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A Spatial Production Economy Explains Gross Metropolitan Product

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A SPATIAL PRODUCTION ECONOMY EXPLAINS GROSS METROPOLITAN PRODUCT

Hiroki Watanabe*

September 11, 2015

Abstract

It has long been known that the city-size distributions are fat tailed, drawing the interest of urban economists. In contrast, not much is known about the distribution of GDP at city level (henceforth referred to as gross metropolitan product, GMP). We build a model of the spatial economy that includes production and confirm the following empirical facts about the GMP counterpart of the city-size distribution. First, both Zipf's and Gibrat's law hold for the distribution of GMP as well. In particular the GMP distribution is well-traced by a lognormal distribution. Second, citywide aggregate production exhibits increasing returns to scale with respect to employment. In particular a 1% increase in employment leads to a 1.117% (or 1.180% in theory) increase in GMP. Agglomeration economies are explained as a result of an endogenous trade-off between externalities and land consumption of consumers.

Keywords: Zipf's Law, Gibrat's Law, GDP by City, Production Economy
JEL classification: D51, E2, R12

1 INTRODUCTION

Four out of five people live in cities, and they do so for various reasons, i.e. better job prospects, decent wage, urban amenities, or family obligations. The resulting size distribution of cities has kept the rapt attention of urban economists, and we now have a growing understanding of what it is and how it came about; however, the story does not end there. No one moves in or out of a city just for the sake of making its size larger or smaller, nor does the city size itself feed its population. The overriding research objective in the literature is the welfare implication of the city-size distribution, but the empirical distribution of GMP has never been analyzed to this date. We will take one step forward to show that the GMP distribution follows a fat-tail distribution and provide a theoretical background behind the relationship between city size and corresponding GMP.

Our major findings are as follows. First, two empirical regularities on the city-size distribution carry over to GMP. Most of GDP are generated in only a few cities just as the city-size distribution, the regularity known as Zipf's law ([[Gab99](#)],

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[Duro7]). In fact only 20% of cities create as much as 78.75% of urban GDP.¹ GMP has a lower Pareto coefficient than city-size counterpart, i.e., its tail end is even heavier than the city-size distribution. Gibrat’s law also extends to GMP, as urban economic growth rates are independent of its GMP size. Second, GMP exhibits increasing returns to employment. That is, New York’s GMP is larger than any other city’s, even size for size. This is consistent with our first finding that the GMP distribution has a heavier tail than the city-size distribution. We build a production economy model and establish that agglomeration economies are due to the trade-off between externalities and housing consumption. We prove that the equilibrium city size has to be such that an additional resident will reduce a housing lot size in the city but make up for it by raising citywide productivity.

In the existing city-size models, with the assumption of free mobility, consumers/workers will update their locations until they exhaust the locational arbitrage opportunities. Thus, regardless of the city size, cities become indifferent to consumers in equilibrium. This does not imply that workers are equally productive or their income will be the same across the board. In practice, per capita GMP varies by location (cf. figure 1 and table 1). People enjoy the same utility level at the end of the day² but what induces interurban migration depends on GMP. People relocate to a city not for the sake of its *size alone* but

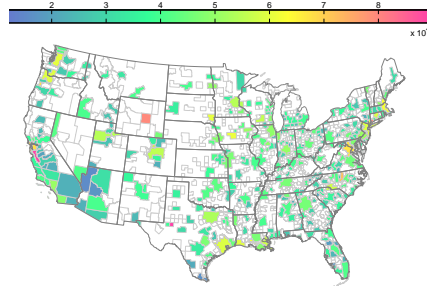


Figure 1. GMP per capita in 2010 (in 2005 USD).

for what its size *has to offer*, one of which is its GMP. The city-size distribution is the result of interurban migration. We will reveal the distribution of GMP and decode the economic forces behind it in this paper.

Figure 2 (in color) is a map of the United States with metropolitan statistical areas (MSA) colored according to their population density and GMP in 2010.³ Figure 2(a) comes with no surprise. It is well documented that the city-size distribution is tail heavy. What is newsworthy is figure 2(b)⁴. GMP shares the same pattern to city size in terms of distribution. Figure 3 represents the probability density function (PDF) and rank-size plot of GMP in 2010. New York accounts for the lion’s share of GDP, followed by Los Angeles, and there are lots of mid-sized cities that are dwarfed by the high-ranked cities.

Our intended contribution is to provide a systematic understanding of the dis-

¹ This relation is known as a 20-80 rule: 20% of agents are accountable for 80% of the results, a typical sign that something of scale-free nature is at work.

² Cities are put in equilibrium either by equating wage (e.g., [Duro7]) or utility level (e.g., [Gab99]).

³ Population data also include micropolitan statistical area along with MSA. For definition of MSA, see <http://www.census.gov/population/metro/about/>.

⁴Data source: Bureau of Economic Analysis.

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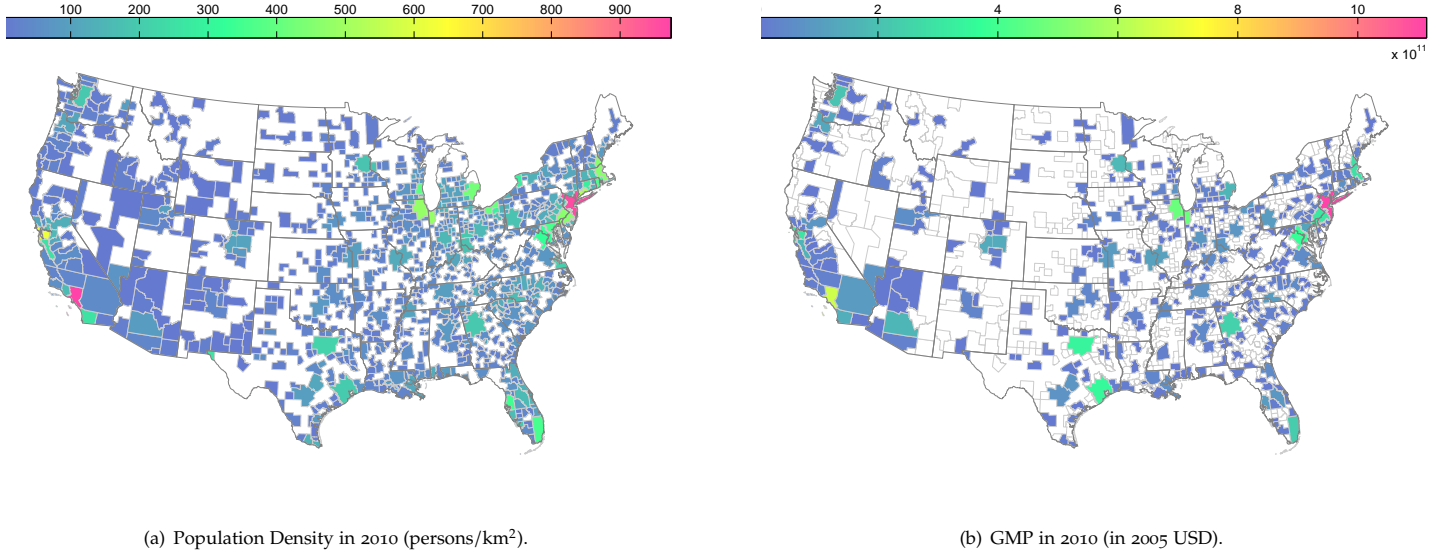


Figure 2.

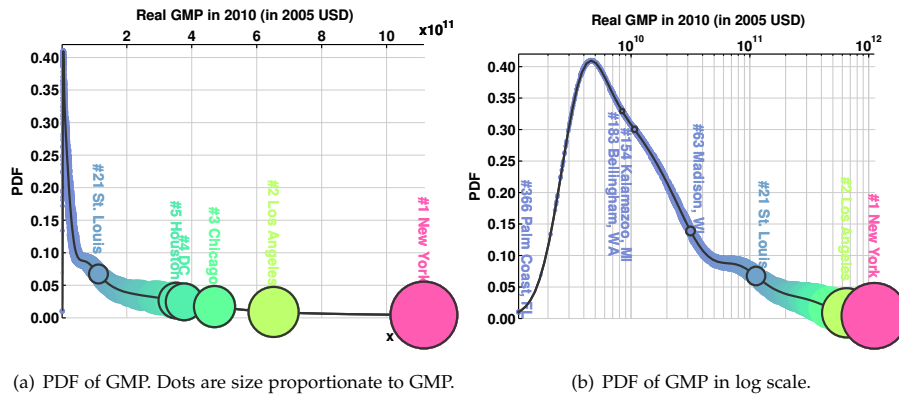


Figure 3. PDF plots of GMP. See table 1 for the explanation of the selected cities above.

tribution of GMP. The conventional range of study on GMP has been limited within a city. Usual questions are in lines of how to promote the urban growth in Detroit, or the effect of overproduced liquid natural gas in Pittsburgh. These fact-finding works and analyses of local economies play a part in the GMP distribution. GMP distribution is, after all, the accumulation of all these local economic activities com-

bined. On the other hand, we found a holistic approach to the GMP distribution missing in the literature: GMP is reported in each city; therefore there will be a GMP distribution. We would like to get an aerial view from coast to coast and address GMP from the general equilibrium perspective.

There are two lines of research related to our project: one on the distribution of city sizes and the other on agglomeration economies. The first line of research studies the distribution of city sizes but not GMP, while the second one studies GMP but not the distribution thereof. We will fill in the gaps in this paper.

Overshadowed by the consuming interest in the city-size distribution, research into the distribution of GMP is nonexistent. Those who quote the citywide production function use it as an intermediate step to reach the equilibrium *city-size* distribution. As a byproduct, we get the equilibrium production level in each city, but predicted *GMP* has never been tested with any empirical data. Their primary objective is to explain the city-size distribution. We will take the GMP distribution as a byword rather than a byproduct.

On the other hand, the second line of work homes in on the question of how much of a boost we get by producing goods and services in a crowd rather than in a rural setting. The question is imperative because if there is no scale economies in cities, then there is no convincing reason to reside in a large but crowded city, barring other centripetal forces such as local public goods or access to a large, diversified labor pool [ABLo7]. A study on citywide *productivity* becomes an essential part of the examination on city size (cf. [Hen74], [KKSSo5]). In fact increasing returns to scale is one of the main⁵ ingredients in the formation of a city (Krugman [Kru91]). See Moomaw [Moo83] for review of earlier work in this literature. More recently, Caliendo et al [?] consider how the economy responds to idiosyncratic changes in regional productivity. The responsiveness of GDP are shown to vary region to region.

Despite having related research agenda, these two lines of work take different approaches to theorizing about their respective target objective. The city-size distribution models based on general equilibrium typically do not include capital stock as part of production function ([Duro7], [Eeco4] for example);⁶ whereas most agglomeration models do. Labor alone serves its purpose to explain the actual city-size distribution without involvement of capital stock. We sided with the city-size distribution models for our purposes. It is easy to measure a city size, but measuring citywide capital stock is not as straightforward as a head count. In fact there are no data on the level of capital stock at city level in the United States. Those studies that quote capital stock use the estimated level based on factors related to capital such as local public goods, housing and state roads, mixed in with predetermined weights ([Seg76]), or estimated retrospectively *from* the pair of labor and GDP per capita at city level ([Sve75]). Capital stock is known to be correlated with city size, which causes a multi-collinearity problem. According to [Seg76], capital stock's contribution to GMP is .116 as opposed to labor's .891. We did not test

⁵But not necessary. Cf. [BKoo].

⁶There are some exceptions. For example, Rossi-Hansberg and Wright [RHWo7] address city-size distribution with capital stock incorporated into the model. Even then, actual capital stock level is not used for empirical testing.

our model on capital stock but it is general enough to incorporate investment if needed.

The data set we use is more inclusive than any previous studies. There are 366 cities with accompanying GMP figures. The largest sample size used so far to test GMP is 30 by Mion and Naticchioni [MN05] according to [MGN09].⁷ For example cities like Beaumont, TX are too small to be included in the data set in [Seg76]. At the time of writing 366 is the largest data size for which GMP is reported.

GMP data also mesh with city-size data to provide an added layer of empirical validation to the existing models on the city-size distribution. The models of city-size distribution are empirically tested on the basis of city size, and the choice of city size as a data set to pitch against a model is obvious because they are built to explain the city-size distribution after all; however, they also need to be crosschecked with other spatial data, including rent or wage in each city. Otherwise a model can only explain city-size distribution but nothing else, which undermines its legitimacy as an urban economic model. GMP is one of those spatial data that complement city-size data to confirm a model's relevance to the reality.

The remainder of the paper is organized as follows: [section 2](#) investigates into the nature of GMP distribution and provides descriptive statistics on GMP along with city size. In [section 3](#) we introduce the spatial production economy model to explain the findings in [section 2](#) before we empirically evaluate our model's performance in [section 4](#). [Section 5](#) concludes our study.

2 GMP ACTUALITIES

We will establish the Zipf's and Gibrat's law for GMP and also identify the relationship between GMP and city size. The US Bureau of Economic Analysis reports annual GDP by MSA along with the US GDP and estimated employment. Descriptive statistics for the employed data are in [table 1](#).

2.1 GIBRAT'S LAW FOR GMP

For starters, we looked into the dynamics: Does a large GMP make a city grow fast? The answer: no. Gibrat's law implies that the size of a city does not have any bearing on its growth rate. The city-size distribution is known to follow Gibrat's law well ([IO03]). It turns out that GMP does the same. We carried out both parametric and non-parametric estimations following [Eeco4] to examine the relationship between GMP and GMP growth rate.

⁷ The aforementioned study [Seg76] has 58 locations but output is limited to the manufacturing sector rather than GMP as a whole. These studies often quote census for manufacturers alone.

Data	Employment	GMP	GMP per capita	GMP growth
Unit	MSA	Million 2005 USD	2005 USD	Annual rate
Year	2010	2010	2010	2010 over 2005
Coverage (MSA/USA)	84.90%	87.60%	87.60%	87.60%
Largest city	#001 New York	#001 New York	#001 Midland, TX	#001 Midland, TX
Largest size	18,919,787	1,147,160	89,350	11.60%
73rd largest city	#073 Akron, OH	#073 Worcester, MA	#073 Waterloo, IA	#073 Dallas, TX
Size share of #1-73	71.80%	78.75%	29.40%	N/A
City near arithmetic mean	#072 North Port, FL	#063 Madison, WI	#150 Greenville, SC	#171 Topeka, KS
Arithmetic mean	707,308	31,763	36,002	.49%
Median city	#183 Laredo, TX	#183 Bellingham, WA	#183 Gainesville, FL	#183 Monroe, LA
Median size	251,539	8,414	34,048	.35%
Smallest city	#366 Carson City, NV	#366 Palm Coast, FL	#366 Palm Coast, FL	#366 Lake Charles, LA
Smallest size	55,212	1,132	11,793	-5.98%
Standard deviation	1,582,442	86,824	11,257	.0212
Skewness	6.669	7.500	1.409	.9088
City near geometric mean	#149 Naples, FL	#154 Kalamazoo, MI	#178 Mobile, AL	N/A
Geometric mean	321,416	11,078	34,460	N/A
Mean of log value	12.68	23.13	10.45	N/A
Standard deviation of log value	1.062	1.216	.2920	N/A
Skewness of log value	1.109	1.103	.2251	N/A

Table 1. Descriptive Statistics. The statistics above the line (shaded in blue) are related to a linear scale and below the line (shaded in green) are related to a log scale. The mean of log value is same as the log of geometric mean. The first 73 cities make up for the upper 20% of the total number of cities.

2.1.1 NON-PARAMETRIC ESTIMATION

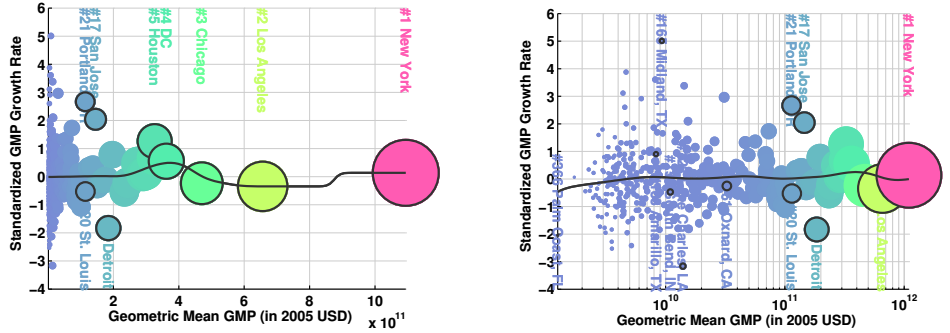
First we estimate the conditional expectation of GMP growth rate $E[g|Y] = m(Y)$ with a Nadaraya-Watson kernel estimator [Wat64]

$$\hat{m}(Y) = \frac{\sum_{i=1}^I g^i K_h(Y - Y^i)}{\sum_{j=1}^I K_h(Y - Y^j)}$$

Y denotes GMP and g denotes its growth rate. Sample size is $I = 366$ with each city indexed by a superscript i . $K_h(\cdot)$ is a scaled kernel with a bandwidth h . We gathered data from 2005 and 2010 to compute growth rates. For non-parametric estimation we standardize the growth rate to take out the nationwide growth rate.⁸ GMP is defined by the geometric mean $Y := \sqrt{Y_{05}Y_{10}}$, assuming exponential growth. Figure 4 plots the growth rate and its kernel estimation. We tried to estimate $\hat{m}(Y)$ first (figure 4(a))⁹. The disperse spread of GMP towards the upper end swings the

⁸ In particular we take the difference between $\frac{\log Y_{10}^i - \log Y_{05}^i}{5}$ and the sample mean, divided by the standard deviation to be the normalized growth rate g^i .

⁹ We had to stretch the bandwidth further than the usual width of $2.727e+10$ to cover up the large gap between New York and Los Angeles. The estimated growth rate is positive for GMP larger than $9e+11$ and



(a) Kernel regression on GMP with bandwidth $h = 6.137e + 10$.

(b) Kernel regression on the log of GMP with bandwidth $h = .3932$.

Figure 4.

estimate from side to side and makes it hard to interpret the relationship. We went for a log of GMP instead and recorded the result in figure 4(b), which now exhibits a discernible pattern. There seems to be a slight inclination to the left and right tails, probably because of a smaller number of observations to the both ends than in the rest of the range. Other than that, our estimate seems to be in support of the Gibrat’s law for GMP. For analysis of variance, see appendix A.1.

2.1.2 PARAMETRIC ESTIMATION

Next, we regress GMP growth rate on GMP.¹⁰ Estimates are reported in table 2. Figure 5(a) seems to indicated that the regression line is pulled upwards partly because of New York.¹¹ To counteract this sensitivity to large cities, we regressed GMP on the log of GMP as well (figure 5(b)).

	Regressor			R ²	Figure
	Intercept	GMP	log(GMP)		
Coefficient	4.482e-03	6.981e-15		8.077e-04	5(a)
t-statistic	3.84	.54			
Coefficient	-2.373e-02		1.230e-03	5.044e-03	5(b)
t-statistic	-1.13		1.36		

Table 2. Ordinary least squared (OLS) estimate of growth rate.

The null is not rejected at the 5% level of confidence on GMP or on the log above solely because of New York and significantly lower than zero because of Los Angeles around $6e+11$ to $9e+11$.

¹⁰ We did not standardize the GMP growth rate for parametric estimation. The intercept will capture the nationwide growth rate.

¹¹ The coefficient on GMP may well have been negative had New York’s growth rate been negative. The estimates’ dependence on New York is not all that welcoming because, while it is large, New York is still just one observation as much as Beaumont, TX is.

thereof. Once again the estimates seem to agree with the Gibrat’s law.

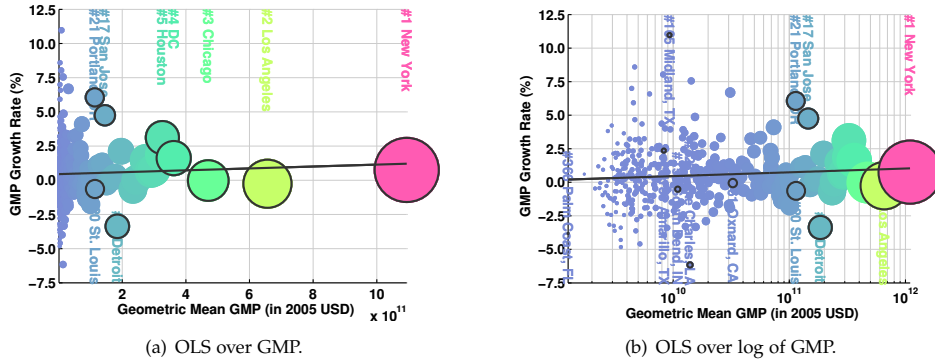


Figure 5.

2.2 ZIPF’S LAW FOR GMP

As we have seen in figure 3, GMP seems to be well traced by a power law. OLS estimation confirms the power-law behavior of GMP, as documented in table 3 and figure 6. The Pareto exponent is -0.9003 on employment¹² whereas we have

	Regressor			R^2	Figure
	Intercept	log(Employment)	log(GMP)		
Coefficient	16.34	-0.9003		.9763	6(a)
t-statistic	174.57	-122.50			
Coefficient			-0.7875	.9756	6(b)
t-statistic			-120.58		

Table 3. Rank-size and rank-GMP regression

-0.7878 on GMP. This is indicative of the fact that the GMP distribution is even more skewed than the corresponding city-size distribution. This is to be theoretically verified with proposition 3.2.

We include OLS just for illustration, with the caveat that it would not work had we had the extensive data. As pointed out by Gabaix and Ioannides [Glo4], the city-size distribution does not sit well with the assumptions on errors in OLS estimation. The same criticism applies to the GMP distribution as well. In addition, due to the limited data range, it is likely that Zipf’s law applies only to the upper tier and that the untruncated GMP distribution deviates from the Zipf’s law for small cities (cf. [Eeco4]). In this case, a distribution other than a Pareto distribution, such as a lognormal or double Pareto lognormal ([GZS10]), is an apt choice to

¹²Employment data are based on population estimates that the Bureau of Economic Analysis uses to compute per capita GMP.

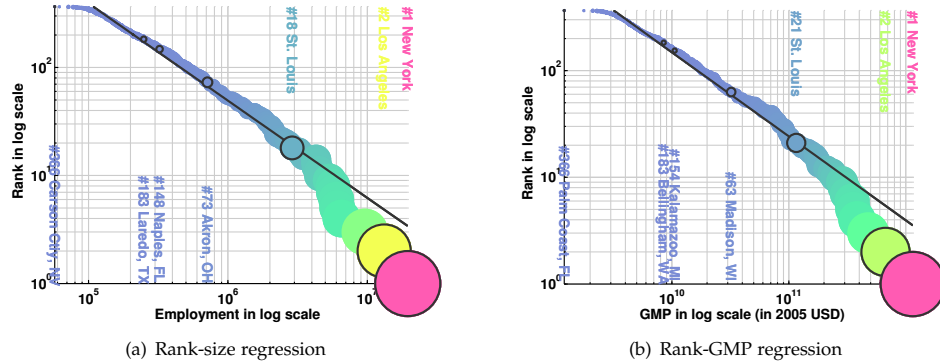


Figure 6.

describe the data. Unfortunately, an exhaustive data set is not available for GMP. In the absence of the lower end of the distribution, the Zipf’s law can be used to describe the remaining mid to upper end of the distribution. Nevertheless, we will reconfirm Zipf’s law for GMP both theoretically (in section 3.2.1) and empirically (in section 3.3) without OLS. The case in point is not whether Zipf’s law describes the upper end of the distribution in particular but that the GMP distribution has a fat tail.

2.3 CITY SIZE AND GMP

Figure 7 shows the relationship between working population and the aggregate product in a city. There seems to be a log-linear relationship between them with coefficient slightly but statistically significantly larger than one, indicating increasing returns to scale between city size and GMP. Table 4 reports the results with figure 7.¹³ The numbers are not too far off from the findings from the second line of work mentioned in section 1. For example Shefer [She73] finds that a 1% rise in input will results in a 1.12% increase in output (note; however, that this is just for the primary metal industry, whereas our numbers are for GMP).

3 MODEL

3.1 SPATIAL PRODUCTION ECONOMY

We construct an intercity general equilibrium model to seek a comprehensive explanation for all the empirical findings in section 2. In particular we develop a production economy with three commodities: composite goods, housing and leisure,¹⁴

¹³Note that

$$\log(Y/L) = \gamma_0 + \gamma_1 \log L \Rightarrow \log Y = \gamma_0 + (\gamma_1 + 1) \log L$$

on a per-capita basis. On aggregate level, $\log Y = \beta_0 + \beta_1 \log L$ so that $\gamma_1 = \beta_1 - 1$, as can be seen in table 4.

¹⁴Alternatively, we can include capital goods but due to lack of data, we limit ourselves to three goods in this economy.

Regressand	Parameter	Value	Intercept	Employment	log(Employment)	R ²	\bar{R}^2	Fig.
GMP	Actual	Coefficient	-6.492e+09	5.409e+04		.9717	.9716	7(a)
		t-value	-7.75	111.81				
log(GMP)	Actual	Coefficient	8.960		1.117	.9528	.9527	7(b)
		t-value	54.02		85.73			
		t-statistic	4.79		-4.81			
log(GMP)	Theoretical	Coefficient	8.165		1.180	.9498	.9497	7(b)
		t-value	47.74		87.79			
GMP per capita	Actual	Coefficient	3.422e+04	2.522e-03		.1257	.1233	7(c)
		t-value	56.69	7.23				
log(GMP per capita)	Actual	Coefficient	8.960		.1173	.1821	.1799	7(d)
		t-value	54.02		9.00			
		t-statistic	4.79		-4.81			
log(GMP per capita)	Theoretical	Coefficient	8.165		.1800	.1302	.1278	7(d)
		t-value	47.74		13.39			
Housing	Actual	Coefficient	-1.736e+09	8489		.9571	.9569	7(e)
		t-value	-9.88	84.60				
log(Housing)	Actual	Coefficient	4.149		1.305	.8769	.8765	7(f)
		t-value	11.98		47.81			
		t-statistic	-4.57		4.59			
log(Housing)	Theoretical	Coefficient	5.732		1.180	.8688	.8684	7(f)
		t-value	16.03		41.88			

Table 4. \bar{R}^2 is an adjusted value of R^2 . For t -value, the null is coefficient equals zero, whereas for t -statistic, the null is coefficient equals theoretical value. Section 4.2 explains theoretical value.

and two types of agents: worker/consumer and landlord.

There are I cities in the economy. S^i residents live in city i , totalling $S = \sum_{i=1}^I S^i$ of urban population nationwide. Each city has a demographic similar to the Alonso model (cf. Berliant and Fujita [BF92]). See figure 8 for one example representation of agents involved in this production economy. Each city has a landlady who owns all the area H in city i . She is retired and lives off her rental income $r^i H$, where r^i marks the city's rental rate (think of her as the first settler in town or a developer). She is an immobile¹⁵ landlady and assumed to consume only composite goods and leisure out of her one unit of allotted time.¹⁶ The remainder of the urban population are mobile, active and identical workers/consumers who supply labor l_R^i out of their one unit of allotted time to produce a basket of goods c_R^i that includes all the goods and services other than housing h_R^i and leisure $(1 - l_R^i)$. Their consumption bundle x^i and endowment e^i are given by

$$x_R^i = \begin{pmatrix} c_R^i \\ h_R^i \\ 1 - l_R^i \end{pmatrix}, \quad x_L^i = \begin{pmatrix} c_L^i \\ h_L^i \\ 1 - l_L^i \end{pmatrix}, \quad e_R^i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_L^i = \begin{pmatrix} 0 \\ H \\ 1 \end{pmatrix},$$

¹⁵ We assume that she cannot change her city of residence so that we can count the rental income toward GMP where it is collected. Otherwise the rental income may be included in the city where she actually lives, which may not be the city whose land she owns if she is an absentee landlady. However, we will not count her toward S^i for notational ease. We will return to the role of her location choice in section 4.2.

¹⁶ Assume that she lives in the city where she is a landlady but in the special lot designated for her outside H to keep our analysis tractable. She needs to live in the city where she is a landlady because of footnote 15.

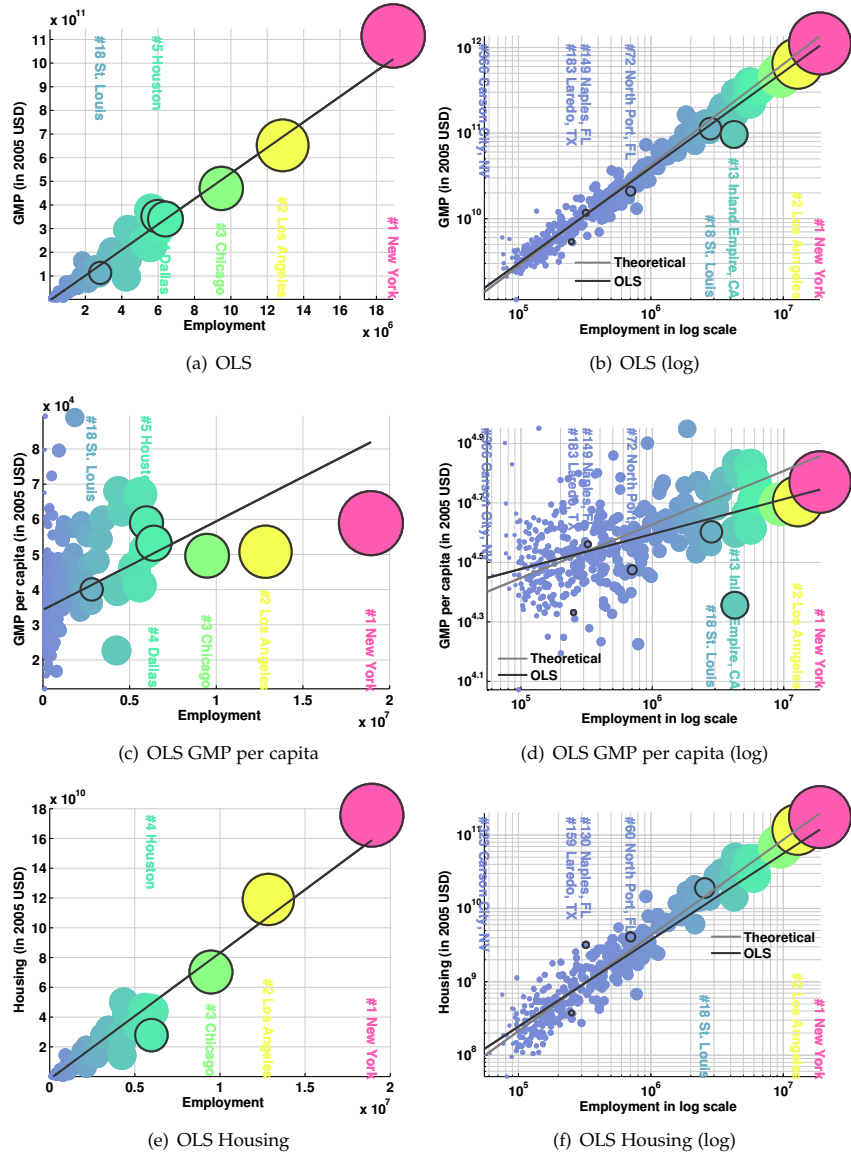


Figure 7.

where subscript R denotes a representative working Resident and L denotes the Landlady.

On the production side, there are many firms in a city who employ one worker each and produce the identical immobile commodity in a perfectly competitive

environment. The production plan y^i of a representative firm is given by

$$y^i = \begin{pmatrix} f(l_F^i, h_F^i; S^i) \\ -h_F^i \\ -l_F^i \end{pmatrix},$$

where l_F^i denotes labor demand and h_F^i denotes land input used as a production site. We let the production function $f(\cdot)$ depend on the city size to allow for externalities within the city such as knowledge spillover effects or congestion to have an impact on productivity of individual firms in the same city.¹⁷

The intracity production economy in city i is identified by

$$\mathcal{P}^i := \left\{ \left(X_N^i, \succeq_N, e_N^i \right)_{N \in \{R, L\}}, Y_F^i \right\},$$

where $X_N^i := \mathbb{R}_+^3$ is a consumption set of a representative worker or the landlady, \succeq_N is a complete preorder over consumption set X_N^i , and Y_F^i is a production set of a representative firm given by

$$Y_F^i := \left\{ y^i = (c_F^i, h_F^i, l_F^i) \in \mathbb{R}_+^3 : c_F^i \leq f(l_F^i, h_F^i; S^i) \right\}.$$

A feasible allocation in \mathcal{P}^i is defined as follows:

DEFINITION 3.1: FEASIBLE ALLOCATION

For given $S^i \in [0, S]$, an allocation $(x_R^i, x_L^i, y^i) \in X_R^i \times X_L^i \times Y_F^i$ in the intracity production economy \mathcal{P}^i is feasible iff

$$x_R^i S^i + x_L^i = y^i S^i + e_R^i S^i + e_L^i. \quad (1)$$

To find GMP we need to compute the *value* of each commodity. Let $p^i := (1, r^i, w^i)'$ be the price on a composite good, lot size and leisure. We take composite goods as a numéraire.¹⁸ There are two equivalent ways to define GMP. From the production point of view, GMP \mathcal{Y}^i is defined by the total value of all the final goods and services produced in the city, $\mathcal{Y}^i = p^i \cdot (y^i + e_R^i S^i + e_L^i)$. From the consumers' end, GMP is the sum of all the expenditures on goods and services, $\mathcal{Y}^i = p^i \cdot (x_R^i S^i + x_L^i)$. They come out to the same number due to Walras' law.

DEFINITION 3.2: GMP

GMP in the intercity production economy \mathcal{P}^i of size S^i is identified by

$$\mathcal{Y}^i := p^i \cdot (y^i + e_R^i S^i + e_L^i) = p^i \cdot (x_R^i S^i + x_L^i). \quad (2)$$

In application GDP does not count leisure time. We consume leisure for the price of the opportunity cost (namely, lost wage), but in practice there is no explicit/accounting trace of market transactions for the consumption of leisure to track down the leisure portion of GDP. In particular we produce and consume $w^i \left\{ (1 - l_R^i) S^i + (1 - l_L^i) \right\}$ worth of leisure, but this part is excluded from recorded

¹⁷ Since we bundle all the goods in a single basket, there is no distinction between localization economies (agglomeration economies within an industry) and urbanization economies (agglomeration economies across the industries within a city) in our model.

¹⁸Note that none of the commodities are tradable beyond the city border in this economy.

GDP and by extension, from GMP as well. We use the data provided by the US Bureau of Economic Analysis, and their figures are based on tax reports. We do not pay tax on leisure consumption. Thus, we shall redefine \mathcal{Y}^i with only the first two entries and take out the last entry (leisure)

$$Y^i := \begin{pmatrix} 1 \\ r^i \end{pmatrix} \cdot \left[\begin{pmatrix} c_R^i \\ h_R^i \end{pmatrix} S^i + \begin{pmatrix} c_L^i \\ h_L^i \end{pmatrix} \right] = \begin{pmatrix} 1 \\ r^i \end{pmatrix} \cdot \left[\begin{pmatrix} f(F^i, H^i; S^i) \\ -h_F^i \end{pmatrix} S^i + \begin{pmatrix} 0 \\ H \end{pmatrix} \right] \quad (3)$$

for statistical purposes.

As we understand from our empirical findings in [section 2](#), Y^i exhibits increasing returns to scale S^i . Then (3) implies that profit less labor compensation $f(\cdot; S^i) - r^i h_F^i$ needs to be increasing in S^i .

To find the equilibrium price vector, first define $\theta^i := (\theta_R^i, \theta_L^i)$ as a vector of a representative resident and landlady's share of profit ($\theta_R^i, \theta_L^i \in [0, 1]$ and $\theta_R^i S^i + \theta_L^i = 1$).

DEFINITION 3.3: INTRACITY EQUILIBRIUM

For a given θ^i and e^i , an intracity equilibrium in city i is a feasible allocation $(x_R^{i*}, x_L^{i*}, y^{i*})$ and price vector p^{i*} such that

1. For $N = R$ and L

$$p^{i*} \cdot x_N^{i*} \leq p^{i*} \cdot e_N^i + \theta_N^i p^{i*} \cdot y^{i*} S^i. \quad (4)$$

2. For $N = R$ and L

$$p^{i*} \cdot x_N^i \leq p^{i*} \cdot e_N^i + \theta_N^i p^{i*} \cdot y^{i*} S^i \quad \Rightarrow \quad x^{i*} \geq_N x^i, \quad (5)$$

for any $x_N^i \in X^i$.

3. For any $y_i \in Y_F^i$,

$$p^{i*} \cdot y^{i*} \geq p^{i*} \cdot y^i. \quad (6)$$

To identify the equilibrium city size, let the intercity production economy $\mathcal{P} := \left\{ (\mathcal{P}^i, S^i)_{i=1}^I, S \right\}$ and define

DEFINITION 3.4: INTERCITY EQUILIBRIUM

For a given ownership matrix $(\theta^i)_{i=1}^I \in [0, 1]^{2I}$ and endowment matrix $(e^i)_{i=1}^I \in \prod_i (X_R^i \times X_L^i)$, an intercity equilibrium in the production economy \mathcal{P} is a list of a feasible allocation matrix $(x_R^{i*}, x_L^{i*}, y^{i*})_{i=1}^I \in \prod_i (X_R^i \times X_L^i \times Y_F^i)$, price matrix $(p^{i*})_{i=1}^I \in \mathbb{R}_+^{3I}$, and size distribution $(S^i)_{i=1}^I \in [0, S]^I$ such that for any i and j with $S^i > 0$ and $S^j > 0$,

1. $(x_R^{i*}, x_L^{i*}, y^{i*}, p^{i*})$ is an intracity equilibrium

- 2.

$$x_R^{i*} \sim_R x_R^{j*}. \quad (7)$$

3. Urban population adds up to

$$\sum_i S^i = S. \quad (8)$$

The second item (7) is due to free mobility of workers. This does not apply to landladies, who are locked in their place of residence to keep the housing portion of GMP where it is generated.

The equilibrium city-size distribution is the size component of an equilibrium in \mathcal{P} and the GMP distribution is (3) computed with an equilibrium in \mathcal{P} .

3.2 APPLICATION

To derive the exact distribution of GMP for empirical testing, consider an application of the spatial production model developed in section 3.1 with production function and labor market in the style of Eeckhout [Eeco4] with the explicit presence of landladies (see figure 8 for a schematic representation of the agents and commodities involved in this example). We will find the analytical solution to the intercity equilibrium, from which we obtain the equilibrium GMP distribution.

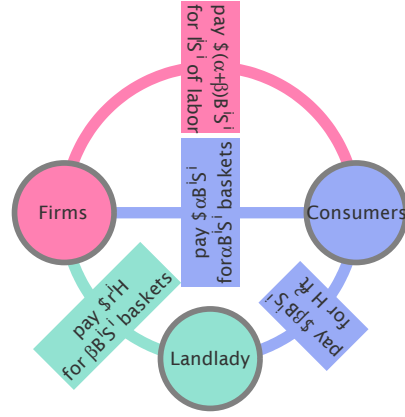


Figure 8. Commodity flow. Leisure is excluded in accordance with the practical definition of GDP adopted by the US Bureau of Economic Analysis.

3.2.1 INTRACITY EQUILIBRIUM

To start off, pick any city i and consider its intracity equilibrium. Firm’s production plan is specified by

$$f(l_F^i, h_F^i; S^i) = A^i a_+(S^i) a_-(S^i) l_F^i, \quad (9)$$

where A^i is a stochastic citywide productivity parameter, $a_+(\cdot) (> 0)$ measures the positive externality shared among the firms operating within the same city, and $a_-(\cdot) (\in (0, 1))$ measures congestion externality. City size is assumed to raise the productivity of all the firms operating in the city. Positive externality enhances with size ($a'_+(\cdot) > 0$). Each consumer supplies l_R^i units of gross labor but congestion externality adversely affects effective labor. The fraction $1 - a_-(S^i)$ of labor will be spent on commuting rather than on production. The level of reduction in effective labor aggravates with the size of a city ($a'_-(\cdot) < 0$). Firms do not pay for the time lost in commuting and workers assume responsibility for the time cost of commuting. That is, firms will pay (ostensible) wages at the rate of ω^i only for the fraction of l_R^i when their worker is present at work, i.e., only for $a_-(S^i) l_R^i$ hours out of l_R^i . On an hourly basis, (effective) wage is knocked down to $w^i(S^i) := \omega^i a_-(S^i)$ for each hour devoted for work, inclusive of commuting time.¹⁹ We will discuss the role of a landlady’s labor supply later. We assume that firms do not require land as input

¹⁹ Hence, the opportunity cost of leisure is $w^i(S^i)$ rather than ω^i .

in accordance with [Eeco4] for simplicity, but land can readily be incorporated into our production economy in section 3.1 as a factor of production.

Profit (6) turns into

$$p^i \cdot y^i = A^i a_+ (S^i) a_- (S^i) l_F^i - \omega^i a_- (S^i) l_F^i = [B^i (S^i) - \omega^i a_- (S^i)] l_F^i, \quad (10)$$

where $B^i(S^i) := A^i a_+ (S^i) a_- (S^i)$. Since production function (9) exhibits constant returns to scale in l_F^i ,

$$p^i \cdot y^{i*} = 0 \quad (11)$$

in equilibrium (otherwise y^{i*} violates profit maximiation condition (6)). Hence, if $l_F^i > 0$, it must follow that

$$B^i(S^i) = \omega^i a_- (S^i) (= w^i (S^i)) \quad (12)$$

in equilibrium.

Note here that aggregate production may exhibit agglomerative economies due to positive externality $a_+(\cdot)$, but internal scale economies are still absent because individual production function is linear in l_F^i . For more on a dialectic between increasing and constant returns to scale, see Rossi-Hansberg and Wright [RHWo7].

Next order of business is the consumers. Represent \succeq_R in \mathcal{P}^i by

$$u_R(c_R^i, h_R^i, l_R^i) = \alpha \log c_R^i + \beta \log h_R^i + \gamma \log(1 - l_R^i), \quad (13)$$

where $\alpha, \beta, \gamma > 0$, and assume $\alpha + \beta + \gamma = 1$ without loss of generality. According to the feasibility condition (1) household income is given by $p^i \cdot e_R^i + \theta_R^i p^i \cdot y^i S^i$. Since firms earn zero profit (11), household income simplifies to labor income $p^i \cdot e_R^i = w^i(S^i) \cdot 1$ alone, with which to buy composite goods c_R^i , housing h_R^i and leisure $(1 - l_R^i)$ at the price of $p^i = (1, r^i, w^i)$. Marshallian demand is

$$x_R^i(p^i, w^i) = \begin{pmatrix} c_R^i(p^i, w^i) \\ h_R^i(p^i, w^i) \\ 1 - l_R^i(p^i, w^i) \end{pmatrix} = \begin{pmatrix} \alpha w^i(S^i) \\ \beta w^i(S^i) / r^i \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha B^i(S^i) \\ \beta B^i(S^i) / r^i \\ \gamma \end{pmatrix} \quad (14)$$

The second equality holds as a result of profit maximization (6) and its consequence (12). Labor supply l_R^i of a typical household will be $1 - \gamma = \alpha + \beta$. Material balance (1) requires that $(1 - l_R^i) S^i + 1 - l_L^i = -l_F^i S^i + 1 \cdot S^i + 1$. Since utility maximization for the retired landlady (5) results in $l_L^i = 0$ in this economy (see (17) below), $l_R^i = l_F^i$, which furthermore implies that the equilibrium production plan will be

$$y^{i*} = \begin{pmatrix} f(l_F^i, h_F^i; S^i) \\ 0 \\ -l_F^i \end{pmatrix} = \begin{pmatrix} (\alpha + \beta) B^i(S^i) \\ 0 \\ -(\alpha + \beta) \end{pmatrix}. \quad (15)$$

Turning to the landlady, represent \succeq_L in \mathcal{P}^i by

$$u_L(c_L^i, h_L^i, l_L^i) = c_L^i \mathbf{1}_{\{l_L^i=0\}}(l_L^i), \quad (16)$$

where $\mathbf{1}_{\{l_L^i=0\}}(\cdot)$ is an indicator function that takes the value of one when $l_L^i = 0$ and zero otherwise. Since she is retired, any hour of labor $l_L^i > 0$ will instantly push her utility level down to zero regardless of an increment in her utility level from an increased consumption of composite goods financed through her labor income.²⁰ That is, her utility level is nonnegative over the plane $l_L^i = 0$ in \mathbb{R}_+^3 and zero elsewhere. Once again, since the share of zero profit (11) earns her nothing, the budget constraint (4) implies that the landlady's income is $p^i \cdot e_L^i = r^i H + w^i (S^i)$. Her Marshallian demand is

$$x_L^i(p^i, w^i) = \begin{pmatrix} c_L^i(p^i, w^i) \\ h_L^i(p^i, w^i) \\ 1 - l_L^i(p^i, w^i) \end{pmatrix} = \begin{pmatrix} r^i H \\ 0 \\ 1 \end{pmatrix}. \quad (17)$$

Then residential utility maximization (14), profit maximization (15) and landlady's utility maximization (17) rewrite feasibility condition (1) as

$$\begin{pmatrix} \alpha B^i(S_R^i) \\ \beta B^i(S_R^i)/r^i \\ 1 - (\alpha + \beta) \end{pmatrix} S^i + \begin{pmatrix} r^i H \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} (\alpha + \beta) B^i(S^i) \\ 0 \\ -(\alpha + \beta) \end{pmatrix} S^i + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} S^i + \begin{pmatrix} 0 \\ H \\ 1 \end{pmatrix}. \quad (18)$$

from which, along with the first order condition (12), we can find the equilibrium price vector in city i as

$$p^{i*} = \begin{pmatrix} 1 \\ r^i \\ w^i \end{pmatrix} = \begin{pmatrix} 1 \\ \beta B^i(S^i) S^i / H \\ B^i(S^i) \end{pmatrix} \quad (19)$$

(see figure 8 to track the commodity flow in equilibrium over \mathcal{P}^i). Observe that the rent r^i goes up if 1) city i draws a good technological shock A^i , 2) positive externality $a_+(S^i)$ intensifies, or 3) city i becomes less crowded ($a_-(S^i)$). Likewise the leisure becomes expensive for the same reasons. Reasons 2) and 3) are triggered by urban growth, whereas reason 1) is independent of S^i .

It is worth pointing out that our economy makes a judicious use of limited labor. The working population is capped at S . The intracity equilibrium and more noticeably, the intercity equilibrium, allocate more people to a city with a good production environment and pull back labor from a city of low productivity. Indirect utility is increasing in $B^i(S^i)$ in equilibrium. Migration dynamics are such that there is an inflow when $B^i(S^i)$ is above the national average and vice versa if it is below. The economy has an auto-rerouting mechanism built into it if population allocation ever deviates from the equilibrium, by incentivizing people to move to a productive city.²¹

²⁰ Alternatively, we could model her as an active worker, which complicates our notations without much gain in insights.

²¹ The allocation of labor is still not efficient though, due to externalities.

With feasibility condition (18) and the equilibrium price (19) we obtain GMP (2) in equilibrium as follows:

$$\begin{aligned}
\mathcal{Y}^i &= p^i \cdot (x_R^{i*} + x_L^{i*}) \\
&= \underbrace{(\alpha + \beta)B(S^i)S^i + \beta B(S^i)S^i}_{\text{Reported portion of GMP } Y^i \text{ (3)}} + B(S^i)(\gamma S^i + 1) \\
&= (1 + \beta)B(S^i)S^i + B(S^i).
\end{aligned} \tag{20}$$

On the second line in (20) are the value of composite goods, housing and leisure for each. Only the first two are included in the actual GMP.

Equilibrium GMP (20) leads to the following:

PROPOSITION 3.1: CITYWIDE SCALE ECONOMIES IN INTRACITY ECONOMY

Consider the equilibrium in an intracity economy \mathcal{P}^i . The reported portion of GMP Y^i exhibits increasing returns to scale in size S^i iff $B^i(S^i)S^i$ exhibits increasing returns to scale in S^i .

Proof. Apparent from (20). □

In reference to section 2.3, observed data seem to suggest that positive externality does outstrip negative externality. However, as we will see in proposition 3.2, citywide scale economies in *intercity* equilibrium will be positive without assuming increasing returns to scale on $B^i(S^i)S^i$.

3.2.2 INTERCITY EQUILIBRIUM

To find the intercity equilibrium in definition 3.4, rewrite indifference principle (7) in terms of a utility function (13) so that indirect utility $u(x_R^{i*}) = u(x_R^{j*})$ for any i and j with $S^i, S^j > 0$. This leads to

$$B^i(S^i)(S^i)^{-\frac{\beta}{\alpha}} = B^j(S^j)(S^j)^{-\frac{\beta}{\alpha}} =: K, \tag{21}$$

where K is a location-invariant constant. According to (20), GMP is $\mathcal{Y}^i = (1 + \beta)K(S^i)^{\frac{\beta}{\alpha}+1} + K(S^i)^{\frac{\beta}{\alpha}}$ and the reported portion of GMP will be

$$Y^i = (\alpha + 2\beta)K(S^i)^{\frac{\beta}{\alpha}+1} \tag{22}$$

in the neighborhood of the equilibrium size, which breaks down into composite goods production/consumption $(\alpha + \beta)K(S^i)^{\frac{\beta}{\alpha}+1}$ and housing production/consumption of $\beta K(S^i)^{\frac{\beta}{\alpha}+1}$. Note that productivity parameter A^i no longer makes an explicit appearance in (22) but GMP is still positively related to it as a high value of A^i is reflected in Y^i through increased S^i . Now we have

PROPOSITION 3.2: CITYWIDE SCALE ECONOMIES IN INTERCITY ECONOMY

If an intercity economy \mathcal{P} is in equilibrium, the reported portion of GMP Y^i exhibits increasing returns to size in the neighborhood of equilibrium size S^i .

Proof. Immediate from (22). □

In comparison to [proposition 3.1](#) it is curious that we have to assume $B^i(S^i)S^i$ to be increasing returns to scale only in \mathcal{P}^i but not in \mathcal{P} . The short answer to this enigma is that free mobility puts the city size where scale economies are at work. For illustrative purposes, assume that $a_+(S^i)a_-(S^i)$ takes the form $(S^i)^{\delta(S^i)}$. Then [proposition 3.1](#) specifically requires $\delta(S^i) > 0$ in \mathcal{P}^i but [proposition 3.2](#) does not because perfect mobility will bring $\delta(S^i)$ above zero anyway. Now assume that \mathcal{P} is in equilibrium. Suppose that in some city i , size S^i rose by one (call this new resident Axel). In this case, housing consumption will be reduced in the city because residents have to make room for Axel's house out of a fixed supply of H , and he also exacerbates congestion in the city. However, since \mathcal{P} is in equilibrium, reduction in utility level from curtailed housing consumption needs to be offset by either c_R^i or l_R^i in compliance with utility equalization (7). Since l_R^i is independent of size (i.e., leisure consumption does not and cannot accommodate the change to the city residents introduced by Axel), compensation must be made through increased consumption in c_R^i alone. Then the question is: Can he produce enough composite goods to leave everyone in city i on the same indifference curve?

It is the answer to this question that makes \mathcal{P} increasing returns to scale in size in equilibrium. The marginal rate of substitution between c_R^i and h_R^i is $\frac{\beta c_R^i}{\alpha h_R^i}$ baskets of composite goods for each ft². Now, Axel carves $\frac{\partial h_R^i}{\partial S^i} = \frac{h_R^i}{S^i}$ ft² from every resident's lot in the city. Thus, each city resident needs to have $\left(\frac{\beta c_R^i}{\alpha h_R^i}\right) \frac{h_R^i}{S^i} = \frac{\beta}{\alpha} c_R^i \frac{1}{S^i}$ more baskets to keep to the countrywide utility level (or else the current allocation will not be an equilibrium). Then the addition of Axel into the city needs to raise the individual production of composite goods as follows:

$$\begin{aligned} \frac{\partial f(\cdot; S^i)}{\partial S^i} &= \frac{\beta}{\alpha} f(\cdot; S^i) \frac{1}{S^i} \\ \Rightarrow \delta(S^i) A^i (S^i)^{\delta(S^i)-1} &= \frac{\beta}{\alpha} A^i (S^i)^{\delta(S^i)-1} \\ \Rightarrow \delta(S^i) &= \frac{\beta}{\alpha} (> 0). \end{aligned} \tag{23}$$

²² If not, for example, if $\delta(S^i) < \frac{\beta}{\alpha}$, then Axel cannot make up for the lost individual housing unit by producing more composite goods through enhanced pooled production externality net of the congestion externality. The knowledge spillover effect he brings in (less the congestion externality he exerts) is not enough to render

²²We took the landlady out of equation because her marginal rate of substitution between composite goods and housing is zero. Cutbacks in housing lot do not affect her at all due to her preferences (16).

the dwindled housing consumption tolerable for the current residents. In this case, city i is better off bumping him out, i.e., it should reduce S^i , contradicting the fact that \mathcal{P} is in equilibrium. And vice versa, city i should be larger if $\delta(S^i) > \frac{\beta}{\alpha}$. Everyone welcomes Axel and wants more residents to move in in this case. Thus, free mobility arbitrages the gap between externality component $\delta(S^i)$ and countrywide constant $\frac{\beta}{\alpha}$ and forces the city to operate in the domain where scale economies are present (otherwise, there will still be an in- or out-flow of people). Note that utility equalization (7) only applies to cities with $S^i > 0$. If a city's aggregate production function does *not* exhibit increasing returns to scale *anywhere* over $0 < S^i \leq S$, then the city will not survive and turns rural in the end. At such a location, $\delta(S^i) < \frac{\beta}{\alpha}$ and all the residents will be drained off to other cities until S^i becomes zero.²³ Thus, increasing returns to scale at the aggregate level are an eligibility requirement to be listed under MSA. See [appendix A.2](#) for further discussion on scale economies in \mathcal{P} as opposed to \mathcal{P}^i .

Note, however, that the preceding argument makes a reference only to the *in-tercity* utility level and it is not a normative assertion of utility maximization with respect to city size. The equilibrium city size does not necessarily coincide with the optimal city size where utility level is maximized. Free mobility simply assigns workers to each city according to exogenously drawn productivity parameter A^i so as to even out the utility level across the country. For instance, Boston could use *more* people to raise its intracity utility level while New York could use *less* people to raise its intracity utility level. Nevertheless, the city size distribution may stay put as long as the utility level is equalized among cities. A worker makes a location choice based on her own utility level without reference to her effect on existing residents' welfare. A social planner may impose an optimal allocation of workers to maximize a population-weighted utility level over the country for example, but he will have to forgo utility equalization (7) to do so, which may not be sustainable in practice.

It is crucial that we unbundle housing consumption from the composite good. If we include housing as part of a composite good, per capita consumption level, and consequently individual production levels will be the same across the cities because the consumption of leisure is the same everywhere and free mobility guarantees an equal utility level. Then aggregate production level becomes directly proportional to the city size. Alternatively, we can unbundle the composite good and create markets for many commodities. In that case we may have increasing returns to scale in *production* but the price of individual commodities tend to negate the variations in output levels and *GMP* will be only proportional to the city size. A positive technological shock enhances the production, which reduces the equilibrium price in a perfectly competitive market. Thus, the *value* of the output will exhibit constant returns to scale, which is not compatible with our findings in [section 2.3](#). We will have to forgo the assumption of perfectly competitive market in this case.

²³ Notice that as the expenditure share of housing β increases, it becomes harder and harder to meet the condition (23) and more cities will be abandoned and fewer cities will survive. As we will see later, (24) confirms that a rise in β will skew the distribution.

3.3 DISTRIBUTION OF GMP

Eeckhout [Eeco4] has shown that S^i follows the lognormal distribution using the central limit theorem. The equilibrium size of a city can be written as a sum of the log of error terms over time. The city size depends on the cumulative effect of multiplicative nature rather than of additive nature [LSAo1], leading to the lognormal distribution (see appendix A.3 for details). In particular $\log(S^i) \sim N(\mu_s, \sigma_s^2)$. In conjunction with (22) we obtain the following:

PROPOSITION 3.3: GMP DISTRIBUTION

The reported portion of GMP follows a lognormal distribution:

$$\log Y \sim N\left(\left(\frac{\beta}{\alpha} + 1\right)\mu_s + \log(\alpha + 2\beta)K, \left\{\left(\frac{\beta}{\alpha} + 1\right)\sigma_s\right\}^2\right) \quad (24)$$

There is a log-linear relationship between GMP and city size (22) and city size follows a lognormal distribution. Naturally, GMP also follows a lognormal distribution by extension. The variance of $\log(Y)$ is inflated by $\frac{\beta}{\alpha} + 1$ due to citywide scale economies (proposition 3.2). This observation is consistent with our findings in section 2.2. GMP (in log scale) spreads further than its city-size counterpart in \mathcal{P} .

Eeckhout [Eeco4] also establishes the Gibrat's law $\frac{d \log S_t^i}{dt} \approx \epsilon$ (t denotes time). See appendix A.3). Then from (22),

$$\frac{d \log Y_t^i}{dt} = \left(\frac{\beta}{\alpha} + 1\right) \frac{d \log S_t^i}{dt} \approx \left(\frac{\beta}{\alpha} + 1\right) \epsilon_t^i,$$

where ϵ_t^i is an i.i.d. random variable. Thus GMP also follows the Gibrat's law. Note that variation in Y^i is inflated by $\frac{\beta}{\alpha} + 1$ and this is coherent with our empirical findings in section 4.1.

4 EMPIRICAL IMPLEMENTATION

4.1 DISTRIBUTION OF GMP

We will put proposition 3.3 to an empirical test in this section. First, rewrite the GMP distribution (24) as $\log Y \sim N(\mu, \sigma^2)$. The maximum likelihood estimator of (24) is $\hat{\mu} = \frac{\sum \log Y^i}{N}$ and $\hat{\sigma}^2 = \frac{\sum (\log Y^i - \hat{\mu})^2}{N}$. We report our estimations in table 5 with supporting density plots in figure 9. Housing portion of GMP, $\beta K (S^i)^{\frac{\beta}{\alpha} + 1}$, is also available and they are expected to follow the lognormal distribution as well.

The overall fit is not too far off. The maximum discrepancy between the empirical and estimated CDF (Kolomogorov-Smirnov statistic) is .1032. We are more than certain that the fit would improve if we used an inclusive data set. Due to truncation to the left of the distribution, the tail end of the distribution does not extend as far as the theory predicts to the left. At this point we do not have GMP data for smaller cities. Hopefully GMP on micropolitan statistical areas or census-designated places will become available someday but we will settle for MSA data for now.

	Employment	GMP	Housing
Data Size I	366	366	323
Censored (Unreported Cities)	0%	0%	11.75%
Censored (Unreported Value)	0%	0%	13.87%
Estimated Mean $\hat{\mu}_S, \hat{\mu}$	12.68	23.13	20.66
Estimated Variance $\hat{\sigma}_S^2, \hat{\sigma}^2$	1.128	1.478	2.133
Theoretical Variance $\left\{ \left(\frac{\beta}{\alpha} + 1 \right) \hat{\sigma}_S^2 \right\}^2$	-	1.571	1.571
Log Likelihood/ I	.2709	.2370	.1968
Kolmogorov-Smirnov Statistic	.1122	.1032	.1031

Table 5.

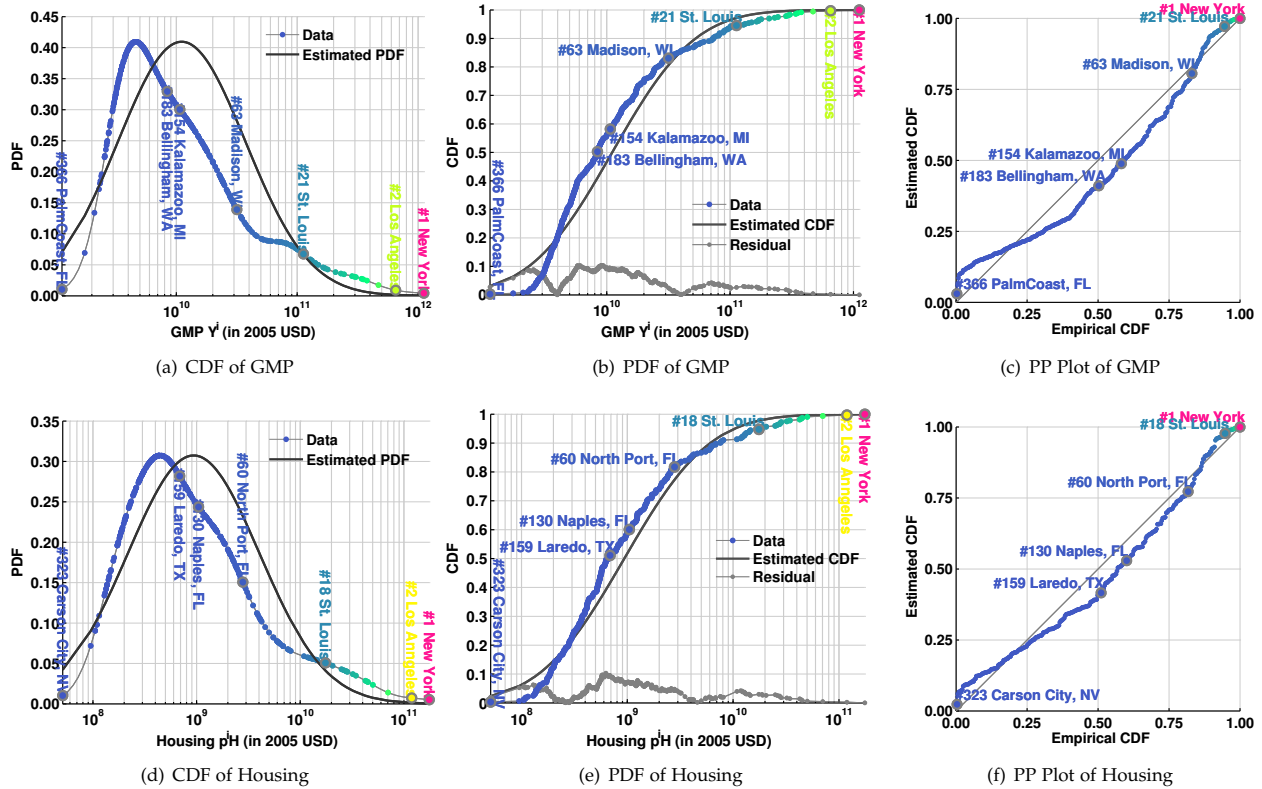


Figure 9.

As we will see in section 4.2, the ratio between β and α is .1800. Following the GMP distribution (24), the theoretically expected value of σ^2 is $(1.180\hat{\sigma}_S^2)^2 = 1.571$ (cf. theoretical variance in table 5). Our expected variance in GMP computed from the expected expenditure shares α and β and estimated variance σ^2 in employment

is very close to the actual variance in GMP (we missed the actual value only by 5.86%). This confirms the validity of the form (24) along with the selection of the utility and production functions in section 3.2. On the other hand, the variance on housing is larger than the theoretical value by 35.86%. We will explore the cause of the large gap in housing in section 4.2.

4.2 SCALE ECONOMIES

According to (22), the ratio between GMP and housing is $\alpha + 2\beta$ to β . The actual ratio is $\$1.162e+13$ to $\$1.603e+12$ among the MSA's, indicating that the expenditure share β of the housing sector is 13.23%.²⁴ Hence, the expected ratio $\frac{\beta}{\alpha} = .1800$. Taking a log of (22),

$$\log Y^i = \log(\alpha + 2\beta)K + \left(\frac{\beta}{\alpha} + 1\right) \log(S^i) = \log(\alpha + 2\beta)K + 1.180 \log(S^i).$$

The actual value of the coefficient is 1.117 in table 4 rather than 1.180, meaning that our model overshoots the coefficient by only 5.64% (or, the economy is off where it should be by 5.64%). The housing portion of GMP $\log \beta K (S^i)^{\frac{\beta}{\alpha}+1}$ also shares the same coefficient in (22). Here the predicted value comes *short* of the actual value by 10.59%. The large discrepancy may be because of the censored data,²⁵ or the landlords' or developers' registered addresses, which may be different from the city where they have their real estate. The fact that imputed rent is excluded and that houses usually last longer than the duration of a fiscal year exacerbates the deviation even further.

5 CONCLUSION AND EXTENSIONS

We have discovered that the GMP distribution shares the same pattern as the city-size distribution, and we sought a systematic illustration of how our local economies are related to their employment and output levels on a national scale. Proposition 3.2 further revealed that GMP grows more than 1% against a 1% growth in employment. Large cities make up for an exceeding share of GDP and they do so more than their city size alone can account for. Consequently, due to agglomeration economies of GMP in employment, the GMP distribution is even more skewed than the city-size distribution.

We constructed a production economy model that endogenously gives rise to agglomeration economies in equilibrium. The interplay between externalities and housing consumption drives the cities to operate at the size where increasing returns to scale are present. Empirical testing verifies our model prediction; however,

²⁴Note that GDP includes real estate sales and excludes imputed rent. Thus, the figure does not necessarily represent the expenditure share in a particular year.

²⁵Housing output values are not reported for all MSA's and the missed portion is not negligible. For example, housing sector in Dallas (rank #4 in employment) is censored out of the data. However, these censored values are included in the nationwide aggregate.

the result could be made better off. Due to data truncation, our predicted distribution does not trace the lower end of the distribution well. Ideally, we would like to test our prediction with an exhaustive data set, which, for the moment, does not exist.

Our objective was to explain the GMP distribution in a consistent manner. As far as we know this is the first attempt to analyze GMP as a distribution. Along the way, we have left several prospects for future work. We assumed a single-input production function. Local output may well be affected by capital, educational attainment, location of the city, access to a rich labor pool, or urban infrastructure, which, obviously vary from city to city. We also packed the consumption goods other than housing and leisure into a single basket. In reality a city comes with various industries. Some of them may exhibit increasing returns to scale and some may not both within and across the industries. Cross-sectional GMP analysis is called for to decode the internal workings of local economies that, on aggregate, exhibits increasing returns to scale and a fat-tail distribution. Lastly, we assumed that all goods are immobile. Data reported by the Bureau of Economic Analysis are based on tax filed in each MSA. As such, the scope of GMP matches the range of production in MSA, but it does not necessarily match the range of consumption in the city. The openness of a spatial economy may be addressed by adding shipping firms to the production economy. One way to do so is to assume that a shipping firm takes composite goods in city i as input and "produces" the same composite good in city j as output, in less than a one-to-one ratio to reflect shipping charges of iceberg form.

A APPENDIX

A.1 ANALYSIS OF VARIANCE

Figure 10 presents the kernel estimate of GMP growth rate. Aside from an increase in variance in the lower mid range and decrease in the mid range, there does not seem to be a systematic correlation between GMP and its variance in growth rate.

A.2 UNCONDITIONAL SCALE ECONOMIES

To further understand the reasoning behind proposition 3.2, conduct comparative statics on $\frac{\beta}{\alpha}$. Imagine that β goes down (or equivalently, α goes up). In intracity equilibrium in \mathcal{P}^i , $B^i(S^i)$ can take any value. In intercity equilibrium in \mathcal{P} , $B^i(S^i)$ is subject to utility equalization condition (21). As the expenditure share of housing β decreases, we can pack lots of people in a city (because they do not care about the lot size much) and produce lots of composite goods (which they do care about). The story ends here in \mathcal{P}^i . In \mathcal{P} , it goes further. Increased city size makes the city appealing because people are willing to swap a large parcel of land for only a few composite goods to squeeze Axel in when β is small. People view a large city with small houses more favorably than a small city with large houses. To offset the rush of people into a large city, the effective wage rate (19) in the city goes

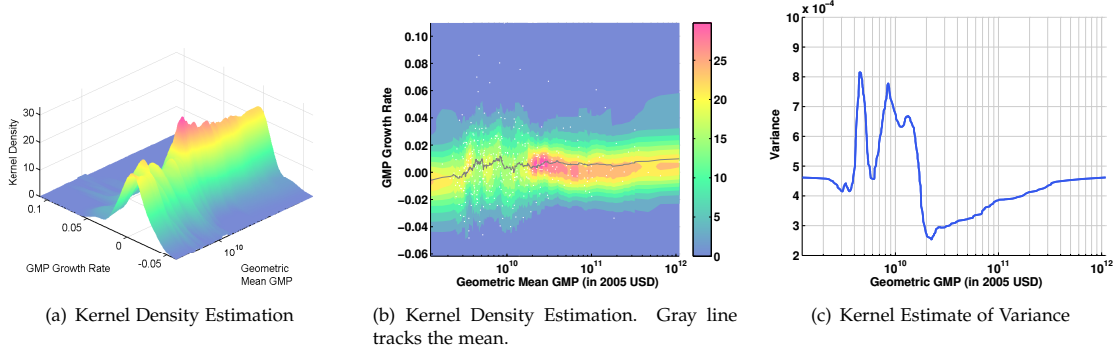


Figure 10.

down in equilibrium to meet the shared utility level (7). \mathcal{P}^i does not factor in the levelling effect of the wage across the cities as much as \mathcal{P} does. Notice that as β becomes smaller, scale economies also weaken because the residents do not care about housing, and they become more willing to give up their land in exchange for even a limited increase in composite goods. Housing market is indispensable in this sense to observe endogenously induced agglomeration economies in \mathcal{P} .

This observation compares to a closed and open monocentric city model (cf. Brueckner [Bru87]). In a closed monocentric city, size is exogenous but utility level is endogenous just as in \mathcal{P}^i . In an open monocentric city, size is endogenous but utility level has to match the national level as in \mathcal{P} . Since the wage rate depends on city size, \mathcal{P} picks the levelling effect of wage but \mathcal{P}^i does not, and therefore, we have to throw in an additional assumption for \mathcal{P}^i .

A.3 THE CITY-SIZE DISTRIBUTION AND GIBRAT'S LAW

Denote discrete time by subscript t and define $\Lambda(S_t^i) := a_+(S_t^i)a_-(S_t^i)(S_t^i)^{-\frac{\beta}{\alpha}}$ and suppose $\Lambda(\cdot)$ is invertible in the neighborhood of equilibrium S^i . Then $A_t^i \Lambda(S_t^i) = K$ from (21) so that $S_t^i = \Lambda^{-1}(K/A_t^i)$. With the law of motion $A_t^i = (1 + \sigma_t^i)A_{t-1}^i$, we have

$$S_t^i = (1 + \epsilon_t^i)S_{t-1}^i, \quad (25)$$

where $1 + \epsilon_t^i := 1/\Lambda^{-1}(1 + \sigma_t^i)$. Then

$$\frac{d \log S_t^i}{dt} \approx \epsilon_t^i,$$

for a small ϵ , leading to the Gibrat's law. Computing the city size recursively, (25) also implies

$$\log S_t^i \approx \log S_0^i + \sum_{\tau=1}^t \epsilon_{\tau}^i,$$

which leads to the lognormal city-size distribution as $t \rightarrow \infty$ by the central limit theorem. See [Eeco4] for details.

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