{k}_{t+1} \right]] \quad (97)$$

$$\text{Observe that } (1-\tau)\theta \frac{y}{k} + 1 - \delta = \gamma/\beta$$

$$\hat{\lambda}_t = E_t[\hat{\lambda}_{t+1} + [\frac{\beta(1-\tau)\theta y}{\gamma k} \hat{y}_{t+1} - \frac{\beta(1-\tau)\theta y}{\gamma k} \hat{k}_{t+1}]] \quad (98)$$

$$\hat{\lambda}_t = E_t \hat{\lambda}_{t+1} + \frac{\beta(1-\tau)\theta y}{\gamma k} E_t \hat{y}_{t+1} - \frac{\beta(1-\tau)\theta y}{\gamma k} E_t \hat{k}_{t+1} \quad (99)$$

$$\hat{\lambda}_t = E_t \hat{\lambda}_{t+1} + \omega_9 E_t \hat{y}_{t+1} - \omega_9 E_t \hat{k}_{t+1} \quad (100)$$

where  $\omega_9 \equiv \frac{\beta(1-\tau)\theta y}{\gamma k}$

## 8.5 Linearized Marginal Rate of Substitution

$$\frac{1}{\alpha c_t} = \frac{1}{w_t} \quad (101)$$

$$\alpha c_t = w_t \quad (102)$$

$$\alpha \ln c_t = \ln w_t \quad (103)$$

$$\alpha \frac{d \ln c_t}{dt} = \frac{d \ln w_t}{dt} \quad (104)$$

$$\alpha \frac{1}{c} \frac{dc_t}{dt} = \frac{1}{w} \frac{dw_t}{dt} \quad (105)$$

$$\alpha \hat{c}_t = \hat{w}_t \quad (106)$$

## 8.6 Linearized Exogenous Technological Process

$$A_{t+1} = A^{(1-\rho)} A_t^\rho e^{\epsilon_{t+1}} \quad (107)$$

$$\ln A_{t+1} = (1-\rho) \ln A + \rho \ln A_t + \epsilon_{t+1} \quad (108)$$

$$\frac{d \ln A_{t+1}}{dt} = (1-\rho) \frac{d \ln A}{dt} + \rho \frac{d \ln A_t}{dt} + \frac{d \epsilon_{t+1}}{dt} \quad (109)$$

$$\frac{1}{A} \frac{d A_{t+1}}{dt} = \rho \frac{1}{A} \frac{d A_t}{dt} + \epsilon_{t+1} \quad (110)$$

where for  $t = 1$   $d \epsilon_{t+1} \approx \ln(e^{\epsilon_{t+1}}/e) = \epsilon_{t+1} - \epsilon = \epsilon_{t+1}$  since  $\epsilon = 0$

$$\hat{A}_{t+1} = \rho \hat{A}_t + \epsilon_{t+1} \quad (111)$$