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## Revealed Preferences and Spatial Segregation

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# Revealed Preferences and Spatial Segregation

Henri Busson

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# 1 Introduction

The relationship income / distance to CBD is basically nonlinear and its form varies a lot across the world. We often consider classic forms such as the typical US city one, where rich people live in the suburbs and the European city where they live downtown. Nevertheless, more complex patterns could be observed like in New York City. In this city, the first zone, the closest to the city center, is rich. People commutes by walking. The second zone is poor; public transportation are used. The middle class lives in the last zone and people commutes with their car.

To explain these stylized facts, we use a model inspired by Turnbull [5]. Instead of employing the classical utility functions, this model applies the revealed preferences theory. In his original paper, Turnbull established several propositions that made his model consistent with the main results obtained in urban economics (for instance that the prices decrease in relation to the distance from the CBD). Despite its consistence, the model gives different predictions than the main studies regarding the relationship between income and distance to CBD.

This model allows us to deal with a great heterogeneity among the consumers (regarding incomes, tastes, transportations modes, amenities consumption) and thus enables many different potential equilibria. This diversity of equilibria permits to represent in a more accurate way cities and patterns and thus justifies the use of this model. Several types of heterogeneities are considered to match the theoretical equilibria and the stylized facts. The first heterogeneity is in income: the location decisions of different households with same tastes are examined. To begin, their commuting costs is assumed to be the same. Then, the case with different commuting modes and costs are examined. In addition, amenities are introduced. Two types of heterogeneity regarding amenities are implemented: each household has a different level of consumption of amenities and the quantities of amenities depend on the distance to CBD.

Moreover, we try to take into account every parameter that could influence location choice in a single framework. Some model focuses on access to public transportation (Glaeser et al. [3], other on amenities Brueckner et al.[4]). But no model has ever succeeded in giving a complete framework for studying the income sorting in urban economics

A new hypothesis is introduced: prices are continuous and differentiable. It is not unrealistic to put this, as location is becoming more distant continuously from city center. We usually use the utility function because the revealed preferences method is highly complex. But in our case, the model is quite simple because we demonstrated that the GARP axiom used in the revealed preferences theory is equivalent to a simple equation. This equation simply states the equality of two derivatives.

Our results are compared every time with the classic method using utility functions and several differences are found.

Firstly, the ratio commuting costs per mile over demand land for rent is found constant between income groups, which is consistent with Wheaton [6]. As this variable is no longer supposed to explain the income sorting, the model focuses on other variables to explain it: income elasticity of housing demand, commuting time or amenity consumption.

The classical models of urban economics (Alonso Muth Mills and its extensions [1]) are insufficient to give a correct representation of the complexity of the distance to CBD relationship. Only one paper find an overlapping of the different types of households (DeBartolome and Ross [2]), the other models predict complete segregation. In our model, non-linearity of the income

CBD relationship is a potential equilibrium. The classic utility functions models focuses on which types of households are overbidding the other types. Here, the land market is in the background of our model. It is specified for every location, what are the consumption baskets that are compatible with the GARP axiom. This method enables us to focus more on the rationality of the consumers. For instance, a rich household will choose to locate downtown because it is more rational regarding his tastes. But another rich household with different tastes may choose to live in the suburbs. Consequently, in our model, some of the richer households will locate closer from CBD than a poorer one while other rich households will locate further away.

In a first section, the previous results found by Turnbull [5] are presented. Then the equivalence between GARP and the equality of derivatives is demonstrated. The equilibrium with homogeneous households is presented. Different types of heterogeneities are introduced. To finish, a general framework is presented.

## 2 Turnbull's method

Turnbull [5] applied the revealed preference theory to the location choice problem. We note  $X_c$  the set of all the consumption available at location  $d_i$ .  $p$  is the price vector.  $x$  is a vector of consumption :  $x = [x_1, \dots, x_n]$ .  $x_1$  represents land consumption. A consumer prefers to consume the  $x$  vector at location  $d_i$  if and only if  $p(d_i)x \geq p(d_i)y_0 \forall y_0 \in X_c$ . We write  $xR^D y_0$ . The bundle  $x$  is compared indirectly with a bundle  $z$  not available at location  $d_i$  but at location  $d_j$ :  $xR^D y, yR^D y_2, \dots, y_n R^D z$  with  $y_n$  available at  $d_j$ . The comparison between  $x$  and  $z$  is noted:  $xRz$ .  $x, d_i$  is chosen by the consumer if  $xRy \forall y$ .  $w$  is the income,  $t$  is the time commuting, therefore  $wt$  is the cost of opportunity of time.  $c$  is the expenses made by commuting.

It is important to notice that a consumer does not choose a particular location  $d_i$  in this model. The model only specify that at a location  $d_i$  which are the consumption bundles compatible with the GARP axiom. Therefore, an individual is characterized by a set of choices  $H$ . The model allow multiple choices for each individual. Each choice is characterized by its distance  $d_i$  and its consumption bundle  $x_i$ .

Turnbull [5] established the following equations, necessary verified at equilibrium: at the chosen location  $d_i$

$$(p_1(d_j) - p_1(d_i))x_1^i \geq wt(d_i) + c(d_i) - wt(d_j) - c(d_j) \forall d_j < d_i \quad (1)$$

$$(p_1(d_i) - p_1(d_j))x_1^i \leq wt(d_j) + c(d_j) - wt(d_i) - c(d_i) \forall d_j > d_i \quad (2)$$

He also proved in his first theorem that prices are monotonic decreasing:

$$d_i > d_j \Rightarrow p(d_i) < p(d_j)$$

Consumers are also assumed to be non satiated, their budget constraint is saturated

$$p(d_i)x_i = m + w(1 - t(d_i)) - c(d_i) \quad (3)$$

$m$  represents the non wage income.

## 2.1 Continuous prices and transportation time

In the rest of the paper, it is assumed that  $p$ ,  $t$  and  $c$  are  $C^2 \forall d_i$ . These are not unrealistic assumptions. These assumptions were already made in previous papers. It is also a convenient mathematical assumption, enabling us to use the derivatives.

## 3 Necessary and sufficient condition of Equilibrium

### 3.1 Sufficient condition of equilibrium

Turnbull [5] showed that (1) and (2) are necessary conditions. In Appendix A, it is also shown they are also sufficient. This leads to an equivalence between GARP and those inequalities: GARP axiom will no longer be needed, instead the inequalities could be taken.

### 3.2 Equality of the derivatives

Another equivalent much easier to manipulate will be given in this paragraph.

Dividing the equations (1)-(2) by  $d_i - d_j$  yields respectively:

$$\begin{aligned} \frac{[p_1(d_j) - p_1(d_i)]}{d_i - d_j} x_1^i &\geq \frac{wt(d_i) + c(d_i) - wt(d_j) - c(d_j)}{d_i - d_j} \forall d_j < d_i \\ \frac{[p_1(d_i) - p_1(d_j)]}{d_i - d_j} x_1^i &\geq \frac{wt(d_j) + c(d_j) - wt(d_i) - c(d_i)}{d_i - d_j} \forall d_j > d_i \end{aligned}$$

(Nb: in the second inequality,  $d_i - d_j < 0$ ). The two inequalities could be rewritten:

$$\begin{aligned} -\frac{[p_1(d_i) - p_1(d_j)]}{d_i - d_j} x_1^i &\geq \frac{wt(d_i) + c(d_i) - wt(d_j) - c(d_j)}{d_i - d_j} \forall d_j < d_i \\ \frac{[p_1(d_i) - p_1(d_j)]}{d_i - d_j} x_1^i &\geq -\frac{wt(d_i) + c(d_i) - wt(d_j) - c(d_j)}{d_i - d_j} \forall d_j > d_i \end{aligned}$$

By taking the limits:

$$p_1'(d_i) = \lim_{d_j \rightarrow d_i} \frac{[p_1(d_j) - p_1(d_i)]}{d_i - d_j}$$

and

$$(wt(d_i) + c(d_i))' = \lim_{d_j \rightarrow d_i} \frac{wt(d_j) + c(d_j) - wt(d_i) - c(d_i)}{d_i - d_j}$$

With the two inequalities, it is obvious that  $x_1^i \times p_1'(d_i) \geq -(wt(d_i) + c(d_i))'$  and  $-x_1^i \times p_1'(d_i) \geq (wt(d_i) + c(d_i))'$

Therefore

$$-x_1^i \times p_1'(d_i) = (wt(d_i) + c(d_i))' \quad (4)$$

It has just been shown that GARP imply an equality in derivatives. This seems very convenient but the question is does this inequality imply GARP?

**When the equality of derivatives are a sufficient condition of equilibrium ?** Assume that, for a consumer,  $-x_1^i \times p_1'(d_i) = (wt(d_i) + c(d_i))'$ .

Two different case will be considered, depending on the sign of the second derivative of housing prices.

**Concave housing prices** First, consider the case with concave housing prices.  $p_1'' < 0$ .

Therefore, for  $d_l < d_i$ ,

$$-x_1^i p_1'(d_l) < -x_1^i p_1'(d_i) = wt'(d_i) + c'(d_i)$$

$$-x_1^i p_1'(d_l) < wt'(d_i) + c'(d_i)$$

If  $d_l$  is taken close enough to  $d_i$ , it is possible to write:

$$-x_1^i \frac{p_1(d_i) - p_1(d_l)}{d_i - d_l} < \frac{wt(d_i) + c(d_i) - wt(d_l) - c(d_l)}{d_i - d_l}$$

$d_i - d_l$  is positive, as a result

$$-x_1^i (p_1(d_i) - p_1(d_l)) < wt(d_i) + c(d_i) - wt(d_l) - c(d_l)$$

This is the contrary of (1), therefore if  $p'' < 0$ , then no equilibrium occurs. Suppose that there is an equilibrium. Therefore (1) is verified and the derivatives are equal. But we also showed that if the derivatives are equal, (1) could not be verified. There is a contradiction. Therefore, the housing prices are necessary convex. Turnbull [5] showed that housing prices were convex in the case of non increasing incremental transport time. It has just been demonstrated in the case of prices continuously differentiable but without making any assumptions on the transport time.

**Convex housing prices** Suppose  $p_1'' > 0$  and  $-x_1^i p_1'(d_i) = wt'(d) + c'(d), \forall d$ . Then it is obvious that  $wt''(d_i) + c''(d_i) \leq 0$

But more importantly, there exists  $k_1$  such that

$$-x_1^i p_1(d) = wt(d) + c(d) + k_1, \forall d \tag{5}$$

Suppose  $d_j < d_i$ ,

$$-x_1^i p_1(d_i) = wt(d_i) + c(d_i) + k_1$$

$$-x_1^j p_1(d_j) = wt(d_j) + c(d_j) + k_1$$

Combining these last two expressions yields

$$-x_1^i p_1(d_i) + x_1^j p_1(d_j) = wt(d_i) + c(d_i) - wt(d_j) + c(d_j)$$

It is easy to verify that expression (1) holds if and only if  $x_1^j \leq x_1^i$ .

$x_1^j \leq x_1^i$  imply  $-x_1^i p_1(d_i) + x_1^j p_1(d_j) \leq x_1^i (p_1(d_j) - p_1(d_i))$

And finally  $x_1^i (p_1(d_j) - p_1(d_i)) \geq wt(d_i) + c(d_i) - wt(d_j) + c(d_j)$ .

To conclude, the price convexity, the assumption that housing consumption increases with distance for the same household and the equality of derivatives is a necessary and sufficient condition of equilibrium under GARP.

The GARP axiom is often considered as very difficult to apply and that is partly why the utility approach have been used. But using equation (6), which is equivalent to GARP, is very convenient and simple. It is a simple equation stating the equality of two derivatives.

### 3.3 Homogeneous households

Assume that all the consumers have the same preferences, incomes, marginal commuting costs. Assume also that  $p$ ,  $t$  and  $c$  are  $C^2$  and  $p'' > 0$ . A household will locate the closest to the central business district at  $d_1 = 0$ . He will choose  $x_1$  such as  $x_1 = -w \frac{t'(0)+c'(0)}{p'(0)}$ . A second household will locate at  $d_2 = x_1$  and will choose  $x_2$  such as  $x_2 = \frac{wt'(x_1)+c'(x_1)}{p'(x_1)}$ . More generally, we have household  $k$  will locate at  $x_{k-1}$  and will choose  $x_k$  such as  $x_k = -\frac{wt'(\sum_{j=1}^{k-1} x_j)+c'(\sum_{j=1}^{k-1} x_j)}{p'(\sum_{j=1}^{k-1} x_j)}$ . There will be a limit to the number of households who will be able to locate : it will depend on the size of the urban areas.

More generally, households do not choose a particular location in this model. The model only precise the consumption bundle compatible with GARP for every locations. For the location  $d_i$ , the housing consumption compatible with GARP is  $x_i = -\frac{wt'(\sum_{j=1}^{i-1} x_j)+c'(\sum_{j=1}^{i-1} x_j)}{p'(\sum_{j=1}^{i-1} x_j)}$ .

**Housing consumption increases with distance** To ensure that this condition of equilibrium is verified, it is simply necessary to assume that the function  $d \rightarrow -\frac{wt'(d)+c'(d)}{p'(d)}$  is increasing. Transportation costs are exogenous. But prices are the results of an endogenous mechanism.

## 4 Two types of households : difference in wages

After studying homogeneous households, it is now necessary to introduce heterogeneity in our model. In this section, to simplify, the monetary costs of commuting are neglected. This is the case in most papers in urban economics. Therefore, the function  $d \rightarrow -\frac{wt'(d)}{p'(d)}$  is considered increasing.

### 4.1 Basic model

Consider a consumer who revealed his preference to locate at  $d_i$  and to consume  $x_i$  :  $y_i = [x_i d_i]$ . He is actually earning  $w$ .  $d_i$  is such that  $wt'(d_i) = -x_i p'_1(d_i)$ .

Consider another consumer, with exactly the same taste, the same marginal costs, but with an income equals to  $w + \Delta w$  with  $\Delta > 0$ . We assume that housing consumption is a normal good; therefore his housing consumption  $x_{rich}^1$  is such that  $x_{rich}^1 \geq x_i^1$ . We define  $\Delta x$  :  $x_{rich}^1 = x_i^1 + \Delta x$ . The rich will choose his housing consumption for every location  $d$  such as  $-\frac{p'_1(d)}{t'(d)} = \frac{w+\Delta w}{x_i^1+\Delta x}$ .

That equality could be rewritten  $-\frac{p'_1(d)}{t'(d)} = \frac{w}{x_i^1} + \Delta \left(\frac{w}{x_i^1}\right)$  with  $\Delta \left(\frac{w}{x_i^1}\right) = \frac{\Delta w - w \frac{\Delta x_i^1}{x_i^1}}{1 + \frac{\Delta x_i^1}{x_i^1}}$ .

It is easy to demonstrate that the sign of  $\Delta\left(\frac{w}{x_i^1}\right)$  entirely depends on the value of the income elasticity of housing consumption.

We define  $f(d) = -\frac{p_1'(d)}{t'(d)}$ . It is known that this function is decreasing. As a result, the location of the richer household depends entirely whether the income elasticity of housing consumption is greater or inferior to 1.

If the income elasticity is greater to 1, then the ratio  $\frac{w}{x_i^1}$  decreases as income rises. Then the solution of the equation  $f(d) = \frac{w}{x_i^1}$  is greater as income rises because  $f$  is decreasing.

#### 4.1.1 Comparison with the classic model using utility functions

The ratio  $\frac{tw}{x_i^1}$  represents the famous ratio: opportunity cost of time over demand for land. In the case of a  $t$  not depending on income, its variation with income is the same as  $\frac{tw}{x_i^1}$ . It is possible to write  $p'(d) = \frac{wt'(d)}{x_i} = \frac{(w+\Delta w)t'(d)}{x_i+\Delta x}$ . At equilibrium,  $p$  being exogenous, for every location  $d$ , both ratios of opportunity cost of time per mile over demand for land are constant across income groups. This result is confirmed by Weathon (1977). It means that, the sorting will not be explained by the differences in this ratio. The sorting will be explained by housing consumption. Each household has its set of choice  $H$ . Each location is associated with a housing consumption. Consequently, a household will choose a particular location  $d$  by choosing a particular housing consumption.

Moreover, both case  $\frac{tw}{x_i^1}$  increasing with income and  $\frac{tw}{x_i^1}$  decreasing with income are compatible with equilibrium in the same city, while it seems impossible in the model using utility functions. Indeed, each richer household has to choose among the set of rational choice  $H$ . Among those choices, some will lead to an increasing  $\frac{tw}{x_i^1}$ , others could lead to a decreasing one.

#### 4.1.2 Two transportation modes

In the previous paragraph, we assumed that a rise in income does not change the time derivative  $t'$ . As in the previous paragraph, we consider two households, with the same tastes excepts they differ in income, housing demands and the richer household may use a different transportation mode. Therefore we note  $t_R$  the time spent in commuting for the rich and  $t_P$  the time spent commuting for the poor households. We note  $\Delta t'$  such as  $t'_R = t'_P + \Delta t'$ .

The equation for the richer household is:

$$-p_1'(d)(x_i^1 + \Delta x) = (w + \Delta w)(t'_P(d) + \Delta t'(d))$$

Rewriting this equation yields

$$\begin{aligned} \frac{-p_1'(d)}{t'_P(d)} &= \frac{(w + \Delta w)}{(x_i^1 + \Delta x_i^1)} \left(1 + \frac{\Delta t'(d)}{t'(d)}\right) \\ \frac{t'_P(d)}{(t'_P(d) + \Delta t'(d))} \frac{-p_1'(d)}{t'_P(d)} &= \frac{(w + \Delta w)}{(x_i^1 + \Delta x_i^1)} \end{aligned}$$

We define  $f(d) = \frac{-p_1'(d)}{t'_P(d)}$  and  $g_R(d) = \frac{t'_P(d)}{(t'_P(d) + \Delta t'(d))} \frac{-p_1'(d)}{t'_P(d)}$ . Note that  $g_R$  could also be written

$$g_R(d) = \frac{t'_P(d)}{(t'_P(d) + \Delta t'(d))} f(d) = h(d) \times f(d)$$



with  $h(d) = \frac{t'_P(d)}{(t'_P(d) + \Delta t'(d))}$

- If  $\forall d, h(d) = 1$ , then the conclusion are exactly the same than in the previous section.
- If  $\forall d, h(d) < 1$ , then  $\forall d, g_R(d) < f(d)$ . The richer household will live downtown if housing consumption is an inferior good (the ratio  $\frac{w}{x_1}$  is increasing with income).

**Proof** We note  $d_I$  the solution of  $f(d_I) = \frac{w + \Delta w}{x_1 + \Delta x}$ ,  $d_P$  the solution of  $f(d_P) = \frac{w}{x_1}$  and  $d_R$  the solution of  $g_R(d_R) = \frac{w + \Delta w}{x_1 + \Delta x}$ . As  $f$  is decreasing and  $\frac{w}{x_1}$  is increasing with income,  $d_I < d_P$ .

It is possible to write  $g_R(d_R) = f(d_I)$ . As  $f$  is decreasing then,  $\forall d > d_I, g_R(d) < f(d) < f(d_I)$ . As a result,  $d_R < d_I < d_P$ .

- If  $\forall d, h(d) > 1$ , then  $\forall d, g_R(d) > g(d)$ . Then, it is impossible to determine the outcome in a general manner.

**Predictions with the derivatives near CBD** There is another way to predict the income sorting with different transportation costs but with an additional hypothesis. We define  $s_r(d) = \frac{w_r t'(d)}{x_r}$  with  $x_r$  the housing consumption chosen by the richer household.  $s_p(d)$  is define almost equivalently for the poorer household.

The distance  $d_r$  (respectively  $d_p$ ) chosen by the ticher household is solution to  $-p'(d) = s_r(d)$  (respectively  $-p'(d) = s_p(d)$ ).

Suppose that  $s_r(d) = s_p(d)$  has no solution. Then if  $p'(0) < s_p(0) < s_r(0)$  or if  $p'(0) > s_p(0) > s_r(0)$ , then the richer household will locate further away from CBD. Moreover if  $p'(0) < s_r(0) < s_p(0)$  or if  $p'(0) > s_r(0) > s_p(0)$ , then the poorer household will locate further away from CBD.

## 4.2 Model with neighborhood effects and amenities

The choice of location depends on amenities. People rather locate nearest them, so it is the opportunity cost of time that matters here. We only have to introduce a new element in the consumption vector (it is the consumption of amenities) and its price is the opportunity cost of time as for the leisure. But unlike the leisures, the consumption of amenities is not uniform on all the locations. We note  $a(d_i)$  the quantity of amenities available at location  $d_i$ . Amenities are considered exogenous in a first place.

We introduce a parameter  $n_i$  which represents the quantity of amenities consumed. However, two different household may not appreciate or use these amenities in the same way. Therefore  $n_i$  represents the intensity of preference for amenities. But depending on its location decision, a household will lose time to commute to consume his desired level of amenities. If a household commutes on a distance  $l$ , his consumption of amenities will be  $\int_0^l a(s) ds$ . Therefore, he will have to commute a length  $l_i$  solution of the equation  $n_i = \int_{d-i-R}^{d_i+Q} a(s) ds$  with  $R + Q = l_i$ . The price of this consumption is therefore  $wt(l_i)$ .

Putting this parameter in the equations, we obtain:

$$[p_1(d_j) - p_1(d_i)]x_1^i + n_i(wt(l_i) - wt(l_j)) \geq wt(d_i) + c(d_i) - wt(d_j) - c(d_j) \forall d_j < d_i$$

$$[p_1(d_i) - p_1(d_j)]x_1^i + n_i(wt(l_j) - wt(l_i)) \leq wt(d_j) + c(d_j) - wt(d_i) - c(d_i) \forall d_j > d_i$$

By taking the derivatives, we obtain:

$$\begin{aligned} -p_1'(d)x_1^i + n_i wt'(l_i) &= wt'(d) \\ \frac{-p_1'(d)}{t'(d)} + n_i \frac{w}{x_1^i} \frac{\partial t(l_i)}{\partial d} \frac{1}{t'(d)} &= \frac{w}{x_1^i} \end{aligned}$$

In order to simplify, we note  $A(d) = \frac{\partial t(l_i)}{\partial d}$ .

$$\frac{-p_1'(d)}{t'(d)} + n_i \frac{w}{x_1^i} \frac{A(d)}{t'(d)} = \frac{w}{x_1^i}$$

A new term appears in the left member :  $n_i \frac{w}{x_1^i} \frac{A(d)}{t'(d)}$ .

We pose  $g^A(d) = \frac{-p_1'(d)}{t'(d)} + n_i \frac{w}{x_1^i} \frac{A(d)}{t'(d)}$ .

We apply the same methodology than in the previous section. Suppose a household that is absolutely equivalent but has an higher income.

The right member will increase with the same condition as in the previous section: if the income elasticity of housing consumption is lower than 1.

The equation for the richer household is:

$$\frac{-p_1'(d)}{t'(d)} + (n_i + \Delta n_i) \frac{w + \Delta w}{x_1^i + \Delta x_1^i} \left( \frac{A(d) + \Delta A(d)}{t'(d)} \right) = \frac{w + \Delta w}{x_1^i + \Delta x_1^i}$$

We note  $g_R^A(d) = \frac{-p_1'(d)}{t'(d)} + (n_i + \Delta n_i) \frac{w + \Delta w}{x_1^i + \Delta x_1^i} \left( \frac{A(d) + \Delta A(d)}{t'(d)} \right)$ .

The function could be rewritten as the sum of the function  $g^A$  defined earlier and another function  $g2$ :

$$g_R^A(d) = \frac{-p_1'(d)}{t'(d)} + (n_i + \Delta n_i) \frac{w + \Delta w}{x_1^i + \Delta x_1^i} \left( \frac{A(d) + \Delta A(d)}{t'(d)} \right) = g^A(d) + g2(d) \text{ with } g2(d) = \Delta n_i \frac{w}{x_1^i} + \Delta \left( \frac{w}{x_1^i} \right) (n_i + \Delta n_i)$$

### 4.3 Generalization

Suppose the equation for a household could be written

$$G(d, Y) = \frac{w}{x_1^i}$$

$G$  represents all the factors depending on the distance  $d$  that could influence its location decision. Notice that  $G$  depends also on the income  $Y$ . For instance, in section(),  $G$  is equal to  $\frac{-p_1'(d)}{t'(d)}$ .

The unknown in this equation is  $d$  if the housing consumption is fixed.

Suppose there exists two households with two different incomes  $Y_R > Y_P$ .

The solution of this equation for the poorer household is noted  $d_P$ :  $G(d_P, Y_P) = \frac{w}{x_1^i}$ .

The solution of this equation for the richer household is noted  $d_R$ :  $G(d_R, Y_R) = \frac{w + \Delta w}{x_1^i + \Delta x_1^i}$ .

The results will depend on the signs of  $\frac{\partial G}{\partial d}$ ,  $\frac{\partial G}{\partial Y}$  and  $\frac{\partial \frac{w}{x_1^i}}{\partial Y}$ .

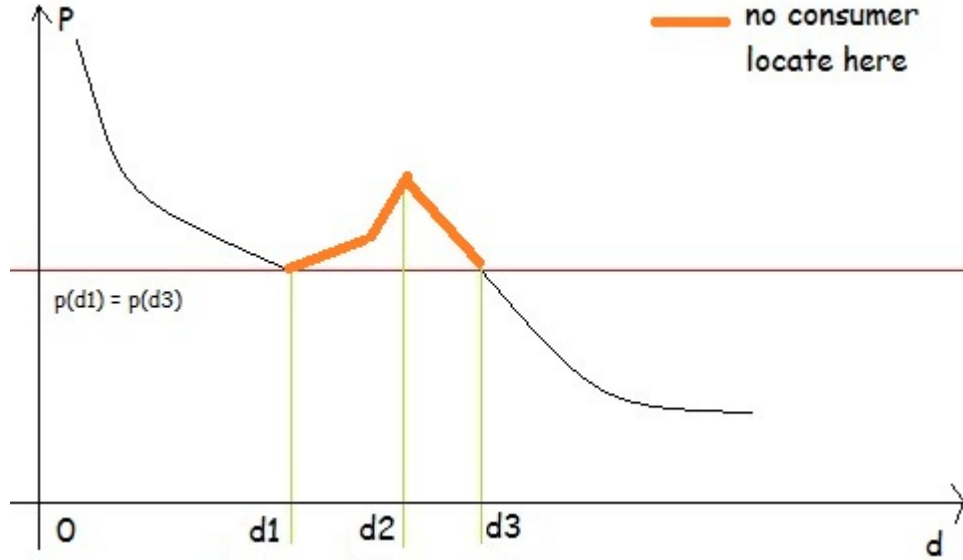
Proposition If  $\frac{\partial G}{\partial Y} > 0$  and  $\frac{\partial \frac{w}{x_1^i}}{\partial Y} < 0$  have opposite signs, then two cases arises. If  $\frac{\partial G}{\partial d} > 0$

Proof  $d_I < d_P$  and  $d_R > d_I$ .

## 5 Discussion on Turnbull's first theorem

In his paper, Turnbull [5] show that housing prices are monotonic declining. But his demonstration made an implicit assumption. He starts by taking a consumer who prefers to locate at  $d_i$  and he shows that  $\forall d_j < d_i, p(d_j) > p(d_i)$ . Then he says to repeat the same procedure for all  $d_i$ . But imagine a case where no consumer prefers to locate at  $d_i$ .

What happens if prices are increasing on an interval  $[d_1, d_2]$ ? First, we note  $d_3$  the location such that  $p(d_3) = p(d_1)$  and  $d_3 > d_2$ . We assume for the moment the existence of  $d_3$ . We also assume that on  $[0, d_1]$  and on  $[d_2, d_r]$ , prices are decreasing ( $d_r$  represents the urban fringe). There is no consumers who prefers to locate on  $[d_1, d_3]$ . Otherwise, it will enter in contradiction with Theorem 1.



It is not true to say that prices are necessary monotonic declining; it is more rigorous to say that only monotonic declining prices are compatible with being chosen as a location.

## Appendix A - Inequalities sufficient condition of GARP axioms

Consider a consumer who prefers the quantity  $x^i$  at the location  $d_i$ . Assume that

$$(p(d_j) - p(d_i))x^i \geq wt(d_i) + c(d_i) - wt(d_j) - c(d_j) \forall d_j \quad (6)$$

and we have also, by non satiation, equation (3) is verified.

It is necessary to demonstrate that GARP is verified under these assumptions. That location is revealed preferred if and only if  $x^i R x^j \forall x_j$

Combining the two expressions (3) – (4), for a particular  $d_j$ , we obtain

$$p(d_j)x^i \geq m + w - wt(d_j) - c(d_j) \quad (7)$$

By non satiation,  $p(d_j)x^j = m + w[1 - t(d_j)] - c(d_j)$ .

We define  $r_{max}$  such as  $\sum_1^{r_{max}} p^l(d_j)x_i^l \leq m + w[1 - t(d_j)] - c(d_j)$  and  $\sum_1^{r_{max}+1} p^l(d_j)x_i^l \geq m + w[1 - t(d_j)] - c(d_j)$ . The existence of  $r_{max}$  is obvious, as long as (5) is assumed. We construct  $x_{inter}$  such as  $\sum_1^{r_{max}} p^l(d_j)x_i^l + p^{r_{max}}(d_j)x_{inter} = m + w[1 - t(d_j)] - c(d_j)$ . The existence of  $x_{inter}$  is easy to be proved : the function  $\sum_1^{r_{max}} p^l(d_j)x_i^l + p^{r_{max}}(d_j)x$  is continuous in  $x$  and the intermediate value theorem guarantees the existence of  $x_{inter}$ .

We define now  $x_{new} = [x^1 = x_i^1, \dots, x^{r_{max}} = x_i^{r_{max}}, x^{r_{max}+1} = x_{inter}, x^{r_{max}+2} = 0, \dots, x^n = 0]$ . Notice that  $p(d_j)x_{new} = m + w[1 - t(d_j)] - c(d_j)$ . Therefore,  $x_{new} R^D x_j$ .  $x_{new}$  is also available at  $d_i$  and it is obvious that  $x_i R^D x_{new}$ . Consequently  $x_i R x_j$ .

As a conclusion the conditions (1) and (2) are a necessary and sufficient condition of equilibrium.

By knowing the the two equations are equivalent to define an equilibrium, instead of using GARP axioms we could use these two equations.

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