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Measuring the frequency dynamics of financial and macroeconomic connectedness

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FINMAP –

**FINANCIAL DISTORTIONS AND MACROECONOMIC
PERFORMANCE: EXPECTATIONS, CONSTRAINTS AND
INTERACTION OF AGENTS**

DATE: 01/13/2016

TITLE

Measuring the frequency dynamics of
financial and macroeconomic
connectedness

by: Jozef Barunik and Tomas Krehlik

ABSTRACT

We propose a general framework for measuring frequency dynamics of connectedness in economic variables based on spectral representation of variance decompositions.

We argue that the frequency dynamics is insightful when studying the connectedness of variables as shocks with heterogeneous frequency responses will create frequency dependent connections of different strength that remain hidden when time domain measures are used. Two applications support the usefulness of the discussion, guide a user to apply the methods in different situations, and contribute to the literature with important findings about sources of connectedness. Giving up the assumption of global stationarity of stock market data and approximating the dynamics locally, we document rich time-frequency dynamics of connectedness in US market risk in the first application. Controlling for common shocks due to common stochastic trends which dominate the connections, we identify connections of global economy at business cycle frequencies of 18 up to 96 months in the second application. In addition, we study the effects of cross-sectional dependence on the connectedness of variables.

Keywords: Connectedness, frequency, spectral analysis, market risk, business cycles

JEL: C18; C58; G15

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Measuring the frequency dynamics of financial and macroeconomic connectedness^{*†}

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Abstract

We propose a general framework for measuring frequency dynamics of connectedness in economic variables based on spectral representation of variance decompositions. We argue that the frequency dynamics is insightful when studying the connectedness of variables as shocks with heterogeneous frequency responses will create frequency dependent connections of different strength that remain hidden when time domain measures are used. Two applications support the usefulness of the discussion, guide a user to apply the methods in different situations, and contribute to the literature with important findings about sources of connectedness. Giving up the assumption of global stationarity of stock market data and approximating the dynamics locally, we document rich time-frequency dynamics of connectedness in US market risk in the first application. Controlling for common shocks due to common stochastic trends which dominate the connections, we identify connections of global economy at business cycle frequencies of 18 up to 96 months in the second application. In addition, we study the effects of cross-sectional dependence on the connectedness of variables.

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^{*}For estimation of the frequency dependent connectedness measures introduced by this paper, we provide the package `frequencyConnectedness` in R software. The package is available on <https://github.com/tomaskrehlik/frequencyConnectedness>

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1 Introduction

Economic markets grew in size and became intertwined during the last decades in an unprecedented fashion. The evolution of economic markets not only caused the change in magnitude of connections but also a change in the structure of the markets and their connections. With this surge, the importance of evaluation of connections among different parts of markets grew, and understanding financial and macroeconomic connectedness became central to many areas of research as risk management, portfolio allocation, and business-cycle analysis. Academics being painfully aware of the unsuitability of the standard correlation-based measures have concentrated on the development of more general frameworks while overlooking the actual sources of connectedness. In this work, we argue that it is crucial to understand the frequency dynamics of the connectedness as shocks to economic activity impact variables at different frequencies with different strength. To consider the long-term, medium-term, and short-term frequency responses of shocks, we propose a general framework which will allow us to measure the connectedness of economic variables at frequency bands of interest.

The distinction between short-term and long-term parts of systems became evident with the dawn of co-integration (Engle and Granger, 1987). Assuming and leveraging co-integration in the system, subsequent literature builds a preliminary notion of disentangling frequencies in connectedness (Gonzalo and Ng, 2001; Blanchard and Quah, 1989; Quah, 1992). Given the decomposition to the long-term common stochastic trend and deviations from trend, one can move the projection in such a way that error to one series will be a shock to long-term trend and the other will be a shock to the deviation from the trend. A shock with strong long-run effect will have high power at low frequencies and in case it transmits to other variables, it points to long-run connectedness. For example, in case of stock markets, low frequency spillovers may be attributed to permanent changes in expectations about future dividends (Balke and Wohar, 2002). While low frequency spillovers are documented by the literature, connectedness at business cycles, or even higher frequencies are not. Hence, we address this call by proposing a general framework for decomposing the connectedness to the frequency band of interest. Similarly to Dew-Becker and Giglio (2013), who set asset pricing into the frequency domain, we view the frequency domain as a natural place for measuring the connectedness between economic variables.

As noted by Diebold and Yilmaz (2009, 2012), and later Diebold and Yilmaz (2014), variance decompositions from approximating models are convenient framework for empirical measurement of connectedness. Precisely, Diebold and Yilmaz (2009) define the measures based on assessing shares of forecast error variation in one variable due to shock arising in another variable in the system. To identify uncorrelated structural shocks from correlated reduced-form shocks, Diebold and Yilmaz (2012) use the generalized variance decomposition of Pesaran and Shin (1998), which moreover allows to define directional connectedness. This approach quickly became popular and recognized by researchers due to its universality.

Being interested in frequency origins of connectedness in variables, one may think about using different forecast horizons of variance decomposition. Staying in time domain, heterogeneous frequency responses of shocks impacting future uncertainty with different strength

will stay hidden, as the effects are simply aggregated through frequencies. To see this, let us consider two examples of bivariate autoregressive process with opposite signs of coefficients. The positive coefficients in the first example will create large connectedness driven by low frequencies of the cross-spectral density. With increasing forecast horizon of variance decompositions, one will measure higher connectedness in the process. In the second example, the negative coefficients of the same magnitude will create equal connectedness as in the first case at all forecasting horizons, although connections come solely from the high frequencies due to anti-persistent nature of the process.

Instead of assessing shares of forecast error variation in a variable due to shock arising in another variable, we are hence interested in assessing shares of forecast error variation in a variable at a specific frequency. This is a natural step to take, as it will show the long-term, medium-term, and short-term impacts of a shock, which can conveniently be summed to total aggregate effect, if needed. For the purpose of frequency dependent measurement, we define spectral representation of generalized forecast error variance decomposition. Instead of impulse responses of the shocks, we work with its Fourier transforms—frequency responses. In the frequency domain, we are simply interested in the portion of forecast error variance at a given frequency that is attributed to shocks in another variable. Our work is inspired by previous research of Geweke (1982, 1984, 1986), and Stiasny (1996) who use related measures in more restrictive environments.

Our spectral representations are useful if we work with stationary data, though, economic time series are often of nonstationary nature. Most prolific example is cointegrated data, which share common stochastic trends commonly expressed as linear combination of the shocks of a system. In words of Granger and Yoon (2002): “*economic data are cointegrated because they respond to shocks together.*” Common stochastic trends makes it even more difficult to understand how the system is connected at various horizons as connections are dominated by the strong relationship. With careful treatment, we extend our notion of frequency dependent measurement of connectedness also to case when data share common stochastic trends and make possible measurement of true business cycles connections.

In addition to introducing the frequency dynamics into the measurement of connectedness and considering important cointegrating dynamics, we also study how cross-sectional correlations impact the connectedness. Higher contemporaneous correlation do not necessarily need to indicate connectedness in a sense literature tries to measure it. A good example is recent crisis of 2007–2008, when stock markets recorded strong cross-sectional correlations biasing the contagion effects estimated by many researchers (Forbes and Rigobon, 2002; Bekaert et al., 2005).

The theoretical discussion is followed by two relevant applications on financial and macroeconomic data that help us to show the usefulness of the framework and guide a user to apply the introduced methods appropriately in different situations. In the first application, we study an important problem of connectedness of market risk. We use the spectral representations of variance decompositions locally to recover the time-frequency dynamics of connectedness in the US stock market, and we document rich dynamics in frequency responses of shocks in volatilities. Dynamics of connectedness is mainly driven by frequencies from one day up to one month, although this does not hold in the period

of turmoils with increased level of uncertainty. In these periods the total connectedness increases, and the increase is mainly due to short-term contemporaneous correlations, as well as causal longer term connectedness.

In the second application, we attempt to measure connectedness of industrial production in G-7 countries at business cycles frequencies. Motivated by the work of Greenwood-Nimmo et al. (2015) who find transmission of shocks to occur gradually with increasing horizons in variance decompositions on similar data, we address the problem with spectral representations. Common shocks driving business cycles, which are confirmed by literature (Stock and Watson, 2005; Canova et al., 2007; Kose et al., 2008) will dominate the variance decompositions with its strong frequency responses. Especially when measuring directional connectedness, effects of common shocks make it difficult to see the directions of connectedness at business cycles. Controlling for these permanent shocks, we measure connectedness at business cycles as commonly defined in the literature, and document frequency dependent connectedness due to shocks with different frequency responses in data.

2 Measuring connectedness in frequency domain

As argued by Diebold and Yilmaz (2014), variance decompositions can intuitively be used for measurement of connectedness between economic variables. A natural way to measure its frequency dynamics is to consider the spectral representation of variance decompositions based on frequency, instead of impulse responses of shocks. Frequency domain being natural place to study the long-run, medium-run, or short-run connectedness shifts our focus from assessing share of variances due to shocks to assessing share of spectra. Stiasny (1996) introduced a first notion of spectral representation for variance decompositions, although in restrictive setting. In our work, we define general spectral representation of variance decompositions, and show how we can use them for defining the frequency dependent connectedness measures.

The spectral representations of variance decompositions can also be viewed as more general way of measuring causality in frequency domain. Geweke (1982) proposes a frequency domain decomposition of the usual likelihood ratio test statistic for Granger causality, and Dufour and Renault (1998); Breitung and Candelon (2006); Yamada and Yanfeng (2014) provide a formal framework for testing causality on various frequencies. Geweke (1984); Granger (1969) develop a multivariate extensions, but all the analysis is done using partial cross-spectra and is therefore silent on indirect causality chains. Hence, this part of literature is also part of our motivation to propose a more general framework.

Before defining the connectedness measures in frequency domain, we briefly discuss the notion of measuring connectedness introduced by Diebold and Yilmaz (2012) using the generalized forecast error variance decompositions (GFEVD), as we build on these ideas in frequency domain later in the text.

2.1 Measuring connectedness with variance decompositions

An intuitive way to measure the connectedness between variables is to consider vector auto-regression (VAR) estimates and its forecast error variance decomposition (FEVD).

Formally, let us have the n -variate process $\mathbf{x}_t = (x_{t,1}, \dots, x_{t,n})$ described by the structural VAR(p) at $t = 1, \dots, T$ as

$$\Phi(L)\mathbf{x}_t = \boldsymbol{\epsilon}_t,$$

where $\Phi(L) = \sum_h \Phi_h L^h$ is $n \times n$ p -th order lag-polynomial and $\boldsymbol{\epsilon}_t$ is white-noise generated by (possibly non-diagonal) covariance matrix Σ . Assuming that the roots of $|\Phi(z)|$ lie outside the unit-circle, the VAR process has following MA(∞) representation

$$\mathbf{x}_t = \Psi(L)\boldsymbol{\epsilon}_t,$$

where $\Psi(L)$ is an $n \times n$ infinite lag polynomial matrix of coefficients.

The usual FEVD derived directly from impulse responses of the system does not have straightforward interpretation as the shocks to variables are not identified. A shock to variable j does not necessarily appear alone, or orthogonally, to shocks to other variables. Hence, to identify the shocks and derive meaningful FEVD we need to employ some identification scheme. Diebold and Yilmaz (2009) use the standard Sims' recursive identification where the errors are standardized by Cholesky decomposition of the covariance matrix. The standardization matrix from Cholesky decomposition, however, depends on the ordering of the variables in the VAR system implying dependence of any measures devised using this identification scheme on reordering of variables. For this reason, Diebold and Yilmaz (2012) use the generalized VAR setting (Pesaran and Shin, 1998) that mitigates the issue by imposing additional assumption of normality of the shocks. Additional convenient feature of this identification scheme is possibility to consider a directional connectedness measures. In our work, we stay within the generalized framework, although the spectral representation can be analogously defined to any other identification scheme and approximating model. With this respect the next sections present general results.

Generalized FEVD can be written in the form¹ (for detailed derivation of the formula see Appendix A)

$$(\boldsymbol{\theta}_H)_{j,k} = \frac{\sigma_{kk}^{-1} \sum_{h=0}^H ((\Psi_h \Sigma)_{j,k})^2}{\sum_{h=0}^H (\Psi_h \Sigma \Psi_h')_{j,j}}, \quad (1)$$

where Ψ_h is a $n \times n$ matrix of coefficients corresponding to lag h , and $\sigma_{kk} = (\Sigma)_{k,k}$. The $(\boldsymbol{\theta}_H)_{j,k}$ denotes the contribution of the k th variable of the system to the variance of forecast error of the element j . Due to one of the notable implications of the generalized VAR framework the effects do not add up to one within columns by definition. To standardize the effects we define

$$(\tilde{\boldsymbol{\theta}}_H)_{j,k} = (\boldsymbol{\theta}_H)_{j,k} / \sum_k (\boldsymbol{\theta}_H)_{j,k}.$$

The connectedness measure is then defined as the share of variances in the forecasts contributed by other than own errors, or equally as ratio of the sum of the off-diagonal

¹Note to notation: $(\mathbf{A})_{j,k}$ denotes the j th row and k th column of matrix \mathbf{A} denoted in bold. $(\mathbf{A})_{j\cdot}$ denotes the full j th row, likewise for columns.

elements to sum of the whole matrix (Diebold and Yilmaz, 2012)

$$\mathcal{C}_H = 100 \frac{\sum_{j \neq k} (\tilde{\boldsymbol{\theta}}_H)_{j,k}}{\sum (\tilde{\boldsymbol{\theta}}_H)_{j,k}} = 100 \left(1 - \frac{\text{Tr} \{ \tilde{\boldsymbol{\theta}}_H \}}{\sum (\tilde{\boldsymbol{\theta}}_H)_{j,k}} \right),$$

where $\text{Tr} \{ \cdot \}$ is the trace operator. Hence, the connectedness is the relative contribution to the forecast variances from the other variables in the system. Note that although \mathcal{C}_H measures connectedness of whole system, directional connectedness measures can be conveniently defined within this framework. We will return to their definition later, already using frequency decompositions.

2.2 Spectral representation for variance decompositions and connectedness measures

Generalized forecast error variance decompositions (GFEVD) are central to measuring connectedness, hence to define frequency dependent measures, we need to consider its spectral counterpart. As can be noted from Equation 1 connectedness measure is based on impulse response functions $\boldsymbol{\Psi}_j$ defined in time domain. As a building block of the presented theory, we consider a frequency response function, $\boldsymbol{\Psi}(e^{-i\omega}) = \sum_h e^{-i\omega h} \boldsymbol{\Psi}_h$, which can be simply obtained from Fourier transform of the coefficients $\boldsymbol{\Psi}$, with $i = \sqrt{-1}$. A spectral density of \mathbf{x}_t at frequency ω can then be conveniently defined as a Fourier transform of MA(∞) filtered series as

$$\mathbf{S}_{\mathbf{x}}(\omega) = \sum_{h=-\infty}^{\infty} E(\mathbf{x}_t \mathbf{x}'_{t-h}) e^{-i\omega h} = \boldsymbol{\Psi}(e^{-i\omega}) \boldsymbol{\Sigma} \boldsymbol{\Psi}'(e^{+i\omega})$$

The power spectrum $\mathbf{S}_{\mathbf{x}}(\omega)$ describes how the variance of the \mathbf{x}_t is distributed over the frequency components ω . Using the spectral representation for covariance, i.e. $E(\mathbf{x}_t \mathbf{x}'_{t-h}) = \int_{-\pi}^{\pi} \mathbf{S}_{\mathbf{x}}(\omega) d\omega$, following definition naturally introduces the frequency domain counterparts of variance decomposition.

Definition 2.1. *Generalized causation spectrum over frequencies $\omega \in (-\pi, \pi)$ is defined as*

$$(\mathbf{f}(\omega))_{j,k} \equiv \frac{\sigma_{kk}^{-1} \left| (\boldsymbol{\Psi}(e^{-i\omega}) \boldsymbol{\Sigma})_{j,k} \right|^2}{(\boldsymbol{\Psi}(e^{-i\omega}) \boldsymbol{\Sigma} \boldsymbol{\Psi}'(e^{+i\omega}))_{j,j}},$$

where $\boldsymbol{\Psi}(e^{-i\omega}) = \sum_h e^{-i\omega h} \boldsymbol{\Psi}_h$ is the Fourier transform of the impulse response $\boldsymbol{\Psi}$.

It is important to note that $(\mathbf{f}(\omega))_{j,k}$ represents the portion of the spectrum of j th variable at frequency ω due to shocks in k th variable. In a sense, we can interpret the quantity as a within frequency causation, as denominator holds spectrum of the j th variable (on-diagonal element of cross-spectral density of \mathbf{x}_t) at given frequency ω . Based on this notion, we will define a within frequency connectedness measures. The *generalized causation*

spectrum can also be related to standard coherence measure, but only one way relation is taken into account in here. Thus the word *causation* as introduced in Stiasny (1996) is justified, as the weighing by respective variances and covariances brings restriction and identification assumptions of the generalized VAR and hence we can interpret the measure causally within the validity of the invoked assumptions on the system. The quantity we consider is different from that of Stiasny (1996), which is reflected in the word *generalized*.

To obtain a natural decomposition of original GFEVD to frequencies, we can simply weight the $(\mathbf{f}(\omega))_{j,k}$ by the frequency share of variance of the j variable. The weighting function can be defined as

$$\Gamma_j(\omega) = \frac{(\Psi(e^{-i\omega})\Sigma\Psi'(e^{+i\omega}))_{j,j}}{\frac{1}{2\pi} \int_{-\pi}^{\pi} (\Psi(e^{-i\lambda})\Sigma\Psi'(e^{+i\lambda}))_{j,j} d\lambda},$$

and represents the power of j th variable at given frequency, which sums through frequencies to a constant value of 2π . Note that while the Fourier transform of the impulse response is in general a complex valued quantity, the generalized causation spectrum is the squared modulus of the weighted complex numbers, hence producing a real quantity.

The following proposition formalizes the discussion, and is central to the development of the connectedness measures in frequency domain.

Proposition 2.1. *Suppose \mathbf{x}_t is wide-sense stationary with $\sigma_{kk}^{-1} \sum_{h=0}^{\infty} |(\Psi_h \Sigma)_{j,k}| < +\infty, \forall j, k$. Then*

$$(\boldsymbol{\theta}_{\infty})_{j,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_j(\omega) (\mathbf{f}(\omega))_{j,k} d\omega.$$

Proof. See Appendix. □

Proposition 2.1 defines the decomposition of GFEVD in the frequency domain. GFEVD at $H \rightarrow \infty$ can then be viewed as weighted average of the generalized causation spectrum $(\mathbf{f}(\omega))_{j,k}$ that gives us the strength of the relationship on given frequency weighted by power of the series on the given frequency. The integral over admissible frequencies reconstructs perfectly the theoretical value of original $\boldsymbol{\theta}_{\infty}$. The proposition is not only important theoretical result, but it also reminds us that when measuring connectedness with $\boldsymbol{\theta}_H$ in time domain, we are looking at information aggregated through frequencies ignoring heterogeneous frequency responses of shocks. It is also important to note that effects at the whole range of frequencies influence $\boldsymbol{\theta}_{\infty}$, which is an important observation when one interprets the measures defined using the spectral representation.

In economic applications, we are interested in assessing the short-term, medium-term, or long-term connectedness rather than connectedness at a given frequency in most of the cases. Hence, it is more convenient to work with frequency bands, which can be arbitrarily chosen. For this purpose, we define the amount of forecast error variance created on an arbitrarily chosen set of frequencies. The quantity is then given by integrating only over the desired frequencies $\omega \in (a, b)$.

Formally, let us have a frequency band $d = (a, b) : a, b \in (-\pi, \pi), a < b$. The generalised

FEVD on some frequency band d is defined as

$$(\boldsymbol{\theta}_d)_{j,k} = \frac{1}{2\pi} \int_d \Gamma_j(\omega) (\mathbf{f}(\omega))_{j,k} d\omega. \quad (2)$$

Because the introduced relationship is an identity and the integral is linear operator, sum over disjoint intervals covering the whole range $(-\pi, \pi)$ will recover the original GFEVD. Following remark formalizes the fact.

Remark 2.1. Denote by d_s an interval on the real line from the set of intervals D that form a partition of the interval $(-\pi, \pi)$, such that $\cap_{d_s \in D} d_s = \emptyset$, and $\cup_{d_s \in D} d_s = (-\pi, \pi)$. Due to the linearity of integral and the construction of d_s we have

$$(\boldsymbol{\theta}_\infty)_{j,k} = \sum_{d_s \in D} (\boldsymbol{\theta}_{d_s})_{j,k}.$$

Using the spectral representation of GFEVD, it is straightforward to define connectedness measures on a given frequency band.

Definition 2.2. Let us define scaled GFEVD on the frequency band $d = (a, b) : a, b \in (-\pi, \pi), a < b$ as

$$(\tilde{\boldsymbol{\theta}}_d)_{j,k} = (\boldsymbol{\theta}_d)_{j,k} / \sum_k (\boldsymbol{\theta}_\infty)_{j,k}.$$

- The frequency connectedness on the frequency band d is then defined as

$$\mathcal{C}_d^{\mathcal{F}} = 100 \left(\frac{\sum (\tilde{\boldsymbol{\theta}}_d)_{j,k}}{\sum (\tilde{\boldsymbol{\theta}}_\infty)_{j,k}} - \frac{\text{Tr}\{\tilde{\boldsymbol{\theta}}_d\}}{\sum (\tilde{\boldsymbol{\theta}}_\infty)_{j,k}} \right).$$

- The within connectedness on the frequency band d is then defined as

$$\mathcal{C}_d^{\mathcal{W}} = 100 \left(1 - \frac{\text{Tr}\{\tilde{\boldsymbol{\theta}}_d\}}{\sum (\tilde{\boldsymbol{\theta}}_d)_{j,k}} \right).$$

The Definition 2.2 works with two notions: the *frequency connectedness* and the *within connectedness*. The *within connectedness* gives us the connectedness effect that happens within the frequency band and is weighted by the power of the series on the given frequency band exclusively. On the other hand the *frequency connectedness* decomposes the original connectedness into distinct parts that in sum give the original connectedness measure \mathcal{C}_∞ . Following remark formalize the notion of reconstruction of the connectedness.

Remark 2.2 (Reconstruction of frequency connectedness). Denote by d_s an interval on the real line from the set of intervals D that form a partition of the interval $(-\pi, \pi)$, such

that $\cap_{d_s \in D} d_s = \emptyset$, and $\cup_{d_s \in D} d_s = (-\pi, \pi)$. We then have that

$$\mathcal{C}_\infty = \sum_{d_s \in D} \mathcal{C}_{d_s}^{\mathcal{F}}. \quad (3)$$

Proof. See Appendix. □

To illustrate the difference between the frequency and within connectedness, remember that the typical spectral shape of economic variables has the most power concentrated on low frequencies (long-term movements or trend). Hence, we could decompose the connectedness into two parts, the one that covers long-term movements and the one that covers the short-term movements. Suppose that 90% of the spectral density is concentrated in the long-term movements. Now, suppose that the connectedness on high frequencies is high, say 80, and low on long-term, say 10. These numbers represent the within connectedness. The total connectedness will be much closer to 10, because the short-term connectedness of size 80 will be down weighted by the very low amount of spectral density on the short-term. Otherwise said, even though the short-term activities are very connected because not much happens in terms of the system activity on the short-term, this connection becomes negligible in the system connectedness. This can be seen clearly in the simulations in the following section.

The concepts of within and frequency connectedness coincide when whole frequency band $d = (-\pi, \pi)$ is considered. This is formalized by the following remark.

Remark 2.3. *Let us have $d = (-\pi, \pi)$. We then have*

$$\mathcal{C}_d^{\mathcal{F}} = \mathcal{C}_d^{\mathcal{W}}. \quad (4)$$

Proof. See Appendix. □

2.3 Estimation of connectedness in frequency domain

The previous theory is devised in terms of theoretical quantities and the method of estimation deserves to follow. Primarily, the standard VAR framework is not the only way how to obtain the estimates of the behavior of the system. More advanced and often more suited approaches can be devised. Diebold et al. (2015) use vector error correction model (VECM) to obtain approximation of the behavior. Estimators that use shrinkage or Bayesian approaches might be a viable alternative in many cases. In our applications, we restrict ourselves to use of the standard VAR and VECM frameworks and leave other methods for future investigation.

Having the estimates, two main issues remain. The first is the approximation of the $\text{MA}(\infty)$ representation of the series and the second is the estimation of the theoretical spectral densities.

The $\text{MA}(\infty)$ representation is used to compute the forecast error variance decomposition that relates percentage of mean squared error (MSE) of forecasts of variable k due to shocks to variable j . Because the computation of these theoretical quantities is based on an infinite

process, we make it feasible by a finite MA(H) approximation. This is possible mainly due to the fact that if the system is stable, there must exist H , such that

$$MSE(x_{k,H}) - MSE(x_{k,H+1}) < \epsilon,$$

meaning that the error due to approximation disappears with growing H . Reverse would mean that innovations would have permanent impact. Hence, we can in principle use two ways to approximate the decomposition. First, we can use an ad hoc selected H that is beyond doubt high enough. Or second, we can use a measure of similarity of matrices to choose the appropriate H during the approximation. The $\hat{\Psi}_h$ coefficients are then computed through standard recursive scheme $\hat{\Psi}_0 = \mathbf{I}$, $\hat{\Psi}_h = \sum_{j=1}^{\max\{h,p\}} \Phi(j) \hat{\Psi}_{h-1}$, where p is the order of VAR and $h \in \{1, \dots, H\}$. Here we note that by studying the quantities in the frequency domain, H serves only as an approximation factor, and have no interpretation as in the time domain. In the applications, we advice to set the H high enough to obtain better approximation of the quantities at all frequencies, especially when small frequency bands are of interest.

Secondly, the spectral quantities are estimated using standard discrete Fourier transforms. The following definition specifies accurately the used estimates of the quantities.

Definition 2.3. *The cross-spectral density on the interval $d = (a, b) : a, b \in (-\pi, \pi), a < b$*

$$\int_d \Psi(e^{-i\omega}) \Sigma \Psi'(e^{+i\omega}) d\omega$$

is estimated as follows

$$\sum_{\omega} \hat{\Psi}(\omega) \hat{\Sigma} \hat{\Psi}'(\omega),$$

for $\omega \in \left\{ \left\lfloor \frac{aH}{2\pi} \right\rfloor, \dots, \left\lfloor \frac{bH}{2\pi} \right\rfloor \right\}$ where

$$\hat{\Psi}(\omega) = \sum_{h=0}^{H-1} \hat{\Psi}_h e^{-2i\pi\omega/H},$$

and $\hat{\Sigma} = \hat{\epsilon}'\hat{\epsilon}/T$.

The decomposition of the impulse response function at the given frequency band is then estimated as

$$\hat{\Psi}(d) = \sum_{\omega} \hat{\Psi}(\omega), \text{ for } \omega \in \left\{ \left\lfloor \frac{aH}{2\pi} \right\rfloor, \dots, \left\lfloor \frac{bH}{2\pi} \right\rfloor \right\}.$$

Using this definition the estimate of the generalized causation spectrum over the interval d is then defined as follows

$$\left(\hat{\mathbf{f}}(d) \right)_{j,k} \equiv \frac{\hat{\sigma}_{kk}^{-1} \left(\left(\hat{\Psi}(\omega) \hat{\Sigma} \right)_{j,k} \right)^2}{\left(\sum_{\omega} \hat{\Psi}(\omega) \hat{\Sigma} \hat{\Psi}'(\omega) \right)_{j,j}}.$$

By employing the estimate of weighting function

$$\hat{\Gamma}_j(d) = \frac{\left(\sum_{\omega} \hat{\Psi}(\omega) \hat{\Sigma} \hat{\Psi}'(\omega) \right)_{j,j}}{\left(\sum_d \sum_{\omega} \hat{\Psi}(\omega) \hat{\Sigma} \hat{\Psi}'(\omega) \right)_{j,j}},$$

we estimate the decomposed GFEVD to a frequency band as

$$\hat{\theta}_{j,k}(d) = \hat{\Gamma}_j(d) \left(\hat{\mathbf{f}}(d) \right)_{j,k}.$$

Then the connectedness measures $\hat{\mathcal{C}}^{\mathcal{W}}$ and $\hat{\mathcal{C}}^{\mathcal{F}}$ at a given frequency band of interest can be readily derived by plugging the $\hat{\theta}_{j,k}(d)$ estimate into the Definition 2.2.²

2.4 Connectedness in cointegrated processes

Spectral representations from previous sections hold if we work with stationary processes. However, in many cases the economic variables we study are nonstationary $I(1)$ processes exhibiting systemic variation. For an $I(1)$ process, $\Psi(1) = \infty$, i.e. the infinite sum of the coefficients is infinite. The source of integration can either be a deterministic trend or non-stationarity driven through the coefficients. In the first case of the deterministic trend, it is important to take it into account while modelling the system (by inclusion of deterministic trend) and if done properly the analysis of GFEVD is the same as in the case of the stationary processes because the trend does not enter the impulse response functions that become stationary. In the second case, where the integration is driven through the stochastic processes one has to think further about the strategy for estimating connectedness.

Most prolific example of such system in economics is the cointegrated process³. Therefore, let us have system defined by

$$\Delta \mathbf{x}_t = \alpha \beta' \mathbf{x}_{t-1} + \sum_{i=1}^p \Xi_i \Delta \mathbf{x}_{t-i} + \epsilon_t, \quad (5)$$

where α, β are matrices of size $n \times l$ holding the loadings and cointegration vectors respectively, Ξ_i are matrices of size $n \times n$ holding the short-run coefficients, and ϵ is white-noise generated by (possibly non-diagonal) covariance matrix Σ . By simple manipulation a VAR(p+1) representation of the system can be derived

$$\Phi(L) \mathbf{x}_t = \epsilon_t.$$

such that $\mathbf{I} - \Phi(z)$ has determinant zero and rank equal to l . In the covariance stationary case, the impulse responses that are needed for our analysis conveniently correspond to the

²The whole estimation is done using the package `frequencyConnectedness` in R software. The package is available on <https://github.com/tomaskrehlik/frequencyConnectedness>.

³For complex exposition see Lütkepohl (2007).

MA representation of the process. The integrated processes however do not possess valid MA representation. Though unorthogonalized impulse response functions can readily be computed recursively in the same fashion as in the case of the stable VAR system providing us with the matrix $\Psi(L)$ where the individual elements hold the unorthogonalized impulse responses in form of the lag polynomials. This familiar notation, yet non-standard, eases the exposition.

With careful treatment, one can think about further extending the ideas presented earlier to nonstationary cases. The main issue one needs to deal with is that unconditional GFEVD will be dominated by information at the zero frequency, as frequency response of common shocks will have very high power at zero frequency. The information at remaining frequencies will be shadowed as the limit of spectrum of the cointegrated process at zero frequency is infinite, $\int_{-\pi}^{\pi} \mathbf{S}_{\mathbf{x}}(\omega) d\omega \rightarrow \infty$, giving the remaining frequencies zero weight $\Gamma_j(\omega) = 0$ in computation of $(\boldsymbol{\theta}_d)_{j,k}$. Even in this case, definition of $(\mathbf{f}(\omega))_{j,k}$ is meaningful, as it shows relative influence of shocks on frequency components of variation. Intuitively, GFEVD will be created solely by long-run variance at zero frequency, as unconditionally the relative contribution of other frequencies are negligible in terms of their variance. Letting $(\mathbf{f}(0))_{j,k} \equiv \lim_{\omega \rightarrow 0} (\mathbf{f}(\omega))_{j,k}$ exist and noting the quantities are symmetric around zero frequency in $[-\pi, \pi]$ domain, the following proposition formalizes this notion.

Proposition 2.2 (Domination of zero frequency). *Suppose \mathbf{x}_t is a cointegrated system defined by Equation 5. Then*

$$(\boldsymbol{\theta}_{\infty})_{j,k} = \frac{1}{\pi} \left(\lim_{z \rightarrow 0^+} \int_z^{z+\epsilon} \Gamma_j(\omega) (\mathbf{f}(\omega))_{j,k} d\omega + \int_{z+\epsilon}^{\pi} \Gamma_j(\omega) (\mathbf{f}(\omega))_{j,k} d\omega \right) = (\mathbf{f}(0))_{j,k},$$

for all ϵ such that $\epsilon \in (0, z)$.

Proof. See the proof in Appendix □

The proposition states that measuring connectedness of cointegrated processes including zero frequency results in measuring the effects of shocks to common stochastic trends (first integral from Proposition 2.2) and ignores the connection at other than zero frequencies (as second integral from Proposition 2.2 is zero). Hence the measure neglects possible connections at remaining frequencies which are of high importance during the move to new equilibrium in the system. Such frequencies are of great interest in studying business cycle connectedness, as will be argued in the next sections.

To measure these connections at business cycle frequencies, we adjust the measure to only take into account the variance and connection contributions at frequencies excluding zero. The following definition formalizes the measure.

Definition 2.4. *Suppose \mathbf{x}_t is a cointegrated system defined by Equation 5. Then*

$$(\boldsymbol{\xi}_{\infty})_{j,k} = \frac{1}{\pi} \int_{\epsilon}^{\pi} \tilde{\Gamma}_j(\omega) (\mathbf{f}(\omega))_{j,k} d\omega,$$

where

$$\tilde{\Gamma}_j(\omega) = \frac{(\Psi(e^{-i\omega}) \Sigma \Psi'(e^{+i\omega}))_{j,j}}{\frac{1}{\pi} \int_{\epsilon}^{\pi} (\Psi(e^{-i\lambda}) \Sigma \Psi'(e^{+i\lambda}))_{j,j} d\lambda}$$

and ϵ is arbitrarily small.

Importantly, we have to note that the causation spectrum remains unchanged but is weighted by different weights. We deliberately choose to use the notation $\boldsymbol{\xi}$ to stress the difference from the traditional GFEVD denoted by $\boldsymbol{\theta}$. Using the measure $\boldsymbol{\xi}$ on the domain (ϵ, π) we can define all the remaining measures of connectedness from previous sections readily.

In essence, the approach of removing the zero frequency from the measures is equivalent to applying an ideal filter to the series. To illustrate this, let us have the ideal filter such that

$$\mathbf{B}(e^{-i\omega}) = \begin{cases} \mathbf{I} & \text{for } \omega \in [-\pi, 0) \cup (0, \pi] \\ 0 & \text{for } \omega = 0 \end{cases}$$

Then

$$\tilde{\mathbf{x}}_t = \mathbf{B}(L)\mathbf{x}_t$$

and the spectral density of the filtered series will be

$$\begin{aligned} \mathbf{S}_{\tilde{\mathbf{x}}}(\omega) &= \mathbf{B}(e^{-i\omega})\mathbf{S}_{\mathbf{x}}(\omega)\mathbf{B}'(e^{+i\omega}) \\ &= \mathbf{B}(e^{-i\omega})\boldsymbol{\Psi}(e^{-i\omega})\boldsymbol{\Sigma}\boldsymbol{\Psi}'(e^{+i\omega})\mathbf{B}'(e^{+i\omega}) \end{aligned}$$

Then by construction, $\tilde{\mathbf{x}}_t$ is stationary, as $\mathbf{B}(1) = \mathbf{B}(e^{-i0}) = 0$ and more importantly,

$$\mathbf{S}_{\tilde{\mathbf{x}}}(\omega) = \begin{cases} \mathbf{S}_{\mathbf{x}}(\omega) & \text{for } \omega \in [-\pi, 0) \cup (0, \pi] \\ 0 & \text{for } \omega = 0 \end{cases}$$

Now, generalized causation spectrum of the filtered series $\tilde{\mathbf{x}}_t$ can be defined as

$$\left(\tilde{\mathbf{f}}(\omega)\right)_{j,k} \equiv \frac{\sigma_{kk}^{-1} \left| (\mathbf{B}(e^{-i\omega})\boldsymbol{\Psi}(e^{-i\omega})\boldsymbol{\Sigma})_{j,k} \right|^2}{(\mathbf{B}(e^{-i\omega})\boldsymbol{\Psi}(e^{-i\omega})\boldsymbol{\Sigma}\boldsymbol{\Psi}'(e^{+i\omega})\mathbf{B}'(e^{+i\omega}))_{j,j}}, \quad (6)$$

which implies that

$$\left(\tilde{\mathbf{f}}(\omega)\right)_{j,k} = \mathbf{f}(\omega) \text{ for } \omega \in [-\pi, 0) \cup (0, \pi].$$

While $\tilde{\mathbf{f}}$ is equal to \mathbf{f} on a frequency band $(0, \pi]$, to obtain connectedness, all we need to do is to alter the definition of weighting function as in Definition 2.4. This is also important part to understand when interpreting estimates. Weighting without zero frequency implies that $\tilde{\Gamma}_j(\omega)$ will be different from $\Gamma_j(\omega)$ causing that these measures will not be directly comparable to full variance decompositions of the system in relative terms.

A natural question arises why we proceed by removing the zero frequency but not filtering the series by appropriate filter⁴ and computing the standard connectedness measures, which might seem more natural? Applying univariate filters to the series and then modeling them as a system might introduce spurious correlations between the series due to cyclical properties of filters (Cogley and Nason, 1995; Murray, 2003). The problem might

⁴There are several widely used filters in the literature, for example Hodrick Prescott, Baxter King.

be mitigated to some extent, but cannot be eradicated due to the nature of the problem, as we always approximate an ideal filter with some error. Although theoretically same approaches, proper statistical modeling with removal of the connection at zero frequencies is superior to applying filters to the series and computing the connectedness measures to the filtered series.

3 Generating the frequency dependent connectedness

To motivate the usefulness of the proposed measures, we study the processes that generate frequency dependent connectedness by simulations. We look at connectedness that are induced through cross-sectional correlations or interactions between bivariate autoregressive (AR) processes. We illustrate the emergence of connectedness and their spectral footprints through change in coefficients in the simplest bivariate VAR(1) case.⁵ Suppose the simplest case that nevertheless illustrates the mechanics generating the data from the following equations

$$\begin{aligned} y_{1,t} &= \beta_1 y_{1,t-1} + s y_{2,t-1} + \epsilon_{1,t} \\ y_{2,t} &= s y_{1,t-1} + \beta_2 y_{2,t-1} + \epsilon_{2,t}, \end{aligned} \tag{7}$$

where $(\epsilon_{1,t}, \epsilon_{2,t}) \sim N(0, \Sigma)$ with $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.

By altering the true coefficients generating the data, we study several cases with known values of theoretical connectedness estimates. We start with a symmetric processes with $\beta = \beta_1 = \beta_2$ with three important cases generating distinctly connected variables $y_{1,t}$ and $y_{2,t}$. First case is the $\beta = \beta_1 = \beta_2 = 0$, when we have two independent processes, which have connectedness zero at all frequencies. Secondly, we study the connectedness of two symmetrically connected AR processes with the parameter $\beta = \beta_1 = \beta_2 = 0.9$ and $s = 0.09$ or $\beta = \beta_1 = \beta_2 = -0.9$ and $s = -0.09$ generating equal total connectedness with different sources from low and high frequencies of cross-spectral densities for positive and negative values of coefficients respectively.

In addition to motivating the importance of frequency dynamics of connectedness, we also show the importance of cross-sectional correlations, which translates to all frequencies, and may bias the connectedness measures. Hence for all cases, we consider two extremes of cross-sectional dependence: no correlation $\rho = 0$ and correlation of $\rho = 0.9$. To show how the cross-sectional correlations impact the connectedness measures, we compute the measures with additional step in estimation, considering only diagonals of the covariance matrix of residuals, hence removing the cross-sectional dependence. In this way, we disentangle the influence of correlations from the true dynamics. In the text we always present only the estimates on the simulated data, and save the true values of measures in the Table 6 in the appendix.

The Table 1 shows the results. We can see that the system connectedness of two unconnected and uncorrelated processes is practically zero on both total and all spectral

⁵Other more elaborate simulation scenarios are outlined in the R package provided with the paper.

β	s	ρ	Connectedness				Connectedness without correlation			
			Total	$(\pi/2, \pi)$	$(\pi/4, \pi/2)$	$(0, \pi/4)$	Total	$(\pi/2, \pi)$	$(\pi/4, \pi/2)$	$(0, \pi/4)$
0.00	0.00	0.00	0.18 (0.16)	0.19 (0.20)	0.19 (0.18)	0.19 (0.21)	0.09 (0.10)	0.10 (0.11)	0.10 (0.11)	0.10 (0.11)
0.00	0.00	0.90	44.68 (0.30)	44.75 (0.36)	44.74 (0.36)	44.72 (0.42)	0.47 (0.55)	0.41 (0.51)	0.41 (0.51)	0.41 (0.50)
0.90	0.09	0.00	37.65 (4.55)	0.69 (0.76)	1.26 (0.76)	37.77 (4.36)	37.24 (4.64)	0.64 (0.93)	1.22 (0.92)	37.84 (4.81)
0.90	0.09	0.90	49.24 (0.33)	43.97 (0.64)	44.15 (0.62)	49.36 (0.27)	35.31 (6.27)	0.37 (0.45)	0.79 (0.53)	35.07 (5.10)
-0.90	-0.09	0.00	39.33 (3.90)	38.91 (4.24)	0.96 (1.19)	0.87 (1.20)	38.97 (4.31)	38.68 (4.21)	0.81 (0.88)	0.71 (0.89)
-0.90	-0.09	0.90	49.40 (0.32)	49.43 (0.23)	43.81 (0.62)	43.77 (0.63)	35.10 (6.28)	35.74 (4.95)	0.44 (0.30)	0.37 (0.29)

Table 1: Simulation results. The first three columns describe parameters for the simulation as described in Equation 7. We set $\beta = \beta_1 = \beta_2$. The results are based on 100 simulations of VAR with the specified parameters of length 1000 with a burnout period of 100. The estimate is computed as mean of the 100 observations and the standard error is sample standard deviation.

parts. In case of correlated noises, the total connectedness with estimated correlation matrix is estimated around 45 with equal footprint on all the scales. Considering only diagonal elements from estimated covariance matrix of residuals, and hence removing the cross-sectional dependence correctly estimates the connectedness zero at all frequencies.

In the case with AR coefficient equal to 0.9, the uncorrelated case shows that the connection between the processes is on the long-run part (as is expected due to the spectral density of the underlying process). On the other hand, introducing correlation increases the total connectedness and most of all obfuscates the source of the dynamics. Considering only diagonals of covariance matrix of estimated residuals, we can see that the correlation in the estimated covariance matrix correctly exposes the underlying dynamics. The remaining case with coefficient equal to -0.9 is very similar to the previous case with the difference that the spectral mass is concentrated on the short frequencies. Otherwise the qualitative results remain the same.

It is important to note that while coefficients with opposite signs of 0.9 and -0.9 generate the time series with equal connectedness, its source is from different parts of spectra. This example motivates the usefulness of our measures, which are able to locate precisely the part of cross-spectra generating the connectedness.

Next we move to the case where the two processes are not symmetric. With the simulation, we want to illustrate two important cases, how the connectedness arises. First, let us keep the parameter s that governs the connection of the two processes through lagged observation constant and change the spectral structure of the processes through the coefficient β_2 . The Table 2 shows that the connectedness in this case is arising due to the increase in spectral similarity of the processes in question. One could take parallel from physics and state if the two processes (time-series) can resonate, even a relatively small interaction coefficient is capable to create strong connectedness.

On the other hand, keeping the structure of the processes constant and increasing the parameter of interconnection increases the connectedness as is documented by Table 5.

This simulation suggests a possible sources of connectedness and motivates the usefulness of our measures. The role of covariance among the processes can be studied through exclusion of the covariance terms, the role of similarity can be examined through individual

β_1	β_2	s	ρ	Connectedness				Connectedness without correlation			
				Connectedness	$(\pi/2, \pi)$	$(\pi/4, \pi/2)$	$(0, \pi/4)$	Connectedness	$(\pi/2, \pi)$	$(\pi/4, \pi/2)$	$(0, \pi/4)$
0.90	0.90	0.09	0.00	36.75 (4.79)	0.74 (1.32)	1.35 (1.27)	38.57 (3.94)	37.06 (4.64)	0.56 (0.45)	1.14 (0.49)	38.03 (4.24)
0.90	0.90	0.09	0.90	49.23 (0.32)	43.99 (0.51)	44.17 (0.49)	49.35 (0.24)	33.30 (5.90)	0.43 (0.46)	0.86 (0.53)	36.03 (4.80)
0.90	0.40	0.09	0.00	5.62 (1.52)	0.43 (0.31)	0.95 (0.32)	7.50 (2.19)	5.69 (1.48)	0.29 (0.07)	0.82 (0.19)	7.36 (1.94)
0.90	0.40	0.09	0.90	46.11 (0.36)	44.10 (0.57)	44.35 (0.53)	46.53 (0.30)	5.40 (1.62)	0.30 (0.06)	0.82 (0.17)	7.38 (2.17)
0.90	0.00	0.09	0.00	2.70 (0.91)	0.42 (0.38)	0.93 (0.35)	4.21 (1.16)	2.54 (0.68)	0.31 (0.08)	0.78 (0.21)	3.95 (1.23)
0.90	0.00	0.09	0.90	45.37 (0.36)	44.37 (0.42)	44.61 (0.39)	45.87 (0.37)	2.66 (0.78)	0.29 (0.06)	0.74 (0.16)	3.77 (1.20)
0.90	-0.90	0.09	0.00	0.53 (0.15)	0.59 (0.38)	0.54 (0.20)	0.53 (0.31)	0.49 (0.10)	0.47 (0.12)	0.45 (0.10)	0.45 (0.11)
0.90	-0.90	0.09	0.90	44.78 (0.30)	44.61 (0.36)	44.76 (0.35)	44.82 (0.35)	0.46 (0.09)	0.46 (0.06)	0.46 (0.11)	0.47 (0.15)

Table 2: Simulation results. The first three columns describe parameters for the simulation as described in Equation 7. The results are based on 100 simulations of VAR with the specified parameters of length 1000 with a burnout period of 100. The estimate is computed as mean of the 100 observations and the standard error is simple sample standard deviation.

spectral densities, however, as mentioned most of the economic series have similar spectral densities (Granger, 1966). Our measures estimate the rich dynamics precisely.

4 Empirical applications

We apply the theory to measure the risk connectedness in financial markets as well connectedness of economic business cycles. The two applications are distinct not only in their economic nature, they differ in the data properties and treatment in measuring connectedness at frequencies. Hence the two applications do not only provide interesting insights on how the data are actually connected, but also guide a user in correct application of the introduced methods in different situations.

Before carrying out the empirical analysis, we describe two additional notions from the connectedness theory that will facilitate the interpretation of the results. In the following part, we introduce the notion of the directional connectedness and the connectedness table.

4.1 Directional connectedness, connectedness table

For the ease of interpretation and deeper understanding, Diebold and Yilmaz (2012) defined the directional connectedness. As previously noted, the connectedness is a quantity describing the size of connection of the whole system with respect to the whole system as a reference. Hence, up to now we are provided with one number that measures the system connectedness. One might be, however, often interested in the connection of the individual elements to the other elements of the system. This can be achieved by looking at the (frequency) elements of GFEVD itself, i.e. the components from which we construct the connectedness measure. The example of connectedness table for n variables is shown in Table 3.

The central part of the table shows the variance decompositions $\tilde{\theta}$ and its frequency counterparts $\tilde{\theta}_{d_s}$ in block elements. Each of the element hold total connectedness in the

	x_1	\dots	x_n	From Others
x_1	$(\tilde{\theta})_{1,1}$ $(\tilde{\theta}_{d_1})_{1,1} \dots (\tilde{\theta}_{d_s})_{1,1}$	\dots	$(\tilde{\theta})_{1,n}$ $(\tilde{\theta}_{d_1})_{1,n} \dots (\tilde{\theta}_{d_s})_{1,n}$	$\mathcal{C}_{\bullet \rightarrow 1}$ $\mathcal{C}_{\bullet \rightarrow 1}^{d_1} \dots \mathcal{C}_{\bullet \rightarrow 1}^{d_s}$
\vdots		\ddots		\vdots
x_n	$(\tilde{\theta})_{n,1}$ $(\tilde{\theta}_{d_1})_{n,1} \dots (\tilde{\theta}_{d_s})_{n,1}$	\dots	$(\tilde{\theta})_{n,n}$ $(\tilde{\theta}_{d_1})_{n,n} \dots (\tilde{\theta}_{d_s})_{n,n}$	$\mathcal{C}_{\bullet \rightarrow n}$ $\mathcal{C}_{\bullet \rightarrow n}^{d_1} \dots \mathcal{C}_{\bullet \rightarrow n}^{d_s}$
To Others	$\mathcal{C}_{1 \rightarrow \bullet}$ $\mathcal{C}_{1 \rightarrow \bullet}^{d_1} \dots \mathcal{C}_{1 \rightarrow \bullet}^{d_s}$	\dots	$\mathcal{C}_{n \rightarrow \bullet}$ $\mathcal{C}_{n \rightarrow \bullet}^{d_1} \dots \mathcal{C}_{n \rightarrow \bullet}^{d_s}$	\mathcal{C} $\mathcal{C}_{d_1}^{\mathcal{F}} \dots \mathcal{C}_{d_s}^{\mathcal{F}}$

Table 3: The directional connectedness table. Each block element of the table shows two rows. Considering s frequency intervals $d_1 \dots, d_s$ such that $\cap_{d_s \in D} d_s = \emptyset$, the first row of each element gives the overall connectedness and the second row gives the decomposition of the first row into frequency bands d_i (it can sum to total connectedness from first row if all frequencies are used, $\cup_{d_s \in D} d_s = (-\pi, \pi)$). The right-most column gives the influence of other variables to the particular variable $\mathcal{C}_{\bullet \rightarrow j}$, the bottom row gives the influence going from the variable to other variables $\mathcal{C}_{k \rightarrow \bullet}$ with corresponding frequency decompositions. Bottom-right element holds total connectedness of the system \mathcal{C} with frequency decompositions.

first row, and decomposition to $d_1 \dots, d_s$ frequency intervals such that $\cap_{d_s \in D} d_s = \emptyset$. Note that in case the frequency bands cover the whole spectrum, i.e. $\cup_{d_s \in D} d_s = (-\pi, \pi)$, the second row will sum to total connectedness in the first row. However, this is not the case if we are studying connectedness of cointegrated system, where we do not consider zero frequency due to reasons outlined in the previous section. Note that in this case, we only replace $\tilde{\theta}_{d_s}$ by $\tilde{\xi}_{d_s}$ in the connectedness table.

Generally, the elements on the diagonal are the forecast variance shares of the own shocks. The reminding elements, $(\tilde{\theta})_{j,k}$ are the contributions of the k th variable to the forecast variance of the j th variable. The elements in the bottom row denoted $\mathcal{C}_{k \rightarrow \bullet}$ give the amount of variance going from the k th variable to all other variables, and is a sum of $(\tilde{\theta})_{j,k}$ through j in a column excluding the own share, i.e. for all $j \neq k$. Likewise, in the last column, the elements $\mathcal{C}_{\bullet \rightarrow j}$ give the amount of variance contributed by all other variables as a sum of $(\tilde{\theta})_{j,k}$ through k in a corresponding row excluding own share, i.e. all $k \neq j$. The corresponding frequency decompositions of the directional connectedness are $\mathcal{C}_{k \rightarrow \bullet}^{d_s}$ and $\mathcal{C}_{\bullet \rightarrow j}^{d_s}$ respectively. Finally, the number denoted \mathcal{C} in the bottom right corner gives the total connectedness with its frequency decompositions $\mathcal{C}_{d_s}^{\mathcal{F}}$.

4.2 Connectedness of risk in major assets of the US market

The question how stock markets are connected has been studied by the literature extensively in past decades. From studies focusing on causality effects, comovement, spillovers, connectedness, and systemic risk, researchers primarily try to answer the question using methods measuring the aggregate effects. In this section, we argue that it is important

to measure frequency sources of connectedness, as shocks to volatility will impact future uncertainty differently. For example fundamental changes in investor’s expectations will impact the market in longer term. These expectations are then transmitted to surrounding assets in the portfolio differently than shocks having short-term impact.

The very early literature measuring the connectedness of stock markets was predominantly interested in contagion effects in market prices during crises. In already seminal paper, Forbes and Rigobon (2002) have, however, shown that if we account for volatilities of the price processes, the contagion effects disappear. This led to a rather strong statement of no-contagion, and interdependence among the markets remained the main effect of interest. Tse and Tsui (2002) concentrated on investigating the connection in the multivariate GARCH framework. They report high cross-correlations on Forex market, national stock market, and Hang Seng sectoral indices. Bae et al. (2003) investigate the co-incidence of the extreme returns across markets and connect this measure by extreme value theory. They evaluate the contagion effects among various parts of the world, such as Latin America, Asia, and the United States, finding high coincidence of negative returns across markets. Engle et al. (2012) provide exhaustive review of empirical literature on volatility spillovers.

A broader picture was later provided by Diebold and Yilmaz (2009) who explicitly investigated volatilities and returns separately, and uncovered contagion effects in volatilities. In the same paper, authors side-step the controversial topic of contagion, which had already been tied predominantly to financial crises and introduce the concept of spillovers that refer to varying interdependency between the markets. Borrowing from both contagion and interdependence notions, Diebold and Yilmaz (2009) define a rigorous framework for measuring spillovers of returns and volatility across markets, coined a connectedness in their subsequent work (Diebold and Yilmaz, 2014). The methodology has been successfully used to measure connectedness effects in the literature by hundreds of studies in few years. Still the literature is silent about origins of the connectedness in stock markets.

4.2.1 Data

Volatility, as one of the mostly studied quantities in the financial literature, is also largely perceived as a risk measure. Hence, considering the connectedness of volatility, we investigate the question how the risk in markets is connected at different frequencies. We study the intra-market connectedness of seven major stocks representing largest sectors within the US economy.⁶ Although the companies might live in the same economy, they are generally tightened by different aspects of it and as argued earlier, these aspects may have different impact at frequencies.

Concretely, we investigate the connectedness between financial, technology, consumer staples that are pro-cyclical, consumer staples that are counter-cyclical, communication, energy, and health sectors. From each sector, we select the most liquid stock to represent the sector,⁷ namely, Bank of America Corporation (BAC), Microsoft Corporation (MSFT),

⁶Sectors are defined in accordance with the Global Industry Classification Standard (GICS) and in a similar manner as in Beber et al. (2011).

⁷We chose the stocks in order to best capture the total capitalization of the sector. Note that this is

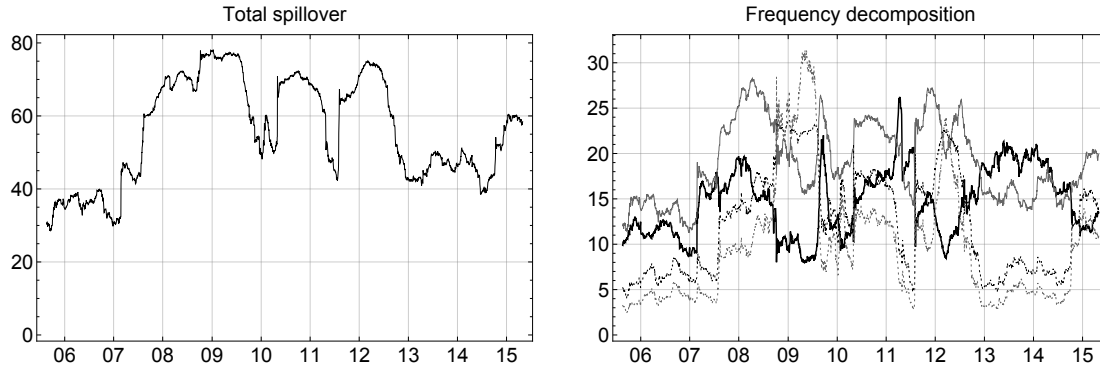


Figure 1: Dynamic frequency connectedness of the US market risk. Left plot represents the total connectedness \mathcal{C} , computed using Diebold and Yilmaz (2012) measure on moving window with length of one year (250 days). Right plot represents the frequency connectedness $\mathcal{C}_{d_s}^F$ with $d_1 \in [1, 5]$ days in solid bold, $d_2 \in (5, 20]$ solid, $d_3 \in (20, 60]$ dotted bold, and $d_4 \in (60, 250]$ dotted lines. Note that all lines through frequency bands d_s sums the total connectedness \mathcal{C} .

Walt Disney Company (DIS), Coca-Cola Company (KO), AT&T (T), Exxon Mobil Corporation (XOM), and Pfizer (PFE) representing the sectors in the same order as named earlier in this paragraph.

For the computation of volatility, we restrict the analysis to daily logarithmic realized volatility computed using 5-minute returns⁸ during the 9:30 a.m. to 4:00 p.m. business hours of the New York Stock Exchange (NYSE). The data are time-synchronized by the same time-stamps, eliminating transactions executed on Saturdays and Sundays, U.S. federal holidays, December 24 to 26, and December 31 to January 2, because of the low activity on these days, which could lead to estimation bias. The data span years 2005 to 2015 providing sample of 2660 trading days. The descriptive statistics of the data can be found in the appendix Table 7. The period under study is informative in terms of market development, sentiment, and expectations since we cover the 2007–2008 financial crisis and its aftermath years. The data were obtained from the TICK Data.⁹

4.2.2 Time-frequency decomposition of US market risk connectedness

One of the issues that has recently gained importance in volatility modelling is giving up the assumption of global stationarity of the data (Stărică and Granger, 2005; Engle and Rangel, 2005) and focusing on local stationarity instead. When studying connectedness of market risk using variance decompositions, it is important to face nonstationarity of realized volatilities as zero frequency may dominate the rest of the frequencies in case we study unconditional connectedness. The discussion gains importance when studying frequency dynamics, as applying our measures blindly to the nonstationary data would result to false inferences.

an approximate extent that varies over time.

⁸Realized volatility for a given day is computed as sum of squared intra-day returns.

⁹<https://www.tickdata.com/>

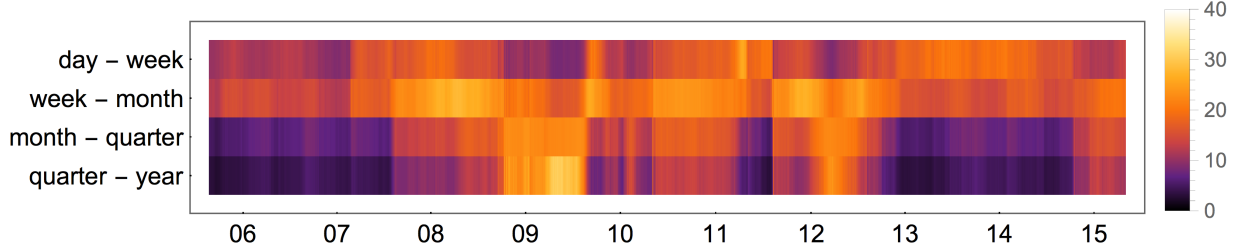


Figure 2: Time-frequency dynamics of connectedness of the US market risk. Frequency connectedness $\mathcal{C}_{d_s}^{\mathcal{F}}$ for $d_1 \in [1, 5]$, $d_2 \in (5, 20]$, $d_3 \in (20, 60]$, and $d_4 \in (60, 250]$ days representing day to week, week to month, month to quarter, and quarter to year connectedness are depicted on vertical axis, while horizontal axis shows time.

Giving up the assumption of global stationarity of the data, we assume the dynamics come from shifts in the unconditional variance of returns. This leads us to convenient approximation of nonstationary data locally by stationary models. In essence, our approach is closely related to the one taken by Stărică and Granger (2005), although we study multivariate system with quite different tools. We use the spectral representations of variance decompositions to recover the time-frequency dynamics of connectedness with moving window of approximately one year (250 trading days), where we confirm stationarity of volatility. Vector auto-regression with two lags is used to capture the dynamics in the window.¹⁰

Focusing on locally stationary structure of the data, we do not report unconditional frequency connectedness table as commonly done in the literature. Instead, we study the time-frequency dynamics of connectedness. Figure 1 reports the time dynamics of the total connectedness of system as measured by time domain variance decompositions in the left part. One can quickly infer that the connectedness was rather low during first two year period increasing dramatically during the 2007–2008 crisis, and varying in the aftermath of the crisis considerably. Right plot of the Figure 1 presents the decomposition of the total connectedness into frequency bands up to one week, one week to one month, one month to one quarter, and one quarter to one year computed as $\mathcal{C}_{d_s}^{\mathcal{F}}$ on the bands corresponding to $d_1 \in [1, 5]$, $d_2 \in (5, 20]$, $d_3 \in (20, 60]$, and $d_4 \in (60, 250]$ days. Note the lowest frequency is bounded at each time point by the window length.

The decomposition shows rich time-frequency dynamics of connections. Focusing on the frequency dynamics, largest portion of connections is created from one week up to one month, although higher frequencies up to one week play similar role in connectedness. The most interesting observations can be made when considering time dynamics of the frequency connections, as we cannot see any clear pattern of some frequency band dominating all others. Instead, we infer rich time-frequency dynamics. While connectedness has been driven mostly by information up to one month (d_1 and d_2) during the first three

¹⁰We have experimented with different lag lengths with no changes in results. This only confirms the appropriateness of the approach, as large changes in time-frequency dynamics due to different lags in the approximating VAR model would point to nonstationarities within windows, where larger number of lags would be approximating the information in the low frequencies. In some sense, this analysis serves as a robustness check. We make these results available upon request.

years of the sample, the structure changed dramatically during the year 2008, and this change lasted until the end of the year 2012. During this period, we observe rich dynamics with lower frequencies playing role in connectedness. Before this dynamics returns to the connectedness structure again during the year 2015, there is two year change with higher frequencies driving the connectedness.

The Figure 2 shows the time-frequency dynamics from a different point of view, which serves as a helpful complementary visualization. In this figure, frequency bands are depicted instead of the value of connectedness by the vertical axis, while horizontal axis holds time. One can view this representation as looking at the three dimensional space of connectedness at time and frequency domains from top, where the third axis showing the strength of connectedness at each time-frequency point is highlighted by color. The heat map representation is useful as one can more clearly see the decomposition of the connectedness into time-frequency space.

Economically, periods with connectedness being created in high frequencies are periods when stock markets seem to process information rapidly, and a shock to one asset in the system will have impact mainly in a short-term (with frequency response at high frequencies mainly). In case the connections come from the opposite part of the cross-spectral density, lower frequencies, it points us to the belief that shocks are being transmitted for longer periods (with frequency response at low frequencies mainly). This behavior may be attributed to fundamental changes in investor's expectations, which impact the market in longer term. These expectations are then transmitted to surrounding assets in the portfolio. Before making further conclusions about the nature of connectedness in US stock market, we look deeper into its sources.

Until now, we have focused on the decomposition of the connections to frequency bands, guaranteeing they will always sum to total connectedness. The frequency components are in fact within spillovers, or causation spectra at frequency band weighted by the variance share at the given band. Hence in case low frequencies hold large amount of information, it will overweight other frequencies. While the frequency decomposition considering power of shocks is important for relative comparison, it is useful to look at unweighted connections as well. Ignoring information outside the considered band, connections within frequency bands can be understood as pure unweighted connections. The Figure 3 shows the within sectoral connectedness of the market in left plot. All frequencies share very similar time dynamics, hence the rich time-frequency decomposition found in previous part is mainly driven by power of frequency responses, as expected.

The main reason why we look at the pure within connectedness is to study the effect of cross-sectional dependence on the connectedness. When using variance decompositions, we are mainly interested in finding causal effects, but these can be biased due to strong contemporaneous relations. To find if there is such a bias in the connections we measure, we adjust the correlation matrix of VAR residuals by the cross-sectional correlations.

The right plot from the Figure 3 shows within connectedness adjusted for this correlation effect. Strikingly, the structure changes dramatically, pointing us to the result that the high frequency connectedness is mainly driven by cross-sectional correlations, while connectedness at lower frequencies is not affected so heavily, mainly during the crisis. One can infer that increase of system connectedness during the crisis is mainly created by increase

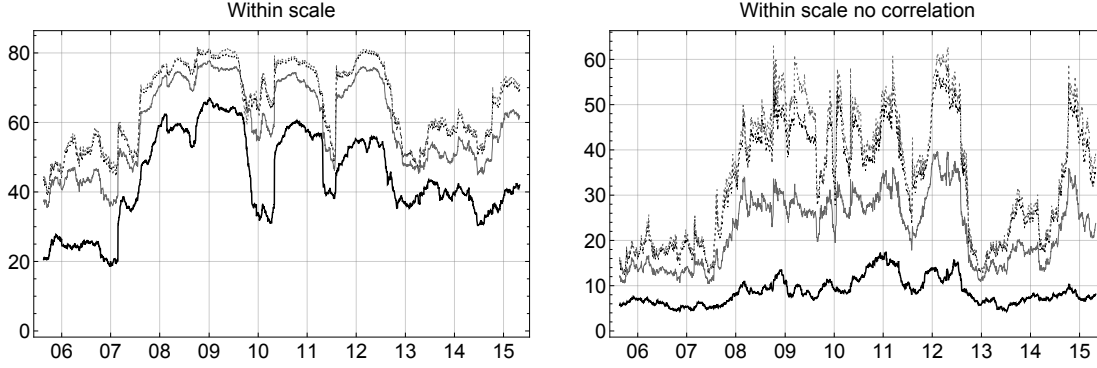


Figure 3: Dynamic within connectedness of the US market risk on frequency bands. Left plot presents the relative connectedness within the frequency band, $C_{d_s}^W$ with $d_1 \in [1, 5]$ days in solid bold, $d_2 \in (5, 20]$ solid, $d_3 \in (20, 60]$ dotted bold, and $d_4 \in (60, 250]$ dotted lines. Right plot presents relative connectedness within the frequency band without the effect of cross-sectional correlations.

in contemporaneous short-term correlations, and causal longer term connectedness.

While studying connectedness of the whole system, the time-frequency dynamics of directional connectedness including pairwise connections, influences “from”, and “to” the considered stocks may be of interest (i.e., all elements of Table 3). Inevitably, reporting all these interesting quantities would substantially inflate this already pregnant text. The main purpose of our work is to introduce the quantities and their proper usage in different situations, hence we leave their full usage to all applications which are sure to come in near future.¹¹

4.3 Macroeconomic connectedness at business cycle frequencies

While spectral representations assuming stationarity serve well the task of measuring connectedness in stock markets, macroeconomic data demand more careful treatment due to common stochastic trends they (may) share. To measure frequency dynamics of connections between macroeconomic variables, one needs to consider the second part of the theoretical discussions provided in earlier sections of this text.

Measurement of the business cycles and its connectedness has been an ongoing research for many decades. An important strand of literature is interested in measuring the convergence of the business cycles, which can be thought of as parallel to measuring its connectedness. For rather exhaustive literature survey of recent results, see De Haan et al. (2008). Recently, the macroeconomic connectedness has been addressed by means of variance decompositions (Diebold et al., 2015; Greenwood-Nimmo et al., 2015), although if we want to see how economies are connected at various business cycles, the spectral quantities devised in earlier sections need to be considered as connections at business cycles may be

¹¹The time-frequency quantities from directional connectedness table are easily computable using the package we provide to the paper.

dominated by common stochastic trends.¹² In the next sections, we contribute to this literature by measuring connectedness at business cycles, which are often hidden by dominant common shocks in the system.

4.3.1 Data about industrial production

For analyzing the connectedness of global business cycles, we use the monthly index of industrial production for the G-7 economies: the U.S., Canada, Japan, Germany, Italy, France, and United Kingdom. The data span years 1978 through 2014, and was acquired through the Federal Research Economic Data originating in the Organization for Economic Co-operation and Development (OECD) dataset. The dataset is plotted in the Figure 4.

Figure 4 suggests that time series of industrial production may contain deterministic and stochastic trends and may be (co)integrated. Augmented Dickey-Fuller test allowing for linear trend does not reject the presence of unit root in logarithms of industrial production levels. Maximum eigenvalue and trace test of Johansen assuming linear trends in log levels of industrial production reject the hypothesis of at most one cointegrating relationship in the data, but larger number of cointegrating relationships is never decisively rejected.¹³

4.3.2 Connectedness of global business cycles

As discussed in the theoretical part of this work, cointegrating relationship dominates the connectedness at all other frequencies. Although industrial production data seem to share no more than one common stochastic trend, we consider vector error correction model to estimate the connectedness at business cycle frequencies. Furthermore, we will estimate the spectral counterparts of connectedness measures excluding the zero frequency, which holds the information about common permanent shocks.

As discussed earlier, the intuition behind excluding the zero frequency when measuring connectedness is that common trends will dominate possible connections at all other frequencies. The literature studying business cycles commonly defined to be no less than six quarters and typically last fewer than 32 quarters, considers the use of filtered data. While studying connectedness at business cycles on filtered data is simple alternative, the use of filters commonly applied in the literature may be problematic as it may induce spurious connectedness. Simple differencing will put heavy weight on high frequency components, and alter timing relationship. More sophisticated filters, like low-pass Hodrick-Prescott filter, or band-pass filters, are more appropriate but may introduce spurious correlations between the series due to cyclical properties of filters (Cogley and Nason, 1995; Murray, 2003; Harvey and Trimbur, 2003).

While the prime concern in the literature using filters is to approximate the ideal filter, our methodology sidesteps this problem, and in some sense can be viewed as equivalent to estimating connectedness at ideally filtered data. To find connections of industrial production at business cycles, exclusion of zero frequency from the spectral representation of variance decompositions serves the purpose.

¹²Common shocks driving business cycles are confirmed by large literature, e.g. Stock and Watson (2005); Canova et al. (2007); Kose et al. (2008)

¹³The results from unit root and cointegration testing are available upon request from authors.

	USA	JPN	GER	ITA	FRA	UK	CAN	From
USA	46.8	4.9	2.1	0.7	6.8	5.1	33.6	53.2
	21.0-8.5-6.0-2.9-2.4	3.4-2.3-1.5-0.8-0.7	3.7-2.5-1.5-0.6-0.3	1.8-0.8-0.4-0.2-0.1	5.7-0.6-0.9-0.9-1.4	1.6-1.1-1.8-1.8-2.7	11.0-3.2-2.9-1.7-1.6	27.1-10.5-8.9-5.9-6.8
	14.7-7.0-5.5-3.4-3.7	3.0-2.8-1.9-0.8-0.6	5.4-3.5-2.3-1.0-0.7	3.9-1.3-0.7-0.3-0.2	6.7-2.0-2.3-2.1-2.9	0.4-1.0-2.2-2.4-3.7	2.4-1.0-2.1-2.4-3.5	21.8-11.7-11.5-9.0-11.7
JPN	9.6	65.8	8.4	1.7	5.3	6.1	3.2	34.2
	3.3-0.2-0.4-0.4-0.7	8.4-6.6-6.6-5.5-7.5	3.2-1.7-1.5-1.2-1.6	2.3-0.7-0.4-0.2-0.2	1.6-1.4-1.3-1.0-1.3	3.0-1.7-3.2-3.2-4.5	3.0-3.3-5.7-5.5-7.8	16.3-8.9-12.4-11.6-16.2
	2.3-0.3-0.1-0.0-0.0	7.9-5.8-6.4-5.5-7.6	1.8-0.7-0.7-0.7-1.0	2.2-0.5-0.3-0.2-0.3	0.9-1.1-1.3-1.1-1.5	4.0-3.2-5.1-4.8-6.7	2.1-4.0-6.2-5.8-8.0	13.3-9.8-13.7-12.5-17.4
GER	5.9	13.9	46.7	7.0	16.6	7.9	2.1	53.3
	12.2-3.5-2.7-1.9-2.4	5.7-4.9-4.6-3.6-4.6	4.4-1.3-1.3-1.0-1.2	2.3-0.4-0.2-0.1-0.1	4.9-0.9-0.4-0.1-0.1	5.8-2.1-2.8-2.6-3.6	4.1-1.9-3.7-3.6-5.1	34.9-13.6-14.4-11.9-15.9
	10.0-3.3-1.8-0.9-0.9	6.1-5.6-5.6-4.4-5.8	8.3-2.0-1.7-1.3-1.6	3.2-0.9-0.6-0.4-0.4	4.6-1.2-0.5-0.2-0.1	3.8-2.4-3.5-3.2-4.5	0.8-1.5-2.7-2.6-3.5	28.4-15.0-14.7-11.7-15.3
ITA	11.8	9.6	14.3	36.2	14.0	9.6	4.5	63.8
	4.3-1.2-1.2-1.1-1.5	4.2-4.8-5.2-4.3-5.7	1.3-1.1-1.1-0.9-1.2	2.2-0.5-0.3-0.2-0.2	1.0-0.6-0.6-0.5-0.6	7.3-3.4-4.5-4.1-5.6	4.6-4.2-6.5-6.0-8.3	22.6-15.3-19.2-16.8-22.8
	2.3-0.5-0.2-0.1-0.1	4.8-5.1-5.8-4.9-6.5	1.3-0.9-0.9-0.8-1.1	4.0-0.8-0.5-0.4-0.5	0.6-0.6-0.7-0.6-0.8	7.3-4.5-6.0-5.3-7.2	2.4-3.8-5.9-5.3-7.3	18.6-15.5-19.6-17.0-23.0
FRA	10.2	9.1	20.9	6.9	34.2	15.1	3.6	65.8
	6.3-2.0-1.8-1.4-1.9	5.1-5.4-5.7-4.6-6.0	1.3-1.1-1.1-0.9-1.2	0.6-0.1-0.1-0.1-0.1	1.6-0.4-0.5-0.4-0.5	7.0-3.3-4.3-3.8-5.2	3.8-3.7-5.8-5.4-7.4	24.1-15.6-18.9-16.2-21.8
	4.2-1.2-0.6-0.3-0.3	6.5-6.1-6.6-5.4-7.1	1.9-1.1-1.1-0.9-1.2	0.9-0.4-0.3-0.2-0.3	2.5-0.4-0.6-0.5-0.7	6.7-4.3-5.7-5.0-6.8	1.5-3.1-5.0-4.5-6.2	21.8-16.1-19.2-16.3-21.9
UK	7.9	3.8	3.3	5.0	5.6	69.6	4.8	30.4
	14.5-4.7-3.5-2.0-1.9	6.1-4.5-2.8-1.2-0.8	3.8-2.0-1.3-0.7-0.5	2.9-0.1-0.0-0.0-0.0	3.1-0.1-0.1-0.1-0.2	13.2-3.9-2.4-1.0-0.6	12.7-4.1-3.1-1.4-0.8	43.2-15.6-10.9-5.2-4.2
	10.0-3.3-2.5-1.6-1.8	10.1-6.9-4.4-1.9-1.3	7.9-4.0-2.8-1.5-1.4	3.9-0.5-0.2-0.1-0.1	2.4-0.2-0.2-0.3-0.4	12.1-4.5-2.7-1.0-0.5	5.4-1.8-1.4-0.6-0.2	39.8-16.7-11.6-5.9-5.2
CAN	19.3	5.8	1.0	0.2	2.8	3.5	67.3	32.7
	21.6-9.1-6.5-3.2-2.6	3.3-1.6-1.2-0.7-0.8	2.7-2.0-1.2-0.5-0.2	1.1-0.3-0.2-0.1-0.1	3.4-0.5-0.8-0.8-1.3	3.5-1.6-1.6-1.3-1.6	13.1-5.0-3.7-1.7-1.2	35.6-15.2-11.5-6.5-6.5
	15.7-7.5-6.0-3.8-4.1	3.1-2.9-1.9-0.9-0.7	5.3-3.9-2.6-1.2-0.8	3.3-1.1-0.6-0.3-0.2	3.7-1.6-2.0-1.9-2.8	1.3-1.4-2.1-2.1-3.0	3.0-1.5-2.4-2.2-3.1	32.5-18.4-15.4-10.1-11.5
To	64.8	47.0	49.9	21.6	51.0	47.3	51.7	47.6
	62.2-20.7-16.1-10.0-11.0	27.8-23.5-21.0-15.1-18.6	16.0-10.4-7.8-4.7-5.1	10.9-2.3-1.2-0.6-0.6	19.7-4.0-3.9-3.5-4.8	28.2-13.1-18.3-16.7-23.2	39.1-20.4-27.9-23.5-31.0	29.1-13.5-13.7-10.6-13.5
	44.6-16.0-11.2-6.7-7.3	33.6-29.5-26.3-18.4-22.0	23.6-14.0-10.5-6.0-6.1	17.5-4.6-2.7-1.5-1.5	18.8-6.8-7.1-6.2-8.5	23.6-16.8-24.5-22.8-31.8	14.6-15.3-23.4-21.1-28.9	25.2-14.7-15.1-11.8-15.1

Table 4: Global business cycles connectedness. Each cell of the table shows three rows. The first row holds total connectedness computed using Diebold and Yilmaz (2012) measure. The second row holds frequency connectedness $\mathcal{C}_{d_s}^{\mathcal{F}}$ with $d_1 \in [1, 18]$, $d_2 \in (18, 36]$, $d_3 \in (36, 72]$, $d_4 \in (72, 96]$ months and d_5 holding the rest excluding the zero frequency. The third row correspond to connectedness excluding the cross-sectional correlations.

Table 4 prints the total connectedness of industrial production using original Diebold and Yilmaz (2012) measure in first row of each element as well as our frequency decompositions measuring connectedness at business cycles at remaining rows. We estimate vector error correction model (VECM) with constant and deterministic trends to compute variance decompositions. Frequency bands are chosen to mimic the usual definition of business cycles at 18, 36, 72, and 96 months.

We begin the discussion with overall connectedness, measured using forecast horizon of 9 months. The choice of forecast horizon is important in this framework, as with longer horizons we will be closer to effects from common shocks due to cointegration of the system. In a similar analysis, Greenwood-Nimmo et al. (2015) note that with increasing horizon directional connectedness intensifies, and rapidly converges to their long-run values. While authors attribute this to transmission of shocks occurring gradually, we add that this behavior is due to long frequency response of the shocks. Increasing horizon will then only approximate better the permanent effects of shocks, moreover the nature of the shocks will not allow to isolate the connectedness at business cycle frequencies, as discussed earlier. Here, using frequency responses of shocks and our spectral representation of measures will be useful.

Before discussing frequency dependent connectedness, we briefly summarize the general connectedness of the system. As expected, global connections are large, reaching value of 47.6. The US proves its largest influence on the rest of countries with directional connectedness of 64.8 for transmitting shocks, but is also quite open to receiving shocks (53.2). The shocks from the US have largest impact on Canada, although it is worth noting that all other countries are influenced largely by the US as well. The rest of the connections we document share similar patterns of connectedness with those confirmed by the literature. Most closely, Diebold et al. (2015) report connectedness of G-6 countries with slight differences, which are attributable to different sample length, and exclusion of Canada from the sample.

A contributive part of the Table 4 is in the second and third row of each element, showing business cycles connectedness. Note that exclusion of the zero frequency from the causation spectra does not allow the decomposition to sum to original measures. We stress, that variance decompositions with forecast horizon, and spectral representations at business cycles frequencies are not directly comparable in relative terms, as they were in the previous analysis on stock market data. Hence it may well happen that connectedness across business cycles will sum to higher number than original connectedness, as it is weighted by frequencies excluding zero. On the other side, frequency connectedness is indeed comparable across different frequency bands approximating business cycles directly.

Table 4 shows sizeable connectedness at business cycle frequencies. Looking at how the system was connected at business cycles, we find that largest connectedness comes from frequencies higher than business cycles (i.e. 1 – 18 months) with value of 29.1, although the remaining business cycles are connected with values of 13.5, 13.7, 10.6, and 13.5 respectively. To summarize, response of shocks is distributed evenly over frequencies, and connectedness at the business cycles frequencies seems to dominate the rest.

Looking at individual countries, we can see that the US contributes to all other countries mostly at shorter than 18 month horizons (with the value of 62.2) although contributions

from business cycles frequencies are not negligible, with 18 – 36 month interval being the most important reading connectedness of 20.7. Interestingly, results are not influenced by cross-sectional correlations (shown by third row of each element in Table 4). On the other hand, Canada seems to contribute by shocks with different frequency response equally. Here, pairwise connection between the US and Canada is notable, as the US reception of shocks from Canada is mostly influenced by contemporaneous correlations. While the US transmits shocks at all business cycle frequencies to Canada, influence from Canada to the US diminishes after controlling for cross-sectional correlations.

The UK and Japan seem to influence all other countries at all business cycles similarly. This means that shocks from these countries with frequency responses at all business cycles exist and are influential. On the other side, both these countries receive less shocks at all business cycles frequencies.

Italy and France transmit shocks with different strength, but the shocks to their industrial production influence other countries only at very short-term, less than business cycles frequencies. On the opposite, both countries receive shocks at all business cycles equally with large strength.

Having discussed the connectedness of G-7 countries at different business cycles, we provided a second important application of the methodology proposed by this paper, which guides users in the situation when measuring frequency dependent connectedness in a cointegrated system. We believe that the application is not autotelic though, as the discussion contributes to the literature measuring business cycles connectedness.

5 Conclusion

In this work, we contribute to the understanding of connectedness between economic variables by proposing to measure its frequency dynamics. Based on the spectral representations of variance decompositions and connectedness measures, we provide a general framework for disentangling the sources of connectedness between economic variables. As shocks to economic activity impact variables at different frequencies with different strength, we view the frequency domain as a natural place for measuring the connectedness between economic variables.

As noted by Diebold and Yilmaz (2009, 2012), and later Diebold and Yilmaz (2014), variance decompositions from approximating models are convenient framework for empirical measurement of connectedness. Diebold and Yilmaz define the measures based on assessing shares of forecast error variation in one variable due to shock arising in another variable in the system. Focusing on frequency responses of shocks instead, we are interested in the portion of the spectrum as counterpart of variance at given frequency that is attributed to shocks in another variable. Moreover, we elaborate on the role of correlation of the residuals in the magnitude and spectral shape of the connectedness.

In the empirical part, we investigate the prolific examples of the connectedness, namely connectedness of stock market risk in the US market and connectedness of the largest world economies at different business cycles. Our results underline the importance of proper measurement of dynamics across time and frequencies and emphasize the important role

of cross-sectional correlation in the connectedness origins.

Monitoring connectedness of stock market risk, we approximate the data locally and obtain rich time-frequency dynamics of connectedness. We conclude that the dynamics is mainly driven by frequencies from one day up to one month, although this does not hold in the period of turmoil with high levels of uncertainty. In these periods the total connectedness increases, and the increase is due to contemporaneous correlations at short-term, and causal longer term connectedness. Economically, periods with connectedness being created in high frequencies are periods when stock markets seem to process information rapidly, and a shock to one asset in the system will have impact mainly in a short-term. In case the connections come from the lower frequencies, it points us to the belief that shocks are being transmitted for longer periods. This behavior may be attributed to fundamental changes in investor's expectations, which impact the market in longer term. These expectations are then transmitted to surrounding assets in the portfolio.

Turning our attention to the macroeconomic application, our measures demonstrate importance of proper assessment of common stochastic trends in data dominating the possible connections at business cycles. Careful treatment of the proposed methods allow us to find the rich frequency dynamics in connections of industrial productions at real business cycles. Application of the methodology to industrial production of G-7 countries reveals different connections at business cycles. This implies that shocks transmitted from countries have different frequency response, creating different connectedness at business cycle frequencies.

Our results open new fascinating routes in understanding connectedness of economic variables with important implications. The further research applying our measures to wide areas of macroeconomics and finance will be important in uncovering the connections of assets within market, or industry, connections across asset classes, international markets, and provide grounds for further research in risk management, portfolio allocation, or business cycle analysis where understanding origins of connections is essential.

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A Derivation of the GFEVD

Let us have the $MA(\infty)$ representation of the GVAR model (details in (Pesaran and Shin, 1998; Dees et al., 2007)) given as

$$\mathbf{x}_t = \Psi(L)\boldsymbol{\epsilon}_t, \quad (8)$$

with the covariance matrix of the errors Σ . Because the errors are assumed to be serially uncorrelated, the total covariance matrix of the forecast error conditional at the information in $t - 1$ is

$$\Omega_H = \sum_{h=0}^H \Psi_h \Sigma \Psi_h'. \quad (9)$$

Next we define the covariance matrix of the forecast error conditional on knowledge of today's shock and future expected shocks to j -th equation. Starting from the conditional forecasting error,

$$\gamma_t^k(H) = \sum_{h=0}^H \Psi_h [\boldsymbol{\epsilon}_{t+H-h} - E(\boldsymbol{\epsilon}_{t+H-h} | \boldsymbol{\epsilon}_{k,t+H-h})], \quad (10)$$

assuming normal distribution, we have

$$\gamma_t^k(H) = \sum_{h=0}^H \Psi_h [\boldsymbol{\epsilon}_{t+H-h} - \sigma_{kk}^{-1}(\Sigma)_{\cdot k} \boldsymbol{\epsilon}_{k,t+H-h}]. \quad (11)$$

Finally the covariance matrix is

$$\Omega_H^k = \sum_{h=0}^H \Psi_h \Sigma \Psi_h' - \sigma_{kk}^{-1} \sum_{h=0}^H \Psi_h (\Sigma)_{\cdot k} (\Sigma)'_{\cdot k} \Psi_h'. \quad (12)$$

Then

$$\Delta_{(j)kH} = (\Omega_H - \Omega_H^k)_{j,j} = \sigma_{kk}^{-1} \sum_{h=0}^H ((\Psi_h \Sigma)_{j,k})^2 \quad (13)$$

is the unscaled H -step ahead forecast error variance of k -th component with respect to j -th component. Scaling the equation yields the desired

$$(\theta_H)_{j,k} = \frac{\sigma_{kk}^{-1} \sum_{h=0}^H ((\Psi_h \Sigma)_{j,k})^2}{\sum_{h=0}^H (\Psi_h \Sigma \Psi_h')_{j,j}} \quad (14)$$

B Proofs

Proposition 2.1. To prove the equality we need the following:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_j(\omega) (\mathbf{f}(\omega))_{j,k} d\omega &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\Psi(e^{-i\omega}) \Sigma \Psi'(e^{+i\omega}))_{j,j}}{\frac{1}{2\pi} \int_{-\pi}^{\pi} (\Psi(e^{-i\lambda}) \Sigma \Psi'(e^{+i\lambda}))_{j,j} d\lambda} \frac{\sigma_{kk}^{-1} |(\Psi(e^{-i\omega}) \Sigma)_{j,k}|^2}{(\Psi(e^{-i\omega}) \Sigma \Psi'(e^{+i\omega}))_{j,j}} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma_{kk}^{-1} |(\Psi(e^{-i\omega}) \Sigma)_{j,k}|^2}{\frac{1}{2\pi} \int_{-\pi}^{\pi} (\Psi(e^{-i\lambda}) \Sigma \Psi'(e^{+i\lambda}))_{j,j} d\lambda} d\omega \\ &= \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma_{kk}^{-1} |(\Psi(e^{-i\omega}) \Sigma)_{j,k}|^2 d\omega}{\frac{1}{2\pi} \int_{-\pi}^{\pi} (\Psi(e^{-i\lambda}) \Sigma \Psi'(e^{+i\lambda}))_{j,j} d\lambda} \\ &= \frac{\sigma_{kk}^{-1} \sum_{h=0}^{\infty} ((\Psi_h \Sigma)_{j,k})^2}{(\sum_{h=0}^{\infty} (\Psi_h \Sigma \Psi_h'))_{k,k}} \\ &= (\theta_{\infty})_{j,k} \end{aligned} \quad (15)$$

Hence, the proof essentially simplifies to proving two things

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma_{kk}^{-1} |(\Psi(e^{-i\omega}) \Sigma)_{j,k}|^2 d\omega = \sigma_{kk}^{-1} \sum_{h=0}^{\infty} ((\Psi_h \Sigma)_{j,k})^2 \quad (16)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (\Psi(e^{-i\lambda}) \Sigma \Psi'(e^{+i\lambda}))_{j,j} d\lambda = \left(\sum_{h=0}^{\infty} (\Psi_h \Sigma \Psi_h') \right)_{k,k} \quad (17)$$

For the following steps we will leverage the standard integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega(u-v)} d\omega = \begin{cases} 1 & \text{for } u = v \\ 0 & \text{for } u \neq v. \end{cases} \quad (18)$$

This integral is mostly useful in cases when we have series $\sum_{h=0}^{\infty} \phi_h \psi_h$ and we want to arrive to spectral representation. Note that $\sum_{h=0}^{\infty} \phi_h \psi_h = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{v=0}^{\infty} \sum_{u=0}^{\infty} \phi_u \psi_v e^{i\omega(u-v)} d\omega$. Leveraging this knowledge we prove the Equation 16

$$\begin{aligned} \sigma_{kk}^{-1} \sum_{h=0}^{\infty} \left((\Psi_h \Sigma)_{j,k} \right)^2 &= \sigma_{kk}^{-1} \sum_{h=0}^{\infty} \left(\sum_{z=1}^n (\Psi_h)_{j,z} (\Sigma)_{z,k} \right)^2 \\ &= \sigma_{kk}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \left(\sum_{x=1}^n (\Psi_u)_{j,x} (\Sigma)_{x,k} \right) \left(\sum_{y=1}^n (\Psi_v)_{j,y} (\Sigma)_{y,k} \right) e^{i\omega(u-v)} d\omega \\ &= \sigma_{kk}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \left(\sum_{x=1}^n (\Psi_u e^{i\omega u})_{j,x} (\Sigma)_{x,k} \right) \left(\sum_{y=1}^n (\Psi_v e^{-i\omega v})_{j,y} (\Sigma)_{y,k} \right) d\omega \\ &= \sigma_{kk}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{u=0}^{\infty} \sum_{x=1}^n (\Psi_u e^{i\omega u})_{j,x} (\Sigma)_{x,k} \right) \left(\sum_{v=0}^{\infty} \sum_{y=1}^n (\Psi_v e^{-i\omega v})_{j,y} (\Sigma)_{y,k} \right) d\omega \\ &= \sigma_{kk}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{x=1}^n (\Psi(e^{i\omega}))_{j,x} (\Sigma)_{x,k} \right) \left(\sum_{y=1}^n (\Psi(e^{-i\omega}))_{j,y} (\Sigma)_{y,k} \right) d\omega \\ &= \sigma_{kk}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left((\Psi(e^{-i\omega}) \Sigma)_{j,k} \right) \left((\Psi(e^{i\omega}) \Sigma)_{j,k} \right) d\omega \\ &= \sigma_{kk}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| (\Psi(e^{-i\omega}) \Sigma)_{j,k} \right|^2 d\omega \end{aligned} \quad (19)$$

We use the switch to the spectral representation of the MA coefficients in the second step. The rest is manipulation with the last step invoking the definition of modulus squared of a complex number to be defined as $|z|^2 = zz^*$. Note that we can use this simplification without loss of generality, because the $MA(\infty)$ representation that is described by the coefficients Ψ_h has always symmetric spectrum.

Next we concentrate on the Equation 17 leveraging similar steps and the positive semidefiniteness of the matrix Σ that ascertains that there exists \mathbf{P} such that $\Sigma = \mathbf{P}\mathbf{P}'$.

$$\begin{aligned}
\sum_{h=0}^{\infty} (\Psi_h \Sigma \Psi_h') &= \sum_{h=0}^{\infty} (\Psi_h P) (\Psi_h P)' \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} (\Psi_u e^{i\omega u} P) (\Psi_v e^{-i\omega v} P)' d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{u=0}^{\infty} (\Psi_u e^{i\omega u} P) \sum_{v=0}^{\infty} (\Psi_v e^{-i\omega v} P)' d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Psi (e^{i\omega}) P) (\Psi (e^{-i\omega}) P)' d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Psi (e^{i\omega}) \Sigma \Psi' (e^{-i\omega})) d\omega
\end{aligned} \tag{20}$$

This completes the proof. \square

Remark 2.2. Using the Remark 2.1 and appropriate substitutions, we have:

$$\begin{aligned}
\sum_{d_z \in D} \mathcal{C}_{d_z}^{\mathcal{F}} &= \sum_{d_z \in D} \left(\frac{\sum (\tilde{\theta}_{d_z})_{j,k}}{\sum (\tilde{\theta}_{\infty})_{j,k}} - \frac{\text{Tr} \{ \tilde{\theta}_{d_z} \}}{\sum (\tilde{\theta}_{\infty})_{j,k}} \right) = 1 - \frac{\sum_{d_z \in D} \text{Tr} \{ \tilde{\theta}_{d_z} \}}{\sum (\tilde{\theta}_{\infty})_{j,k}} = \\
&= 1 - \frac{\text{Tr} \{ \sum_{d_z \in D} \tilde{\theta}_{d_z} \}}{\sum (\tilde{\theta}_{\infty})_{j,k}} = \mathcal{C}_{\infty}
\end{aligned} \tag{21}$$

where the next to last equality follows from the linearity of the trace operator. \square

Remark 2.3. Using the definition of the connectedness, we have

$$\mathcal{C}_{(-\pi, \pi)}^{\mathcal{W}} = \mathcal{C}_{\infty} \tag{22}$$

$$\mathcal{C}_{(-\pi, \pi)}^{\mathcal{F}} = \frac{(\tilde{\theta}_{(-\pi, \pi)})_{j,k}}{n} - \frac{\text{Tr} \{ \tilde{\theta}_{\infty} \}}{\sum (\tilde{\theta}_{\infty})_{j,k}} = \frac{n}{n} - \frac{\text{Tr} \{ \tilde{\theta}_{\infty} \}}{\sum (\tilde{\theta}_{\infty})_{j,k}} = 1 - \frac{\text{Tr} \{ \tilde{\theta}_{\infty} \}}{\sum (\tilde{\theta}_{\infty})_{j,k}} = \mathcal{C}_{\infty} \tag{23}$$

\square

Proposition 2.2. Let us take the original statement

$$\lim_{z \rightarrow 0^+} \int_z^{z+\epsilon} \Gamma_j(\omega) (\mathbf{f}(\omega))_{j,k} d\omega + \int_{z+\epsilon}^{\pi} \Gamma_j(\omega) (\mathbf{f}(\omega))_{j,k} d\omega$$

and evaluate the individual summands.

Regarding the second summand, let us first note that in case of the cointegrated processes the unconditional variance is infinite. Hence, the $\Gamma_j(\omega) = 0$ for any $\omega \in (\epsilon, \pi), 0 <$

$\epsilon < \pi$. Moreover, the causation spectrum does not exist for the integrated series, however, we can use the spectrum of the ideally filtered series which exists and is equal to the original spectrum on the interval $(0, \pi]$.

Hence, we have

$$\int_{z+\epsilon}^{\pi} \Gamma_j(\omega) (\mathbf{f}(\omega))_{j,k} d\omega = \int_{z+\epsilon}^{\pi} \Gamma_j(\omega) (\tilde{\mathbf{f}}(\omega))_{j,k} d\omega = \int_{z+\epsilon}^{\pi} 0 (\tilde{\mathbf{f}}(\omega))_{j,k} d\omega = 0$$

due to the fact that the causation spectrum $\tilde{\mathbf{f}}(\omega)$ is finite on the domain $(z + \epsilon, \pi)$.

For the part including limit let us suppose the ideal filter such that the spectrum of the filtered series in 0 is well-defined rewrite

$$\begin{aligned} \lim_{z \rightarrow 0^+} \int_z^{z+\epsilon} \Gamma_j(\omega) (\tilde{\mathbf{f}}(\omega))_{j,k} d\omega &= \lim_{z \rightarrow 0^+} \int_z^{z+\epsilon} \frac{(\Psi(e^{-i\omega}) \Sigma \Psi'(e^{+i\omega}))_{j,j}}{\frac{1}{\pi} \int_z^{\pi} (\Psi(e^{-i\lambda}) \Sigma \Psi'(e^{+i\lambda}))_{j,j} d\lambda} \frac{\sigma_{kk}^{-1} |(\Psi(e^{-i\omega}) \Sigma)_{j,k}|^2}{(\Psi(e^{-i\omega}) \Sigma \Psi'(e^{+i\omega}))_{j,j}} d\omega \\ &= \lim_{z \rightarrow 0^+} \int_z^{z+\epsilon} \frac{\sigma_{kk}^{-1} |(\Psi(e^{-i\omega}) \Sigma)_{j,k}|^2}{\frac{1}{\pi} \int_z^{\pi} (\Psi(e^{-i\lambda}) \Sigma \Psi'(e^{+i\lambda}))_{j,j} d\lambda} d\omega \\ &= \pi \lim_{z \rightarrow 0^+} \frac{\int_z^{z+\epsilon} \sigma_{kk}^{-1} |(\Psi(e^{-i\omega}) \Sigma)_{j,k}|^2 d\omega}{\int_z^{\pi} (\Psi(e^{-i\lambda}) \Sigma \Psi'(e^{+i\lambda}))_{j,j} d\lambda} \\ &= \pi \lim_{z \rightarrow 0^+} \frac{\frac{d}{dz} \int_z^{z+\epsilon} \sigma_{kk}^{-1} |(\Psi(e^{-i\omega}) \Sigma)_{j,k}|^2 d\omega}{\frac{d}{dz} \int_z^{\pi} (\Psi(e^{-i\lambda}) \Sigma \Psi'(e^{+i\lambda}))_{j,j} d\lambda} \\ &= \pi \lim_{z \rightarrow 0^+} \frac{\sigma_{kk}^{-1} |(\Psi(e^{-iz}) \Sigma)_{j,k}|^2}{(\Psi(e^{-iz}) \Sigma \Psi'(e^{+iz}))_{j,j}} \\ &= \pi \lim_{z \rightarrow 0^+} \frac{|B(z)|^2 \sigma_{kk}^{-1} |(\Psi(e^{-iz}) \Sigma)_{j,k}|^2}{|B(z)|^2 (\Psi(e^{-iz}) \Sigma \Psi'(e^{+iz}))_{j,j}} \\ &= \pi \lim_{z \rightarrow 0^+} \frac{\sigma_{kk}^{-1} |(B(z) \Psi(e^{-iz}) \Sigma)_{j,k}|^2}{(B(-z) \Psi(e^{-iz}) \Sigma \Psi'(e^{+iz}) B'(z))_{j,j}} \\ &= \pi \frac{\sigma_{kk}^{-1} |(B(0) \Psi(e^{-i0}) \Sigma)_{j,k}|^2}{(B(0) \Psi(e^{-i0}) \Sigma \Psi'(e^{+i0}) B'(0))_{j,j}} = \pi (\mathbf{f}(0))_{j,k} \end{aligned} \tag{24}$$

The proof uses the l'Hospital rule. By this we have proven the second equality. It remains to prove that

$$\lim_{H \rightarrow \infty} \frac{\sigma_{kk}^{-1} \sum_{h=0}^H \left((\tilde{\Psi}_h \Sigma)_{j,k} \right)^2}{\sum_{h=0}^H (\tilde{\Psi}_h \Sigma \tilde{\Psi}_h')_{j,j}} = \frac{\sigma_{kk}^{-1} |(B(0) \Psi(e^{-i0}) \Sigma)_{j,k}|^2}{(B(0) \Psi(e^{-i0}) \Sigma \Psi'(e^{+i0}) B'(0))_{j,j}},$$

where $\tilde{\Psi}_h = \sum_{i=1}^h \Delta_h$, where Δ_h is the h -lag of coefficients estimated on the differenced series. Note that from the theoretical ideal filter $\mathbf{B}(\omega)$ we move to the standard differencing filter $\mathbf{B}(\omega) = |1 - e^{-i\omega}|^2$ in the following discussion. This is needed as the GFEVD is a time-domain entity and we need appropriate non-forward looking filter to be able to pass from the time-domain to the frequency domain. This is, however, of minor importance as we use the filter to appropriately estimate the original coefficients and any non-forward looking filter would perform similarly. Although, the differencing is handy in terms of ease of manipulation.

$$\begin{aligned} \sum_{h=0}^H \left(\sum_{i=0}^h \Delta_i \Sigma \right)^2 &= \sum_{h=0}^z \left(\sum_{i=0}^h \Delta_i \Sigma \right)^2 + \sum_{h=z+1}^H \left(\sum_{i=0}^h \Delta_i \Sigma \right)^2 \\ &= \sum_{h=0}^z \left(\sum_{i=0}^{\infty} \Delta_i \Sigma - \sum_{i=h+1}^{\infty} \Delta_i \Sigma \right)^2 + \sum_{h=z+1}^H \left(\sum_{i=0}^{\infty} \Delta_i \Sigma - \sum_{i=h+1}^{\infty} \Delta_i \Sigma \right)^2 \end{aligned} \quad (25)$$

Now, let us set $z = \lfloor \sqrt{H} \rfloor$.

$$\begin{aligned} \sum_{h=0}^H \left(\sum_{i=0}^h \Delta_i \Sigma \right)^2 &= \sum_{h=0}^{\lfloor \sqrt{H} \rfloor} \left(\sum_{i=0}^{\infty} \Delta_i \Sigma - \sum_{i=h+1}^{\infty} \Delta_i \Sigma \right)^2 + \sum_{h=\lfloor \sqrt{H} \rfloor+1}^H \left(\sum_{i=0}^{\infty} \Delta_i \Sigma - \sum_{i=h+1}^{\infty} \Delta_i \Sigma \right)^2 \\ &= \sum_{h=0}^{\lfloor \sqrt{H} \rfloor} \left(\left(\sum_{i=0}^{\infty} \Delta_i \Sigma \right)^2 - 2 \left(\sum_{i=0}^{\infty} \Delta_i \Sigma \right) \left(\sum_{i=h+1}^{\infty} \Delta_i \Sigma \right) + \left(\sum_{i=h+1}^{\infty} \Delta_i \Sigma \right)^2 \right) \\ &\quad + \sum_{h=\lfloor \sqrt{H} \rfloor+1}^H \left(\left(\sum_{i=0}^{\infty} \Delta_i \Sigma \right)^2 - 2 \left(\sum_{i=0}^{\infty} \Delta_i \Sigma \right) \left(\sum_{i=h+1}^{\infty} \Delta_i \Sigma \right) + \left(\sum_{i=h+1}^{\infty} \Delta_i \Sigma \right)^2 \right) \\ &= (H+1) \left(\sum_{i=0}^{\infty} \Delta_i \Sigma \right)^2 \\ &\quad + \sum_{h=0}^{\lfloor \sqrt{H} \rfloor} \left(\left(\sum_{i=h+1}^{\infty} \Delta_i \Sigma \right)^2 - 2 \left(\sum_{i=0}^{\infty} \Delta_i \Sigma \right) \left(\sum_{i=h+1}^{\infty} \Delta_i \Sigma \right) \right) \\ &\quad + \sum_{h=\lfloor \sqrt{H} \rfloor+1}^H \left(\left(\sum_{i=h+1}^{\infty} \Delta_i \Sigma \right)^2 - 2 \left(\sum_{i=0}^{\infty} \Delta_i \Sigma \right) \left(\sum_{i=h+1}^{\infty} \Delta_i \Sigma \right) \right) \end{aligned} \quad (26)$$

Now, taking the $1/H$ limit of the expression, the first element converges to $(\sum_{i=0}^{\infty} \Delta_i \Sigma)^2$,

the second term converges to zero because $\left(\left(\sum_{i=h+1}^{\infty} \Delta_i \Sigma \right)^2 - 2 \left(\sum_{i=0}^{\infty} \Delta_i \Sigma \right) \left(\sum_{i=h+1}^{\infty} \Delta_i \Sigma \right) \right)$ is bounded, and the third term converges to zero because $\frac{1}{H} \left(\sum_{i=h+1}^{\infty} \Delta_i \Sigma \right)$ converges to zero for $H \rightarrow \infty$. It suffices to notice that $\left(\sum_{i=0}^{\infty} \Delta_i \Sigma \right)^2 = (B(0) \Psi(e^{-i0}) \Sigma)^2$

The same argument would apply to the individual elements of the denominator.

□

C Supplementary Tables and Figures

β_1	β_2	s	ρ	Connectedness as by DY12				Connectedness as by DY12, nullified correlation			
				Connectedness	$(\pi/2, \pi)$	$(\pi/4, \pi/2)$	$(0, \pi/4)$	Connectedness	$(\pi/2, \pi)$	$(\pi/4, \pi/2)$	$(0, \pi/4)$
0.40	-0.40	0.00	0.00	0.15 (0.12)	0.20 (0.21)	0.20 (0.19)	0.19 (0.21)	0.08 (0.08)	0.07 (0.07)	0.07 (0.07)	0.07 (0.07)
0.40	-0.40	0.00	0.90	44.76 (0.35)	44.78 (0.35)	44.78 (0.35)	44.79 (0.36)	0.11 (0.11)	0.13 (0.18)	0.13 (0.17)	0.12 (0.17)
0.40	-0.40	0.20	0.00	3.37 (0.60)	3.49 (0.99)	3.30 (0.78)	3.20 (1.08)	3.48 (0.64)	3.44 (0.67)	3.47 (0.68)	3.49 (0.69)
0.40	-0.40	0.20	0.90	45.03 (0.34)	43.88 (0.48)	45.49 (0.32)	45.98 (0.31)	3.53 (0.60)	3.55 (0.54)	3.55 (0.62)	3.55 (0.67)
0.40	-0.40	0.59	0.00	23.11 (0.93)	23.07 (1.55)	22.86 (1.17)	22.75 (1.90)	23.14 (0.96)	22.93 (1.18)	22.93 (1.00)	22.97 (1.27)
0.40	-0.40	0.59	0.90	46.86 (0.29)	41.53 (0.70)	47.75 (0.17)	48.43 (0.16)	24.38 (0.97)	23.00 (0.90)	22.80 (1.14)	22.74 (1.45)
0.40	-0.40	-0.20	0.00	3.48 (0.64)	3.55 (0.89)	3.49 (0.80)	3.46 (1.05)	3.42 (0.65)	3.38 (0.69)	3.35 (0.67)	3.34 (0.68)
0.40	-0.40	-0.20	0.90	45.03 (0.31)	45.84 (0.28)	44.45 (0.37)	43.45 (0.46)	3.56 (0.61)	3.43 (0.65)	3.41 (0.57)	3.40 (0.54)
0.40	-0.40	-0.59	0.00	23.07 (0.94)	23.25 (1.49)	22.98 (1.08)	22.83 (1.68)	23.16 (1.04)	23.23 (1.30)	23.15 (0.95)	23.16 (1.25)
0.40	-0.40	-0.59	0.90	46.88 (0.29)	48.31 (0.17)	45.31 (0.34)	39.56 (0.78)	24.70 (0.92)	23.07 (1.56)	23.09 (0.86)	23.13 (0.85)

Table 5: Simulation results. The first three columns describe parameters for the simulation as described in equation 7. The results are based on 100 simulations of VAR with the specified parameters of length 1000 with a burnout period of 100. The estimate is computed as mean of the 100 observations and the standard error is simple sample standard deviation. The numbers are multiplied by hundred.

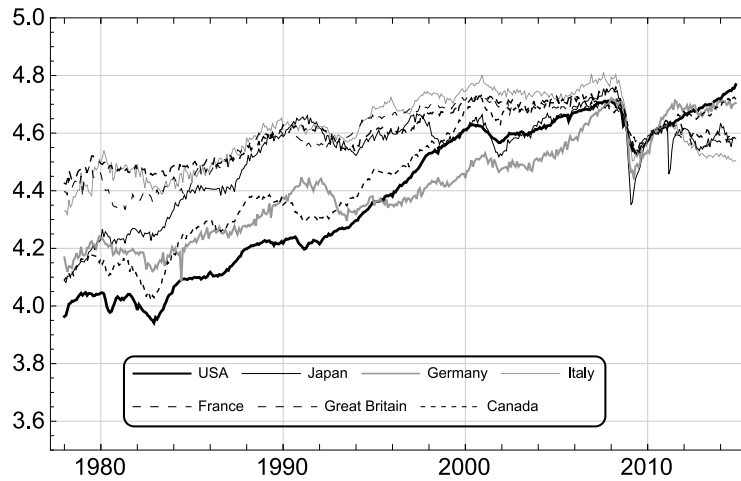


Figure 4: Industrial production (in logarithms) for the G-7 countries as used in the estimation.

β_1	β_2	s	ρ	Connectedness	$(\pi/2, \pi)$	$(\pi/4, \pi/2)$	$(0, \pi/4)$
0.0	0.0	0.00	0.0	0.00	0.00	0.00	0.00
0.0	0.0	0.00	0.9	44.75	44.75	44.75	44.75
0.9	0.9	0.09	0.0	40.50	0.30	0.90	41.15
0.9	0.9	0.09	0.9	49.47	44.25	44.41	49.51
-0.9	-0.9	0.09	0.0	40.50	40.77	0.34	0.24
-0.9	-0.9	0.09	0.9	41.28	41.01	45.22	45.22
0.9	0.4	0.09	0.0	5.66	0.32	0.88	7.48
0.9	0.4	0.09	0.9	46.09	44.25	44.48	46.56
0.9	0.0	0.09	0.0	2.59	0.32	0.80	3.97
0.9	0.0	0.09	0.9	45.40	44.25	44.51	45.98
0.9	-0.9	0.09	0.0	0.45	0.45	0.45	0.45
0.9	-0.9	0.09	0.9	44.76	44.26	44.97	45.26
0.4	-0.4	0.00	0.0	0.00	0.00	0.00	0.00
0.4	-0.4	0.00	0.9	44.75	44.75	44.75	44.75
0.4	-0.4	0.20	0.0	3.33	3.33	3.33	3.33
0.4	-0.4	0.20	0.9	45.01	43.52	45.62	46.28
0.4	-0.4	0.59	0.0	23.08	23.08	23.08	23.08
0.4	-0.4	0.59	0.9	46.87	40.94	47.86	48.64
0.4	-0.4	-0.20	0.0	3.33	3.33	3.33	3.33
0.4	-0.4	-0.20	0.9	45.01	46.05	44.27	43.00
0.4	-0.4	-0.59	0.0	23.08	23.08	23.08	23.08
0.4	-0.4	-0.59	0.9	46.87	48.51	45.13	38.84

Table 6: The true values for connectedness in the VAR settings.

	BAC	DIS	KO	MSFT	PFE	T	XOM
Mean	0.018	0.012	0.009	0.012	0.012	0.011	0.012
Median	0.013	0.010	0.008	0.010	0.010	0.009	0.010
Standard Dev.	0.018	0.008	0.005	0.007	0.006	0.008	0.007
Skewness	3.820	3.980	4.339	3.367	3.537	4.209	5.455
Kurtosis	20.506	26.242	31.091	18.852	21.759	29.954	57.341

Table 7: Descriptive statistics of the BPV realised volatilities of the sample.