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First-Price Auctions with Resale**

by

Charles Z. Zheng

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Existence of Monotone Equilibria in First-Price Auctions with Resale*

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Abstract

Existence of a monotone pure-strategy perfect Bayesian equilibrium is proved for a multistage game of first-price auctions with interbidder resale, with any finite number of ex ante different bidders. Endogenous gains at resale complicate the winner's curse and upset previous fixed-point methods to prove existence of monotone equilibria. This paper restructures the fixed-point approach with respect to comparative statics of the resale mechanisms strategically chosen after the auction. Despite speculation possibilities and the discontinuity-inducing uniform tie-breaking rule, at our equilibrium any bid that stands a chance to win is strictly increasing in the bidder's use value.

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1 Introduction

Analyses of economic institutions are based on existence of equilibria of the underlying games. Among them first-price auctions, widely used in practice, are of particular theoretical interest because of a discontinuity problem, arising at tying bids, that may upset standard arguments of equilibrium existence. To solve this discontinuity problem sophisticated methods based on fixed-point theorems have been developed, one guaranteeing existence of monotone pure-strategy equilibria due to Athey [1], McAdams [10], Reny and Zamir [16], and Reny [15], and the other for mixed-strategy equilibria, augmented with endogenous tie-breaking rules, due to Jackson, Simon, Swinkels and Zame [5].¹ However, neither method has been applied to dynamic games such as auctions with resale.² With resale, foundational assumptions need to be reexamined with respect to the continuation play at resale. For example, a main hurdle for the fixed-point approach to monotone equilibria is the winner's curse, which has been handled in the literature by bounding it with sufficiently strong primitive assumptions. But resale would endogenize the winner's curse and renders it unbounded a priori, as a bidder could magnify the winner's curse for the rivals by acting as a high-bidding speculator so that his rivals might want to lose now and buy the good at resale. This paper contributes to the monotone pure-strategy fixed-point approach by restructuring it with respect to comparative statics of resale thereby proving existence of a perfect Bayesian equilibrium, with strictly increasing bid functions, for a two-stage game of a first-price auction with resale.

Let us start by looking into the main steps of the monotone fixed-point approach to see how they may fail given resale possibilities. After that, the rest of the Introduction will outline how these steps are replaced by new arguments based on analysis of resale.

The general idea of this fixed-point approach, dating back to the general equilibrium literature,³ is to approximate the original economy by some sequence of finite economies where equilibria exist and then prove that a limit point of the sequence of such *approximation equilibria* is an equilibrium of the original one. For auctions, the main impediment to such passing-to-limit arguments is a discontinuity problem caused by the possibility of ties. For

¹ Kotowski [6] has a recent application of the fixed-point methods in auctions with budget constraints.

² The conceptual awkwardness of the no-resale assumption has been noted by Zheng [19] and Hafalir and Krishna [4]. The possibility of resource misallocation, which may occur at equilibrium in first-price auctions among ex ante different bidders given the no-resale assumption, induces bidders to attempt resale.

³ For example, Werner [18] and Magill and Quinzii [9].

instance, in a three-bidder case depicted by Figure 1, each bidder i plays an equilibrium bidding strategy β_i^m , a nondecreasing function from his type t_i to a bid, in the approximation auction game indexed by m ; when the sequence $(\beta_1^m, \beta_2^m, \beta_3^m)_{m=1}^\infty$ converges to its limit, a nonvanishing mass of bids, submitted by bidder 1 of types in $[a_1, z_1]$ and bidder 2 of types in $[a_2, z_2]$, are clustered within an interval collapsing into the point x (while bidder 3's types that bid within the cluster vanish into a point z_3). The crucial stage of the fixed-point approach is to demonstrate a contradiction to the approximation equilibria by arguing that some types of at least one of the bidders, say some elements in $[a_2, z_2]$, strictly prefer to deviate from their β_2^m -bids within the cluster at x to a bid say x' slightly above the cluster. This no-tie argument, due to Athey [1] and now standard within the fixed-point literature,

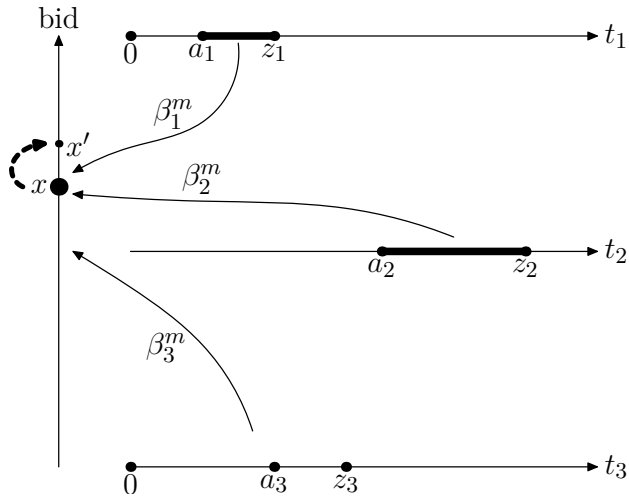


Figure 1: A tying situation

can be summarized into two steps, illustrated here from bidder 2's viewpoint:⁴

- i. One needs to prove that, as bidder 2's type increases from a_2 to z_2 , his preference to winning strictly increases and eventually, with sufficiently high types, he strictly prefers to win conditional on the *winning event* that he can win with the β_2^m -bids within the cluster at x , which roughly corresponds to the event " $(t_1, t_3) \in [0, a_1] \times [0, z_3]$ ".
- ii. For the desired contradiction it suffices to show that the types obtained in the previous step strictly prefer to deviate to x' from their β_2^m -bids within the cluster at x . This was done by proving that their expected net gains from winning cannot decrease when

⁴ The two steps correspond to Claims 1 and 2 in the Appendix of Athey [1].

they consider only the event in which the deviation is pivotal, i.e., that bidder 2 cannot prefer less to win when the conditioned event moves from the winning event “ $(t_1, t_3) \in [0, a_1] \times [0, z_3]$ ” up to the *pivotal event* “ $(t_1, t_3) \in [a_1, z_1] \times [0, z_3]$ ”.

To see the troubles, consider an independent private values model where t_i is bidder i 's use value of the good for sale. Step (i) can fail because a bidder with high types, say the elements of $[a_2, z_2]$ in Figure 1, may eventually acquire and consume the good whether he wins it now or buys it later at resale. Then the type t_2 in bidder 2's payoff as a winner is canceled out by the t_2 in his payoff as a loser, so his net gain from winning does not increase in t_2 , and $[a_2, z_2]$ need not contain a type that strictly prefers to win, contrary to Step (i).

To consider a case where Step (ii) is unsalvageable, suppose within this paragraph that, in Figure 1, bidder 1's bids within the cluster at x are above bidder 2's within the cluster, so that bidder 1 wins when they both bid in the cluster. Thus, the winning event for bidder 2, when he bids within the cluster, corresponds to “ $(t_1, t_3) \in [0, a_1] \times [0, z_3]$ ”, while the pivotal event for bidder 2's deviation from the cluster to x' corresponds to “ $(t_1, t_3) \in [a_1, z_1] \times [0, z_3]$ ”. Athey's Step (ii) would argue that bidder 2's preference to win does not decrease when the conditioned event moves from the winning event to the pivotal one. Given resale, however, the opposite can be true. For instance, let the probability of $[a_3, z_3]$ be so large that, conditional on the winning event $[0, a_1] \times [0, z_3]$, if bidder 2 loses then with a large probability he buys the good from the types $[a_3, z_3]$ of bidder 3. By contrast, conditional on the pivotal event $[a_1, z_1] \times [0, z_3]$, if bidder 2 loses, he buys the good from bidder 1 with types in $[a_1, z_1]$. Since $[a_3, z_3]$ is higher than $[a_1, z_1]$ in strong-set order, the resale price offered to bidder 2, in expectation, is higher in the winning event (where bidder 3 is the reseller) than in the pivotal event (where bidder 1 is the reseller). Thus, when the conditioned event moves up to the pivotal one, bidder 2's expected payoff from losing, or roughly speaking the *winner's curse*, becomes higher. On the other hand, bidder 2's payoff from winning is invariant to his rivals' types because, from Figure 1, $a_2 > z_3 > z_1$ and hence if he wins then he will consume the good to obtain its use value t_2 . Consequently, when he takes into account that his deviation is pivotal, bidder 2 prefers strictly less to win, contrary to Step (ii).

The fundamental reason why Athey's no-tie argument does not work here is that a monotonicity assumption in the literature may fail given resale. The assumption stipulates that a bidder's ex post net payoff from winning is nondecreasing in his rivals' types (e.g., A.1.iii of Reny and Zamir). With resale, by contrast, a winner's payoff may fail to be

nondecreasing in his rivals' types because the optimal resale mechanism may resell the good to a subsidized bidder who pays a lower price than someone else, so the winner's ex post resale revenue could decrease when a subsidized bidder's type rises to buy the good from him at resale. A loser's payoff may fail to be nonincreasing because a loser's gain from trading with reseller j may be larger than that with reseller k . Thus, when j has a slightly higher type to become the reseller instead of k , this bidder's ex post payoff increases. Hence the ex post net gain from winning may fail to be nondecreasing in the rivals' types.

In addition to the no-tie argument, two other important conditions, which did not appear difficult in the received literature, become problematic given resale. One is single crossing, crucial to guarantee existence of the aforementioned approximation equilibria. The other is payoff security, which is needed to deliver the passing-to-limit result in the literature. The single-crossing condition says that if a bidder prefers a high bid to a low one then the preference remains so when his type gets higher. The literature obtained this condition by assuming it for every possible profile of realized types (e.g., A.1.iv of Reny and Zamir). With resale, the assumption fails when an increase of a bidder's type turns him from a speculator to a consumer, with sufficiently high types of his rivals.⁵ The payoff-security condition says that bidding slightly above an atom of the rivals' bids does not make a bidder worse-off than bidding at the atom. In the literature, verification of this condition is simply Step (ii),⁶ which as illustrated above can fail with resale.

This paper is devoted to overcoming these challenges that resale presents to the fixed-point approach. To capture the endogenous nature of resale, we assume that the resale mechanism is a reseller-optimal auction à la Myerson [13] based on post-auction beliefs. Athey's critical steps are restructured with respect to new comparative statics properties of the Myerson resale auction, with initial bids or post-auction beliefs being the parameters.

The existence proof starts by establishing an increasing-difference theorem (Theorem 1), which through its single-crossing implication ensures existence of the aforementioned approximation equilibria. It is based on two comparative statics properties of the Myerson resale mechanism (Propositions 1 and 2). Then comes the critical step, the no-tie argument,

⁵ While the higher bid brings about higher revenues for the speculator-type since he charges higher resale prices due to the higher posterior about the willingness-to-pay of his clientele, the consumer-type, who benefits from none of such revenue effect, strictly prefers the lower bid, which costs him less. This also upsets a slightly weaker single-crossing assumption proposed by Quah and Strulovici [14, Th. 4(c), p28].

⁶ For example, the displayed formula (A.5) in Reny and Zamir [16].

to prove that ties do not occur at a limit point of a sequence of such approximating equilibria (Theorem 2). With its counterpart in the received literature hindered by resale, our no-tie argument is complicated and relies on new properties of endogenous resale uncovered in this paper. Then a passing-to-limit argument delivers the existence theorem (Theorem 3).

The first step of our no-tie argument is to prove that, if a tie at the limit occurs then there exists a *dominant* bidder whose probability of winning the tie converges to one (Lemma 8). In Figure 1, for instance, the infimum a_1 of bidder 1's types that bid within the cluster at x is less than all elements of $[a_2, z_2]$, bidder 2's types bidding in the cluster. Consequently, with types being use values of the good, conditional on the pivotal event " $(t_2, t_3) \in [a_2, z_2] \times [0, z_3]$ " of the bid increase from the cluster to x' , bidder 1 would have zero gain from trading with the reseller player 2. I.e., bidder 1 would suffer zero winner's curse with the bid increase. On the other hand, the bid increase generates a *revenue effect* by adding a mass of high types $[a_2, z_2]$ to bidder 1's clientele thereby increasing his expected resale revenue by a positive amount (Lemma 9, due to a property of the optimal resale mechanism proved in §A.2.1).⁷ Thus, bidder 1 with types nearby a_1 would strictly prefer to deviate unless within the cluster his bids are almost exclusively on the top layer so that he mostly outbids the tying rivals. Hence bidder 1 is the dominant bidder.

To derive a contradiction from the supposed occurrence of a tie, our next step is to prove that some bidder who is supposed to bid just below the dominant rival within the tying cluster, such as bidder 2 in Figure 1, strictly prefers to deviate to a bid slightly above the cluster. The proof, from §5.2.1 to §5.2.5, is nontrivial because the winner's curse for bidder 2 is not negligible. Contrary to the case of bidder 1, even the infimum a_2 of the atom-bidding types of bidder 2 can gain from buying the good at resale from some atom-bidding types of bidder 1, as $a_2 > a_1$. This nontrivial winner's curse is handled in two substeps. First, we prove that if $[a_2, z_2]$ contains some sufficiently high types then for such types of bidder 2 the winner's curse is more than outweighed by the "winner's blessing" (payoff from winning conditional on the pivotal event). Then he strictly prefers the deviation (§5.2.3, due to a

⁷ Note that the revenue effect is null in no-resale models. In other words, notwithstanding zero winner's curse, Athey's no-tie argument still cannot be replicated to prove that bidder 1 strictly prefers the higher bid. Even if her Step (i) works, so that bidder 1's preference to win strictly increases in his type on $[a_1, z_1]$ conditional on his winning event, his preference may still be reversed when the conditioned event switches to the *pivotal* event. That is because his ex post payoff from winning may fail to be nondecreasing in his rivals' types, as explained above regarding the monotonicity assumption.

property of the optimal resale mechanism proved in §A.3). Second, in the other case, we find some types in $[a_2, z_2]$ for whom the winner’s curse is nearly balanced by the winner’s blessing. This is done by deducing the viability of bidder 2’s deviation from the profitability of bidder 1’s on-path action despite information asymmetry between them (§5.2.4).⁸ Then the revenue effect of the deviation, as in the case for bidder 1 in the previous paragraph, implies bidder 2’s strict incentive to deviate (§5.2.5), which delivers the no-tie theorem.

In the received literature, a no-tie theorem would have sufficed the passing-to-limit argument, as the aforementioned payoff-security condition is implied by simply repeating Step (ii) in Athey’s argument. Not so with resale, because as explained previously the monotonicity assumption may fail. With the monotonicity assumption, Athey’s Step (ii) is accomplished without relying on any equilibrium condition. Without this assumption, our no-tie argument relies on the condition that the deviant bidder 2 is supposed to bid at the cluster according to the approximation equilibria (so that the deviation to x' in Figure 1 costs him only an infinitesimal increase of payment). But such an equilibrium condition is not available when the payoff-security condition is being considered.

To avoid this problem I assume that the reserve price of the initial auction is zero. Then the no-tie theorem implies that the approximation equilibria at the limit allow for only *inconsequential* atoms, which stand no chance to win (Lemma 14). To complete the passing-to-limit argument, therefore, it suffices to handle such atoms. Here complications can occur when a bidder can change the inconsequentiality of an atom with a unilateral deviation, which could cause discontinuity at the limit. This problem is solved by Lemma 15. Then the existence proof is complete.

This existence theorem is more general than previous results in first-price auctions with resale in that it allows for any finite number of differently distributed bidders while the previous literature assumed either two bidders or at most two kinds of bidders ex ante, with bidders of the same kind drawn from the same distribution. Notwithstanding some remarkable results in this literature, such as Garratt and Tröger [2] in mixed strategies and Hafalir and Krishna [4], Lebrun [7, 8] and Virág [17] in pure strategies, the two-distribution

⁸ The deduction, consisting of Lemmas 11 and 12, is based on two nontrivial facts. First, bidder 2 can nearly mimic bidder 1’s optimal resale mechanism in the event of the tie, largely due to the fact that bidder 1 is the dominant rival. Second, the expected revenue produced by a fixed Myerson auction does not decrease when the weight of a bidder’s type is pushed upward (Lemma 22, proved here despite the fact that the ex post revenue generated by a Myerson auction need not be nondecreasing in a bidder’s type).

assumption has been crucial to their differential equations method.

Nevertheless, the existence theorem is still restricted by the aforementioned assumption of zero reserve price, as well as several other assumptions such as the privacy of a loser’s bid in the initial auction, common infimum for bidders’ prior supports, and a reseller’s power to choose resale mechanisms. These assumptions, however, are common in the current auction-resale literature such as those cited above as well as Zheng [19] and Garratt, Tröger and Zheng [3].⁹ Now that the existence proof has shown it feasible to extend the fixed-point approach beyond its previous confines of no-resale single-stage models, investigations of its further expansion, including dispensability of these assumptions, are at hand.

2 The Model

2.1 The Auction-Resale Game

There are two periods, a finite set I of bidders, and an indivisible good. For each $i \in I$, bidder i ’s *type*, or use value of the good, is independently drawn from a commonly known distribution F_i , with the realized value privately known to i . In period one, every bidder i submits as his bid an element of $\{l\} \cup B_i$, where $l < 0$ denotes the *losing bid*, amounting to nonparticipation in the period-one auction, and $B_i \subseteq [r, \infty)$ is the set of *serious bids* admissible for bidder i , with reserve price $r \geq 0$ for all bidders. Ties are broken randomly and uniformly with equal probabilities. If the good is sold then, after the winner is selected, the highest bid and the winner’s identity are announced publicly, with nothing else disclosed,¹⁰ and the winner pays for the good at the price equal to his winning bid. Then period two starts and the period-one winner chooses a selling mechanism that offers resale to the other bidders in I , called *losing bidders*. A selling mechanism is any game form to be played by the losing bidders. After the players have acted given this mechanism, the entire game ends.

Every bidder is assumed risk-neutral in his payoff, defined to be his use value, if he is the final owner of the good, plus the net monetary transfer he receives from others. Discounting

⁹ Zheng [19] did not assume common infimum of the priors but made some other assumptions. Hafalir and Krishna [4] and Lebrun [7, 8] considered some other disclosure policies and weaker bargaining power of the reseller based on the two-distribution assumption and take-it-or-leave offers as the resale mechanism.

¹⁰ If the action of a losing bidder is also disclosed, pure-strategy equilibrium is unlikely to exist unless the loser gets to choose the resale mechanism.

is assumed away for simplicity.

Assume for every bidder i the prior F_i has differentiable and strictly positive density f_i on its support $T_i := [0, \bar{t}_i]$, with prior virtual utility $t_i - (1 - F_i(t_i))/f_i(t_i)$ having strictly positive derivative with respect to t_i on T_i . Denote $T_{-i} := \prod_{k \in I \setminus \{i\}} T_k$ and $T := \prod_{k \in I} T_k$.¹¹

A profile $(\beta_i)_{i \in I}$ of bid functions, with $\beta_i : T_i \rightarrow \{l\} \cup B_i$ for each $i \in I$, is said *monotone* if and only if β_i is a weakly increasing function for each $i \in I$, i.e., everyone's period-one bid is weakly increasing in his use value of the good.

2.2 Boldfaced Symbols for Random Variables

Denote bidder i 's type by \mathbf{t}_i as the random variable and t_i as the realized value. Denote $\mathbf{t}_{-i} := (\mathbf{t}_k)_{k \in I \setminus \{i\}}$ and $t_{-i} := (t_k)_{k \in I \setminus \{i\}}$ as the random vector and the realization for the type profile across rivals of i . Analogously, denote $\mathbf{t} := (\mathbf{t}_i, \mathbf{t}_{-i}) := (\mathbf{t}_k)_{k \in I}$, $t := (t_i, t_{-i}) := (t_k)_{k \in I}$, $\mathbf{t}_{-(i,j)} := (\mathbf{t}_k)_{k \in I \setminus \{i,j\}}$ and $t_{-(i,j)} := (t_k)_{k \in I \setminus \{i,j\}}$. Denote $\mathbb{E}[g(\mathbf{x})]$ for the expected value of any function g of the random variable or random vector \mathbf{x} , with the random variable/vector boldfaced, based on the prior distributions. Denote $\mathbb{E}[g(\mathbf{x}) | E]$ for the expected value conditional on event E , $\mathbf{1}[E]$ for the indicator function of event E , and $\Pr\{E\} := \mathbb{E}[\mathbf{1}[E]]$.

3 The Endogenous Payoff Functions

We shall derive a bidder's expected payoff in the auction-resale game from a continuation equilibrium at the resale stage, which implements a reseller-optimal auction à la Myerson [13].

3.1 Continuation Equilibrium at Resale

3.1.1 Atoms and Inverse Images of Bids

If $\beta_i : T_i \rightarrow \mathbb{R}$ is a weakly increasing function, denote for any $b \geq \beta_i(0)$

$$\beta_i^{-1}(b) := \{t_i \in T_i : \beta_i(t_i) = b\},$$

$$\beta_{i,\text{inf}}^{-1}(b) := \sup\{t_i \in T_i : \beta_i(t_i) < b\}, \quad (1)$$

$$\beta_{i,\text{sup}}^{-1}(b) := \sup\{t_i \in T_i : \beta_i(t_i) \leq b\}. \quad (2)$$

¹¹ The assumption that bidders have a common infimum of their prior supports is used in Lemmas 15 and 23. The positive-derivative assumption of prior virtual utilities is slightly stronger than the usual one that requires only strict monotonicity. The strengthening is needed in Lemmas 12 and 15.

We adopt the convention of letting $\sup S := \inf S := 0$ when a subset S of T_i is empty. Note that if $\beta_i^{-1}(b) \neq \emptyset$ then $\beta_{i,\inf}^{-1}(b) = \inf \beta_i^{-1}(b)$ and $\beta_{i,\sup}^{-1}(b) = \sup \beta_i^{-1}(b)$.

For any bidder i , an *atom of β_i* means a bid $b \in B_i$ such that $\beta_i^{-1}(b)$ is a nondegenerate interval, i.e., $\beta_{i,\inf}^{-1}(b) < \beta_{i,\sup}^{-1}(b)$. An *atom of β_{-i}* , with $\beta_{-i} := (\beta_j)_{j \neq i}$, means an atom of β_j for some $j \in I \setminus \{i\}$. Likewise, an atom of $\beta := (\beta_j)_{j \in I}$ means an atom of β_j for some $j \in I$.

3.1.2 Public Histories and Posterior Beliefs

If bidder i wins with bid b_i in period one (so $b_i > l$, i.e., $b_i \in B_i$) then (i, b_i) denotes the commonly known *public history*. Given any public history (i, b_i) , with every losing bidder k ($k \neq i$) having played according to β_k , the posterior distribution $F_k(\cdot \mid i, b_i, \beta)$ of \mathbf{t}_k is derived from Bayes's rule based on the observation that k has been defeated either because $\beta_k(\mathbf{t}_k) < b_i$ or because $\beta_k(\mathbf{t}_k) = b_i$ and k did not win the tie-breaking lottery.

Lemma 1 *For any public history (i, b_i) , any monotone profile β , and any $k \neq i$, the density $f_k(\cdot \mid i, b_i, \beta)$ of $F_k(\cdot \mid i, b_i, \beta)$ is finite and strictly positive on its support $[0, \beta_{k,\sup}^{-1}(b_i)]$; if b_i is not an atom of β_k then $f_k(\cdot \mid i, b_i, \beta)$ is continuous on this posterior support; else $f_k(\cdot \mid i, b_i, \beta)$ is continuous at all but one point in the posterior support.*

Proof Appendix C. ■

3.1.3 Posterior Virtual Utilities

For each losing bidder $k \in I \setminus \{i\}$ in public history (i, b_i) , define $V_{k,b_i,\beta} : T_k \rightarrow \mathbb{R}$ by

$$V_{k,b_i,\beta}(t_k) := V_k(t_k \mid b_i, \beta) := \begin{cases} t_k - \frac{1 - F_k(t_k \mid i, b_i, \beta)}{f_k(t_k \mid i, b_i, \beta)} & \text{if } t_k \leq \beta_{k,\sup}^{-1}(b_i) \\ \beta_{k,\sup}^{-1}(b_i) & \text{if } t_k \geq \beta_{k,\sup}^{-1}(b_i), \end{cases} \quad (3)$$

and define the *posterior virtual utility* function for losing bidder $k \neq i$ to be either $V_{k,b_i,\beta}$ if b_i is not an atom of β_k , or the ironed version of $V_{k,b_i,\beta}$ according to Myerson's [13] procedure if b_i is an atom of β_k . By the previous and the next lemmas, $V_{k,b_i,\beta}$ fails to be monotone and hence ironing is needed precisely when the winning bid b_i is an atom of β_k . Denote k 's posterior virtual utility by $\bar{V}_{k,i,b_i,\beta}(t_k)$ or $\bar{V}_k(t_k \mid i, b_i, \beta)$.¹²

¹² When the winning bid b_i is an atom of β_k , the posterior distribution of \mathbf{t}_k depends on i by Eq. (75). Hence the notation i for the winner in the ironed posterior virtual utility function $\bar{V}_{k,i,b_i,\beta}$ cannot be dropped.

Lemma 2 *There exists $\lambda > 0$ such that, for any public history (i, b_i) , any monotone profile β , and any $k \neq i$, if b_i is not an atom of β_k , then:*

a. *for any $t_k \in T_k$, $\bar{V}_{k,i,b_i,\beta}(t_k) = V_{k,b_i,\beta}(t_k)$ and, if $t_i \in [0, \beta_{k,\text{sup}}^{-1}(b_i)]$,*

$$V_{k,b_i,\beta}(t_k) = t_k - \frac{F_k(\beta_{k,\text{sup}}^{-1}(b_i)) - F_k(t_k)}{f_k(t_k)}; \quad (4)$$

b. *$V_{k,b_i,\beta}$ is strictly increasing on $[0, \beta_{k,\text{sup}}^{-1}(b_i)]$, at a rate greater than or equal to λ , and is constant on $[\beta_{k,\text{sup}}^{-1}(b_i), \bar{t}_k]$;*

c. *if $b'_i > b_i$ and b'_i is not an atom of β_k , then $V_{k,b_i,\beta} \geq V_{k,b'_i,\beta}$ on $[0, \beta_{k,\text{sup}}^{-1}(b_i)]$;*

d. *$V_{k,b_i,\beta}$ is continuous on T_k ;*

Proof Appendix C. ■

3.1.4 Resale Mechanisms

Given any public history (i, b_i) , by Lemma 1, Myerson's [13] characterization of optimal auctions is applicable to the auction-design problem for our reseller i .¹³ Thus, the mechanism $M_i(b_i, t_i, \beta)$ defined below is optimal for the bidder-turned reseller i with type $t_i \in T_i$:

a. each losing bidder $k \neq i$ independently submits a report, say t_k , of his type;

b. for any $t_{-i} \in T_{-i}$, i resells the good to a bidder $k \neq i$ such that

$$\bar{V}_k(t_k | i, b_i, \beta) = \max \left\{ t_i, \max_{j \neq i} \bar{V}_j(t_j | i, b_i, \beta) \right\};$$

if there are more than one such bidders then i picks one of them through an equal-probability lottery; if no such k exists then i keeps the good;

c. for any $k \neq i$, if bidder k is resold the good then the payment k delivers to i equals

$$p_{k,i,b_i,\beta}(t_{-k}) := \inf \left\{ t'_k \in T_k : \bar{V}_{k,i,b_i,\beta}(t'_k) \geq \max \left\{ t_i, \max_{j \in I \setminus \{i,k\}} \bar{V}_{j,i,b_i,\beta}(t_j) \right\} \right\}; \quad (5)$$

if k is not resold the good then k pays zero to i .

¹³ Myerson [13] assumed continuous density throughout a bidder's support while our posterior density may be discontinuous at one point (Lemma 1). But this difference does not affect Myerson's result. Also see Footnote 9 of Garratt, Tröger and Zheng [3] for an explanation why Myerson's result is applicable here despite the possibility that the reseller may be privately informed of her type.

Following directly from Myerson's result, we have—

Lemma 3 *For any public history (i, b_i) , any $t_i \in T_i$ and any monotone profile β , if the posterior belief of \mathbf{t}_j is $F_j(\cdot \mid i, b_i, \beta)$ for each $j \neq i$, then it is a continuation equilibrium for player i to choose $M_i(b_i, t_i, \beta)$ and everyone else to participate and be truthful.*

For any public history (i, b_i) , if b_i is not an atom of β_{-i} , then Lemma 2 implies that, for any losing bidder $k \neq i$, the posterior virtual utility function $\bar{V}_{k,i,b_i,\beta}$ is equal to the strictly increasing function $V_{k,b_i,\beta}$ on the posterior support $[0, \beta_{k,\text{sup}}^{-1}(b_i)]$ of \mathbf{t}_k , hence for any t_{-k} such that bidder k of type t_k wins in $M_i(b_i, t_i, \beta)$ (i.e., $\max\{t_i, \max_{j \in I \setminus \{i,k\}} V_{j,b_i,\beta}(t_j)\} \leq \beta_{k,\text{sup}}^{-1}(b_i)$), Eq. (5) is simplified to, with $V_{k,b_i,\beta}^{-1}$ denoting the inverse function of $V_{k,b_i,\beta}$,

$$p_{k,i,b_i,\beta}(t_{-k}) = V_{k,b_i,\beta}^{-1} \left(\max \left\{ t_i, \max_{j \in I \setminus \{i,k\}} V_{j,b_i,\beta}(t_j) \right\} \right). \quad (6)$$

3.2 The Payoff from the Auction

3.2.1 The Indicator Function for Winning

The uniform tie-breaking rule corresponds to a random vector $(\rho_i)_{i \in I}$ subject to two conditions: (i) for any realization $(\rho_i)_{i \in I}$, $\rho_i \in \{1, \dots, |I|\}$ for any $i \in I$, and $\rho_i \neq \rho_j$ for any $i \neq j$; and (ii) any such realization has the same probability. The interpretation is that if $\rho_i > \rho_j$ then bidder i beats j in the coin toss when their bids are tied.

For any realization $(\rho_k)_{k \in I}$ of the uniform tie-breaking lottery, any $i \in I$, any $J \subseteq I \setminus \{i\}$, and any profile $(b_k)_{k \in J \cup \{i\}}$ of bids across bidders in $J \cup \{i\}$, write

$$(i, b_i) \succ_{(\rho_k)_{k \in I}} (b_k)_{k \in J}, \quad \text{or briefly} \quad (i, b_i) \succ (b_k)_{k \in J},$$

if and only if

$$b_i \in B_i \text{ and } \left[b_i > \max_{k \in J} b_k \text{ or } \left[b_i = \max_{k \in J} b_k \text{ and } \forall k \in \arg \max_{j \in J} b_j : \rho_i > \rho_k \right] \right].$$

And write $(i, b_i) \not\succeq (b_k)_{k \in J}$ if and only if $(i, b_i) \succ (b_k)_{k \in J}$ is not true.

For example, $\mathbf{1} \left[(i, b_i) \succ (\beta_k(t_k))_{k \in I \setminus \{i\}} \right]$ is the indicator function for the event that bidder i wins, possibly after tie-breaking, with bids b_i from i and $\beta_k(t_k)$ from each rival k .

3.2.2 Ex Post Payoff for a Winner

For any public history (i, b_i) and any $(t_i, t_{-i}) \in T_i \times T_{-i}$, define $W_i(t_{-i} | b_i, t_i, \beta)$ to be the payoff for player i when i wins at the initial auction with bid b_i and offers resale via the Myerson auction $M_i(b_i, t_i, \beta)$ according to the continuation equilibrium specified in Lemma 3, when rivals of i abide by the monotone profile β_{-i} in period one and the profile of realized types across other players happens to be t_{-i} . That bidder i wins with bid b_i implies $b_i \in B_i$. For the case $b_i \notin B_i$, i.e., $b_i = l$, define $W_i(t_{-i} | l, t_i, \beta) := 0$.

If a serious bid b_i (i.e., $b_i \in B_i$) is not an atom of β_{-i} , one can derive from Lemmas 2 and 3 that, for all $t_{-i} \in \prod_{k \neq i} [0, \beta_{k, \text{sup}}^{-1}(b_i)]$ except a set of measure zero and for any $t_i \in T_i$,

$$\begin{aligned} W_i(t_{-i} | b_i, t_i, \beta) &= t_i \mathbf{1} \left[t_i > \max_{k \neq i} V_k(t_k | b_i, \beta) \right] \\ &+ \sum_{j \neq i} p_{j, i, b_i, \beta}(t_{-j}) \mathbf{1} \left[V_j(t_j | b_i, \beta) > \max \left\{ t_i, \max_{k \notin \{i, j\}} V_k(t_k | b_i, \beta) \right\} \right]. \end{aligned} \quad (7)$$

3.2.3 Ex Post Payoff for a Losing Bidder

For any distinct bidders $i \neq j$ and any $(t_i, t_{-i}) = (t_i, t_j, t_{-(i,j)}) \in T_i \times T_j \times T_{-(i,j)}$ such that $\beta_j(t_j) \in B_j$ and $\beta_j(t_j) \geq \beta_k(t_k)$ for all $k \in I \setminus \{i, j\}$, define $L_{ij}(t_{-i} | t_i, \beta)$ to be the payoff for player i when bidder j wins at the initial auction with bid $\beta_j(t_j)$ and offers resale via mechanism $M_j(\beta_j(t_j), t_j, \beta)$ according to the continuation equilibrium, when everyone is supposed by other players to abide by the monotone profile β in period one and the profile of realized types across bidders happens to be $(t_i, t_j, t_{-(i,j)})$.

Note that $L_{ij}(t_{-i} | t_i, \beta)$ is invariant to i 's period-one bid b_i , due to the fact that reseller j in choosing resale mechanisms does not know the bids from the losing bidders.

If $\beta_j(t_j)$ is not an atom of β_{-j} then, as in the previous case for W_i , for any $i \neq j$, for all $t_{-(i,j)} \in \prod_{k \notin \{i, j\}} [0, \beta_{k, \text{sup}}^{-1}(\beta_j(t_j))]$ but a set of measure zero, and for any $t_i \in T_i$,

$$L_{ij}(t_{-i} | t_i, \beta) = (t_i - p_{i, j, \beta_j(t_j), \beta}(t_{-i})) \mathbf{1} \left[V_{i, \beta_j(t_j), \beta}(t_i) > \max \left\{ t_j, \max_{k \notin \{i, j\}} V_{k, \beta_j(t_j), \beta}(t_k) \right\} \right]. \quad (8)$$

Before the auction outcome is announced in period one, bidder i does not know who is the winner, but he knows that, at any realized type profile $t \in T$, if he loses the auction then the winner is selected from $I \setminus \{i\}$ with each rival $k \in I \setminus \{i\}$ bidding $\beta_k(t_k)$. Thus, i 's ex post payoff from losing, given any realized type profile $(t_i, t_{-i}) \in T$, is equal to

$$L_i(t_{-i} | t_i, \beta) := \sum_{j \neq i} \Pr \left\{ (j, \beta_j(t_j)) \succ (\beta_k(t_k))_{k \in I \setminus \{i, j\}} \right\} L_{ij}(t_{-i} | t_i, \beta). \quad (9)$$

3.2.4 Interim Expected Payoff

Denote $U_i(b_i, t_i, \beta)$ for type- t_i bidder i 's expected payoff in the entire game from bidding b_i in period one followed by the continuation equilibrium specified by Lemma 3, provided that everyone else abides by the monotone profile β at period one. Thus,

$$U_i(b_i, t_i, \beta) = \mathbb{E} \left[\mathbf{1} \left[(i, b_i) \succ (\beta_k(\mathbf{t}_k))_{k \in I \setminus \{i\}} \right] (W_i(\mathbf{t}_{-i} \mid b_i, t_i, \beta) - b_i - L_i(\mathbf{t}_{-i} \mid t_i, \beta)) \right] + \mathbb{E} [L_i(\mathbf{t}_{-i} \mid t_i, \beta)], \quad (10)$$

where the boldfaced letters inside the expectation operator \mathbb{E} denote the random variables.

Since W_i and L_i are derived from the continuation equilibrium at resale, we obtain a perfect Bayesian equilibrium if the period-one bid functions best reply one another:

Lemma 4 *If a monotone profile $(\beta_i)_{i \in I}$ of period-one bid functions constitutes a Nash equilibrium, across almost all bidder-types, with respect to the interim expected payoff functions $(U_i(\cdot, \cdot, \beta))_{i \in I}$ given by Eq. (10), then $(\beta_i)_{i \in I}$ coupled with the continuation play characterized in Lemma 3 constitutes a perfect Bayesian equilibrium of the auction-resale game.*

4 Increasing Difference

Based on comparative statics of the continuation equilibrium, the first theorem says that the difference in a bidder's expected payoff due to an increase in his period-one bid is weakly increasing in his type, provided that ties occur with zero probability, i.e.,

$$\forall i \in I : \forall j \in I \setminus \{i\} : \forall b_i \in B_i : b_i \text{ is not an atom of } \beta_j. \quad (11)$$

Theorem 1 (increasing difference) *For any bidder i , any monotone profile β of bid functions satisfying Eq. (11), and any $b'_i, b''_i \in B_i \cup \{l\}$ such that $b''_i > b'_i$, $U_i(b''_i, t_i, \beta) - U_i(b'_i, t_i, \beta)$ is a weakly increasing function of t_i throughout T_i .*

This property is due to a relationship between period-one bids and the final allocation after resale (Propositions 1 and 2), which say that higher period-one bids imply higher probabilities of being the final owner of the good. This relationship implies the increasing difference property through the payoff-equivalence routine in mechanism design. With notations and lemmas introduced in §4.1–§4.3, the proof of the theorem is completed in §4.4. Eq. (11) is needed to ensure that the posterior virtual utility functions are well-behaved.

4.1 Final Allocations

For any bidder i , any monotone profile β of bid functions, and any $t := (t_k)_{k \in I} \in T$, define:

- $Q_i(b_i, t, \beta)$ to be the probability with which i is the final owner in the continuation equilibrium (Lemma 3) conditional on the public history (i, b_i) , when $b_i \in B_i$ and the realized type profile is t (if $b_i \notin B_i$, i.e., $b_i = l$, then define $Q_i(b_i, t, \beta) := 0$);
- $q_{ij}(t, \beta)$ to be the probability with which i is the final owner in the continuation equilibrium (Lemma 3) conditional on the public history $(j, \beta_j(t_j))$, when $\beta_j(t_j) \in B_j$ and the realized type profile is t (if $\beta_j(t_j) = l$ then define $q_{ij}(t, \beta) := 0$);
- $q_i(t, \beta)$ to be the probability with which i is the final owner when some rival of i wins the period-one auction and offers resale according to the continuation equilibrium, i.e.,

$$q_i(t, \beta) = \sum_{j \neq i} \Pr \left\{ (j, \beta_j(t_j)) \succ (\beta_k(t_k))_{k \in I \setminus \{i, j\}} \right\} q_{ij}(t, \beta). \quad (12)$$

If $b_i \in B_i$ is not an atom of β_{-i} , then one can derive from Lemmas 2 (Claims a and b) and 3 that, for all $t_{-i} \in \prod_{k \neq i} [0, \beta_{k, \text{sup}}^{-1}(b_i)]$ but a set of measure zero and for any $t_i \in T_i$,

$$Q_i(b_i, t, \beta) = \mathbf{1} \left[t_i \geq \max_{k \in I \setminus \{i\}} V_k(t_k \mid b_i, \beta) \right]. \quad (13)$$

Analogously, for any $t_j \in T_j$ with $\beta_j(t_j) \in B_j$, if $\beta_j(t_j)$ is not an atom of β_{-j} then for any $i \neq j$, for all $t_{-(i,j)} \in \prod_{k \notin \{i, j\}} [0, \beta_{k, \text{sup}}^{-1}(\beta_j(t_j))]$ but a set of measure zero and for any $t_i \in T_i$,

$$q_{ij}(t, \beta) = \mathbf{1} \left[V_i(t_i \mid \beta_j(t_j), \beta) \geq \max \left\{ t_j, \max_{k \in I \setminus \{i, j\}} V_k(t_k \mid \beta_j(t_j), \beta) \right\} \right]. \quad (14)$$

4.2 The Envelope Condition

For any bidder i , define (with boldfaced letters denoting random variables):

$$\bar{W}_i(b_i, t_i, \beta) := \mathbb{E} \left[W_i(\mathbf{t}_{-i} \mid b_i, t_i, \beta) \mid (i, b_i) \succ (\beta_j(\mathbf{t}_j))_{j \in I \setminus \{i\}} \right], \quad (15)$$

$$\bar{L}_i(b_i, t_i, \beta) := \mathbb{E} \left[L_i(\mathbf{t}_{-i} \mid t_i, \beta) \mid (i, b_i) \not\succeq (\beta_j(\mathbf{t}_j))_{j \in I \setminus \{i\}} \right], \quad (16)$$

$$\bar{Q}_i(b_i, t, \beta) := \mathbb{E} \left[Q_i(b_i, t_i, \mathbf{t}_{-i}, \beta) \mid (i, b_i) \succ (\beta_j(\mathbf{t}_j))_{j \in I \setminus \{i\}} \right], \quad (17)$$

$$\bar{q}_i(b_i, t_i, \beta) := \mathbb{E} \left[q_i(t_i, \mathbf{t}_{-i}, \beta) \mid (i, b_i) \not\succeq (\beta_j(\mathbf{t}_j))_{j \in I \setminus \{i\}} \right]. \quad (18)$$

The next lemma follows from the Milgrom-Segal envelope theorem [11].

Lemma 5 For any $i \in I$, any $b_i \in B_i \cup \{l\}$, and any monotone profile β , the functions $\bar{W}_i(b_i, \cdot, \beta)$ and $\bar{L}_i(b_i, \cdot, \beta)$ are absolutely continuous and, for any $t_i \in T_i$,

$$\bar{W}_i(b_i, t_i, \beta) = \bar{W}_i(b_i, 0, \beta) + \int_0^{t_i} \bar{Q}_i(b_i, \tau_i, \beta) d\tau_i, \quad (19)$$

$$\bar{L}_i(b_i, t_i, \beta) = \int_0^{t_i} \bar{q}_i(b_i, \tau_i, \beta) d\tau_i. \quad (20)$$

Proof Appendix D. ■

4.3 Initial Bids and the Final Allocation

The comparative statics in Propositions 1 and 2 are about ex post probabilities conditional on the profile of realized types across all bidders, not to be confused with expected probabilities.

Proposition 1 For any $i \in I$ and any monotone profile β satisfying Eq. (11), if $b'_i > b_i$ then $Q_i(b'_i, t, \beta) \geq Q_i(b_i, t, \beta)$ for any $t_i \in T_i$ and almost every $t_{-i} \in \prod_{k \neq i} [0, \beta_{k, \text{sup}}^{-1}(b'_i)]$.

Proof Appendix D. ■

Proposition 1 says that if a bidder wins the initial auction then his probability of eventually keeping the good cannot be lower had he submitted any higher bid. The intuition is that a higher winning bid would make the winner think more highly about the losing bidders' willingness to pay and hence set higher reserve prices. Consequently, given the same realized types, his mechanism results in no resale with a higher probability.

Proposition 2 For any bidders $i \neq j$ and any monotone profile β satisfying Eq. (11), $Q_i(b_i, t, \beta) \geq q_i(t, \beta)$ for any $t_i \in T_i$ and almost every $t_{-i} \in T_{-i}$ such that $b_i \geq \max_{k \neq i} \beta_k(t_k)$.¹⁴

Proof Appendix D. ■

Proposition 2 says that a bidder is more likely to become the final owner of the good when he is the reseller than when he is a potential buyer at resale. This is similar to an elementary economics fact that a monopolist who cannot perfectly discriminate its potential buyers would under-supply its goods. The monopolist at resale, our reseller would not resell the good without a price markup above her own use value, while potential buyers are willing to pay for it at any price not exceeding their use values.

¹⁴ Proposition 2 extends Lemma 1 of Garratt, Tröger and Zheng [3] to the ex post perspective.

4.4 Proof of Theorem 1

By Eqs. (10), (15) and (16),

$$U_i(b_i, t_i, \beta) = \mathbb{E} [\mathbf{1} [b_i \succ \mathbf{t}_{-i}]] (\overline{W}_i(b_i, t_i, \beta) - b_i) + \mathbb{E} [\mathbf{1} [b_i \not\succ \mathbf{t}_{-i}]] \overline{L}_i(b_i, t_i, \beta), \quad (21)$$

where $b_i \succ \mathbf{t}_{-i}$ is a shorthand for i 's winning event $(i, b_i) \succ (\beta_k(\mathbf{t}_k))_{k \in I \setminus \{i\}}$, and $b_i \not\succ \mathbf{t}_{-i}$ its complement. For any $b''_i > b'_i$, let $\Delta U_i(t_i) := U_i(b''_i, t_i, \beta) - U_i(b'_i, t_i, \beta)$. By Eq. (21),

$$\begin{aligned} \Delta U_i(t_i) &= \mathbb{E} [\mathbf{1} [b''_i \succ \mathbf{t}_{-i}]] (\overline{W}_i(b''_i, t_i, \beta) - b''_i) - \mathbb{E} [\mathbf{1} [b'_i \succ \mathbf{t}_{-i}]] (\overline{W}_i(b'_i, t_i, \beta) - b'_i) \\ &\quad + \mathbb{E} [\mathbf{1} [b''_i \not\succ \mathbf{t}_{-i}]] \overline{L}_i(b''_i, t_i, \beta) - \mathbb{E} [\mathbf{1} [b'_i \not\succ \mathbf{t}_{-i}]] \overline{L}_i(b'_i, t_i, \beta). \end{aligned}$$

Differentiate this equation with respect to t_i and then plug into the right-hand side the envelope equations (19) and (20) and the equations (17) and (18) for \overline{Q} and \overline{q} to obtain

$$\begin{aligned} \frac{\partial}{\partial t_i} \Delta U_i(t_i) &= \mathbb{E} [\mathbf{1} [b''_i \succ \mathbf{t}_{-i}] Q_i(b''_i, t_i, \mathbf{t}_{-i}, \beta) - \mathbf{1} [b'_i \succ \mathbf{t}_{-i}] Q_i(b'_i, t_i, \mathbf{t}_{-i}, \beta)] \\ &\quad + \mathbb{E} [\mathbf{1} [b''_i \not\succ \mathbf{t}_{-i}] q_i(t_i, \mathbf{t}_{-i}, \beta) - \mathbf{1} [b'_i \not\succ \mathbf{t}_{-i}] q_i(t_i, \mathbf{t}_{-i}, \beta)]. \end{aligned}$$

The right-hand side, after rearrangements, with notation β suppressed, is equal to

$$\underbrace{\mathbb{E} [\mathbf{1} [b'_i \succ \mathbf{t}_{-i}] (Q_i(b''_i, t_i, \mathbf{t}_{-i}) - Q_i(b'_i, t_i, \mathbf{t}_{-i}))]}_{=:X} + \underbrace{\mathbb{E} [\mathbf{1} [b'_i \not\succ \mathbf{t}_{-i}, b''_i \succ \mathbf{t}_{-i}] (Q_i(b''_i, t_i, \mathbf{t}_{-i}) - q_i(t_i, \mathbf{t}_{-i}))]}_{=:Y}.$$

For any t_{-i} at which the indicator function inside the integral X is nonzero, $b'_i \geq \max_{j \neq i} \beta_j(t_j)$ and hence Proposition 1 applies; for any t_{-i} at which the indicator inside Y is nonzero, $b''_i \geq \max_{j \neq i} \beta_j(\mathbf{t}_j)$ and hence Proposition 2 applies. Thus, both X and Y are nonnegative. Hence $\frac{\partial}{\partial t_i} \Delta U_i(t_i) \geq 0$ for any t_i interior to T_i . This, coupled with the fact that $\Delta U_i(t_i)$ is absolutely continuous in t_i (since U_i by Eq. (21) is a linear combination of \overline{W}_i and \overline{L}_i , each absolutely continuous in t_i by Lemma 5), implies the monotonicity of ΔU_i . ■

5 Equilibria of the Approximation Games

Based on Theorem 1, if the bid spaces in the initial auction are replaced by some discrete spaces, a monotone equilibrium exists. To obtain equilibrium in the original game, we shall prove that the equilibrium property of such approximation equilibria is passed onto the limit when the discrete bid spaces converge to the original one. A critical step of the proof is to show that ties occur with zero probability at the limit (Theorem 2). As explained in the Introduction, our no-tie argument is significantly different from that in the literature.

5.1 The Approximation Games

For any $m = 1, 2, \dots$, define an m -approximation game by replacing for any bidder i the space B_i of serious bids with a discrete set B_i^m such that

$$i \neq j \implies B_i^m \cap B_j^m = \emptyset, \quad (22)$$

$$m < m' \implies B_i^m \subseteq B_i^{m'}, \quad (23)$$

$$\min \{|b_i - b'_i| : b_i, b'_i \in B_i^m; b_i \neq b'_i\} = 2^{-m}, \quad (24)$$

$$\lim_{m \rightarrow \infty} \min B_i^m = r, \quad \lim_{m \rightarrow \infty} \sup B_i^m = \infty.$$

The main condition is Eq. (22), devised by Reny and Zamir [16] because their single-crossing condition, like our increasing-difference theorem, applies only to non-atom bids.¹⁵ The condition ensures that, in any m -approximation game, a bidder's serious bid is never an atom of a rival's bid function. Consequently, a bidder's winning event is simplified:

$$(i, b_i) \succ (\beta_k(\mathbf{t}_k))_{k \in I \setminus \{i\}} \iff b_i > \max_{j \neq i} \beta_j(\mathbf{t}_j). \quad (25)$$

Another consequence is that the posterior virtual utility functions are simplified to Eq. (4) due to Lemma 2.a. More importantly, Theorem 1 applies, so $U_i(b_i, t_i, \beta)$ has the increasing difference property in any m -approximation game.

For any $m = 1, 2, \dots$, a profile $(\beta_i^m)_{i \in I}$ of functions $\beta_i^m : T_i \rightarrow \{l\} \cup B_i^m$ is an m -equilibrium if and only if, for any bidder i and any $t_i \in T_i$,

$$\forall b_i^m \in B_i^m \cup \{l\} : U_i(\beta_i^m(t_i), t_i, \beta^m) \geq U_i(b_i^m, t_i, \beta^m). \quad (26)$$

If, in addition, β_i^m is weakly increasing for every i , then the m -equilibrium is said *monotone*. The next proposition follows from Kakutani's fixed point theorem applied to each m -approximation game based on the single-crossing property implied by Theorem 1. The proof is the same as Athey's [1, Theorem 1] and hence omitted.

Proposition 3 *For any $m = 1, 2, \dots$, there exists a monotone m -equilibrium.*

¹⁵ Not needed here is the other perturbation devised by Athey [1] and adopted by Reny and Zamir, that a bidder has to submit the losing bid l when his type belongs to $[0, 1/m)$. They need the perturbation to ensure a revealed-preference result. It would be redundant in this paper because our revealed-preference result is ensured by an upcoming notion of consequentiality, which is needed anyway for our no-tie argument.

By revealed preference, at any m -equilibrium a bidder never bids more than his expected payoff as a winner if he stands a positive probability of winning:

Lemma 6 *For any $m = 1, 2, \dots$, if $(\beta_i^m)_{i \in I}$ is an m -equilibrium then for any $i \in I$ and any $t_i \in T_i$ such that $\Pr \{ \beta_i^m(t_i) > \max_{k \neq i} \beta_k^m(\mathbf{t}_k) \} > 0$, we have $\overline{W}_i(\beta_i^m(t_i), t_i, \beta^m) - \beta_i^m(t_i) \geq 0$.*

Proof Applying Ineq. (26) to the case $b_i^m = l$ and using Eqs. (10) and (25), we have

$$\Pr \left\{ \beta_i^m(t_i) > \max_{k \neq i} \beta_k^m(\mathbf{t}_k) \right\} (\overline{W}_i(\beta_i^m(t_i), t_i, \beta^m) - \beta_i^m(t_i) - \overline{L}_i(\beta_i^m(t_i), t_i, \beta^m)) \geq 0.$$

By the hypothesis $\Pr \{ \beta_i^m(t_i) > \max_{k \neq i} \beta_k^m(\mathbf{t}_k) \} > 0$, the term in the bracket “ (\dots) ” is nonnegative. Then the conclusion of the lemma follows from $\overline{L}_i(\beta_i^m(t_i), t_i, \beta^m) \geq 0$, which is true because i can choose not to participate in the resale mechanism. ■

5.2 Impossibility of Ties at the Limit

Given a monotone profile β of bid functions, call a serious bid b_* *consequential* if $\Pr \{ \beta_k(\mathbf{t}_k) \leq b_* \} > 0$ for every bidder $k \in I$, and *inconsequential* if otherwise. A *tie* of β means a serious bid that is an atom for at least two distinct bidders according to their bid functions in β .

Theorem 2 (no tie) *If a sequence $(\beta^m)_{m=1}^\infty$ of monotone m -equilibria converges pointwise almost everywhere to a monotone profile β^* , then β^* admits no consequential tie.*

To prove Theorem 2, suppose to the contrary that β^* admits a consequential tie b_* . We shall derive a contradiction to the equilibrium property of the sequence $(\beta^m)_{m=1}^\infty$. As a preliminary, the next lemma provides a minute picture of the clusters of rivaling bids collapsing to the atom b_* as $m \rightarrow \infty$.

Lemma 7 *If a sequence $(\beta^m)_{m=1}^\infty$ of monotone profiles converges pointwise a.e. to a monotone profile β^* and if J is the set of bidders such that a serious bid b_* is an atom of β_j^* for all $j \in J$, then there exist subsequence $(\beta^{m_n})_{n=1}^\infty$ and sequence $(\delta_n)_{n=1}^\infty \rightarrow 0$ such that, with*

$$\begin{aligned} a_i &:= \sup \{ t_i \in T_i : \beta_i^*(t_i) < b_* \}, \\ z_i &:= \sup \{ t_i \in T_i : \beta_i^*(t_i) \leq b_* \}, \end{aligned} \tag{27}$$

$$a_i^n := \inf \{ t_i \in T_i : \beta_i^{m_n}(t_i) > b_* - \delta_n \}, \tag{28}$$

$$z_i^n := \sup \{ t_i \in T_i : \beta_i^{m_n}(t_i) < b_* + \delta_n \} \tag{29}$$

for each i , we have:

$$\forall i \in J : \quad \forall t_i \in (a_i^n, z_i^n) : b_* - \delta_n < \beta_i^{m_n}(t_i) < b_* + \delta_n, \quad (30)$$

$$\forall i \in J : \quad \lim_{n \rightarrow \infty} \Pr \{t_i \in T_i \setminus (a_i^n, z_i^n) : b_* + \delta_n \leq \beta_i^{m_n}(t_i) \leq b_* + \delta_n + 2^{-m_n}\} = 0, \quad (31)$$

$$\forall i \in I : \quad a_i = \lim_{n \rightarrow \infty} a_i^n, \quad z_i = \lim_{n \rightarrow \infty} z_i^n, \quad (32)$$

$$\forall k \notin J : \quad \lim_{n \rightarrow \infty} \Pr \{t_k \in T_k : b_* - \delta_n \leq \beta_k^{m_n}(t_k) \leq b_* + \delta_n + 2^{-m_n}\} = 0. \quad (33)$$

Proof Appendix E.1. ■

With the $(\delta_n)_{n=1}^\infty$ in Lemma 7, the collapsing interval $(b_* - \delta_n, b_* + \delta_n)$ is the range of the β^{m_n} -bids for those types of bidder i in (a_i^n, z_i^n) , says Ineq. (30). Along the subsequence $(\beta^{m_n})_{n=1}^\infty$, Eq. (31) says that the probability with which the types outside (a_i^n, z_i^n) would bid within $(b_* - \delta_n, b_* + \delta_n)$ vanishes, Eq. (32) says that (a_i^n, z_i^n) converges to (a_i, z_i) , and Eq. (33) says that if β_k^* has no atom at b_* then the probability with which player k bids in $(b_* - \delta_n, b_i^n)$, with b_i^n being any bidder i 's lowest grid point above $b_* + \delta_n$, goes to zero. Given the subsequence $(\beta^{m_n})_{n=1}^\infty$ identified in Lemma 7, for each n denote

$$\bar{\beta}^n := \beta^{m_n}.$$

By Eq. (33) and the consequentiality of b_* , we have

$$\forall k \notin J : \quad \lim_{n \rightarrow \infty} \Pr \{\bar{\beta}_k^n(\mathbf{t}_k) < b_* - \delta_n\} > 0. \quad (34)$$

For any $n \in \{1, 2, \dots\}$, any i , any $t_i^n \in T_i$ and any bids b_i^n and c_i^n in $B_i^{m_n}$ with $b_i^n > c_i^n$, the expected-payoff difference for a type- t_i^n bidder i caused by his bid increase from c_i^n to b_i^n in the m_n -equilibrium β^{m_n} is

$$\Delta U_i^n(t_i^n) := U_i(b_i^n, t_i^n, \bar{\beta}^n) - U_i(c_i^n, t_i^n, \bar{\beta}^n). \quad (35)$$

To prove Theorem 2 by contradiction, it suffices to find a bidder i and a sequence $(t_i^n, c_i^n, b_i^n)_{n=1}^\infty$ such that $\limsup_n \Delta U_i^n(t_i^n) > 0$ and, for any sufficiently large n , the $\bar{\beta}^n$ -inverse-image of c_i^n is nondegenerate and contains t_i^n . Then for all sufficiently large n , $\Delta U_i^n(t_i^n) > 0$ and, with $\Delta U_i^n(\cdot)$ continuous (Lemma 5), the strict inequality extends to a neighborhood of t_i^n , which contradicts the fact that $\bar{\beta}^n$ constitutes an m_n -equilibrium.

To this end, decompose $\Delta U_i^n(t_i^n)$ into three parts (proved in Appendix E.2):

$$\Delta U_i^n(t_i^n) = \Delta W_i^n(t_i^n) - \Delta b^n + \Delta \Pi_i^n(t_i^n), \quad (36)$$

where

$$\Delta W_i^n(t_i^n) := \Pr \left\{ b_i^n > \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) \right\} (\bar{W}_i(b_i^n, t_i^n, \bar{\beta}^n) - \bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n)), \quad (37)$$

$$\Delta b^n := (b_i^n - c_i^n) \Pr \left\{ b_i^n > \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) \right\},$$

$$\Delta \Pi_i^n(t_i^n) := \Pr \left\{ b_i^n > \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) > c_i^n \right\} (\bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n) - c_i^n - \bar{L}_i^n(t_i^n)), \quad (38)$$

$$\bar{L}_i^n(t_i^n) := \mathbb{E} \left[L_i(\mathbf{t}_{-i} \mid t_i^n, \bar{\beta}^n) \mid b_i^n > \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) > c_i^n \right].$$

Eq. (36) says that $\Delta U_i^n(t_i^n)$ consists of the *revenue effect* $\Delta W_i^n(t_i^n)$, *payment effect* Δb^n , and *pivotal effect* $\Delta \Pi_i^n(t_i^n)$, which includes $\bar{L}_i^n(t_i^n)$, the *winner's curse* in our context.

5.2.1 Step 1: Locating a Deviant Bidder

Recall the set J of tying rivals specified in Lemma 7. Pick an element $j \in J$ such that

$$\forall k \in J : a_j \leq a_k. \quad (39)$$

With B_j^{mn} discrete, there exists

$$c_j^n := \min \{ \bar{\beta}_j^n(t_j) : t_j \in (a_j^n, z_j^n) \}. \quad (40)$$

Lemma 8 $\lim_{n \rightarrow \infty} \Pr \{ c_j^n < \max_{k \in J \setminus \{j\}} \bar{\beta}_k^n(\mathbf{t}_k) \leq b_* + \delta_n \} = 0$.

Lemma 8 is proved in Appendix E.3. It can be understood from the viewpoint of those types of bidder j nearby a_j . If the lemma were not true, there would be a mass of rivaling bids within $(b_* - \delta_n, b_* + \delta_n)$ that outbid such types of bidder j , and the mass would not vanish along the sequence of the approximation equilibria. On one hand, with valuation nearly equal to a_j and with Ineq. (39), such types of bidder j would have almost zero gain from buying the good from these rival-types at resale, i.e., the winner's curse for such types of bidder j to jump over these rival-types is negligible. On the other hand, if such a low-value bidder j outbids these rival-types, he would profit from reselling to them, again due to Ineq. (39); with the mass of these rival-types nonvanishing, this expected profit is bounded away from zero. Both sides considered, bidder j with types nearby a_j would deviate to a bid slightly above $(b_* - \delta_n, b_* + \delta_n)$ if Lemma 8 does not hold.

For any $n = 1, 2, \dots$ and any $i \in J \setminus \{j\}$, with c_j^n defined in Eq. (40), let

$$c_i^n := \max \left\{ \bar{\beta}_i^n(t_i) : t_i \in \left[0, (\bar{\beta}^n)_{i,\inf}^{-1}(c_j^n) \right] \right\}. \quad (41)$$

For any sufficiently large n , $\left[0, (\bar{\beta}^n)_{i,\text{inf}}^{-1}(c_j^n)\right) \neq \emptyset$ due to Lemma 8 and the hypothesis that b_* is consequential; with $B_i^{m_n}$ discrete, c_i^n exists.

Since $J \setminus \{j\}$ is finite, there exists $i \in J \setminus \{j\}$ with $c_i^{n_\gamma} = \max_{k \in J \setminus \{j\}} c_k^{n_\gamma}$ for all γ in an infinite subsequence $(n_\gamma)_{\gamma=1}^\infty$. For this i , $\lim_{\gamma \rightarrow \infty} \Pr \{c_i^{n_\gamma} < \max_{k \in J \setminus \{j, i\}} \bar{\beta}_k^{n_\gamma}(\mathbf{t}_k) < c_j^{n_\gamma}\} = 0$. Combining this with Lemma 8 and Eq. (41) and relabeling subsequence $(n_\gamma)_{\gamma=1}^\infty$, we have

$$\lim_{n \rightarrow \infty} \Pr \left\{ c_i^n < \max_{k \in J \setminus \{j\}} \bar{\beta}_k^n(\mathbf{t}_k) < b_* + \delta_n \right\} = 0. \quad (42)$$

Thus, as $n \rightarrow \infty$, the m_n -equilibrium bids from all players other than bidder j vanish from $(c_i^n, b_* + \delta_n)$. By $c_i^n < c_j^n$, the interval $(c_i^n, b_* + \delta_n)$ is almost exclusively occupied by the bids from bidder j with types in (a_j^n, z_j^n) , which converges to the nondegenerate (a_j, z_j) since b_* is an atom of β_j^* . This coupled with Eq. (34) (consequentiality of b_*) implies

$$\lim_{n \rightarrow \infty} \Pr \left\{ c_i^n < \max_{k \in J \setminus \{i\}} \bar{\beta}_k^n(\mathbf{t}_k) < b_* + \delta_n \right\} > 0. \quad (43)$$

By construction, $c_i^n < c_j^n < b_* + \delta_n$; by Eq. (42), the mass of i 's bids in $(c_i^n, b_* + \delta_n)$ vanishes while, with $i \in J$, a nonvanishing mass of i 's bids remains in $(b_* - \delta_n, b_* + \delta_n)$. Thus, for all large n , $c_i^n > b_* - \delta_n$ and hence

$$b_* - \delta_n < c_i^n < c_j^n < b_* + \delta_n. \quad (44)$$

By Eq. (41), the $\bar{\beta}_i^n$ -inverse-image of c_i^n is nondegenerate. To complete the proof by contradiction, it suffices to prove existence of a sequence $(t_i^n)_{n=1}^\infty$ such that each t_i^n belongs to this inverse image and $\limsup_n \Delta U_i^n(t_i^n) > 0$, with $\Delta U_i^n(t_i^n)$ the expected-payoff difference rendered by the deviation from c_i^n to

$$b_i^n := \min \{b_i \in B_i^{m_n} : b_i \geq b_* + \delta_n\}. \quad (45)$$

To this end, we calculate the three components of $\Delta U_i^n(t_i^n)$ according to Eq. (36). Among them, the payment effect Δb^n is $O(\delta_n)$ (hence $O(1/n)$ by Lemma 7) because of Ineq. (44) and $b_i^n - c_i^n \leq 2^{m_n} + b_* + \delta_n - c_i^n$, which follows directly from Eq. (45). Thus, we need only to calculate the revenue effect $\Delta W_i^n(t_i^n)$ and pivotal effect $\Delta \Pi_i^n(t_i^n)$.

5.2.2 Step 2: The Revenue Effect of the Deviation

By a revealed-preference argument, one can prove $\Delta W_i^n \geq 0$ (Proposition 4, Appendix A.2.1). The next lemma asserts further that the revenue effect is bounded away from zero if bidder i has potential gain of trade with his rivals when he wins with the higher bid.

Lemma 9 *If $t_i^n \rightarrow_n t_i$ such that $0 < t_i < \max_{k \neq i} z_k$, then $\limsup_{n \rightarrow \infty} \Delta W_i^n(t_i^n) > 0$.*

Proof Appendix E.5. ■

By Eq. (32), z_k is the limit of the supremum z_k^n of bidder k 's types that bid below b_i^n in the m_n -equilibrium. Hence the condition " $t_i^n \rightarrow_n t_i$ such that $t_i < \max_{k \neq i} z_k$ " implies that, for all approximation equilibria sufficiently far along the sequence, bidder i can profit from reselling the good to his rivals if he wins with the bid b_i^n . By Eq. (43), the mass of rival-types surpassed by the bid increase does not vanish along the sequence. Hence the bid increase brings about a nonvanishing increase of resale probability and expected revenue at resale, given the possible gain of resale hypothesized in this lemma.

5.2.3 Step 3: Pivotal Effect Case One: Bypassing the Middleman

Two cases need to be considered on the pivotal effect $\Delta \Pi_i^n(t_i^n)$. In the first case, bidder i 's type is so high that, in the event of tying at b_* and he loses to bidder j , he buys the good nearly for sure from bidder j . Essentially a middleman, bidder j charges this type of i a price markup in addition to the period-one price. In making the bid increase thereby surpassing j , bidder i avoids paying the price markup, which constitutes the pivotal effect in this case.

More precisely, for any $k \in I$ and any $x \in T_k$, define

$$\mathcal{V}_{k,x}(t_k) := \begin{cases} t_k - (F_k(x) - F_k(t_k))/f_k(t_k) & \text{if } 0 \leq t_k \leq x \\ x & \text{if } t_k \geq x. \end{cases} \quad (46)$$

By Lemma 25 (Appendix E.4, due to Eq. (42)), when bidder j wins with a bid b in the collapsing $(c_j^n, b_* + \delta_n)$, every losing bidder k 's posterior virtual utility function converges to \mathcal{V}_{k,z_k} as $n \rightarrow \infty$. Hence the precise meaning of our first case is that at the limit bidder i outranks everyone else in terms of $(\mathcal{V}_{k,z_k})_{k \neq j}$, i.e., $\mathcal{V}_{i,z_i}(t_i) \geq \max_{k \neq i} z_k$ as in the next lemma.

Lemma 10 *If $t_i^n \rightarrow_n t_i$ such that $\mathcal{V}_{i,z_i}(t_i) \geq \max_{k \neq i} z_k$, then $\lim_{n \rightarrow \infty} \Delta \Pi_i^n(t_i^n) > 0$.*

Proof Appendix E.6. ■

Since the types of j that bid in $(c_j^n, b_* + \delta_n)$ would nearly for sure resell the good to bidder i when i 's type happens to satisfy the hypothesis of the lemma, the expected payment extracted from such a high type of bidder i is larger than j 's expected resale revenue by a nonvanishing margin, as i could be of low types according to j 's posterior belief (Lemma 23, Appendix A.3). With j 's expected resale revenue never below his period-one winning bid

(Lemma 6), this nonvanishing margin implies a nonvanishing markup between the current price for the good and the expected payment that the high-type bidder i would need to deliver to reseller j . This markup constitutes the pivotal effect of the bid increase.

5.2.4 Step 4: Pivotal Effect Case Two: Becoming the Middleman

Here comes the other case for the pivotal effect, where bidder i 's type is not high enough to nearly for sure buy the good at resale from bidder j . Different than the previous case, bidder j 's period-one bid, which is approximately the current price in the event that bidder i 's deviation is pivotal, could be higher than the price that j will charge i at resale: Even if the revenue extracted from i is less than what j pays at period one, j can still profit from the revenues extracted from the other potential buyers.¹⁶ Then the deviant bidder i suffers a winner's curse in the magnitude of the period-one price minus the lower price at resale.

The solution stems from an idea of turning the table: In the same way that j 's loss from dealing with i is balanced by j 's revenues extracted from other bidders, i 's winner's curse is balanced by the revenues from the same clientele if i becomes the reseller status instead of j . Denote

$$\Omega_i^n := \left\{ t_{-i} \in T_{-i} : \max_{k \notin \{i,j\}} \bar{\beta}_k^n(t_k) < b_i^n; c_i^n < \bar{\beta}_j^n(t_j) < b_i^n \right\}, \quad (47)$$

$$\psi_i^n(t_i^n) := \bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n) - c_i^n - \mathbb{E}[L_i(\mathbf{t}_{-i} | t_i^n, \bar{\beta}^n) | \Omega_i^n]. \quad (48)$$

Hence Ω_i^n is the pivotal event of i 's bid increase, and $\psi_i^n(t_i^n)$ his expected payoff from winning minus his winning bid and minus his winner's curse.

Lemma 11 *If $\bar{\beta}_i^n(t_i^n) = c_i^n$ for each n and $(t_i^n)_{n=1}^\infty$ converges, then*

$$\lim_{n \rightarrow \infty} \psi_i^n(t_i^n) \geq \lim_{n \rightarrow \infty} \mathbb{E}[\mathbf{1}[t_j < \mathcal{V}_{i,z_i^n}(t_i^n)] (W_j(t_i^n, \mathbf{t}_{-(i,j)} | \bar{\beta}_j^n(\mathbf{t}_j), \mathbf{t}_j, \bar{\beta}^n) - \bar{\beta}_j^n(\mathbf{t}_j)) | \Omega_i^n]. \quad (49)$$

¹⁶ For example, suppose that in the continuation game where bidder j is the reseller, $t_j = 2$, t_i is uniformly distributed on $[0, 4]$, and t_k uniformly distributed on $[0, 10]$. In j 's optimal resale mechanism, the maximum of bidder i 's expected payment (when $t_i = 4$) is equal to

$$\frac{6}{10} \times 3 + \int_6^7 (t_k - 3) dt_k / 10 = 2.15,$$

while the reseller j 's expected payoff equals

$$\frac{3}{4} \times \frac{6}{10} \times 2 + \frac{1}{4} \times \frac{6}{10} \times 3 + \frac{3}{4} \times \frac{4}{10} \times 6 + \int_3^4 \int_{t_i+3}^{10} (t_i + 3) \frac{dt_k}{10} \frac{dt_i}{4} + \int_6^7 \int_{t_k-3}^4 (t_k - 3) \frac{dt_i}{4} \frac{dt_k}{10} \approx 3.76.$$

Thus, at period one, it is possible for bidder j to submit a bid strictly between 2.15 and 3.76.

Proof Appendix E.7. ■

Lemma 12 *There exists a sequence $(t_i^n)_{n=1}^\infty$ such that $\bar{\beta}_i^n(t_i^n) = c_i^n$ for each n and*

$$\limsup_{n \rightarrow \infty} \mathbb{E} [\mathbf{1} [t_j < \mathcal{V}_{i,z_i^n}(t_i^n)] (W_j(t_i^n, \mathbf{t}_{-(i,j)} \mid \bar{\beta}_j^n(\mathbf{t}_j), \mathbf{t}_j, \bar{\beta}^n) - \bar{\beta}_j^n(\mathbf{t}_j)) \mid \Omega_i^n] \geq 0. \quad (50)$$

Proof Appendix E.8. ■

To explain the two lemmas, let us temporarily pretend that, when j 's bid is clustered around the tie, bidder i somehow knows exactly what j 's bid b_j is equal to. Consider a resale mechanism M^n that i could offer if he wins at period one: First, i announces his own type t_i^n and then asks bidder j whether t_i^n is above the reserve price that j would have offered i had j been the winner, which implies that j would always resell the good had j won. If bidder j says No, then i offers resale to all bidders via the Myerson auction that i should have chosen on the $\bar{\beta}^n$ -equilibrium path. If bidder j answers Yes, by contrast, i offers resale to all but bidder j via j 's resale mechanism, where i 's own announced type, together with the losing bidders', are discounted to their virtual utilities. Here i can replicate j 's resale mechanism because of our temporary assumption that i knows j 's bid b_j . Reseller i 's uncertainty about j 's type makes no difference, because i excludes j in this case.

While the mechanism M^n is suboptimal to i , it generates enough expected revenue to cover the winner's curse and winning bid. To see why, note that the winner's curse is null if bidder j , presumed honest, answers No to i 's question. In that case, bidder i 's net gain is just his expected revenue as a reseller minus his winning bid. Since his resale mechanism in that case coincides with the Myerson auction that he should have chosen on path, the expected revenue it generates is the same as his on-path expected revenue, which can cover the winning bid by a revealed-preference argument (Lemma 6).

Thus, consider the case where bidder j answers Yes to i 's question. In that case, i 's resale mechanism M^n either keeps the good to i himself or resells the good to some bidder k other than j . Similarly, had bidder i lost to j at period one then j would resell the good to either bidder i or some other losing bidder k but would never keep the good to j herself. The events for these final outcomes are identical between M^n and j 's resale mechanism, since the two mechanisms coincide when j honestly answers Yes. Let us calculate i 's gain and loss from outbidding j in these two events:

final owner	gain	foregone trade with j	current price	net gain
i	i 's use value t_i^n	$t_i^n - p_{ij}^n$	c_i^n	$p_{ij}^n - c_i^n$
k	p_{ki}^n	0	c_i^n	$p_{ki}^n - c_i^n$

Here p_{ij}^n denotes the resale price that i would need to pay j had j won, and p_{ki}^n the resale price at which k buys from i in M^n . Since M^n replicates j 's resale mechanism, $p_{ki}^n = p_{kj}^n$. Thus, whether the final owner is i himself or some $k \notin \{i, j\}$, i 's net gain from outbidding j is nearly the same as j 's profit had j won (with $c_i^n \approx b_* \approx b_j$), which is nonnegative by a revealed-preference argument for bidder j .

In sum, whether j answers Yes or No to i 's question, i 's payoff from outbidding j can nearly offset the winner's curse (foregone gain of buying from j) and the current price. This is the combined implication of Ineqs. (49) and (50), where the indicator function $\mathbf{1}[\mathbf{t}_j < \mathcal{V}_{i, z_i^n}(t_i^n)]$ corresponds to j 's affirmative answer.

Two problems in the above heuristic argument need to be repaired. First, the argument was based on a false assumption that i somehow knows j 's bid $\bar{\beta}_j^n(\mathbf{t}_j)$. However, removing this assumption does not upset our conclusion. By Eq. (42), when i 's bid increase is pivotal, bidder j 's bid $\bar{\beta}_j^n(\mathbf{t}_j)$ ranges in the interval $(c_i^n, b_* + \delta_n)$ where the others rarely bid, hence the resale mechanism selected by j as a reseller stays mostly constant. Therefore, i can nearly replicate j 's mechanism with the pretended winning bid b_j being any element in this interval.

The second problem is that each player accounts its own profits and loss based on its private information, hence nonnegative expected profit from j 's viewpoint need not imply nonnegative expected profit conditional on the realized type of bidder i . Lemma 12 solves this problem by observing that there exist types t_i^n of bidder i conditional on which j 's expected profit is nonnegative. In order for such t_i^n to be those whose $\bar{\beta}_i^n$ -bids equal c_i^n , essentially the highest among i 's bids that belong to the tying cluster, we need such t_i^n to exist at the high end of bidder i 's posterior support. That is ensured by comparative statics of the Myerson auction (Lemma 22, Appendix A.2.2).

5.2.5 Step 5: Completing the Proof of Theorem 2

There are only two possible cases: either (i) $z_i < \max_{k \neq i} z_k$ or (ii) $z_i \geq \max_{k \neq i} z_k$.

In Case (i), by Lemma 12, there exists a sequence $(t_i^n)_{n=1}^\infty$ such that $\bar{\beta}_i^n(t_i^n) = c_i^n$ for each n and Ineq. (50) holds. Extracting a converging subsequence if necessary, we may assume without loss of generality that $t_i^n \rightarrow_n t_i$ for some t_i . Then Lemma 11 says that

Ineq. (49) holds. Combining both inequalities we have

$$\limsup_{n \rightarrow \infty} \Delta \Pi_i^n(t_i^n) = \limsup_{n \rightarrow \infty} \Pr(\Omega_i^n) \limsup_{n \rightarrow \infty} \psi_i^n(t_i^n) \geq 0. \quad (51)$$

For any n , since $c_i^n < b_* + \delta_n$ by Ineq. (44), $t_i^n \leq z_i^n$. Hence $t_i \leq z_i < \max_{k \neq i} z_k$. Thus, Lemma 9 implies $\limsup_{n \rightarrow \infty} \Delta W_i^n(t_i^n) > 0$. Plugging into Eq. (36) this strict inequality, as well as Ineq. (44) and Eq. (51), we have $\limsup_{n \rightarrow \infty} \Delta U_i^n(t_i^n) > 0$.

In Case (ii), where $z_i \geq \max_{k \neq i} z_k$, Lemma 10 implies that $\lim_{n \rightarrow \infty} \Delta \Pi_i^n(z_i) > 0$. Plugging this into Eq. (36) and noting $\Delta W_i^n(z_i) \geq 0$ (Proposition 4, Appendix A.2.1) and Eq. (44), we obtain $\lim_{n \rightarrow \infty} \Delta U_i^n(z_i) > 0$. With $\Delta U_i^n(t_i)$ continuous in t_i (Lemma 5), there exists $\alpha < z_i$ such that $\lim_{n \rightarrow \infty} \Delta U_i^n(t_i^n) > 0$ for all $t_i^n \in (\alpha, z_i]$. By Eq. (42), the distance between z_i^n and the supremum of the inverse image $(\bar{\beta}^n)_i^{-1}(c_i^n)$ converges to zero; thus, with $z_i^n \rightarrow_n z_i$ by Eq. (32), this supremum converges to z_i . Hence we can pick a sequence $(t_i^n)_{n=1}^\infty$ such that $t_i^n \in (\bar{\beta}^n)_i^{-1}(c_i^n)$ for each n and $t_i^n \rightarrow_n z_i$. Then for all sufficiently large n , $t_i^n \in (\alpha, z_i]$ and hence $\limsup_{n \rightarrow \infty} \Delta U_i^n(t_i^n) > 0$. Therefore, the desired contradiction $\limsup_{n \rightarrow \infty} \Delta U_i^n(t_i^n) > 0$ is obtained, which completes the proof of Theorem 2. ■

Slightly modifying the construction of c_i^n , we can extend the above proof to obtain—

Corollary 1 *If $b_* > r$ then b_* is not a consequential atom of β^* .*

Proof Appendix E.9. ■

6 Equilibrium of the Original Game

We shall complete the existence proof by showing that the m -equilibrium condition, Ineq. (26), converges to the equilibrium condition for the original game, Ineq. (59) in the following. This is not as automatic as in the received literature, without their payoff-security condition explained in the Introduction. We break down the argument into two cases, depending on whether an atom is involved at the limit. When no atom is involved, the convergence of a bidder's expected payoff follows from the convergence of his winner's payoff, loser's payoff and winning status, all due to the convergence of posterior virtual utilities (Lemmas 16, 17 and 19). When an atom is involved, it is either inconsequential or equal to the reserve price r (Corollary 1). Assuming $r = 0$, we shall show that the atom is necessarily inconsequential (Lemma 14) and, furthermore, no bidder can change its inconsequentiality with a unilateral

deviation (Lemma 15). Then a bidder's expected payoff from bidding at the atom becomes a loser's payoff, which converges as in the former case (Lemma 18).

Theorem 3 (existence) *Assume that the space of serious bids is $[0, \infty)$ for any bidder i . Then the auction-resale game defined in §2.1 admits a monotone perfect Bayesian equilibrium; furthermore, at this equilibrium, any bidder's period-one bid that can win with strictly positive probability is a strictly increasing function of the bidder's use value.*

By Proposition 3, for each $m = 1, 2, \dots$ there exists a monotone m -equilibrium $\beta^m := (\beta_i^m)_{i \in I}$ of the m -approximation game. Taking a convergent subsequence of $(\beta^m)_{m=1}^\infty$ if necessary, we can assume without loss that for any bidder i there exists a weakly increasing bid function $\beta_i^* : T_i \rightarrow B_i \cup \{l\}$ such that β_i^m converges to β_i^* pointwise almost everywhere on T_i . The rest of the proof proceeds with the following lemmas.

Lemma 13 *If $b_* > l$ is an inconsequential atom of β^* , then there exists $k \in I$ such that b_* is not an atom of β_k^* and, for any such k ,*

$$\Pr \{\beta_k^*(\mathbf{t}_k) \geq b_*\} = \Pr \{\beta_k^*(\mathbf{t}_k) > b_*\} = 1, \quad (52)$$

$$\lim_{m \rightarrow \infty} b^m = b \implies \lim_{m \rightarrow \infty} \Pr \{\beta_k^m(\mathbf{t}_k) > b^m\} = 1. \quad (53)$$

Proof By definition of consequentiality, with b_* not consequential, there exists $k \in I$ for whom b_* is not an atom of β_k^* and Eq. (52) holds for any such k . Since b_* is not an atom of β_k^* , Eq. (52) implies $\lim_{m \rightarrow \infty} \Pr \{\beta_k^m(\mathbf{t}_k) > b_*\} = 1$, which implies Eq. (53). ■

Lemma 14 *If $[0, \infty)$ is the set of serious bids then no serious bid is a consequential atom of β^* .*

Proof Suppose not, say $b_* > l$ is a consequential atom of β_j^* . With zero reserve price, either (i) $b_* > 0$ or (ii) $b_* = 0$. Case (i) is impossible by Corollary 1. Thus, consider Case (ii) and let the bid zero be a consequential atom of β_j^* , with $(\bar{\beta}^n)_{n=1}^\infty$ the subsequence and $(\delta_n)_{n=1}^\infty$ the shrinking radius of the bid cluster at zero, specified by Lemma 7. Let

$$b_i^n := \min\{b_i \in B_i^{m_n} : b_i > \delta_n\}.$$

By Theorem 2,

$$\lim_{n \rightarrow \infty} \Pr \left\{ l < \max_{k \neq j} \bar{\beta}_k^n(\mathbf{t}_k) < b_i^n \right\} = 0. \quad (54)$$

For any $i \neq j$, with zero being a consequential bid, there exists a $z_i > 0$ such that, for all sufficiently large n , $\bar{\beta}_i^n(t_i) = l$ for all $t_i \in (0, z_i)$. Pick any such i and t_i . Bidder i 's expected-payoff difference between bidding b_i^n and submitting l , by Eq. (10), is equal to

$$\begin{aligned} \Delta U_i^n(t_i) &= \mathbb{E} \left[\mathbf{1} \left[b_i^n > \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) \right] \left(W_i(\mathbf{t}_{-i} \mid b_i^n, t_i, \bar{\beta}^n) - b_i^n - L_i(\mathbf{t}_{-i} \mid t_i, \bar{\beta}^n) \right) \right] \\ &\geq \mathbb{E} \left[\mathbf{1} \left[b_i^n > \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) \right] \left(t_i - L_{ij}(\mathbf{t}_{-i} \mid t_i, \bar{\beta}^n) \right) \right] - O(1/n), \end{aligned}$$

where the inequality uses the facts $\bar{W}_i \geq t_i$ to replace \bar{W}_i with t_i , $b_i^n \rightarrow_n 0$ to remove b_i^n , and Eqs. (33) and (54) to replace L_i with L_{ij} . Since $\lim_{n \rightarrow \infty} \Pr \{ b_i^n > \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) \} \geq \prod_{k \neq i} F_k(z_k) > 0$, to prove $\lim_n \Delta U_i^n(t_i) > 0$ it suffices to show

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[t_i - L_{ij}(\mathbf{t}_{-i} \mid t_i, \bar{\beta}^n) \mid b_i^n > \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) \right] > 0. \quad (55)$$

For any $t_{-i} \in T_{-i}$, $L_{ij}(t_{-i} \mid t_i, \bar{\beta}^n)$ is equal to either zero or $t_i - p_{i,j}^n(t_{-i})$, with $p_{i,j}^n(t_{-i})$ the resale price offered by j when j is the reseller. By its optimality, $p_{i,j}^n(t_{-i})$ is at least as high as the reserve price offered to i by the zero type of j based on the posterior distribution of \mathbf{t}_i with support $\left[0, (\bar{\beta}^n)_{i, \text{sup}}^{-1}(l) \right]$, which converges to the nondegenerate $[0, z_i]$ as $n \rightarrow \infty$. Hence the reserve price converges to some $p > 0$. Thus, at the limit, the integrand on the left-hand side of Ineq. (55) is no less than a strictly positive constant, either t_i or p . Hence we obtain the desired contradiction that bidder i of type t_i would deviate for large n . ■

Lemma 15 *If a serious bid b_* is an inconsequential atom of β^* then there are at least two bidders whose β^* -bidding functions do not have b_* as an atom.*

Proof Appendix F. ■

Lemma 15 eliminates a case where a serious bid b_* is an atom of β_k^* for all k but a single bidder i , so that b_* is inconsequential only because i 's β_i^* -bid is above b_* almost surely. In this case, if i chooses to bid below b_* instead of abiding by β_i^* then the bid b_* would have positive winning probability and cause possible discontinuity in i 's expected payoff. The lemma asserts impossibility of this case, with a proof similar in spirit to that of Lemma 10: Should the case occur, then in m -equilibria for sufficiently large m , we can choose a bidder j with sufficiently high types who would almost always outrank his rivals in the resale mechanism offered by those types of bidder i who would have won without j 's deviation. These types of i , if undefeated, act merely as middlemen for j and impose on j

a price markup (Lemma 23, Appendix A.3), which bidder j could have bypassed with a higher bid. Different from Lemma 10, however, these types of bidder i do not constitute a nonvanishing mass. Therefore, much of the proof of the lemma is to fine-tune the magnitude of j 's deviation so that his expected net gain is strictly positive.

Lemma 16 *For any $k \in I$, if $(b^m)_{m=1}^\infty$ is a sequence of serious bids that converges to some b such that neither is b an atom of β_k^* nor is b^m an atom of β_k^m ($\forall m$), then*

$$\forall t_k \in \left[0, (\beta^*)_{k,\text{sup}}^{-1}(b)\right) : \lim_{m \rightarrow \infty} V_k(t_k | b^m, \beta^m) = V_k(t_k | b, \beta^*). \quad (56)$$

Proof Since neither b^m is an atom of β_k^m nor b an atom of β_k^* , Lemma 2.a is applicable. Thus, for any $t_k \in \left[0, (\beta^*)_{k,\text{sup}}^{-1}(b)\right)$, $V_k(t_k | b, \beta^*)$ obeys Eq. (4) with $(\beta^*)_{k,\text{sup}}^{-1}(b)$ being the $\beta_{k,\text{sup}}^{-1}(b)$ there; for large enough m , such t_k also belongs to $\left[0, (\beta^m)_{k,\text{sup}}^{-1}(b^m)\right]$ and hence $V_k(t_k | b^m, \beta^m)$ also obeys Eq. (4) with $(\beta^m)_{k,\text{sup}}^{-1}(b^m)$ being the $\beta_{k,\text{sup}}^{-1}(b)$ there. Since b is not an atom of β_k^* , the mass of t_k between $(\beta^m)_{k,\text{sup}}^{-1}(b^m)$ and $(\beta^*)_{k,\text{sup}}^{-1}(b)$ vanishes as $b^m \rightarrow b$. Thus, $F_k\left((\beta^m)_{k,\text{sup}}^{-1}(b^m)\right) \rightarrow_m F_k\left((\beta^*)_{k,\text{sup}}^{-1}(b)\right)$. Hence Eq. (56) follows. ■

Lemma 17 *For any $i \in I$, any $t_i \in T_i$ and any measurable subset $S \subseteq T_{-i}$,*

$$\lim_{m \rightarrow \infty} \mathbb{E}[L_i(t_{-i} | t_i, \beta^m) \mathbf{1}[S]] = \mathbb{E}[L_i(t_{-i} | t_i, \beta^*) \mathbf{1}[S]]. \quad (57)$$

Proof By definition of L_i in Eq. (9), $L_i(t_{-i} | t_i, \beta^*)$ equals zero unless t_{-i} belongs to the set

$$S' := \{t_{-i} \in T_{-i} : \beta_j^*(t_j) > l \text{ for some } j \neq i\}.$$

Since $\beta^m \rightarrow \beta^*$ and l is isolated from $[0, \infty)$, $\lim_{m \rightarrow \infty} L_i(t_{-i} | t_i, \beta^m) = 0$ by Eq. (9) unless $t_{-i} \in S'$. Thus, it suffices to prove (57) with the integration domain S replaced with $S \cap S'$.

First, consider any $t_{-i} \in S'$ at which the highest bid $\beta_j^*(t_j)$ among rivals of i is not an atom of β_{-j}^* . Then Eq. (56) holds for all $k \neq j$ with the role (b^m, b) played by $(\beta_j^m(t_j), \beta_j^*(t_j))$, and Eqs. (6) and (8) hold with respect to β^* . Eqs. (6) and (8) also hold with respect to β^m due to Eq. (22). Thus,

$$\lim_{m \rightarrow \infty} L_i(t_{-i} | t_i, \beta^m) = L_i(t_{-i} | t_i, \beta^*). \quad (58)$$

Second, consider the other kind of t_{-i} in S' , the elements of

$$S'' := \left\{ t_{-i} \in S' : \exists j \neq i \left[\beta_j^*(t_j) \geq \max_{k \notin \{i,j\}} \beta_k^*(t_k); \beta_j^*(t_j) \text{ is an atom of } \beta_{-j}^* \right] \right\}.$$

Since there are at most countably many atoms of β_{-j}^* , we can discard any $t_{-i} \in S''$ such that $\beta_j^*(t_j)$ is not an atom of β_j^* , as all such t_{-i} constitute only a zero-measure subset of T_{-i} . Thus, suppose that $\beta_j^*(t_j)$ is an atom of both β_j^* and β_{-j}^* . Then $\beta_j^*(t_j)$ would be a tie, which is impossible by Theorem 2, unless $\beta_j^*(t_j)$ is inconsequential. Now that $\beta_j^*(t_j)$ is inconsequential, Lemma 15 (applicable because $\beta_j^*(t_j) > l$, as $t_{-i} \in S'$) implies that there are at least two bidders whose bid functions in β^* do not have $\beta_j^*(t_j)$ as an atom. One of them is a bidder k different than the i in this lemma, and Eqs. (52) and (53) imply $\Pr \{\beta_k^*(\mathbf{t}_k) > \beta_j^*(t_j)\} = 1$ and $\lim_{m \rightarrow \infty} \{\beta_k^m(\mathbf{t}_k) > \beta_j^m(t_j)\} = 1$. The first equation says that those $t_{-(i,j)}$ at which $\beta_j^*(t_j)$ wins against β_{-j}^* constitute a zero-measure set, and the second says that the measure of those $t_{-(i,j)}$ at which $\beta_j^m(t_j)$ wins against β_{-j}^m shrinks to zero as $m \rightarrow \infty$. Thus,

$$\lim_{m \rightarrow \infty} \mathbb{E}[L_i(\mathbf{t}_{-i} \mid t_i, \beta^m) \mathbf{1}[S'']] = 0 = \mathbb{E}[L_i(\mathbf{t}_{-i} \mid t_i, \beta^*) \mathbf{1}[S'']].$$

Eq. (57) is obtained by summing this equation with the integration of Eq. (58) across all $t_{-i} \in S \cap S' \setminus S''$. ■

Lemma 18 *For any $i \in I$, any $t_i \in T_i$ and any m , let $b_i^m \in B_i^m$ and $b_i^m \rightarrow_m b_i$. If b_i is an inconsequential atom of β_{-i}^* , then $\lim_{m \rightarrow \infty} U_i(\beta_i^m(t_i), t_i, \beta^m) \geq U_i(b_i, t_i, \beta^*)$.*

Proof Since $b_i^m \rightarrow_m b_i$ and $b_i^m \in B_i^m$ for each m , $b_i > l$. Since b_i is an inconsequential atom, Lemma 15 implies that there exists $k \neq i$ for whom b_i is not an atom of β_k^* , and Lemma 13 implies that Eqs. (52) and (53) holds. Thus, with β^m an m -equilibrium and $b_i^m \in B_i^m$,

$$\begin{aligned} \lim_{m \rightarrow \infty} U_i(\beta_i^m(t_i), t_i, \beta^m) &\geq \lim_{m \rightarrow \infty} U_i(b_i^m, t_i, \beta^m) \stackrel{(53)}{=} \lim_{m \rightarrow \infty} \mathbb{E} L_i(\mathbf{t}_{-i} \mid t_i, \beta^m) \\ &\stackrel{(57)}{=} \mathbb{E} L_i(\mathbf{t}_{-i} \mid t_i, \beta^*) \stackrel{(52)}{=} U_i(b_i, t_i, \beta^*), \end{aligned}$$

where the first and last equalities also use Eq. (10). ■

Lemma 19 *For any $i \in I$, any $t_i \in T_i$ and any m , let $b_i^m \in B_i^m$ and $b_i^m \rightarrow_m b_i$. If b_i is not an atom of β_{-i}^* then $\lim_{m \rightarrow \infty} U_i(b_i^m, t_i, \beta^m) = U_i(b_i, t_i, \beta^*)$.*

Proof By hypothesis, b_i is not an atom of β_{-i}^* , nor b_i^m an atom of β_{-i}^m , due to Eq. (22). Thus, from Eq. (10) and suppressing the symbol t_i , we have

$$\begin{aligned} U_i(b_i^m, \beta^m) &= \mathbb{E} \left[\mathbf{1} \left[b_i^m > \max_{k \neq i} \beta_k^m(\mathbf{t}_k) \right] (W_i(\mathbf{t}_{-i} \mid b_i^m, \beta^m) - b_i^m - L_i(\mathbf{t}_{-i} \mid \beta^m)) \right] + \mathbb{E}[L_i(\mathbf{t}_{-i} \mid \beta^m)], \\ U_i(b_i, \beta^*) &= \mathbb{E} \left[\mathbf{1} \left[b_i > \max_{k \neq i} \beta_k^*(\mathbf{t}_k) \right] (W_i(\mathbf{t}_{-i} \mid b_i, \beta^*) - b_i - L_i(\mathbf{t}_{-i} \mid \beta^*)) \right] + \mathbb{E}[L_i(\mathbf{t}_{-i} \mid \beta^*)]. \end{aligned}$$

Here $W_i(\mathbf{t}_{-i} \mid b_i^m, \beta^m)$ and $W_i(\mathbf{t}_{-i} \mid b_i, \beta^*)$ obey Eq. (7) with virtual utility functions $(V_k(\cdot \mid b_i^m, \beta^m))_{k \neq i}$ and $(V_k(\cdot \mid b_i, \beta^*))_{k \neq i}$, because the condition for (7) is guaranteed by the indicator functions $\mathbf{1}[b_i^m > \max_{k \neq i} \beta_k^m(\mathbf{t}_k)]$ and $\mathbf{1}[b_i > \max_{k \neq i} \beta_k^*(\mathbf{t}_k)]$. Since $b_i^m \rightarrow_m b_i$ and b_i is not an atom of β_{-i}^* , Eq. (56) applies to the case $b = b_i$. Thus, by Eqs. (6) and (7),

$$\lim_{m \rightarrow \infty} W_i(t_{-i} \mid b_i^m, \beta^m) = W_i(t_{-i} \mid b_i, \beta^*) \quad \text{a.e. } t_{-i} \in T_{-i}.$$

As $\beta^m \rightarrow \beta^*$ and b_i is not an atom of β_{-i}^* , we also have

$$\lim_{m \rightarrow \infty} \mathbf{1} \left[b_i^m > \max_{k \neq i} \beta_k^m(t_k) \right] = \mathbf{1} \left[b_i > \max_{k \neq i} \beta_k^*(t_k) \right] \quad \text{a.e. } t_{-i} \in T_{-i}.$$

Combining these two equations with Eq. (57) yields the conclusion of the lemma. ■

Lemma 20 *If $[0, \infty)$ is the set of serious bids, then for any $i \in I$ and almost every $t_i \in T_i$*

$$\forall b_i \in [0, \infty) \cup \{l\} : U_i(\beta_i^*(t_i), t_i; \beta^*) \geq U_i(b_i, t_i, \beta^*). \quad (59)$$

Proof We know that $\beta_i^m(t_i) \rightarrow_m \beta_i^*(t_i)$ for almost every $t_i \in T_i$. Pick any such t_i . For any $b_i \in B$, Lemma 14 says that b_i is either not an atom, or an inconsequential atom, of β_{-i}^* . In the first case, with $\cup_{m=1}^{\infty} B_i^m$ dense in B due to Eq. (24), there is a sequence $(b_i^m)_{m=1}^{\infty}$ converging to b_i with $b_i^m \in B_i^m$ for each m . By Lemma 19 and revealed preference of the m -equilibrium bid $\beta_i^m(t_i)$,

$$\lim_{m \rightarrow \infty} U_i(\beta_i^m(t_i), t_i, \beta^m) \geq U_i(b_i, t_i, \beta^*). \quad (60)$$

In the second case, Lemma 18 implies the same inequality. The same inequality holds for $b_i = l$ by revealed preference of $\beta_i^m(t_i)$. Thus, Ineq. (60) holds for any $b_i \in B \cup \{l\}$.

Thus, it suffices to show $\lim_{m \rightarrow \infty} U_i(\beta_i^m(t_i), t_i, \beta^m) = U_i(\beta_i^*(t_i), t_i, \beta^*)$. If $\beta_i^*(t_i) > l$ and is not an atom of β_{-i}^* , this equation follows directly from Lemma 19. If $\beta_i^*(t_i) > l$ and is an atom of β_{-i}^* , then Lemma 14 says that $\beta_i^*(t_i)$ is inconsequential; hence Eqs. (52) and (53), applied to the case $b = \beta_i^*(t_i)$, imply that

$$\Pr \left\{ \beta_i^*(t_i) \geq \max_{k \neq i} \beta_k^*(\mathbf{t}_k) \right\} = \lim_{m \rightarrow \infty} \Pr \left\{ \beta_i^m(t_i) \geq \max_{k \neq i} \beta_k^m(\mathbf{t}_k) \right\} = 0,$$

which by Eq. (10) implies

$$\lim_{m \rightarrow \infty} U_i(\beta_i^m(t_i), t_i, \beta^m) = \lim_{m \rightarrow \infty} \mathbb{E}L_i(\mathbf{t}_i \mid t_i, \beta^m) \stackrel{(57)}{=} \mathbb{E}L_i(\mathbf{t}_i \mid t_i, \beta^*) = U_i(\beta_i^*(t_i), t_i, \beta^*).$$

If $\beta_i^*(t_i) = l$, then the formula displayed above follows trivially. Thus,

$$U_i(\beta_i^*(t_i), t_i, \beta^*) = \lim_{m \rightarrow \infty} U_i(\beta_i^m(t_i), t_i, \beta^m) \geq U_i(b_i, t_i, \beta^*)$$

for any $b_i \in [0, \infty) \cup \{l\}$, as desired. ■

Lemma 20 implies that $(\beta_i^*)_{i \in I}$ constitutes a Nash equilibrium given the interim payoff functions $(U_i)_{i \in I}$ defined in Eq. (10). Then Lemma 4 implies that $(\beta_i^*)_{i \in I}$ coupled with the continuation equilibrium constitutes a perfect Bayesian equilibrium of the original auction-resale game. In addition, Lemma 14 implies strict monotonicity of each β_i^* for all bids that have a strictly positive probability of winning at β^* . This completes the proof of Theorem 3.

7 Conclusion

Fixed-point approaches have been foundational to theoretical investigations of discontinuous games especially certain auction mechanisms. Incorporation of post-auction resale into such approaches not only would make them more realistically relevant but also is theoretically compelling because, as noted in the literature, resources can be misallocated in certain asymmetric auctions, triggering the incentive for resale. The possibility of resale brings about new challenges to the fixed-point approaches. The value-correlation across bidders, previously assumed exogenous, becomes endogenously determined by resale, which is itself endogenous. The discontinuity problem of tying bids gets compounded to the discontinuity of post-auction beliefs and that of the payoffs at resale. Yet these challenges turn out to be surmountable, as demonstrated in this paper, extending the fixed-point approach for monotone equilibria beyond its previous confines of single-stage no-resale models.

To capture its endogenous nature, this paper models resale by assuming that the winner in an auction gets to choose any selling mechanism to offer resale, hence at equilibrium resale is offered through the Myerson auctions. The extent to which the properties of such resale mechanisms instrumental to our existence proof may be generalized to other resale settings is left for future investigations. Nevertheless, there is a merit, at least for the first endeavor, to endogenize resale mechanism as in our model. It shows us the power of mechanism design, as a modeling technique, to pin down resale mechanisms among the myriad of secondary-market arrangements often hard to observe. Just as the rational choice axiom reduces individual behaviors to regularity, the endogenous treatment of resale mechanisms generates subtle comparative statics in equilibrium with forward-looking bidding behaviors.

A Comparative Statics of the Myerson Auction

Here are some properties of the optimal auction characterized by Myerson [13] with respect to the distributions of the bidder-types. In our context, the Myerson auction corresponds to the equilibrium resale mechanism selected by a reseller whose winning bid is not an atom of the other bidders' strategies, and the distributions the post-auction beliefs.

A.1 Notations and Preliminaries

Let $i_* \in I$ denote the current seller and $I \setminus \{i_*\}$ the set of potential buyers. For any $i \in I \setminus \{i_*\}$ and any $\zeta_i \leq \bar{t}_i$, let the distribution F_{i,ζ_i} of i 's type t_i is derived from the prior F_i via

$$F_{i,\zeta_i}(t_i) := F_i(t_i)/F_i(\zeta_i)$$

for all t_i in the support $[0, \zeta_i]$, and likewise for the density f_{i,ζ_i} . Assume, on the support of F_i , that F_i has strictly positive density f_i and the prior virtual utility $t_i - (1 - F_i(t_i))/f_i(t_i)$ is strictly increasing in t_i . Define the posterior virtual utility $\mathcal{V}_{i,\zeta_i}(t_i)$ by Eq. (46) if $t_i \leq \zeta_i$ and by $\mathcal{V}_{i,\zeta_i}(t_i) := \zeta_i$ if $t_i \geq \zeta_i$. Then \mathcal{V}_{i,ζ_i} is strictly increasing and continuous on $[0, \zeta_i]$. Let seller i_* 's realized type t_{i_*} be given. For any $i \neq i_*$ and any $t_{-i} := (t_{i_*}, (t_k)_{k \notin \{i, i_*\}})$, denote

$$v_i(t_{-i}) := \max \left\{ t_{i_*}, \max_{k \notin \{i, i_*\}} \mathcal{V}_{k,\zeta_k}(t_k) \right\}. \quad (61)$$

Given positive vector $\zeta := (\zeta_k)_{k \neq i_*}$, let $\mathbb{M}(\zeta)$ denote the Myerson auction based on $(\mathcal{V}_{k,\zeta_k})_{k \neq i_*}$, which for each realized type profile (t_{i_*}, t_{-i_*}) sells only to a bidder k for whom $\mathcal{V}_{k,\zeta_k}(t_k) \geq v_k(t_{-k})$ at price $\mathcal{V}_{k,\zeta_k}^{-1}(v_k(t_{-k}))$ and charges everyone else zero price. For any $t_{-i_*} \in \prod_{k \in I \setminus \{i_*\}} T_k$, denote $R(t_{i_*}, t_{-i_*}, \zeta)$ for seller i_* 's ex post payoff (“revenue”) generated by mechanism $\mathbb{M}(\zeta)$ when the realized type profile is t_{-i_*} . Let

$$\bar{R}(\zeta) := \mathbb{E} [R(t_{i_*}, \mathbf{t}_{-i_*}, \zeta) \mid \mathbf{t}_{-i_*} \leq \zeta_{-i_*}],$$

where, for any points x and y in the same euclidean space, $x \leq y$ means $x_k \leq y_k$ for each coordinate k , and $x \not\leq y$ means “not $x \leq y$ ”. With every \mathcal{V}_{k,ζ_k} strictly increasing on $[0, \zeta_k]$, the seller's optimization problem belongs to the regular case of Myerson [13]. Thus,

$$\bar{R}(\zeta) = \mathbb{E} \left[\max \left\{ t_{i_*}, \max_{k \neq i_*} \mathcal{V}_{k,\zeta_k}(t_k) \right\} \mid \mathbf{t}_{-i_*} \leq \zeta_{-i_*} \right]. \quad (62)$$

A.2 Monotonicity of Expected Revenues

Different from the case in the received literature, such a monotonicity condition does not follow from the affiliation inequalities in Milgrom and Weber [12], because the ex post revenue $R(t_{-i_*}, \zeta)$ need not be nondecreasing in t_{-i_*} , as explained in the Introduction. The monotone properties obtained here are based on the fact that the expected revenue is maximized.

A.2.1 With Respect to Supremums of Type-Supports

With monotone bidding strategies in the initial auction, a higher winning bid implies higher supremums of the posterior supports of the losing bidders. If the reseller adjusts her resale mechanism accordingly, the effect on her expected revenues is quantified below.

Proposition 4 *If $\zeta \leq \zeta'$ and $\zeta \neq \zeta'$, then $\bar{R}(\zeta) \leq \bar{R}(\zeta')$ and:*

a. *if $\max_{k \neq i_*} \zeta_k \leq t_{i_*} < \max_{k \neq i_*} \zeta'_k$, then*

$$\bar{R}(\zeta') - \bar{R}(\zeta) \geq \left(1 - \frac{F_k \left(\mathcal{V}_{k, \zeta'_k}^{-1}(t_{i_*}) \right)}{F_k(\zeta'_k)} \right) \left(\mathcal{V}_{k, \zeta'_k}^{-1}(t_{i_*}) - t_{i_*} \right); \quad (63)$$

b. *if $t_{i_*} < \max_{k \neq i_*} \zeta_k$ then*

$$\bar{R}(\zeta') - \bar{R}(\zeta) \geq \gamma \Pr \{ \exists k \in I \setminus \{i_*\} : \zeta_k < t_k \leq \zeta'_k \}, \quad (64)$$

where

$$\gamma := \frac{\prod_{k \neq i} F_k(\zeta_k) - \prod_{k \neq i} F_k(\min \{ \mathcal{V}_{k, \zeta_k}^{-1}(t_{i_*}), \zeta_k \})}{\left(\prod_{k \neq i} F_k(\zeta'_k) \right) \left(\prod_{k \neq i} F_k(\zeta_k) \right)} t_{i_*}. \quad (65)$$

Proof There are only two possible cases: (i) $t_{i_*} \geq \max_{k \neq i_*} \zeta_k$ and (ii) $t_{i_*} < \max_{k \neq i_*} \zeta_k$.

Case (i): $t_{i_*} \geq \max_{k \neq i_*} \zeta_k$. Then there is no gain of trade and hence $\bar{R}(\zeta) = t_{i_*}$. By the fact that $\bar{R}(\zeta') \geq t_{i_*}$, we have $\bar{R}(\zeta') \geq \bar{R}(\zeta)$. Now suppose, in addition, that $t_{i_*} < \zeta'_k$ for some $k \neq i_*$, which is the case in Claim (a) of the proposition. Then there is a strictly positive probability that sale happens at mechanism $\mathbb{M}(\zeta')$; in the case of sale, seller i_* 's payoff is at least as large as the reserve price $\mathcal{V}_{k, \zeta'_k}^{-1}(t_{i_*})$. Hence Ineq. (63) follows.

Case (ii): $t_{i_*} < \max_{k \neq i_*} \zeta_k$. Note that the mechanism $\mathbb{M}(\zeta)$ is ex post incentive feasible for any potential buyer $k \in I \setminus \{i_*\}$: Conditional on any $t_{-k} \in T_{-k}$, k 's winning probability in $\mathbb{M}(\zeta')$ is nondecreasing in t_k since k 's virtual utility $\mathcal{V}_{k, \zeta_k}(t_k)$ is so, and k 's payment

$\mathcal{V}_{k,\zeta_k}^{-1}(v(t_{-k}))$, denoted $p_k(t_{-k})$ here, satisfies the envelope equation conditional on t_{-k} . Thus, when the support supremums are $(\zeta'_k)_{k \neq i_*}$ instead of $(\zeta_k)_{k \neq i_*}$, $\mathbb{M}(\zeta)$ is still incentive feasible. Hence let $\widehat{R}(\zeta \mid \zeta')$ denote seller i_* 's expected payoff generated by $\mathbb{M}(\zeta)$ at the truth-telling equilibrium given distributions $(F_{k,\zeta'_k})_{k \neq i_*}$. By revealed preference from i_* 's viewpoint,

$$\overline{R}(\zeta') - \overline{R}(\zeta) \geq \widehat{R}(\zeta \mid \zeta') - \overline{R}(\zeta). \quad (66)$$

Denote

$$\begin{aligned} A &:= \left\{ t_{-i} \in T_{-i} : t_{-i_*} \leq \zeta; t_{i_*} > \max_{k \neq i_*} \mathcal{V}_{k,\zeta_k}(t_k) \right\}, \\ B &:= \left\{ t_{-i} \in T_{-i} : t_{-i_*} \not\leq \zeta; t_{-i_*} \leq \zeta'; t_{i_*} > \max_{k \neq i} \mathcal{V}_{k,\zeta_k}(t_k) \right\}, \\ C &:= \{ t_{-i} \in T_{-i} : t_{-i_*} \leq \zeta \} \setminus A, \\ D &:= \{ t_{-i} \in T_{-i} : t_{-i_*} \not\leq \zeta; t_{-i_*} \leq \zeta' \} \setminus B. \end{aligned}$$

Thus, $A \cup C$ is the support of \mathbf{t}_{-i_*} given ζ ; within $A \cup C$, A is the event in which i_* does not sell the good at mechanism $\mathbb{M}(\zeta)$. Analogously, $A \cup B \cup C \cup D$ is the support of \mathbf{t}_{-i_*} given ζ' , and $A \cup B$ the event of no-sale at mechanism $\mathbb{M}(\zeta)$.

Let $\pi(A)$, $\pi(B)$, $\pi(C)$ and $\pi(D)$ denote the prior probabilities of these sets. Since $t_{i_*} < \zeta_k$ for some $k \neq i$ in this case, $\pi(C) > 0$ (since $\mathcal{V}_{k,\zeta_k} \leq \zeta_k$). Let us compare the performance of mechanism $\mathbb{M}(\zeta)$ conditional on ζ with its performance conditional ζ' :

	$A \cup B$	C	D
probability given ζ	$\frac{\pi(A)}{\pi(A)+\pi(C)}$	$\frac{\pi(C)}{\pi(A)+\pi(C)}$	0
probability given ζ'	$\frac{\pi(A)+\pi(B)}{\pi(A)+\pi(B)+\pi(C)+\pi(D)}$	$\frac{\pi(C)}{\pi(A)+\pi(B)+\pi(C)+\pi(D)}$	$\frac{\pi(D)}{\pi(A)+\pi(B)+\pi(C)+\pi(D)}$
ex post payoff for i	t_{i_*}	$\sum_{j \neq i_*} q_{ji_*}(t) p_j(t_{-j})$	$\sum_{j \neq i_*} q_{ji_*}(t) p_j(t_{-j})$

In the cells on the last row and the third and fourth columns, the ex post payoff for seller i_* is equal to $\sum_{j \neq i_*} q_{ji_*}(t) p_j(t_{-j})$, where $q_{ji_*}(t)$ denotes the probability with which i_* sells the good to j in the mechanism $\mathbb{M}(\zeta)$. According to the payment rule in $\mathbb{M}(\zeta)$, this sum of payments is at least as large as the reserve price $\mathcal{V}_{j,\zeta_j}^{-1}(t_{i_*})$ for any j who wins; this reserve price is strictly greater than t_{i_*} as $t_{i_*} < \zeta'_j$ in this case. Therefore,

$$\begin{aligned} & \widehat{R}(\zeta \mid \zeta') - \overline{R}(\zeta) \\ & > t_{i_*} \left(\frac{\pi(D)}{\pi(A) + \pi(B) + \pi(C) + \pi(D)} - 0 + \frac{\pi(A) + \pi(B)}{\pi(A) + \pi(B) + \pi(C) + \pi(D)} - \frac{\pi(A)}{\pi(A) + \pi(C)} \right) \\ & = \frac{\pi(C) (\pi(B) + \pi(D))}{(\pi(A) + \pi(B) + \pi(C) + \pi(D)) (\pi(A) + \pi(C))} t_{i_*}. \end{aligned}$$

which is equal to $\pi(B) + \pi(D)$ multiplied by the γ defined by Eq. (65). Note that $\pi(B) + \pi(D) = \Pr \{\exists k \in I \setminus \{i_*\} : \zeta_k < \mathbf{t}_k \leq \zeta'_k\}$. Thus, Ineq. (66) implies (64). ■

A.2.2 With Respect to an Upward Push of a Bidder's Type-Support

Recall that $R(t_{i_*}, t_{-i_*}, \zeta)$ denotes the ex post payoff for the type- t_{i_*} seller i_* generated by the Myerson auction $\mathbb{M}(\zeta)$ when the realized type profile across $k \neq i_*$ is t_{-i_*} . Here we consider how a perturbation of a bidder's type distribution may affect the expected value of $R(t_{i_*}, \mathbf{t}_{-i_*}, \zeta)$. For any $i \neq i_*$ and any $t_i \in T_i$, let

$$\varphi_i(t_i, t_{i_*}) := \mathbb{E} \left[R(t_i, t_{i_*}, \mathbf{t}_{-(i, i_*)}, \zeta) \mid \mathbf{t}_{-(i, i_*)} \leq \zeta_{-(i, i_*)} \right]. \quad (67)$$

Given the posterior beliefs determined by ζ , $\mathbb{E}[\varphi_i(\mathbf{t}_i, t_{i_*}) \mid \mathbf{t}_i \in [0, \zeta_i]]$ is equal to the expected payoff for i_* generated by $\mathbb{M}(\zeta)$ and is the maximum expected payoff for the type- t_{i_*} seller i_* among all incentive feasible mechanisms. Within this subsection, ζ is held fixed and is suppressed from our notations, hence \mathcal{V}_k means \mathcal{V}_{k, ζ_k} ; also suppressed is the symbol t_{i_*} .

Lemma 21 *For any $t_{i_*} \in T_{i_*}$ and any $i \neq i_*$, $\varphi_i(\cdot, t_{i_*})$ is continuous on $[0, \zeta_i]$.*

Proof Let $\zeta_i \geq x'' > x' \geq 0$. Suppose the value of t_i increases from x' to x'' . Given any t_{-i} , this change affects the seller's ex post payoff $R(t_i, t_{-(i, i_*)}, \zeta)$ in only two cases:

- i. $\mathcal{V}_{i, \zeta_i}(x') < v_i(t_{-i}) \leq \mathcal{V}_i(x'')$. Let k_* be the bidder whose virtual utility is the highest when $t_i = x'$. Then, when t_i increases from x' to x'' , the winner in $\mathbb{M}(\zeta)$ switches from k_* to i and the revenue for i_* changes from $\mathcal{V}_{k_*}^{-1}(v_{k_*}(t_{-(i, k_*)}, t_i = x'))$ to $\mathcal{V}_i^{-1}(\mathcal{V}_{k_*}(t_{k_*}))$.
- ii. $\mathcal{V}_i(x'') < v_i(t_{-i})$ and $\mathcal{V}_i(x'')$ equals the second highest among $\left((\mathcal{V}_k(t_k))_{k \notin \{i, i_*\}}, \mathcal{V}_i(x'') \right)$. Let k^* be the bidder whose virtual utility is the highest when $t_i = x''$ (hence also the highest when $t_i = x'$). When t_i increases from x' to x'' , k^* remains to be the winner in $\mathbb{M}(\zeta)$ but her payment increases from $\mathcal{V}_{k^*}^{-1}(v_{k^*}(t_{-(i, k^*)}, t_i = x'))$ to $\mathcal{V}_{k^*}^{-1}(\mathcal{V}_i(x''))$. By Eq. (61) the definition of v_{k^*} and continuity of \mathcal{V}_{k^*} , this increase in revenue

$$\begin{aligned} \mathcal{V}_{k^*}^{-1}(\mathcal{V}_i(x'')) - \mathcal{V}_{k^*}^{-1} \left(\max \left\{ t_{i_*}, \mathcal{V}_i(x'), \max_{k \notin \{i, i_*, k^*\}} \mathcal{V}_k(t_k) \right\} \right) &\leq \mathcal{V}_{k^*}^{-1}(\mathcal{V}_i(x'')) - \mathcal{V}_{k^*}^{-1}(\mathcal{V}_i(x')) \\ &= O(x'' - x'). \end{aligned}$$

Note that $\varphi_i(x'') - \varphi_i(x')$ is equal to the expected value of the sum of the two revenue-differences, in Cases (i) and (ii), across all $t_{-(i, i_*)} \in \prod_{k \notin \{i, i_*\}} [0, \zeta_k]$. As noted above, the

case-(ii) difference is $O(x'' - x')$, and so is its expected value. The probability measure of those $t_{-(i,i_*)}$ that belong to Case (i) is also $O(x'' - x')$, because Case (i) occurs only if $\mathcal{V}_i(x') < v_i(t_{-i}) \leq \mathcal{V}_i(x'')$, which belongs to the event

$$\bigcup_{j \notin \{i, i_*\}} \left\{ t_{-(i, i_*)} \in \prod_{k \notin \{i, i_*\}} [0, \zeta_k] : \mathcal{V}_i(x') < \max_{k \notin \{i, i_*\}} \mathcal{V}_k(t_k) = \mathcal{V}_j(t_j) \leq \mathcal{V}_i(x'') \right\},$$

whose measure is $O(x'' - x')$ since \mathcal{V}_j is strictly increasing and \mathcal{V}_i continuous. Thus, $\varphi_i(x'') - \varphi_i(x') = O(x'' - x')$. ■

Lemma 22 For any $t_{i_*} \in T_{i_*}$, any $i \neq i_*$ and any $\alpha \in [0, \zeta_i)$,

$$\mathbb{E}[\varphi_i(\mathbf{t}_i, t_{i_*}) \mid \mathbf{t}_i \in [\alpha, \zeta_i]] \geq \mathbb{E}[\varphi_i(\mathbf{t}_i, t_{i_*}) \mid \mathbf{t}_i \in [0, \zeta_i]]. \quad (68)$$

Proof Denote $\bar{\varphi}_i := \mathbb{E}[\varphi_i(\mathbf{t}_i) \mid \mathbf{t}_i \in [0, \zeta_i]]$. Suppose, to the contrary of the lemma, that $\mathbb{E}[\varphi_i(\mathbf{t}_i) \mid \mathbf{t}_i \in [\alpha, \zeta_i]] < \bar{\varphi}_i$ for some $\alpha \in [0, \zeta_i)$. Then trivially $\alpha > 0$. Let $S := \{\mathbf{t}_i \in [\alpha, \zeta_i] : \varphi_i(\mathbf{t}_i) < \bar{\varphi}_i\}$. Then the measure of S is strictly positive, so $\inf S < \zeta_i$. Let

$$x := \begin{cases} \inf S & \text{if } \inf S > \alpha \\ \sup \{\mathbf{t}_i \in [0, \alpha] : \varphi_i(\mathbf{t}_i) \geq \bar{\varphi}_i\} & \text{if } \inf S = \alpha. \end{cases}$$

By definition of x , if $x < \mathbf{t}_i < \alpha$ then $\varphi_i(\mathbf{t}_i) < \bar{\varphi}_i$; and if $\alpha < \mathbf{t}_i < x$ then $\varphi_i(\mathbf{t}_i) \geq \bar{\varphi}_i$. Thus,

$$\mathbb{E}[\varphi_i(\mathbf{t}_i) \mid \mathbf{t}_i \in [x, \zeta_i]] < \bar{\varphi}_i. \quad (69)$$

Hence $x > 0$. By continuity of φ_i (Lemma 21), $\varphi_i(x) = \bar{\varphi}_i$.

Now consider a mechanism \tilde{M} which is the same as $M(\zeta)$ except that bidder i 's virtual utility function \mathcal{V}_i is replaced by a function $\tilde{\mathcal{V}}_i$ defined by

$$\tilde{\mathcal{V}}_i(t_i) := \begin{cases} \mathcal{V}_i(t_i) & \text{if } t_i \leq x \\ \mathcal{V}_i(x) & \text{if } t_i \geq x. \end{cases}$$

\tilde{M} is incentive compatible: $\tilde{\mathcal{V}}_i$ is nondecreasing and hence bidder i 's probability of winning is nondecreasing in his type; the monotonicity of the other bidders' winning probabilities is unaffected. The payment rule satisfies the envelope formula because the payment is defined according to the formula based on $(\tilde{\mathcal{V}}_i, (\mathcal{V}_k)_{k \notin \{i, i_*\}})$. Individual rationality of \tilde{M} is obvious.

Thus, we may assume that bidders participate and are truthful in \tilde{M} . When bidder i 's type is any $t_i \in [0, x]$, \tilde{M} acts in the same way as $M(\zeta)$, generating the same expected

revenue $\varphi_i(t_i)$ conditional on t_i . When $t_i > x$, by contrast, \tilde{M} acts as $\mathbb{M}(\zeta)$ except that t_i is treated as x , so the expected revenue conditional on t_i becomes $\varphi_i(x)$.

Thus, the expected revenue generated by \tilde{M} is equal to a convex combination between $\mathbb{E}[\varphi_i(\mathbf{t}_i) \mid \mathbf{t}_i \in [0, x]]$ and $\varphi_i(x)$. As noted above, $\varphi_i(x) = \bar{\varphi}_i$. Also $\mathbb{E}[\varphi_i(\mathbf{t}_i) \mid \mathbf{t}_i \in [0, x]] > \bar{\varphi}_i$ by Ineq. (69). This coupled with the fact $x > 0$ implies that the expected revenue yielded by \tilde{M} is greater than $\bar{\varphi}_i$. But that contradicts the fact that $\bar{\varphi}_i$ is maximum among all equilibrium-feasible mechanisms. ■

Corollary 2 *For any measurable $S \subseteq T_{i^*}$ and any $i \neq i^*$, $\mathbb{E}[\varphi_i(t_i, \mathbf{t}_{i^*}) \mid \mathbf{t}_{i^*} \in S]$ is continuous in t_i on $[0, \zeta_i]$, and for any $\alpha \in [0, \zeta_i)$*

$$\mathbb{E}[\varphi_i(\mathbf{t}_i, \mathbf{t}_{i^*}) \mid \mathbf{t}_i \in [\alpha, \zeta_i]; \mathbf{t}_{i^*} \in S] \geq \mathbb{E}[\varphi_i(\mathbf{t}_i, \mathbf{t}_{i^*}) \mid \mathbf{t}_i \in [0, \zeta_i]; \mathbf{t}_{i^*} \in S]. \quad (70)$$

Proof Since $\mathbb{E}[\varphi_i(t_i, \mathbf{t}_{i^*}) \mid \mathbf{t}_{i^*} \in S]$ is an integral of $\varphi(t_i, \mathbf{t}_{i^*})$ across t_{i^*} and the integrand $\varphi(t_i, \mathbf{t}_{i^*})$ is continuous in t_i by Lemma 21, the integral is also continuous in t_i . For any $\alpha \in [0, \zeta_i)$, Ineq. (68) holds. Integrating (68) across all $t_{i^*} \in S$, we obtain Ineq. (70). ■

A.3 An Upper Bound of Expected Revenues

This upper bound is the expected payment made by the highest possible bidder-type.

Lemma 23 *If $i \in I \setminus \{i^*\}$ and $\zeta_i = \max_{k \neq i^*} \zeta_k > t_{i^*}$, then*

$$\bar{R}(\zeta) < \mathbb{E} \left[\mathcal{V}_{i, \zeta_i}^{-1} \left(\max \left\{ t_{i^*}, \max_{k \notin \{i, i^*\}} \mathcal{V}_{k, \zeta_k}(\mathbf{t}_k) \right\} \right) \mid \mathbf{t}_{-(i, i^*)} \leq \zeta_{-(i, i^*)} \right]. \quad (71)$$

Proof By Eqs. (61) and (62),

$$\begin{aligned} \bar{R}(\zeta) &= \mathbb{E} \left[\max \{ \mathcal{V}_{i, \zeta_i}(\mathbf{t}_i), v_i(t_{i^*}, \mathbf{t}_{-(i, i^*)}) \} \mid \mathbf{t}_{-i^*} \leq \zeta_{-i^*} \right] \\ &= \underbrace{\mathbb{E} \left[v_i(t_{i^*}, \mathbf{t}_{-(i, i^*)}) \mathbf{1} [v_i(t_{i^*}, \mathbf{t}_{-(i, i^*)}) \geq \mathcal{V}_{i, \zeta_i}(\mathbf{t}_i)] \mid \mathbf{t}_{-i^*} \leq \zeta_{-i^*} \right]}_{=: X} \\ &\quad + \underbrace{\mathbb{E} \left[\mathcal{V}_{i, \zeta_i}(\mathbf{t}_i) \mathbf{1} [v_i(t_{i^*}, \mathbf{t}_{-(i, i^*)}) < \mathcal{V}_{i, \zeta_i}(\mathbf{t}_i)] \mid \mathbf{t}_{-i^*} \leq \zeta_{-i^*} \right]}_{=: Y}. \end{aligned}$$

First, we calculate X . For any t_{-i^*} in the integration domain of X , the hypothesis $\zeta_i \geq \max\{t_{i^*}, \max_{k \notin \{i, i^*\}} \zeta_k\}$ implies $v_i(t_{i^*}, t_{-(i, i^*)}) \leq \zeta_i$ and hence $\mathcal{V}_{i, \zeta_i}^{-1}(v_i(t_{i^*}, t_{-(i, i^*)})) \geq v_i(t_{i^*}, t_{-(i, i^*)})$. Furthermore, there is a positive-measure subset of the integration domain in which this inequality is strict: When t_{-i^*} is nearly zero so that every bidder's virtual utility

is negative, $v_i(t_{i_*}, t_{-(i,i_*)}) = t_{i_*}$ while $\mathcal{V}_{i,\zeta_i}^{-1}(v_i(t_{i_*}, t_{-(i,i_*)})) = \mathcal{V}_{i,\zeta_i}^{-1}(t_{i_*})$, strictly larger than t_{i_*} by Eq. (46). This subset is of positive measure in T_{-i_*} since the priors have no gap. Thus,

$$X < \mathbb{E} \left[\mathcal{V}_{i,\zeta_i}^{-1}(v_i(t_{i_*}, \mathbf{t}_{-(i,i_*)})) \mathbf{1} \left[v_i(t_{i_*}, \mathbf{t}_{-(i,i_*)}) \geq \mathcal{V}_{i,\zeta_i}(\mathbf{t}_i) \right] \mid \mathbf{t}_{-i_*} \leq \zeta_{-i_*} \right]. \quad (72)$$

To calculate Y , denote $q_i(t_{-i_*}) := \mathbf{1} \left[v_i(t_{i_*}, t_{-(i,i_*)}) < \mathcal{V}_{i,\zeta_i}(t_i) \right]$, the probability with which bidder i buys the good at player i_* 's mechanism $\mathbb{M}(\zeta)$. Denote

$$\bar{q}_i(t_i) := \mathbb{E} \left[q_i(t_i, \mathbf{t}_{-(i,i_*)}) \mid \mathbf{t}_{-(i,i_*)} \leq \zeta_{-(i,i_*)} \right].$$

By Eq. (46),

$$Y = \mathbb{E} \left[\left(t_i - \frac{1 - F_{i,\zeta_i}(t_i)}{f_{i,\zeta_i}(t_i)} \right) q_i(t_i, \mathbf{t}_{-(i,i_*)}) \mid \mathbf{t}_{-i_*} \leq \zeta_{-i_*} \right] = \int_0^{\zeta_i} \bar{q}_i(t_i) \left(t_i - \frac{1 - F_{i,\zeta_i}(t_i)}{f_{i,\zeta_i}(t_i)} \right) f_{i,\zeta_i}(t_i) dt_i.$$

Going through the integration-by-parts routine in reverse order, we have

$$\begin{aligned} Y &= \int_0^{\zeta_i} t_i \bar{q}_i(t_i) f_{i,\zeta_i}(t_i) dt_i - \int_0^{\zeta_i} \int_{t_i}^{\zeta_i} \bar{q}_i(t_i) f_{i,\zeta_i}(t'_i) dt'_i dt_i \\ &= \int_0^{\zeta_i} t_i \bar{q}_i(t_i) f_{i,\zeta_i}(t_i) dt_i - \int_0^{\zeta_i} \int_0^{t_i} \bar{q}_i(t_i) dt_i f_{i,\zeta_i}(t'_i) dt'_i \\ &= \int_0^{\zeta_i} \left(t_i \bar{q}_i(t_i) - \int_0^{t_i} \bar{q}_i(t_i) dt_i \right) f_{i,\zeta_i}(t_i) dt_i, \end{aligned}$$

which by the envelope-theorem routine is equal to the ex ante expected payment of type ζ_i in player i_* 's mechanism $\mathbb{M}(\zeta)$ conditional on the event that $\mathbf{t}_k \in [0, \zeta_k]$ for all $k \neq i_*$. Thus, by the definition of the payment rule in that mechanism,

$$Y = \mathbb{E} \left[\mathcal{V}_{i,\zeta_i}^{-1}(v_i(t_{i_*}, \mathbf{t}_{-(i,i_*)})) \mathbf{1} \left[v_i(t_{i_*}, \mathbf{t}_{-(i,i_*)}) < \mathcal{V}_{i,\zeta_i}(\mathbf{t}_i) \right] \mid \mathbf{t}_{-i_*} \leq \zeta_{-i_*} \right].$$

This combined with Ineq. (72) gives the desired inequality:

$$\bar{R}(\zeta) < \mathbb{E} \left[\mathcal{V}_{i,\zeta_i}^{-1}(v_i(t_{i_*}, \mathbf{t}_{-(i,i_*)})) \mid \mathbf{t}_{-i_*} \leq \zeta_{-i_*} \right] = \mathbb{E} \left[\mathcal{V}_{i,\zeta_i}^{-1}(v_i(t_{i_*}, \mathbf{t}_{-(i,i_*)})) \mid \mathbf{t}_{-(i,i_*)} \leq \zeta_{-(i,i_*)} \right]. \quad \blacksquare$$

B A Semicontinuity Property of Monotone Functions

For any weakly increasing function $g : [a, z] \rightarrow \mathbb{R}$, define $g_{\inf}^{-1}(y)$ and $g_{\sup}^{-1}(y)$ by Eqs. (1) and (2), with g taking the role of β_k .

Lemma 24 Let $g : [a, z] \rightarrow \mathbb{R}$ be a weakly increasing function with $a < z$. For any $y \geq g(a)$ and any $\epsilon > 0$, there exists $\delta > 0$ such that if $y - \delta < y' < y + \delta$ then

$$g_{\inf}^{-1}(y) - \epsilon < g_{\inf}^{-1}(y') \leq g_{\sup}^{-1}(y') < g_{\sup}^{-1}(y) + \epsilon. \quad (73)$$

Proof First, suppose $g(a) < y < g(z)$. Let $\epsilon > 0$. As g is weakly increasing, we can shrink ϵ so that $g_{\inf}^{-1}(y) - \frac{\epsilon}{2}$ and $g_{\sup}^{-1}(y) + \frac{\epsilon}{2}$ each belong to $[a, z]$. By definitions of $g_{\inf}^{-1}(y)$ and $g_{\sup}^{-1}(y)$,

$$g\left(g_{\inf}^{-1}(y) - \frac{\epsilon}{2}\right) < y < g\left(g_{\sup}^{-1}(y) + \frac{\epsilon}{2}\right). \quad (74)$$

Let

$$\delta := \min \left\{ \frac{g\left(g_{\sup}^{-1}(y) + \frac{\epsilon}{2}\right) + y}{2} - y, y - \frac{g\left(g_{\inf}^{-1}(y) - \frac{\epsilon}{2}\right) + y}{2} \right\}.$$

Then $\delta > 0$. Pick any y' such that $y - \delta < y' < y + \delta$. Either (i) $g^{-1}(y') = \emptyset$ or (ii) $g^{-1}(y') \neq \emptyset$. In case (i), since

$$g\left(g_{\inf}^{-1}(y) - \epsilon/2\right) < \frac{g\left(g_{\inf}^{-1}(y) - \epsilon/2\right) + y}{2} \leq y - \delta < y',$$

by Eq. (1) we have $g_{\inf}^{-1}(y') \geq g_{\inf}^{-1}(y) - \epsilon/2 > g_{\inf}^{-1}(y) - \epsilon$. In case (ii), if $x \in g^{-1}(y)$ and $x < g_{\inf}^{-1}(y) - \epsilon/2$, then monotonicity of g implies

$$y' = g(x) \leq g\left(g_{\inf}^{-1}(y) - \epsilon/2\right) < \frac{g\left(g_{\inf}^{-1}(y) - \epsilon/2\right) + y}{2} \leq y - \delta,$$

contradicting the fact that $y' > y - \delta$; thus, $g_{\inf}^{-1}(y') \geq g_{\inf}^{-1}(y) - \epsilon/2 > g_{\inf}^{-1}(y) - \epsilon$. Analogously, we can show $g_{\sup}^{-1}(y') < g_{\sup}^{-1}(y) + \epsilon$. Thus, (73) holds if $g(a) < y < g(z)$.

Next consider the case where $y \geq g(z)$. If $y > g(z)$ then, for any sufficiently small $\delta > 0$, $y - \delta > g(z)$ and hence $y - \delta < y' < y + \delta$ implies $g_{\inf}^{-1}(y') = g_{\sup}^{-1}(y') = z = g_{\inf}^{-1}(y) = g_{\sup}^{-1}(y)$. If $y = g(z)$ then we just replace the upper bound $g\left(g_{\sup}^{-1}(y) + \frac{\epsilon}{2}\right)$ in Ineq. (74) by any number bigger than y , and then the calculation in the previous paragraph follows. The case where $y \leq g(a)$ is analogous. ■

C Posterior Densities and Virtual Utilities

Proof of Lemma 1 Denote $\pi_k(i, b_i)$ for the probability of the event that bidder k , conditional on submitting the highest bid b_i (thereby tying with i and possibly others), loses the

tie-breaking lottery. Then (with the convention of letting $\prod_{j \in \emptyset} x_j := 1$)

$$\pi_k(i, b_i) = \sum_{S \subseteq I \setminus \{i, k\}} \frac{|S| + 1}{|S| + 2} \left(\prod_{j \in S} (F_j(\beta_{j, \sup}^{-1}(b_i)) - F_j(\beta_{j, \inf}^{-1}(b_i))) \right) \left(\prod_{j \in I \setminus (S \cup \{i, k\})} F_j(\beta_{j, \inf}^{-1}(b_i)) \right).$$

Note that $\pi_k(i, b_i)$ is independent of t_k . By Bayes's rule,

$$F_k(t_k | i, b_i, \beta) = \begin{cases} \frac{F_k(t_k)}{F_k(\beta_{k, \inf}^{-1}(b_i)) + (F_k(\beta_{k, \sup}^{-1}(b_i)) - F_k(\beta_{k, \inf}^{-1}(b_i)))\pi_k(i, b_i)} & \text{if } t_k \leq \beta_{k, \inf}^{-1}(b_i) \\ \frac{F_k(\beta_{k, \inf}^{-1}(b_i)) + (F_k(t_k) - F_k(\beta_{k, \inf}^{-1}(b_i)))\pi_k(i, b_i)}{F_k(\beta_{k, \inf}^{-1}(b_i)) + (F_k(\beta_{k, \sup}^{-1}(b_i)) - F_k(\beta_{k, \inf}^{-1}(b_i)))\pi_k(i, b_i)} & \text{if } \beta_{k, \inf}^{-1}(b_i) \leq t_k \leq \beta_{k, \sup}^{-1}(b_i). \end{cases} \quad (75)$$

Thus, the density

$$f_k(t_k | i, b_i, \beta) = \begin{cases} \frac{f_k(t_k)}{F_k(\beta_{k, \inf}^{-1}(b_i)) + (F_k(\beta_{k, \sup}^{-1}(b_i)) - F_k(\beta_{k, \inf}^{-1}(b_i)))\pi_k(i, b_i)} & \text{if } t_k < \beta_{k, \inf}^{-1}(b_i) \\ \frac{f_k(t_k)\pi_k(i, b_i)}{F_k(\beta_{k, \inf}^{-1}(b_i)) + (F_k(\beta_{k, \sup}^{-1}(b_i)) - F_k(\beta_{k, \inf}^{-1}(b_i)))\pi_k(i, b_i)} & \text{if } \beta_{k, \inf}^{-1}(b_i) < t_k \leq \beta_{k, \sup}^{-1}(b_i); \end{cases} \quad (76)$$

at the point $\beta_{k, \inf}^{-1}(b_i)$, the right density is equal to $\pi_k(i, b_i)$ times the left density. Thus, $f_k(\cdot | i, b_i, \beta)$ exists and is strictly positive on the posterior support and is continuous unless $\beta_{k, \inf}^{-1}(b_i) \neq \beta_{k, \sup}^{-1}(b_i)$, in which case $\beta_{k, \inf}^{-1}(b_i)$ is the only discontinuity point. ■

Proof of Lemma 2 By hypothesis, b_i is not an atom of β_k . Thus, $\beta_{k, \sup}^{-1}(b_i) = \beta_{k, \inf}^{-1}(b_i)$ and Eqs. (75) and (76) together imply Eq. (4) for any $t_k \in [0, \beta_{k, \sup}^{-1}(b_i)]$. For any $t_k \in (0, \beta_{k, \sup}^{-1}(b_i))$, the derivative of $V_{k, b_i, \beta}$, or $\frac{d}{dt_i} \left(t_i - \frac{F_k(\beta_{k, \sup}^{-1}(b_i)) - F_k(t_k)}{f_k(t_k)} \right)$ by Eq. (4), is no less than either 2 or $\frac{d}{dt_i} \left(t_i - \frac{1 - F_k(t_k)}{f_k(t_k)} \right)$, which, strictly positive on the compact T_i by assumption, is bigger than a $\lambda_k > 0$ constant to t_i . Thus, Claims (b) follows, with $\lambda := \max_{k \in I} \min\{\lambda_k, 2\}$, and so does Claim (d). With $V_{k, b_i, \beta}$ nondecreasing on T_k according to Claim (b), the ironing procedure is unnecessary and hence Claim (a) follows. To prove Claim (c), pick any $b'_i > b_i$ such that b'_i is not an atom of β_k . Hence both b'_i and b_i satisfy Eq. (4). Let $\Delta V_k(t_k | b'_i, b_i) := V_{k, b'_i, \beta}(t_k) - V_{k, b_i, \beta}(t_k)$. By Eqs. (3) and (4),

$$\Delta V_k(t_k | b'_i, b_i) = \begin{cases} -\frac{F_k(\beta_{k, \sup}^{-1}(b'_i)) - F_k(\beta_{k, \sup}^{-1}(b_i))}{f_k(t_k)} & \text{if } t_k \leq \beta_{k, \sup}^{-1}(b_i) \\ V_{k, b'_i, \beta}(t_k) - \beta_{k, \sup}^{-1}(b_i) & \text{if } \beta_{k, \sup}^{-1}(b_i) \leq t_k \leq \beta_{k, \sup}^{-1}(b'_i) \\ \beta_{k, \sup}^{-1}(b'_i) - \beta_{k, \sup}^{-1}(b_i) & \text{if } t_k \geq \beta_{k, \sup}^{-1}(b'_i), \end{cases} \quad (77)$$

where $\beta_{k, \sup}^{-1}(b_i) \leq \beta_{k, \sup}^{-1}(b'_i)$ because β_k is weakly increasing. The uppermost branch of Eq. (77) implies Claim (c). (The other branches will be used in Lemma 15.) ■

D Details of the Increasing Difference Theorem

Proof of Lemma 5 First, we prove Eq. (19). In the continuation equilibrium, in choosing a resale mechanism, player i the reseller effectively inputs an alleged type \hat{t}_i into the formula in Lemma 3 that outputs a mechanism optimal for \hat{t}_i , which determines i 's expected probability of being the final owner, $\bar{Q}_i(b_i, \hat{t}_i, \beta)$, and i 's expected revenue, denoted by $R_i(b_i, \hat{t}_i, \beta)$. Then i 's expected payoff in period 2 is $t_i \bar{Q}_i(b_i, \hat{t}_i, \beta) + R_i(b_i, \hat{t}_i, \beta)$. Optimality of the resale mechanism means this expected payoff is maximized when $\hat{t}_i = t_i$. Then the envelope theorem of Milgrom and Segal [11, Theorem 2] implies (19).

Next we prove Eq. (20). From i 's viewpoint, the continuation equilibrium in the event that i does not win the period-one auction is equivalent to an incentive feasible direct revelation mechanism that solicits from i a report of his type and then plays the continuation equilibrium on his behalf. Thus, the envelope theorem again implies

$$\bar{L}_i(b_i, t_i, \beta) = \bar{L}_i(b_i, 0, \beta) + \int_0^{t_i} \bar{q}_i(b_i, \tau_i, \beta) d\tau_i.$$

The resale mechanism, optimal for the reseller, leaves zero surplus to the zero type of any other bidder. Thus, $\bar{L}_i(b_i, 0, \beta) = 0$ and Eq. (20) follows. ■

Propositions 1 If $b'_i \notin B_i$ then the conclusion is vacuously true, since in that case $Q_i(b'_i, t, \beta) = Q_i(l, t, \beta) = 0$ by definition of Q_i . Thus, let $b'_i \in B_i$. Since $b''_i > b'_i$, we also have $b''_i \in B_i$. Then Eq. (11) implies that neither b''_i nor b'_i is an atom of β_{-i} . Thus, for all $t_{-i} \in \prod_{k \neq i} [0, \beta_{k, \text{sup}}^{-1}(b'_i)]$ but a set of measure zero and for any $t_i \in T_i$, Eq. (13) holds whether $b_i = b''_i$ or $b_i = b'_i$. Consider any such t_{-i} , and we shall prove that $Q_i(b''_i, t, \beta) \geq Q_i(b'_i, t, \beta)$. To avoid triviality, suppose $Q_i(b'_i, t, \beta) > 0$. Then Eq. (13) implies, for each $k \in I \setminus \{i\}$:

$$t_i \geq V_k(t_k | b'_i, \beta) \geq V_k(t_k | b''_i, \beta),$$

where the second inequality is due to the fact that $V_k(t_k | b_i, \beta)$ is weakly decreasing in b_i (Lemma 2.c, applicable because $t_{-i} \in \prod_{k \neq i} [0, \beta_{k, \text{sup}}^{-1}(b'_i)]$). Thus, $Q_i(b''_i, t, \beta) = 1$ by Eq. (13), with b''_i playing the role of b_i there. Hence $Q_i(b''_i, t, \beta) \geq Q_i(b'_i, t, \beta)$. ■

Proposition 2 Pick any $t \in T$. To avoid triviality, suppose that $q_i(t, \beta) > 0$. Then Eq. (12) implies that $q_{ij}(t, \beta) > 0$ for some $j \neq i$ such that $l < \beta_j(t_j) = \max_{k \neq i} \beta_k(t_k)$. By hypothesis $b_i \geq \max_{k \neq i} \beta_k(t_k)$, $b_i > l$. Thus, both b_i and $\beta_j(t_j)$ are serious bids, hence

by Eq. (11) b_i is not an atom of β_{-i} , nor $\beta_j(t_j)$ an atom of β_{-j} . Thus, Eq. (13) holds for all $t_{-i} \in \prod_{k \neq i} [0, \beta_{k,\text{sup}}^{-1}(b_i)]$ but a set of measure zero, and Eq. (14) holds for all $t_{-(i,j)} \in \prod_{k \notin \{i,j\}} [0, \beta_{k,\text{sup}}^{-1}(\beta_j(t_j))]$ but a set of measure zero. By the choice of j , the condition $t_{-(i,j)} \in \prod_{k \notin \{i,j\}} [0, \beta_{k,\text{sup}}^{-1}(\beta_j(t_j))]$ is satisfied; by the hypothesis $b_i \geq \max_{k \neq i} \beta_k(t_k)$, the condition $t_{-i} \in \prod_{k \neq i} [0, \beta_{k,\text{sup}}^{-1}(b_i)]$ is met. Thus, Eqs. (13) and (14) both hold for almost every $t_{-i} \in T_{-i}$ such that $q_{ij}(t, \beta) > 0$ and $b_i \geq \max_{k \neq i} \beta_k(t_k)$. Thus, $q_{ij}(t, \beta) > 0$ implies

$$V_i(t_i | \beta_j(t_j), \beta) \geq \max \left\{ t_j, \max_{k \in I \setminus \{i,j\}} V_k(t_k | \beta_j(t_j), \beta) \right\}. \quad (78)$$

By Eq. (13), it suffices to prove $t_i \geq \max_{k \in I \setminus \{i\}} V_k(t_k | b_i, \beta)$. To this end, given (78) and the fact $t_i \geq V_i(t_i | \beta_j(t_j), \beta)$ and $t_j \geq V_j(t_j | b_i, \beta)$ due to Eq. (3), it suffices to prove

$$\forall k \in I \setminus \{i, j\} : V_k(t_k | \beta_j(t_j), \beta) \geq V_k(t_k | b_i, \beta). \quad (79)$$

Thus, pick any $k \notin \{i, j\}$. By hypothesis, $b_i \geq \beta_j(t_j)$. By the choice of j , $t_k \in [0, \beta_{k,\text{sup}}^{-1}(\beta_j(t_j))]$, so Ineq. (79) follows from Lemma 2.c with the (b'_i, b_i) there being $(b_i, \beta_j(t_j))$ here. Hence $Q_i(b_i, t, \beta) \geq q_{ij}(t, \beta)$. This being true for any $j \neq i$ who may win at the realized profile t_{-i} , Eq. (12) implies $Q_i(b_i, t, \beta) \geq q_i(t, \beta)$. ■

E Details of the No-Tie Theorem

E.1 Proof of Lemma 7

Denote $x := b_*$. Let $\epsilon > 0$. Since the density f_i of \mathbf{t}_i is positive on its compact support $[0, \bar{t}_i]$ for every $i \in I$, there exists $\tilde{\eta}(\epsilon) > 0$ such that

$$0 < \eta < \tilde{\eta}(\epsilon) \implies \forall i \in I : \forall y \in T_i : \text{Prob} \{y \leq \mathbf{t}_i \leq y + 4\eta\} < \frac{\epsilon}{\max_{j \in I} \bar{t}_j}. \quad (80)$$

Pick any $\eta > 0$ such that

$$\eta < \min \left\{ \tilde{\eta}(\epsilon), \epsilon, \min_{k \in J} (z_k - a_k) \right\}. \quad (81)$$

For any $\delta > 0$, let

$$N_i^m(x; \delta) := (x - \delta, x + \delta) \cap B_i^m.$$

By Lemma 24 and monotonicity of β_i^* , there exists a $\tilde{\delta}(\epsilon) > 0$ such that

$$\forall i \in I : (\beta_i^*)^{-1} \left(N_i^m(x; 2\tilde{\delta}(\epsilon)) \right) \subseteq (a_i - \eta, z_i + \eta). \quad (82)$$

As $\beta^m \rightarrow \beta^*$ pointwise almost everywhere, $\beta^m \rightarrow \beta^*$ uniformly except on a set $E^* := \Pi_{i \in I} E_i^*$ such that each E_i^* has Lebesgue measure less than η (Littlewood's third principle or Egoroff's theorem). Thus, for any $\delta > 0$ such that

$$\delta \leq \tilde{\delta}(\epsilon), \quad (83)$$

there exists $\tilde{m}(\delta)$ with

$$2^{-\tilde{m}(\delta)} < \tilde{\delta}(\epsilon)/2 \quad (84)$$

such that for every integer $m \geq \tilde{m}(\delta)$ and for every $i \in I$, we have

$$\forall t_i \in T_i \setminus E_i^* : |\beta_i^m(t_i) - \beta_i^*(t_i)| < \delta/2, \quad (85)$$

$$(\beta_i^*)^{-1}(N_i^m(x; \delta/2)) \subseteq (a_i - \eta, z_i + \eta), \quad (86)$$

where (86) follows from (82) and (83).

Now we construct an infinite subsequence $(\beta^{m_n})_{n=1}^\infty$. For each $n = 1, 2, \dots$, let $\epsilon_n := 1/n$. With ϵ_n taking the role of ϵ , there exists η_n as the left-hand side of Ineq. (81) and $\tilde{\delta}(\epsilon_n)$ specified in Eq. (82). Let

$$\delta_n := \min \left\{ 1/n, \tilde{\delta}(\epsilon_n), x - l \right\}. \quad (87)$$

Hence there exists an $\tilde{m}(\delta_n)$ satisfying Ineq. (84). Let

$$m_n := \min \{ m = 1, 2, \dots : m \geq \tilde{m}(\delta_n); m \geq m_{n-1} + 1 \}.$$

Note that $n' > n \Rightarrow m_{n'} > m_n$. Hence subsequence $(\beta^{m_n})_{n=1}^\infty$ is constructed. Also Eqs. (85) and (86) are satisfied when (m_n, δ_n, η_n) plays the role of (m, δ, η) .

First, we claim that, for each $i \in I$,

$$\forall t_i \in (\beta_i^*)^{-1}(N_i^{m_n}(x; \delta_n/2)) \setminus E_i^* : x - \delta_n < \beta_i^{m_n}(t_i) < x + \delta_n, \quad (88)$$

$$(\beta_i^{m_n})^{-1}(N_i^{m_n}(x; \delta_n + 2^{-m_n})) \setminus E_i^* \subseteq (a_i - \eta_n, z_i + \eta_n). \quad (89)$$

To prove (88), pick any $t_i \in (\beta_i^*)^{-1}(N_i^{m_n}(x; \delta_n/2)) \setminus E_i^*$. Then

$$\beta_i^{m_n}(t_i) \stackrel{(85)}{<} \beta_i^*(t_i) + \delta_n/2 < x + \delta_n/2 + \delta_n/2 = x + \delta_n,$$

and analogously $\beta_i^{m_n}(t_i) > x - \delta_n$. To prove (89), suppose $t_i \leq a_i - \eta_n$. Then (82) and monotonicity of β_i^* imply $\beta_i^*(t_i) \leq x - 2\tilde{\delta}(\epsilon_n)$; according to (85), either $t_i \in E_i^*$, or

$$\beta_i^{m_n}(t_i) < \beta_i^*(t_i) + \delta_n/2 \leq x - 2\tilde{\delta}(\epsilon_n) + \delta_n/2 \stackrel{(83)}{\leq} x - 2\tilde{\delta}(\epsilon_n) + \tilde{\delta}(\epsilon_n)/2 = x - \tilde{\delta}(\epsilon_n) - \tilde{\delta}(\epsilon_n)/2 < x - \delta_n - 2^{-m_n},$$

with the last inequality due to (83) and (84). Analogously, $t_i \geq z_i + \eta_n$ implies either $t_i \in E_i^*$ or $\beta_i^{m_n}(t_i) > x + \delta_n + 2^{-m_n}$. Hence (89) follows.

Second, we show that, for each $i \in J$, a_i^n and z_i^n defined by Eqs. (28) and (29) exist and “ $a_i < z_i^n$ and $z_i > a_i^n$ ” holds. By definition of a_i and z_i , $(a_i, z_i) \subseteq (\beta_i^*)^{-1}(N_i^{m_n}(x; \delta_n/2))$. Since the Lebesgue measure of E_i^* is less than η_n , which by (81) is smaller than $z_i - a_i$, there exists $t_i \in (a_i, z_i) \setminus E_i^* \subseteq (\beta_i^*)^{-1}(N_i^{m_n}(x; \delta_n/2)) \setminus E_i^*$. Hence (88) implies that the sets on the right-hand sides of Eqs. (28) and (29) are nonempty. Thus, a_i^n and z_i^n exist. By the choice of this t_i and Ineq. (85), we have $\beta_i^{m_n}(t_i) < \beta_i^*(t_i) + \delta_n/2 = x + \delta_n/2$. Thus, by definition of z_i^n , $t_i \leq z_i^n$. Hence $a_i < z_i^n$, otherwise $t_i > a_i \geq z_i^n$, a contradiction. Analogously, $z_i > a_i^n$.

Third, (30) follows from Eqs. (28) and (29) and the fact that $\beta_i^{m_n}$ is nondecreasing.

Fourth, we prove (32). Recall that $(a_i, z_i) \subseteq (\beta_i^*)^{-1}(N_i^{m_n}(x; \delta_n/2))$ and $a_i^n < z_i$. Thus, if $a_i < a_i^n$ then (88) implies that $(a_i, a_i^n) \subseteq E_i^*$; with E_i^* of Lebesgue measure less than η_n , we have $a_i^n - a_i < \eta_n$. Analogously we have $z_i - z_i^n < \eta_n$. Also, if $a_i^n < a_i - \eta_n$, then (89) implies $(a_i^n, a_i - \eta_n) \subseteq E_i^*$ and hence the interval cannot be longer than η_n ; hence $a_i - a_i^n < 2\eta_n$. Analogously we have $z_i^n - z_i < 2\eta_n$. Thus, since $\eta_n < \epsilon_n$ by (81), we have (32).

Fifth, we prove (31) and (33). For any $k \in I \setminus J$, $(\beta_k^*)^{-1}(x)$ is either singleton or empty, hence $a_k = z_k$ by definition. Thus, it follows from (89) that $(\beta_k^{m_n})^{-1}(N_k^{m_n}(x; \delta_n + 2^{-m_n}))$ is either contained in $(a_k - \eta_n, a_k] \cup [z_k, z_k + \eta_n)$ or contained in E_k^* . Since the Lebesgue measure of neither set is bigger than $2\eta_n$, (80) implies (33). Likewise, (80) implies (31) for any $i \in J$ because, by (89), $\{t_i \in T_i : x + \delta_n \leq \beta_i^{m_n}(t_i) < x + \delta_n + 2^{-m_n}\}$ is contained in $E_i^* \cup [z_i^n, z_i + \eta_n)$.

E.2 Proof of the Decomposition Equation (36)

Eq. (36) is the same as the following equation: for any bids $b'_i, b_i \in B_i$ with $b'_i > b_i$,

$$\begin{aligned} & U_i(b'_i, t_i, \beta) - U_i(b_i, t_i, \beta) \\ &= \mathbb{E}[\mathbf{1}[b'_i \succ \mathbf{t}_{-i}]] (\overline{W}_i(b'_i, t_i, \beta) - \overline{W}_i(b_i, t_i, \beta)) - (b'_i - b_i) \Pr\{b'_i \succ \mathbf{t}_{-i}\} \\ & \quad + \Pr\{b'_i \succ \mathbf{t}_{-i}, b_i \not\succ \mathbf{t}_{-i}\} (\overline{W}_i(b_i, t_i, \beta) - b_i - \overline{L}_i(b'_i, b_i, t_i, \beta)), \end{aligned} \quad (90)$$

where $b_i \succ \mathbf{t}_{-i}$ is a shorthand for i 's winning event, $b_i \not\succ \mathbf{t}_{-i}$ its complement, and

$$\overline{L}_i(b'_i, b_i, t_i, \beta) := \mathbb{E}[L_i(\mathbf{t}_{-i} \mid t_i, \beta) \mid b'_i \succ \mathbf{t}_{-i}, b_i \not\succ \mathbf{t}_{-i}].$$

To prove Eq. (90), note that Eq. (10) says, with the symbols (t_i, β) suppressed,

$$U_i(b_i) = \mathbb{E}[\mathbf{1}[b_i \succ \mathbf{t}_{-i}] (W_i(\mathbf{t}_{-i} | b_i) - b_i - L_i(\mathbf{t}_{-i}))] + \mathbb{E}[L_i(\mathbf{t}_{-i})].$$

Then for any bids $b'_i > b_i$,

$$\begin{aligned} U_i(b'_i) - U_i(b_i) &= \mathbb{E}[\mathbf{1}[b'_i \succ \mathbf{t}_{-i}] W_i(\mathbf{t}_{-i} | b'_i)] - \mathbb{E}[\mathbf{1}[b_i \succ \mathbf{t}_{-i}] W_i(\mathbf{t}_{-i} | b_i)] \\ &\quad - b'_i \mathbb{E}[\mathbf{1}[b'_i \succ \mathbf{t}_{-i}]] + b_i \mathbb{E}[\mathbf{1}[b_i \succ \mathbf{t}_{-i}]] + \mathbb{E}[(\mathbf{1}[b_i \succ \mathbf{t}_{-i}] - \mathbf{1}[b'_i \succ \mathbf{t}_{-i}]) L_i(\mathbf{t}_{-i})] \\ &\stackrel{(15)}{=} \mathbb{E}[\mathbf{1}[b'_i \succ \mathbf{t}_{-i}] \bar{W}_i(b'_i)] - \mathbb{E}[\mathbf{1}[b_i \succ \mathbf{t}_{-i}] \bar{W}_i(b_i)] \\ &\quad - b'_i \mathbb{E}[\mathbf{1}[b'_i \succ \mathbf{t}_{-i}]] + b_i \mathbb{E}[\mathbf{1}[b_i \succ \mathbf{t}_{-i}]] - \mathbb{E}[\mathbf{1}[b_i \not\succ \mathbf{t}_{-i}, b'_i \succ \mathbf{t}_{-i}] L_i(\mathbf{t}_{-i})]. \end{aligned}$$

Then Eq. (90) follows from breaking apart $\mathbf{1}[b_i \succ \mathbf{t}_{-i}] = \mathbf{1}[b'_i \succ \mathbf{t}_{-i}] - \mathbf{1}[b_i \not\succ \mathbf{t}_{-i}, b'_i \succ \mathbf{t}_{-i}]$.

E.3 Proof of Lemma 8

Suppose that the lemma is not true. Then, extracting a subsequence and relabeling superscripts if necessary, we may assume without loss of generality that

$$\lim_{n \rightarrow \infty} \Pr \left\{ c_j^n < \max_{k \in J \setminus \{j\}} \bar{\beta}_k^n(\mathbf{t}_k) \leq b_* + \delta_n \right\} > 0. \quad (91)$$

By definition of c_j^n in Eq. (40) and monotonicity of $\bar{\beta}_j^n$,

$$\forall n : \exists \epsilon_n \in \left(0, \min \left\{ 1/n, \max_{k \in J} (z_k - a_j)/2 \right\} \right) : \forall t_j \in (a_j^n, a_j^n + \epsilon_n) : \bar{\beta}_j^n(t_j) = c_j^n. \quad (92)$$

(The above choice of ϵ_n is feasible because by Ineq. (39) $z_k > a_k \geq a_j$ for every $k \in J$.) Let $n \mapsto t_j^n$ be any choice function such that for each n

$$t_j^n \in (a_j^n, a_j^n + \epsilon_n).$$

We shall derive a contradiction by proving that for some sufficiently large n the type- t_j^n bidder j strictly prefers to deviate from his m_n -equilibrium bid c_j^n to the bid

$$b_j^n := \min \{ b_j \in B_j^{m_n} : b_j \geq b_* + \delta_n \}.$$

To prove this claim, first we establish

$$\lim_{n \rightarrow \infty} \Delta \Pi_j^n(t_j^n) \geq 0. \quad (93)$$

By Eq. (38), $\Delta\Pi_j^n(t_j^n)$ is equal to a probability times $\psi_j^n(t_j^n) := \overline{W}_j(c_j^n, t_j^n, \bar{\beta}^n) - c_j^n - \overline{L}_j^n(t_j^n)$. Hence it suffices to show $\lim_{n \rightarrow \infty} \psi_j^n(t_j^n) \geq 0$. To this end, we first claim that

$$\lim_{n \rightarrow \infty} \mathbb{E} [L_j(\mathbf{t}_{-j} \mid t_j^n, \bar{\beta}^n) \mid \mathbf{t}_{-j} \in \Omega_j^n] = 0, \quad (94)$$

where $\Omega_j^n := \{t_{-j} \in T_{-j} : c_j^n < \max_{k \neq j} \bar{\beta}_k^n(\mathbf{t}_k) < b_j^n\}$. By Lemma 7, as $n \rightarrow \infty$, the difference between Ω_j^n and the $\tilde{\Omega}_j^n$ defined below vanishes:

$$\begin{aligned} \tilde{\Omega}_j^n &:= \cup_{k \in J \setminus \{j\}} \tilde{\Omega}_{jk}^n, \quad \text{where for each } k \in J \setminus \{j\} \\ \tilde{\Omega}_{jk}^n &:= \left\{ t_{-j} \in T_{-j} : t_k \in (a_k^n, z_k^n); c_j^n < \bar{\beta}_k^n(t_k) = \max_{k' \neq j} \bar{\beta}_{k'}^n(\mathbf{t}_{j'}) < b_j^n; \forall h \notin J [\bar{\beta}_h^n(t_h) < c_j^n] \right\}. \end{aligned}$$

Thus, Ineq. (94) is unchanged when its integrand $L_j(\mathbf{t}_{-j} \mid t_j^n, \bar{\beta}^n)$ is replaced by

$$\sum_{k \in J \setminus \{j\}} \mathbf{1} \left[\bar{\beta}_k^n(\mathbf{t}_k) > \max_{j' \notin \{i, j\}} \bar{\beta}_{j'}^n(\mathbf{t}_{j'}) \right] L_{jk}(\mathbf{t}_{-j} \mid t_j^n, \bar{\beta}^n).$$

For any $k \in J \setminus \{j\}$, bidder j 's payoff L_{jk} from losing the auction to bidder k cannot exceed $t_j^n - t_k$. Since $t_j^n < a_j^n + \epsilon_n < a_j^n + 1/n$ by the choice of t_j^n and ϵ_n (Eq. (92)), and since $a_i^n \rightarrow_n a_i$ for each $i \in J$, we have for any $t_k \in (a_k^n, z_k^n)$:

$$t_j^n < a_j^n + \epsilon_n \leq a_j + O(1/n) \stackrel{(39)}{\leq} a_k + O(1/n) \leq a_k^n + O(1/n) < t_k + O(1/n).$$

Thus, $0 \leq L_{jk}^n(t_{-j} \mid t_j^n, \bar{\beta}^n) \leq t_j^n - t_k = O(1/n)$ for any $t_{-j} \in \tilde{\Omega}_{jk}^n$. That proves (94). Thus,

$$\lim_{n \rightarrow \infty} \psi_j^n(t_j^n) = \lim_{n \rightarrow \infty} (\overline{W}_j(c_j^n, t_j^n, \bar{\beta}^n) - c_j^n) \geq 0,$$

with the second inequality due to the fact $\bar{\beta}_j^n(t_j^n) = c_j^n$ and Lemma 6. That proves Ineq. (93).

Second, by Eq. (92),

$$t_j^n < a_j^n + \epsilon_n \leq a_j + \epsilon_n + O(1/n) < a_j + \max_{k \in J} (z_k - a_j)/2 + O(1/n) = \max_{k \in J} (z_k + a_j)/2 + O(1/n),$$

hence $\lim_{n \rightarrow \infty} t_j^n \leq \max_{k \in J} (z_k + a_j)/2 < \max_{k \in J} z_k \leq \max_{k \neq j} z_k$. Thus, Lemma 9 implies $\limsup_{n \rightarrow \infty} \Delta W_j^n(t_j^n) > 0$. Plugging this inequality, Ineq. (93), and $\lim_{n \rightarrow \infty} (b_j^n - c_j^n) = 0$ ($c_j^n \in (b_* - \delta_n, b_* + \delta_n)$ since $t_j^n \in (a_j^n, z_j^n)$) into Eq. (36), we have $\limsup_{n \rightarrow \infty} \Delta U_j^n(t_j^n) > 0$. Thus, there are sufficiently large n for which the type- t_j^n bidder j strictly prefers deviating to b_j^n from his m_n -equilibrium bid c_j^n . This contradiction proves the lemma.

E.4 The Dominant Rival's Resale Mechanisms

The next lemma helps us to predict the resale mechanism employed by the dominant bidder j specified in Lemma 8 when he wins with a bid clustered at b_* . In that event, j 's winning bid ranges within a neighborhood where his rivals rarely bid, hence his posterior belief about the others stays mostly constant to his winning bid, and so are the posterior virtual utility functions and payment rules at resale. We shall use the notation $\mathcal{V}_{k,x}$ defined in Eq. (46).

Lemma 25 *Let j be specified by Lemma 8, c_i^n specified by Eqs. (41) and (42), and b_i^n specified by Eq. (45). If x^n is someone's winning bid in $[c_i^n, b_i^n]$ for each $n = 1, 2, \dots$, then for any $k \in I \setminus \{j\}$, with z_k specified in Eq. (27),*

$$\lim_{n \rightarrow \infty} (\bar{\beta}^n)_{k,\text{sup}}^{-1}(x^n) = z_k, \quad (95)$$

$$\forall t_k \in T_k : \lim_{n \rightarrow \infty} V_k(t_k | x^n, \bar{\beta}^n) = \mathcal{V}_{k,z_k}(t_k), \quad (96)$$

and, if in addition $z_k > \max\{t_j, \max_{k' \notin \{j,k\}} \mathcal{V}_{k',z_{k'}}(t_{k'})\}$, then

$$\lim_{n \rightarrow \infty} p_{k,j,x^n,\bar{\beta}^n}(t_j, t_{-(j,k)}) = \mathcal{V}_{k,z_k}^{-1}\left(\max\left\{t_j, \max_{k' \notin \{j,k\}} \mathcal{V}_{k',z_{k'}}(t_{k'})\right\}\right). \quad (97)$$

Proof Let $k \in I \setminus \{j\}$. Since $x^n \in [c_i^n, b_i^n]$, Eqs. (31), (33) and (42) together imply that the probability measure of the interval between $(\bar{\beta}^n)_{k,\text{sup}}^{-1}(x^n)$ and z_k^n (defined in Eq. (29)) vanishes. Thus, Eq. (95) follows from the no-gap assumption of F_k and the fact $z_k^n \rightarrow_n z_k$ by Eq. (32). To prove (96), note that x^n , a winning bid in the m_n -approximation game, is not an atom of the losers' strategies, due to Eq. (22). Thus, $V_k(t_k | x^n, \bar{\beta}^n)$ obey Eq. (4), with x^n being the b_i there, if $t_k \leq (\bar{\beta}^n)_{k,\text{sup}}^{-1}(x^n)$ and is equal to $(\bar{\beta}^n)_{k,\text{sup}}^{-1}(x^n)$ if $t_k \geq (\bar{\beta}^n)_{k,\text{sup}}^{-1}(x^n)$. Then Eq. (96) follows from Eqs. (46) and (95). To prove Eq. (97), let its condition $z_k > \max\{t_j, \max_{k' \notin \{j,k\}} \mathcal{V}_{k',z_{k'}}(t_{k'})\}$ be satisfied. Then Eqs. (95) and (96) imply

$$(\bar{\beta}^n)_{i,\text{sup}}^{-1}(x^n) > \max\left\{t_j, \max_{k \notin \{i,j\}} V_{k,x^n,\bar{\beta}^n}(t_k)\right\}$$

for sufficiently large n . Thus, since j 's winning bid is not an atom of $\bar{\beta}_{-j}^n$, the conditions for Eq. (6) are satisfied. Plug Eq. (96) for all $k' \neq j$ into Eq. (6) and we obtain Eq. (97). ■

E.5 Proof of Lemma 9

Since $b_i^n \geq b_* + \delta_n$ by definition of b_i^n , the probability with which b_i^n wins is no less than $\Pr\{b_* + \delta_n > \max_{k \neq i} \bar{\beta}_k^n(t_k)\}$, and $\limsup_{n \rightarrow \infty} \Pr\{b_* + \delta_n > \max_{k \neq i} \bar{\beta}_k^n(t_k)\} > 0$ by the con-

sequentiality of b_* and the convergence $\bar{\beta}^n \rightarrow \beta^*$. Thus, by Eq. (37), it suffices to show

$$\limsup_{n \rightarrow \infty} (\bar{W}_i(b_i^n, t_i^n, \bar{\beta}^n) - \bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n)) > 0.$$

To this end, denote $y_k^n := (\bar{\beta}^n)_{k, \text{sup}}^{-1}(c_i^n)$ for each $k \in I$. Extracting a convergent subsequence and relabeling superscripts if necessary, we may assume without loss of generality

$$\forall k \in I \exists y_k \in T_k : \lim_{n \rightarrow \infty} y_k^n = y_k. \quad (98)$$

By Eq. (46), $\mathcal{V}_{k,x}^{-1}(t_i)$ is continuous in x for any $t_i \in [0, x]$. Thus, if $t_i < z_k$ then

$$\forall k \in I \setminus \{i\} : \lim_{n \rightarrow \infty} \mathcal{V}_{k, y_k^n}^{-1}(t_i) = \mathcal{V}_{k, y_k}^{-1}(t_i). \quad (99)$$

The rest of the proof uses Proposition 4, with $(y_k^n)_{k \neq i}$ and $(z_k^n)_{k \neq i}$ playing the role of ζ and ζ' there. The proposition is applicable because the winning bids c_i^n and b_i^n are not atoms of $\bar{\beta}_{-i}^n$, hence the posterior virtual utility functions obey Eq. (4). There are only two cases: either (i) $t_i \geq \max_{j \neq i} y_j$ or (ii) $t_i < y_j$ for some $j \neq i$.

Case (i): This implies, by Eq. (98), $t_i^n + 1/n \geq \max_{j \neq i} y_j^n$ for sufficiently large n . Since $t_i < \max_{k \neq i} z_k$ by hypothesis of the lemma, Eq. (32) implies $t_i^n < \max_{k \neq i} z_k^n$ for sufficiently large n . Thus, $\max_{j \neq i} y_j^n - 1/n \leq t_i^n < \max_{k \neq i} z_k^n$ for sufficiently large n . For any such n , $\bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n) \leq t_i^n + 1/n$ because the public history (i, c_i^n) implies $\mathbf{t}_j \leq y_j^n$ for each $j \neq i$. Thus, mimicking the reasoning for Ineq. (63), we have, for some $j \neq i$ with $z_j > t_i$,

$$\bar{W}_i(b_i^n, t_i^n, \bar{\beta}^n) - \bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n) \geq \left(1 - \frac{F_j(\mathcal{V}_{j, z_j^n}^{-1}(t_i^n))}{F_j(z_j^n)} \right) \left(\mathcal{V}_{j, z_j^n}^{-1}(t_i^n) - t_i^n - 1/n \right).$$

Since $t_i^n \rightarrow t_i$ and $z_j^n \rightarrow_n z_j$, Eq. (99) and the continuity of $\mathcal{V}_{j, z_j}^{-1}$ together imply that the right-hand side converges to $\left(1 - F_j(\mathcal{V}_{j, z_j}^{-1}(t_i)) / F_j(z_j) \right) \left(\mathcal{V}_{j, z_j}^{-1}(t_i) - t_i \right)$, which is strictly positive since $\mathcal{V}_{j, z_j}^{-1}(t_i) > t_i$ due to $z_j > t_i$ (which implies $z_j > \mathcal{V}_{j, y_j}^{-1} > t_i$ by Eq. (46)).

Case (ii): By Eq. (98), for infinitely many n , $t_i^n < y_j^n$ and hence Ineq. (64) holds, i.e.,

$$\bar{W}_i(b_i^n, t_i^n, \bar{\beta}^n) - \bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n) \geq \gamma_n \Pr \left\{ c_i^n < \max_{j \in J \setminus \{i\}} \bar{\beta}_j^n(\mathbf{t}_j) < b_i^n \right\},$$

where according to Eq. (65)

$$\gamma_n = \frac{\prod_{k \neq i} F_k(y_k^n) - \prod_{k \neq i} F_k(\min \{ \mathcal{V}_{k, y_k^n}^{-1}(t_i^n), y_k^n \})}{\left(\prod_{k \neq i} F_k(z_k^n) \right) \left(\prod_{k \neq i} F_k(y_k^n) \right)} t_i^n.$$

By Ineq. (43), $\limsup_{n \rightarrow \infty} \Pr \{c_i^n < \max_{j \in \mathcal{J} \setminus \{i\}} \bar{\beta}_j^n(t_j) < b_i^n\} > 0$; by Eq. (99),

$$\limsup_{n \rightarrow \infty} \gamma_n = \frac{\prod_{k \neq i} F_k(y_k) - \prod_{k \neq i} F_k(\min \{\mathcal{V}_{k,y_k}^{-1}(t_i), y_k\})}{\left(\prod_{k \neq i} F_k(z_k)\right) \left(\prod_{k \neq i} F_k(y_k)\right)} t_i > 0,$$

where the inequality is due to the fact $t_i < y_j$ (which implies $t_i < \mathcal{V}_{j,y_j}^{-1} < y_j$ by Eq. (46)). Hence again $\limsup_{n \rightarrow \infty} (\bar{W}_i(b_i^n, t_i^n, \bar{\beta}^n) - \bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n)) > 0$, as desired.

E.6 Proof of Lemma 10

By Eqs. (38), (47) and (48), the pivotal effect equals $\Delta \Pi_i^n(t_i^n) = \Pr(\Omega_i^n) \psi_i^n(t_i^n)$. By Ineq. (43), $\limsup_n \Pr(\Omega_i^n) > 0$. Thus, it suffices to show that $\limsup_n \psi_i^n(t_i^n) > 0$.

To this end, we start by calculating the price markup. Pick any $t_j \in (a_j, z_j)$. For all sufficiently large n , $t_j \in (a_j^n, z_j^n)$ by Eq. (32), hence Eqs. (30) and (44) imply $c_i^n < \bar{\beta}_j^n(t_j) < b_i^n$. Note that Eq. (97) holds for $k = i$ because $z_i > \max \{t_j, \max_{k' \notin \{j,k\}} \mathcal{V}_{k',z_{k'}}(t_{k'})\}$, which is due to the t_j chosen above coupled with any $t_{-(i,j)} \in \prod_{k' \notin \{i,j\}} [0, z_{k'})$, as $\max_{k' \neq i} z_{k'} \leq z_i$ by hypothesis of the lemma. Integrating Eq. (97) across all such $t_{-(i,j)}$ gives

$$\begin{aligned} & \mathbb{E} \left[\mathcal{V}_{i,z_i}^{-1} \left(\max \left\{ t_j, \max_{k \notin \{i,j\}} \mathcal{V}_{k,z_k}(\mathbf{t}_k) \right\} \right) \middle| \mathbf{t}_{-(i,j)} \leq z_{-(i,j)} \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[p_{i,j,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_j, \mathbf{t}_{-(i,j)}) \middle| \mathbf{t}_{-(i,j)} \leq z_{-(i,j)} \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[p_{i,j,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_j, \mathbf{t}_{-(i,j)}) \middle| \bar{\beta}_j^n(t_j) > \max_{k \notin \{i,j\}} \bar{\beta}_k^n(\mathbf{t}_k) \right], \end{aligned}$$

with the second line due to Eq. (95) applied to the case $x^n = \bar{\beta}_j^n(t_j)$. By Eq. (62),

$$\bar{W}_j(\bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n) = \mathbb{E} \left[\max \left\{ t_j, \max_{k \neq j} \mathcal{V}_{k,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(\mathbf{t}_k) \right\} \middle| \bar{\beta}_j^n(t_j) > \max_{k \neq j} \bar{\beta}_k^n(\mathbf{t}_k) \right].$$

By Eqs. (95) and (96),

$$\lim_{n \rightarrow \infty} \bar{W}_j(\bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n) = \mathbb{E} \left[\max \left\{ t_j, \max_{k \neq j} \mathcal{V}_{k,z_k}(\mathbf{t}_k) \right\} \middle| \mathbf{t}_{-j} \leq z_{-j} \right].$$

Thus,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\mathbb{E} \left[p_{i,j,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_j, \mathbf{t}_{-(i,j)}) \middle| \bar{\beta}_j^n(t_j) > \max_{k \notin \{i,j\}} \bar{\beta}_k^n(\mathbf{t}_k) \right] - \bar{W}_j(\bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n) \right) \\ &= \mathbb{E} \left[\mathcal{V}_{i,z_i}^{-1} \left(\max \left\{ t_j, \max_{k \notin \{i,j\}} \mathcal{V}_{k,z_k}(\mathbf{t}_k) \right\} \right) \middle| \mathbf{t}_{-(i,j)} \leq z_{-(i,j)} \right] - \mathbb{E} \left[\max \left\{ t_j, \max_{k \neq j} \mathcal{V}_{k,z_k}(\mathbf{t}_k) \right\} \middle| \mathbf{t}_{-j} \leq z_{-j} \right], \end{aligned}$$

which is strictly positive by Ineq. (71), applicable since $z_i \geq \max_{k \neq i} z_k$. The strict positivity remains valid when $\bar{W}_j(\bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n)$ is replaced by $\bar{\beta}_j^n(t_j)$, as $\bar{\beta}_j^n(t_j) \leq \bar{W}_j(\bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n)$ by Lemma 6. Thus,

$$\lim_{n \rightarrow \infty} \left(\mathbb{E} \left[p_{i,j,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_j, \mathbf{t}_{-(i,j)}) \left| \bar{\beta}_j^n(t_j) > \max_{k \notin \{i,j\}} \bar{\beta}_k^n(\mathbf{t}_k) \right. \right] - \bar{\beta}_j^n(t_j) \right) > 0.$$

This being true for all $t_j \in (a_j, z_j)$, integration across such t_j gives the price markup

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[p_{i,j,\bar{\beta}_j^n(\mathbf{t}_j),\bar{\beta}^n}(\mathbf{t}_j, \mathbf{t}_{-(i,j)}) - \bar{\beta}_j^n(\mathbf{t}_j) \left| \bar{\beta}_j^n(\mathbf{t}_j) > \max_{k \notin \{i,j\}} \bar{\beta}_k^n(\mathbf{t}_k); \mathbf{t}_j \in (a_j, z_j) \right. \right] > 0. \quad (100)$$

Next, we calculate bidder i 's winner's curse. By the hypothesis $\mathcal{V}_{i,z_i}(t_i) \geq \max_{k \neq i} z_k$ and $t_i^n \rightarrow_n t_i$, as well as Eqs. (32) and (96), for any $t_j \in (a_j, z_j)$

$$0 \geq \lim_{n \rightarrow \infty} \left(\max_{k \neq i} z_k^n - V_{i,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_i^n) \right) \geq \lim_{n \rightarrow \infty} \left(\max \left\{ t_j, \max_{k \notin \{i,j\}} V_{i,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_k) \right\} - V_{i,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_i^n) \right).$$

Hence the probability with which i wins in j 's resale mechanism goes to one in the event where i 's bid increase is pivotal, i.e., the event Ω_i^n defined in Eq. (47). Thus,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E} \left[L_i(\mathbf{t}_{-i} \mid t_i^n, \bar{\beta}^n) \mid \mathbf{t}_{-i} \in \Omega_i^n \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[L_{ij}(\mathbf{t}_{-i} \mid t_i^n, \bar{\beta}^n) \left| \bar{\beta}_j^n(\mathbf{t}_j) > \max_{k \notin \{i,j\}} \bar{\beta}_k^n(\mathbf{t}_k); \mathbf{t}_{-i} \in \Omega_i^n \right. \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[t_i^n - p_{i,j,\bar{\beta}_j^n(\mathbf{t}_j),\bar{\beta}^n}(\mathbf{t}_j, \mathbf{t}_{-(i,j)}) \left| \bar{\beta}_j^n(\mathbf{t}_j) > \max_{k \notin \{i,j\}} \bar{\beta}_k^n(\mathbf{t}_k); \mathbf{t}_j \in (a_j, z_j) \right. \right], \end{aligned} \quad (101)$$

with the first line due to Eqs. (31) and (42), and the substitution of the conditioned event on the second line due to $a_j^n \rightarrow_n a_j$ and $z_j^n \rightarrow_n z_j$ (Eq. (32)).

Finally is the pivotal effect $\limsup_n \psi_i^n(t_i^n)$. By definition, $\limsup_n \psi_i^n(t_i^n)$ is

$$\lim_{n \rightarrow \infty} \left(\bar{W}_i^n(c_i^n, t_i^n, \bar{\beta}^n) - \mathbb{E} \left[L_i(\mathbf{t}_{-i} \mid t_i^n, \bar{\beta}^n) \mid \mathbf{t}_{-i} \in \Omega_i^n \right] - c_i^n \right),$$

which, by Eq. (101) and the fact $\bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n) \geq t_i^n$, is greater than or equal to the left-hand side of (100) and hence is strictly positive, as desired.

E.7 Proof of Lemma 11

Step 1: A resale mechanism for bidder i In the m_n -approximation game, bidder i upon winning can offer resale via the following game form M^n for bidders $k \neq i$, with the notation $(z_i^n, \mathcal{V}_{i,z_i^n})$ defined in Eqs. (29) and (46):

a. Every bidder $k \neq i$ picks an element from $[0, z_k^n]$, say t_k , and reports it as k 's type.

b. If $t_j \geq \mathcal{V}_{i, z_i^n}(t_i^n)$ then—

i. i resells the good to a bidder $k \in I \setminus \{i\}$ for whom

$$V_{k, c_i^n, \bar{\beta}^n}(t_k) \geq \max \left\{ t_i^n, \max_{h \notin \{i, k\}} V_{h, c_i^n, \bar{\beta}^n}(t_h) \right\}$$

at the price $V_{k, c_i^n, \bar{\beta}^n}^{-1} \left(\max \left\{ t_i^n, \max_{h \notin \{i, k\}} V_{h, c_i^n, \bar{\beta}^n}(t_h) \right\} \right)$;

ii. if no such k exists then i keeps the good.

c. If $t_j < \mathcal{V}_{i, z_i^n}(t_i^n)$ then—

i. i resells the good to a bidder $k \in I \setminus \{i, j\}$ for whom

$$V_{k, c_i^n, \bar{\beta}^n}(t_k) \geq \max \left\{ \mathcal{V}_{i, z_i^n}(t_i^n), \max_{h \notin \{i, j, k\}} V_{h, c_i^n, \bar{\beta}^n}(t_h) \right\}$$

at the price $V_{k, c_i^n, \bar{\beta}^n}^{-1} \left(\max \left\{ \mathcal{V}_{i, z_i^n}(t_i^n), \max_{h \notin \{i, j, k\}} V_{h, c_i^n, \bar{\beta}^n}(t_h) \right\} \right)$;

ii. if no such k exists then i keeps the good.

We claim that the mechanism M^n is ex post incentive feasible. It is ex post individually rational because a bidder $k \neq i$ can stay out by reporting his type being zero, thereby reporting a negative virtual utility. This gives him zero probability to win in Case (b), as $t_i^n \geq 0$. This also renders zero winning probability in Case (c), where $t_j \leq \mathcal{V}_{i, z_i^n}(t_i^n)$ implies $\mathcal{V}_{i, z_i^n}(t_i^n) \geq 0$. Thus, in either case bidder k can stay out thereby ensuring zero payoff.

The mechanism M^n is also ex post incentive compatible. For any $k \neq i$ and any t_{-k} , if bidder k 's true type is t_k , then by the rules in (b) and (c) his payoff conditional on winning is positive if and only if $t_k > V_{k, c_i^n, \bar{\beta}^n}^{-1}(v_k^n(t_{-k}))$, where

$$v_k^n(t_{-k}) := \begin{cases} \max \left\{ t_i^n, \max_{h \notin \{i, k\}} V_{h, c_i^n, \bar{\beta}^n}(t_h) \right\} & \text{if } t_j \geq \mathcal{V}_{i, z_i^n}(t_i^n) \\ \max \left\{ \mathcal{V}_{i, z_i^n}(t_i^n), \max_{h \notin \{i, j, k\}} V_{h, c_i^n, \bar{\beta}^n}(t_h) \right\} & \text{if } t_j < \mathcal{V}_{i, z_i^n}(t_i^n). \end{cases}$$

With $V_{k, c_i^n, \bar{\beta}^n}$ strictly increasing, $t_k > V_{k, c_i^n, \bar{\beta}^n}^{-1}(v_k^n(t_{-k}))$ is equivalent to $V_{k, c_i^n, \bar{\beta}^n}(t_k) > v_k^n(t_{-k})$, i.e., the event that k wins in M^n after reporting truthfully. Thus, having a positive payoff from winning in M^n is equivalent to the event that he should win after truthtelling. Since

the payoff from not winning in M^n is equal to zero, this implies incentive compatibility for any bidder $k \in I \setminus \{i\}$.¹⁷

Step 2: Bidder i 's expected payoff as a reseller Denote $\hat{w}^n(t_{-i})$ for i 's ex post payoff generated by the participation and truthtelling equilibrium in M^n when the realized type profile across $k \neq i$ is t_{-i} . By revealed preference,

$$\bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n) \geq \mathbb{E} \left[\hat{w}^n(\mathbf{t}_{-i}) \mid \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) < c_i^n \right].$$

Denote

$$\begin{aligned} X^n &:= \mathbb{E} \left[\mathbf{1} [t_j \geq \mathcal{V}_{i,z_i^n}(t_i^n)] (\hat{w}^n(\mathbf{t}_{-i}) - c_i^n) \mid \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) < c_i^n \right], \\ Y^n &:= \mathbb{E} \left[\mathbf{1} [t_j < \mathcal{V}_{i,z_i^n}(t_i^n)] (\hat{w}^n(\mathbf{t}_{-i}) - c_i^n) \mid \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) < c_i^n \right]. \end{aligned}$$

Then

$$\bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n) - c_i^n \geq X^n + Y^n.$$

To calculate X^n , note from its definition that it is an integral on the set of t_{-i} such that $t_j \geq \mathcal{V}_{i,z_i^n}(t_i^n)$. At such t_{-i} , mechanism M^n follows its rule (b), which coincides with the resale mechanism $M_i(c_i^n, t_i^n, \bar{\beta}^n)$ that the type- t_i^n bidder i would choose upon winning with bid c_i^n . (The posterior virtual utility functions conditional on the public history (i, c_i^n) are $(V_{k,c_i^n,\bar{\beta}})_{k \neq i}$ because Lemma 2.a applies, as c_i^n is not an atom of $\bar{\beta}_{-i}^n$.) Thus,

$$X^n = \mathbb{E} \left[\mathbf{1} [t_j \geq \mathcal{V}_{i,z_i^n}(t_i^n)] (W_i(\mathbf{t}_{-i} \mid c_i^n, t_i^n, \bar{\beta}^n) - c_i^n) \mid \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) < c_i^n \right].$$

If $\Pr \{t_j \geq \mathcal{V}_{i,z_i^n}(t_i^n) \mid \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) < c_i^n\} = 0$ then $X^n = 0$; else then Lemma 22 implies

$$\mathbb{E} \left[W_i(\mathbf{t}_{-i} \mid c_i^n, t_i^n, \bar{\beta}^n) \mid t_j \geq \mathcal{V}_{i,z_i^n}(t_i^n); \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) < c_i^n \right] \geq \mathbb{E} \left[W_i(\mathbf{t}_{-i} \mid c_i^n, t_i^n, \bar{\beta}^n) \mid \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) < c_i^n \right].$$

Thus,

$$X^n \geq \Pr \left\{ t_j \geq \mathcal{V}_{i,z_i^n}(t_i^n) \mid \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) < c_i^n \right\} \mathbb{E} \left[W_i(\mathbf{t}_{-i} \mid c_i^n, t_i^n, \bar{\beta}^n) - c_i^n \mid \max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) < c_i^n \right];$$

furthermore, since $\bar{\beta}_i^n(t_i^n) = c_i^n$, the second factor on the right-hand side according to Lemma 6 is nonnegative. Thus, $X^n \geq 0$ and hence

$$\bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n) - c_i^n \geq Y^n. \quad (102)$$

¹⁷ When $\mathcal{V}_{i,z_i^n}(t_i^n) < t_j < V_{j,c_i^n,\bar{\beta}^n}^{-1}(t_i^n)$, bidder j gets zero payoff whether he reports his type to be above $\mathcal{V}_{i,z_i^n}(t_i^n)$ or below $\mathcal{V}_{i,z_i^n}(t_i^n)$, but he cannot profit from lying.

To calculate Y^n , note from its definition that it is an integral on the set of t_{-i} such that $t_j < \mathcal{V}_{i,z_i^n}(t_i^n)$. At such t_{-i} , mechanism M^n follows its rule (c), under which t_j has no effect on the outcome of M^n , hence i 's ex post payoff $\hat{w}^n(t_{-i})$ from M^n is constant to t_j . Since the indicator $\mathbf{1}[t_j < \mathcal{V}_{i,z_i^n}(t_i^n)]$ is weakly decreasing in t_j , with $\hat{w}^n(t_{-i})$ independent of t_j and nonnegative, $\mathbf{1}[t_j < \mathcal{V}_{i,z_i^n}(t_i^n)] \hat{w}^n(t_{-i})$ is a weakly decreasing function of t_j for any $t_{-(i,j)}$. Hence the integral of this function cannot increase when we move upward some mass of \mathbf{t}_j , by replacing the conditioned event $\max_{k \neq i} \bar{\beta}_k^n(\mathbf{t}_k) < c_i^n$ with the one in the following:

$$Y^n \geq \mathbb{E} \left[\mathbf{1} [t_j < \mathcal{V}_{i,z_i^n}(t_i^n)] (\hat{w}^n(\mathbf{t}_{-i}) - c_i^n) \left| c_i^n < \bar{\beta}_j^n(t_j) < b_i^n; \max_{k \notin \{i,j\}} \bar{\beta}_k^n(\mathbf{t}_k) < c_i^n \right. \right].$$

By Eq. (42), we can replace the above conditioned event by Ω_i^n defined in Eq. (47). Thus,

$$\lim_{n \rightarrow \infty} Y^n \geq \lim_{n \rightarrow \infty} \mathbb{E} \left[\mathbf{1} [t_j < \mathcal{V}_{i,z_i^n}(t_i^n)] (\hat{w}^n(\mathbf{t}_{-i}) - c_i^n) \mid \mathbf{t}_{-i} \in \Omega_i^n \right]. \quad (103)$$

Step 3: The winner's curse By Eq. (42), the probability with which bidders $k \notin \{i, j\}$ wins given Ω_i^n vanishes as $n \rightarrow \infty$. Thus, the loser's payoff for i comes mainly from i 's payoff from losing to j , i.e., when bidder j with some type t_j wins with bid $\bar{\beta}_j^n(t_j) \in (c_i^n, b_i^n)$. In that event, bidder j chooses the resale mechanism $M_j(\bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n)$, which is determined by posterior virtual utility functions $\left(V_{k, \bar{\beta}_j^n(t_j), \bar{\beta}^n} \right)_{k \neq j}$ (due to Lemma 2.a, applicable because the winning bid $\bar{\beta}_j^n(t_j)$ is not an atom of $\bar{\beta}_{-j}^n$, by Eq. (22)). Recall that the probability with which bidder i gets to buy the good from j is denoted by $q_{ij}(t_i^n, t_{-i}, \bar{\beta}^n)$, with the price denoted by $p_{i,j, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_{-i})$. Thus,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E} [L_i(\mathbf{t}_{-i} \mid t_i^n, \bar{\beta}^n) \mid \mathbf{t}_{-i} \in \Omega_i^n] \\ \stackrel{(42)}{=} & \lim_{n \rightarrow \infty} \mathbb{E} \left[\left(t_i^n - p_{i,j, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_{-i}) \right) q_{ij}(t_i^n, \mathbf{t}_{-i}, \bar{\beta}^n) \mid \mathbf{t}_{-i} \in \Omega_i^n \right] \\ = & \lim_{n \rightarrow \infty} \mathbb{E} \left[\mathbf{1} [t_j < \mathcal{V}_{i,z_i^n}(t_i^n)] \left(t_i^n - p_{i,j, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_{-i}) \right) q_{ij}(t_i^n, \mathbf{t}_{-i}, \bar{\beta}^n) \mid \mathbf{t}_{-i} \in \Omega_i^n \right], \end{aligned} \quad (104)$$

where the second equality holds because $\mathbf{t}_j \geq \mathcal{V}_{i,z_i^n}(t_i^n)$ implies that the probability with which bidder i can buy the good from j , and hence i 's payoff at resale, vanishes as n enlarges.

Step 4: Y^n balances the winner's curse Combining (103) with (104) yields

$$\begin{aligned} & \lim_{n \rightarrow \infty} (Y^n - \mathbb{E} [L_i(\mathbf{t}_{-i} \mid t_i^n, \bar{\beta}^n) \mid \mathbf{t}_{-i} \in \Omega_i^n]) \\ \geq & \lim_{n \rightarrow \infty} \mathbb{E} \left[\mathbf{1} [t_j < \mathcal{V}_{i,z_i^n}(t_i^n)] \left(\hat{w}^n(t_i^n, \mathbf{t}_{-i}) - c_i^n - \left(t_i^n - p_{i,j, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_{-i}) \right) q_{ij}(t_i^n, \mathbf{t}_{-i}, \bar{\beta}^n) \right) \mid \mathbf{t}_{-i} \in \Omega_i^n \right]. \end{aligned}$$

To calculate the right-hand side, let t_{-i} range within Ω_i^n such that $t_j < \mathcal{V}_{i,z_i^n}(t_i^n)$. Then mechanism M^n operates under rule (c), and i 's payoff $\hat{w}^n(t_i^n, \mathbf{t}_{-i})$ is equal to either t_i^n if

$$\mathcal{V}_{i,z_i^n}(t_i^n) > \max_{k \notin \{i,j\}} V_{k,c_i^n, \bar{\beta}^n}(t_k), \quad (105)$$

or the payment

$$V_{k,c_i^n, \bar{\beta}^n}^{-1} \left(\max \left\{ \mathcal{V}_{i,z_i^n}(t_i^n), \max_{h \notin \{i,j,k\}} V_{h,c_i^n, \bar{\beta}^n}(t_h) \right\} \right) \quad (106)$$

from some bidder $k \notin \{i, j\}$ if

$$V_{k,c_i^n, \bar{\beta}^n}(t_k) > \max \left\{ \mathcal{V}_{i,z_i^n}(t_i^n), \max_{h \notin \{i,j,k\}} V_{h,c_i^n, \bar{\beta}^n}(t_h) \right\}. \quad (107)$$

By Eq. (96) and $z_i^n \rightarrow_n z_i$, $t_i^n \rightarrow_n t_i$ and continuity of the mapping $x \mapsto \mathcal{V}_{i,x}(t_i)$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{V}_{i,z_i^n}(t_i) &= \mathcal{V}_{i,z_i}(t_i), \\ \lim_{n \rightarrow \infty} V_{k', \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_{k'}) &= \mathcal{V}_{k', z_{k'}}(t_{k'}) = \lim_{n \rightarrow \infty} V_{k', c_i^n, \bar{\beta}^n}(t_{k'}), \\ \lim_{n \rightarrow \infty} V_{i, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_i^n) &= \mathcal{V}_{i,z_i}(t_i) = \lim_{n \rightarrow \infty} \mathcal{V}_{i,z_i^n}(t_i^n). \end{aligned}$$

Then for all sufficiently large n , the event (105) is approximated by

$$V_{i, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_i^n) > \max_{k \notin \{i,j\}} V_{k, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_k)$$

(which means if j wins j would resell to i since $t_j < \mathcal{V}_{i,z_i^n}(t_i^n) \approx V_{i, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_i^n)$), and the event (107) is approximated by

$$V_{k, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_k) > \max \left\{ V_{i, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_i^n), \max_{h \notin \{i,j,k\}} V_{h, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_h) \right\}$$

(which means j would resell to k), with the payment (106) approximated by $p_{k,j, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_{-k})$.

Thus, for all $t_{-i} \in \Omega_i^n$ with $t_j < \mathcal{V}_{i,z_i^n}(t_i^n)$ except a subset whose measure is $O(1/n)$,

$$\hat{w}^n(t_i^n, t_{-i}) + O(1/n) = t_i^n q_{ij}(t_i^n, t_{-k}, \bar{\beta}^n) + \sum_{k \in I \setminus \{i,j\}} q_{kj}(t_k, t_{-k}, \bar{\beta}^n) p_{k,j, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_{-k})$$

and hence

$$\begin{aligned} & \hat{w}^n(t_i^n, t_{-i}) - \left(t_i^n - p_{i,j, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_{-i}) \right) q_{ij}(t_i^n, t_{-i}, \bar{\beta}^n) + O(1/n) \\ &= \sum_{k \in I \setminus \{j\}} q_{kj}(t_k, t_{-k}, \bar{\beta}^n) p_{k,j, \bar{\beta}_j^n(t_j), \bar{\beta}^n}(t_{-k}), \end{aligned}$$

which is equal to $W_j(t_i^n, t_{-(i,j)} | \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n)$ because $t_j < \mathcal{V}_{i,z_i^n}(t_i^n)$ implies $t_j < V_{i,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_i^n)$ for all sufficiently large n , at which bidder j , upon winning, always resells the good. Thus,

$$\begin{aligned} & \lim_{n \rightarrow \infty} (Y^n - \mathbb{E}[L_i(\mathbf{t}_{-i} | t_i^n, \bar{\beta}^n) | \mathbf{t}_{-i} \in \Omega_i^n]) \\ & \geq \lim_{n \rightarrow \infty} \mathbb{E}[\mathbf{1}[t_j < \mathcal{V}_{i,z_i^n}(t_i^n)] (W_j(t_i^n, \mathbf{t}_{-(i,j)} | \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n) - c_i^n) | \mathbf{t}_{-i} \in \Omega_i^n]. \end{aligned}$$

Combining this with Eq. (102), as well as the facts $\lim_{n \rightarrow \infty} c_i^n = b_*$ by Eq. (44) and $b_* = \lim_{n \rightarrow \infty} \bar{\beta}_j^n(t_j)$ for all t_j such that $\bar{\beta}_j^n(t_j) \in (c_i^n, b_i^n)$, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} (\bar{W}_i(c_i^n, t_i^n, \bar{\beta}^n) - c_i^n - \mathbb{E}[L_i(\mathbf{t}_{-i} | t_i^n, \bar{\beta}^n) | \mathbf{t}_{-i} \in \Omega_i^n]) \\ & \geq \lim_{n \rightarrow \infty} \mathbb{E}[\mathbf{1}[t_j < \mathcal{V}_{i,z_i^n}(t_i^n)] (W_j(t_i^n, \mathbf{t}_{-(i,j)} | \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n) - \bar{\beta}_j^n(t_j)) | \mathbf{t}_{-i} \in \Omega_i^n], \end{aligned}$$

which is Eq. (49). This proves the lemma.

E.8 Proof of Lemma 12

Pick any $t_j \in (a_j, z_j)$. By Lemma 6,

$$\bar{\beta}_j^n(t_j) \leq \mathbb{E} \left[W_j(\mathbf{t}_{-j} | \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n) \left| \max_{k \neq j} \bar{\beta}_k^n(\mathbf{t}_k) < \bar{\beta}_j^n(t_j) \right. \right].$$

Taking the limit and using Eq. (95), we have

$$\lim_{n \rightarrow \infty} \bar{\beta}_j^n(t_j) \leq \mathbb{E}_{\mathbf{t}_i} \left[\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbf{t}_{-(i,j)}} \left[W_j(\mathbf{t}_{-j} | \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n) \left| \max_{k \notin \{i,j\}} \bar{\beta}_k^n(\mathbf{t}_k) < \bar{\beta}_j^n(t_j) \right. \right] \middle| \mathbf{t}_i \leq z_i \right].$$

By Eq. (22) the winning bid $\bar{\beta}_j^n(t_j)$ is not an atom of $\bar{\beta}_{-j}^n$, so the $W_j(\mathbf{t}_{-j} | \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n)$ on the right-hand side obeys Eq. (7) and hence is determined by the virtual utility functions $\left(V_{k,\bar{\beta}_j^n(t_j),\bar{\beta}^n} \right)_{k \neq j}$. By Lemma 25, for each $k \neq j$, $V_{k,\bar{\beta}_j^n(t_j),\bar{\beta}^n} \rightarrow_n \mathcal{V}_{k,z_k}$, which is the virtual utility function given distribution $F_k(\cdot)/F_k(z_k)$. Thus, $W_j(\mathbf{t}_{-j} | \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n)$ converges to the type- t_j reseller j 's expected payoff from the Myerson auction $\mathbb{M}(z)$ defined in §A.1. I.e., with the φ_i defined in Eq. (67) where ζ_k is z_k here,

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbf{t}_{-(i,j)}} \left[W_j(t_i, \mathbf{t}_{-(i,j)} | \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n) \left| \max_{k \notin \{i,j\}} \bar{\beta}_k^n(\mathbf{t}_k) < \bar{\beta}_j^n(t_j) \right. \right] = \varphi_i(t_i, t_j). \quad (108)$$

Denote $\bar{\varphi}_i(t_j) := \mathbb{E}_{\mathbf{t}_i}[\varphi_i(\mathbf{t}_i, t_j) | \mathbf{t}_i \in [0, z_i]]$, so the above-displayed inequality means

$$\lim_{n \rightarrow \infty} \bar{\beta}_j^n(t_j) \leq \bar{\varphi}_i(t_j). \quad (109)$$

Denote

$$\begin{aligned}\varphi_*(t_i) &:= \mathbb{E}[\mathbf{1}[t_j < z_i]\varphi_i(t_i, \mathbf{t}_j) \mid \mathbf{t}_j \in (a_j, z_j)], \\ \bar{\varphi}_* &:= \mathbb{E}[\mathbf{1}[t_j < z_i]\bar{\varphi}_i(\mathbf{t}_j) \mid \mathbf{t}_j \in (a_j, z_j)].\end{aligned}$$

We construct a sequence $(t_i^n)_{n=1}^\infty$ such that $\bar{\beta}_i^n(t_i^n) = c_i^n$ for each n , $t_i^n \rightarrow_n z_i$, and

$$\lim_{n \rightarrow \infty} \varphi_*(t_i^n) \geq \bar{\varphi}_*. \quad (110)$$

To this end, recall $(\bar{\beta}^n)_i^{-1}(c_i^n) \neq \emptyset$ by Eq. (41). By Eq. (95), $(\bar{\beta}^n)_{i,\text{sup}}^{-1}(c_i^n)$ the supremum of this inverse image converges to z_i as $n \rightarrow \infty$. Hence there exists $(t_i^n)_{n=1}^\infty$ such that $t_i^n \in (\bar{\beta}^n)_i^{-1}(c_i^n)$ for each n and $t_i^n \rightarrow_n z_i$. Thus, by continuity of φ_* (Corollary 2), $\lim_{n \rightarrow \infty} \varphi_*(t_i^n) = \varphi_*(z_i)$, and $\varphi_*(z_i) \geq \bar{\varphi}_*$ (otherwise $\varphi_* < \bar{\varphi}_*$ on an interval $(\alpha', z_i]$ for some $\alpha' < z_i$ by continuity of φ_* , which contradicts Ineq. (70) of Corollary 2). Hence Ineq. (110) holds.

Plugging the definitions of φ_* and $\bar{\varphi}_*$ into (110), we have

$$\mathbb{E}_{\mathbf{t}_j} \left[\mathbf{1}[t_j < z_i] \left(\lim_{n' \rightarrow \infty} \varphi_i(t_i^{n'}, \mathbf{t}_j) - \bar{\varphi}_i(\mathbf{t}_j) \right) \mid \mathbf{t}_j \in (a_j, z_j) \right] \geq 0.$$

Replace $\varphi_i(t_i^{n'}, \mathbf{t}_j)$ with the left-hand side of Eq. (108), switch the positions of the integration and the limit operators and then use Ineq. (109) to obtain

$$\lim_{n' \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E} \left[\mathbf{1}[t_j < z_i] \left(W_j \left(t_i^{n'}, \mathbf{t}_{-(i,j)} \mid \bar{\beta}_j^n(\mathbf{t}_j), \mathbf{t}_j, \bar{\beta}^n \right) - \bar{\beta}_j^n(\mathbf{t}_j) \right) \left| \begin{array}{l} \mathbf{t}_j \in (a_j, z_j); \\ \max_{k \notin \{i,j\}} \bar{\beta}_k^n(\mathbf{t}_k) < \bar{\beta}_j^n(\mathbf{t}_j) \end{array} \right. \right] \geq 0.$$

In the above integral, for any n and t_{-i} , $W_j(t_i, \mathbf{t}_{-(i,j)} \mid \bar{\beta}_j^n(\mathbf{t}_j), \mathbf{t}_j, \bar{\beta}^n)$ is a uniformly bounded and uniformly equicontinuous function of t_i . Uniform boundedness follows from the compactness of the prior supports of use values. Uniform equicontinuity follows from the fact that, by Eq. (22), W_j obeys Eq. (7), where for any $k \neq j$ the virtual utility V_k and the resale price $p_{j,k}$ are uniformly equicontinuous in t_k : V_k is so by Eq. (4); $p_{j,k}$ is so because it obeys Eq. (6), where the derivative of the increasing inverse function V_k^{-1} is bounded from above by $1/\lambda > 0$ (Lemma 2.b). Thus, in the sequence $((t_i^{n'}, \bar{\beta}^n)_{n=1}^\infty)_{n'=1}^\infty$, the diagonal subsequence in which the superscripts of $t_i^{n'}$ and $\bar{\beta}_j^n$ coincide converges to the above double limit. Thus,

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left[\mathbf{1}[t_j < z_i] \left(W_j \left(t_i^n, \mathbf{t}_{-(i,j)} \mid \bar{\beta}_j^n(\mathbf{t}_j), \mathbf{t}_j, \bar{\beta}^n \right) - \bar{\beta}_j^n(\mathbf{t}_j) \right) \left| \begin{array}{l} \mathbf{t}_j \in (a_j, z_j); \\ \max_{k \notin \{i,j\}} \bar{\beta}_k^n(\mathbf{t}_k) < \bar{\beta}_j^n(\mathbf{t}_j) \end{array} \right. \right] \geq 0.$$

On the left-hand side, since $a_j^n \rightarrow_n a_j$ and $z_j^n \rightarrow_n z_j$, the part $\mathbf{t}_j \in (a_j, z_j)$ in the conditioned event can be replaced by $\mathbf{t}_j \in (a_j^n, z_j^n)$, and the entire conditioned event can be

replaced by the pivotal event Ω_i^n by Eqs. (42) and (47). Since z_i is a limit point of $(t_i^n)_{n=1}^\infty$, with \mathcal{V}_{i,z_i} continuous and $\mathcal{V}_{i,z_i}(z_i) = z_i$, the indicator function $\mathbf{1}[t_j < z_i]$ can be replaced by $\mathbf{1}[t_j < \mathcal{V}_{i,z_i}(t_i^n)]$. Hence we obtain Ineq. (50), and the lemma is proved.

E.9 Proof of Corollary 1

Suppose not, say $b_* > r$ is a consequential atom of β_j^* . By Theorem 2, no bidder other than j has an atom at b_* . Let $(\delta_n)_{n=1}^\infty$ and $((a_j, z_j), (a_j^n, z_j^n)_{n=1}^\infty)$ be those specified in Lemma 7. For any $i \neq j$, define c_i^n by Eq. (41). Since $b_* > r$, $\lim_{n \rightarrow \infty} (b_* - \delta_n - \max_{i \neq j} c_i^n) = 0$; otherwise, since the auction is first-price, bidder j with types in (a_j^n, z_j^n) would deviate to a bid $d_j^n \in B_j^{m_n}$ such that $\max_{i \neq j} c_i^n < d_j^n < b_* - \delta_n$. With $I \setminus \{j\}$ finite, there exist an $i \in I \setminus \{j\}$ and an infinite subsequence $(n_k)_{k=1}^\infty$ along which $c_i^{n_k} = \max_{j' \in I \setminus \{j\}} c_{j'}^{n_k}$ for all k . As no one but j has an atom at b_* , Eq. (42) holds. Since $\lim_{n \rightarrow \infty} (b_* - \delta_n - \max_{i \neq j} c_i^n) = 0$, $c_i^{n_k} < b_* + \delta_{n_k}$ and $b_* + \delta_{n_k} - c_i^{n_k} = O(\delta_{n_k})$. Furthermore, b_* is consequential by hypothesis of the lemma. The rest of the proof is identical to the proof of Theorem 2 starting from Eq. (42).

F Proof of Lemma 15

Let a serious bid b_* be an inconsequential atom of β^* . By Lemma 13, there exists a bidder i for whom b_* is not an atom of β_i^* . Since $\cup_{m=1}^\infty B_i^m$ is dense in the space of serious bids due to Eq. (24), there is a sequence $(b_i^m)_{m=1}^\infty$ converging to b_* with $b_i^m \in B_i^m$ for each m . Then Eqs. (52) and (53) imply that

$$\Pr\{\beta_i^*(\mathbf{t}_i) > b_*\} = \lim_{m \rightarrow \infty} \Pr\{\beta_i^m(\mathbf{t}_i) > b_i^m\} = 1. \quad (111)$$

To prove by contradiction, suppose b_* is an atom of β_k^* for all $k \neq i$. Denote for any m

$$\begin{aligned} \underline{b}_i^m &:= \min\{b \in B_i^m : \beta_i^m = b \text{ on some } (x, x') \subseteq T_i \text{ with } x < x'\}, \\ \underline{b}_i^* &:= \inf\{\beta_i^*(t'_i) : t'_i > 0\}. \end{aligned}$$

For each $k \neq i$ and any m , denote

$$\begin{aligned} z_k^m &:= (\beta^m)_{k,\text{sup}}^{-1}(\underline{b}_i^m), \\ z_k &:= (\beta^*)_{k,\text{sup}}^{-1}(\underline{b}_i^*). \end{aligned}$$

Since $\beta^m \rightarrow \beta^*$, $\underline{b}_i^m \rightarrow_m \underline{b}_i^*$ and z_k is a limit point of $(z_k^m)_{m=1}^\infty$. Extracting a converging subsequence and relabeling if necessary, assume that $z_k^m \rightarrow_m z_k$ for each $k \neq i$. Note that $z_k > 0$ for each $k \neq i$, as b_* is an atom of β_k^* .

Note that \underline{b}_i^* is not an atom of β_i^* . Otherwise, since $\underline{b}_i^* \geq b_*$ (Eq. (111)) and b_* is an atom of β_{-i}^* , \underline{b}_i^* would be a consequential atom, contradicting Lemma 14. By the same token, for any $k \neq i$, the interval $(\underline{b}_i^*, \infty)$ contains no atom of β_k^* .

Choose a $j \neq i$ such that $z_j = \max_{k \neq i} z_k$. To derive a desired contradiction, we shall prove that some types of bidder j that are supposed to bid below \underline{b}_i^m at the m -equilibrium would rather deviate to a slightly higher bid for large m .

Step 1: The price markup By definition of \underline{b}_i^m and monotonicity of β_i^m , there is a sequence $(t_i^m)_{m=1}^\infty$ such that $\beta_i^m(t_i^m) = \underline{b}_i^m$ for each m and $t_i^m \rightarrow_m 0$. Then for all sufficiently large m , $z_j^m > t_i^m$.

Since $z_j = \max_{k \neq i} z_k$ and $z_k^m \rightarrow_m z_k$ for all $k \neq i$, for each m there exists $\epsilon_m \geq 0$ such that $z_k^m - \epsilon_m \leq z_j^m$ for each $k \notin \{i, j\}$ and $\epsilon_m \rightarrow 0$. Thus, for each $t_{-(i,j)} \in T_{-(i,j)}$ such that $t_k \leq z_k^m - \epsilon_m$ for each $k \notin \{i, j\}$, if bidder i wins with bid \underline{b}_i^m (hence $t_k \leq z_k^m$ for all $k \neq i$), we have $z_j^m > t_i^m$ (for all large m) and

$$V_{k, \underline{b}_i^m, \beta^m}(t_k) \leq t_k \leq z_k^m - \epsilon_m \leq z_j^m \stackrel{(3)}{=} \max V_{j, \underline{b}_i^m, \beta^m},$$

hence the resale price $p_{j, i, \underline{b}_i^m, \beta^m}(t_i^m, t_{-(i,j)})$ for j obeys Eq. (6). With the notation in (46),

$$p_{j, i, \underline{b}_i^m, \beta^m}(t_i^m, \mathbf{t}_{-(i,j)}) = \mathcal{V}_{j, z_j^m}^{-1} \left(\max \left\{ t_i^m, \max_{k \notin \{i, j\}} \mathcal{V}_{k, z_k^m}(\mathbf{t}_k) \right\} \right).$$

Since $z_k^m \rightarrow_m z_k$, $\epsilon_m \rightarrow 0$, $t_i^m \rightarrow_m 0$, and the functions \mathcal{V}_{k, z_k^m} and $x \mapsto \mathcal{V}_{k, x}^{-1}(v)$ are continuous,

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbb{E} \left[p_{j, i, \underline{b}_i^m, \beta^m}(t_i^m, \mathbf{t}_{-(i,j)}) \mid \forall k \notin \{i, j\} : \mathbf{t}_k \leq z_k^m - \epsilon_m \right] \\ &= \mathbb{E} \left[\mathcal{V}_{j, z_j}^{-1} \left(\max \left\{ 0, \max_{k \notin \{i, j\}} \mathcal{V}_{k, z_k}(\mathbf{t}_k) \right\} \right) \mid \forall k \notin \{i, j\} : \mathbf{t}_k \leq z_k \right]. \end{aligned}$$

By Eq. (62) and the fact $\mathcal{V}_{k, z_k^m} \rightarrow_m \mathcal{V}_{k, z_k}$ for each $k \neq i$,

$$\lim_{m \rightarrow \infty} \overline{W}_i(\underline{b}_i^m, t_i^m, \beta^m) = \mathbb{E} \left[\max \left\{ 0, \max_{k \neq i} \mathcal{V}_{k, z_k}(\mathbf{t}_k) \right\} \mid \forall k \neq i : \mathbf{t}_k \leq z_k \right].$$

These two equations combined with Ineq. (71), which is due to $z_j = \max_{k \neq i} z_k > 0$, imply

$$\lim_{m \rightarrow \infty} \left(\mathbb{E} \left[p_{j, i, \underline{b}_i^m, \beta^m}(t_i^m, \mathbf{t}_{-(i,j)}) \mid \forall k \notin \{i, j\} : \mathbf{t}_k \leq z_k^m - \epsilon_m \right] - \overline{W}_i(\underline{b}_i^m, t_i^m, \beta^m) \right) > 0.$$

Since $\beta_i^m(t_i^m) = \underline{b}_i^m$, $\overline{W}_i(\underline{b}_i^m, t_i^m, \beta^m) \geq \underline{b}_i^m$ by Lemma 6, the above inequality implies

$$\lim_{m \rightarrow \infty} \left(\mathbb{E} \left[p_{j,i,\underline{b}_i^m, \beta^m}(t_i^m, \mathbf{t}_{-(i,j)}) \mid \forall k \notin \{i, j\} : \mathbf{t}_k \leq z_k^m - \epsilon_m \right] - \underline{b}_i^m \right) > 0.$$

Thus, there exists $\eta \in (0, z_j/\lambda)$ such that

$$\lim_{m \rightarrow \infty} \left(\mathbb{E} \left[p_{j,i,\underline{b}_i^m, \beta^m}(t_i^m, \mathbf{t}_{-(i,j)}) \mid \forall k \notin \{i, j\} : \mathbf{t}_k \leq z_k^m - \epsilon_m \right] - \underline{b}_i^m \right) - 5\eta > 0, \quad (112)$$

with λ the positive constant specified in Lemma 2.b.

Step 2: Construct a deviation for bidder j For any m , if $b', b \in B_i^m$ then for each $k \neq i$, neither bids are atom of β_k^m , by Eq. (22). Thus, if in addition $b' > b$ then Eq. (77) holds. Consequently, with $f_k > 0$ on the compact T_k for all k ,

$$|V_{k,b,\beta^m}(t_k) - V_{k,b',\beta^m}(t_k)| = O \left(F_k \left((\beta^m)_{k,\text{sup}}^{-1}(b') \right) - F_k \left((\beta^m)_{k,\text{sup}}^{-1}(b) \right) \right).$$

Thus, there exists $\xi > 0$ such that, for any m and any $k \neq i$,

$$\Pr \{ b < \beta_k^m(\mathbf{t}_k) < b' \} < \xi \implies \|V_{k,b',\beta^m} - V_{k,b,\beta^m}\|_{\text{sup}} < \eta\lambda/|I|, \quad (113)$$

Since the limit \underline{b}_i^* of $(\underline{b}_i^m)_{m=1}^\infty$ is not an atom of the limit β_i^* of $(\beta_i^m)_{m=1}^\infty$, and $(\underline{b}_i^*, \infty)$ contains no atom of the limit β_{-i}^* of $(\beta_{-i}^m)_{m=1}^\infty$, by Lemma 24 there exists $\delta \in (0, \eta/2)$ for which

$$\lim_{m \rightarrow \infty} \left(\sup \{ t_i : \beta_i^m(t_i) \leq \underline{b}_i^m + \delta \} \right) < \eta\lambda, \quad (114)$$

$$\lim_{m \rightarrow \infty} \Pr \{ \exists k \neq i : \underline{b}_i^m < \beta_k^m(\mathbf{t}_k) \leq \underline{b}_i^m + \delta \} < \min \left\{ \xi, \eta, \frac{\eta}{\max_{k \in I} \bar{t}_k} \right\}. \quad (115)$$

For each m , let

$$\begin{aligned} b_j^m &:= \min \{ b \in B_j^m : b > \underline{b}_i^* + \delta \}, \\ c_j^m &:= \max \{ \beta_j^m(t_j) : t_j < z_j^m \}, \\ \Delta U_j^m(t_j) &:= U_i(b_j^m, t_j, \beta^m) - U_i(c_j^m, t_j, \beta^m). \end{aligned}$$

Step 3: Bidder j 's strict incentive to deviate For each m , by definition of c_j^m , there is a nondegenerate interval (x^m, z_j^m) such that $\beta_j^m(t_j) = c_j^m$ for all $t_j \in (x^m, z_j^m)$. Also note $c_j^m \leq \beta_j^m(z_j^m) < \underline{b}_i^m$ by monotonicity of β_j^m , the definition of z_j^m , and Eq. (22). Thus, since $\underline{b}_i^m \rightarrow_m \underline{b}_i^*$, $c_j^m \leq \underline{b}_i^* < b_j^m$ for all sufficiently large m . We shall derive the desired contradiction by proving that for sufficiently large m some elements of (x^m, z_j^m) strictly prefer to deviate from their m -equilibrium bid c_j^m to the bid b_j^m . By continuity of ΔU_j^m (Lemma 5), it suffices to show $\lim_{m \rightarrow \infty} \Delta U_j^m(z_j^m) > 0$.

Substep 3.a: The probability of winning Now that $c_j^m < \underline{b}_i^m$ for large m ,

$$\lim_{m \rightarrow \infty} \Pr \{ \beta_i^m(\mathbf{t}_i) > c_j^m \} \geq \lim_{m \rightarrow \infty} \Pr \{ \beta_i^m(\mathbf{t}_i) \geq \underline{b}_i^m \} = 1,$$

with the equality due to the definition of \underline{b}_i^m . Hence

$$\lim_{m \rightarrow \infty} U_j(c_j^m, t_j, \beta^m) = \lim_{m \rightarrow \infty} \mathbb{E} L_j(\mathbf{t}_{-j} \mid t_j, \beta^m)$$

for any $t_j \in T_j$ by Eq. (10). Thus, again by Eq. (10),

$$\lim_{m \rightarrow \infty} \Delta U_j^m(z_j) = \lim_{m \rightarrow \infty} \Pr \left\{ b_j^m > \max_{k \neq j} \beta_k^m(\mathbf{t}_k) \right\} \bar{\Pi}_j^m(z_j^m),$$

where

$$\bar{\Pi}_j^m(z_j^m) := \mathbb{E} \left[W_j(\mathbf{t}_{-j} \mid b_j^m, z_j^m, \beta^m) - b_j^m - L_i(\mathbf{t}_{-j} \mid z_j^m, \beta^m) \Big| b_j^m > \max_{k \neq j} \beta_k^m(\mathbf{t}_k) \right]. \quad (116)$$

By definition of \underline{b}_i^* being $\inf\{\beta_i^*(t'_i) : t'_i > 0\}$, $\Pr \{ \underline{b}_i^* \leq \beta_i^*(\mathbf{t}_i) \leq \underline{b}_i^* + \delta \} > 0$. Consequently, by definition of b_j^m as well as the fact that $z_k > 0$ for all $k \neq i$,

$$\lim_{m \rightarrow \infty} \Pr \left\{ b_j^m > \max_{k \neq j} \beta_k^m(\mathbf{t}_k) \right\} \geq \Pr \{ \underline{b}_i^* \leq \beta_i^*(\mathbf{t}_i) \leq \underline{b}_i^* + \delta \} \prod_{k \notin \{i,j\}} F_k(z_k) > 0.$$

Thus, it suffices to show $\lim_{m \rightarrow \infty} \bar{\Pi}_j^m(z_j^m) > 0$.

Substep 3.b: The resale prices By (115), for any large enough m and any $k \neq i$,

$$\underline{b}_i^m \leq \beta_i^m(t_i) \leq \underline{b}_i^m + \delta \Rightarrow \Pr \{ \underline{b}_i^m < \beta_k^m(\mathbf{t}_k) < \beta_i^m(t_i) \} < \xi \stackrel{(113)}{\Rightarrow} \|V_{k, \beta_i^m(t_i), \beta^m} - V_{k, \underline{b}_i^m, \beta^m}\|_{\text{sup}} < \eta \lambda / |I|.$$

Thus, by (114) and the fact $t_i^m \rightarrow_m 0$, we have for any sufficiently large m , any $t_{-(i,j)} \in T_{-(i,j)}$ and any t_i such that $\underline{b}_i^m \leq \beta_i^m(t_i) \leq \underline{b}_i^m + \delta$,

$$\max \left\{ t_i, \max_{k \notin \{i,j\}} V_{k, \beta_i^m(t_i), \beta^m}(t_k) \right\} - \max \left\{ t_i^m, \max_{k \notin \{i,j\}} V_{k, \underline{b}_i^m, \beta^m}(t_k) \right\} > -2\eta \lambda. \quad (117)$$

By Eq. (6), the two terms on the left-hand side of (117) can be inverted into j 's resale prices via the inverses of his posterior virtual utility functions. Thus, with the derivatives of the inverses bounded from above by $1/\lambda$ (Lemma 2.b). Hence

$$p_{j, i, \beta_i^m(t_i), \beta^m}(t_i, t_{-(i,j)}) - p_{j, i, \underline{b}_i^m, \beta^m}(t_i^m, t_{-(i,j)}) > -2\eta \quad (118)$$

for any $t_{-(i,j)} \in T_{-(i,j)}$ such that $t_k \leq z_k^m - \epsilon_m$ for each $k \notin \{i, j\}$ and any t_i such that $\underline{b}_i^m \leq \beta_i^m(t_i) \leq \underline{b}_i^m + \delta$. The applicability of Eq. (6) to the resale price $p_{j, i, \underline{b}_i^m, \beta^m}(t_i^m, t_{-(i,j)})$

has been explained at Step 1, and its applicability to $p_{j,i,\beta_i^m(t_i),\beta^m}(t_i, t_{-(i,j)})$, with $\beta_i^m(t_i)$ playing the role b_i , is because for large m we have $t_i < z_j^m$ ($\underline{b}_i^m \leq \beta_i^m(t_i) \leq \underline{b}_i^m + \delta$ implies via (114) that $t_i < \eta\lambda < z_i^m$ for large m) and, for each $k \notin \{i, j\}$,

$$V_{k,\beta_i^m(t_i),\beta^m}(t_k) \leq t_k \leq z_k^m - \epsilon_m \leq z_j^m = (\beta^m)_{j,\sup}^{-1}(\underline{b}_i^m) \leq (\beta^m)_{j,\sup}^{-1}(\beta_i^m(t_i)).$$

Integrating (118) across the $(t_i, t_{-(i,j)})$ quantified above, we have, for all large m ,

$$\begin{aligned} & \mathbb{E} \left[p_{j,i,\beta_i^m(t_i),\beta^m}(t_i, \mathbf{t}_{-(i,j)}) \mid \underline{b}_i^m \leq \beta_i^m(\mathbf{t}_i) < b_j^m; \forall k \notin \{i, j\} [\mathbf{t}_k \leq z_k^m - \epsilon_m] \right] \\ & \geq \mathbb{E} \left[p_{j,i,\underline{b}_i^m,\beta^m}(t_i^m, \mathbf{t}_{-(i,j)}) \mid \forall k \notin \{i, j\} [\mathbf{t}_k \leq z_k^m - \epsilon_m] \right] - 2\eta. \end{aligned}$$

This combined with Ineq. (112) and $\delta < \eta/2$ (which implies $b_j^m < \underline{b}_i^* + \eta$ by the definition of b_j^m ; then by $\underline{b}_i^m \rightarrow_m \underline{b}_i^*$ we have $b_j^m < \underline{b}_i^m + \eta$ for all large m) gives

$$\begin{aligned} & \lim_{m \rightarrow \infty} \left(\mathbb{E} \left[p_{j,i,\beta_i^m(t_i),\beta^m}(t_i, \mathbf{t}_{-(i,j)}) \mid \underline{b}_i^m \leq \beta_i^m(\mathbf{t}_i) < b_j^m; \forall k \notin \{i, j\} [\mathbf{t}_k \leq z_k^m - \epsilon_m] \right] - b_j^m \right) \\ & \geq \lim_{m \rightarrow \infty} \left(\mathbb{E} \left[p_{j,i,\underline{b}_i^m,\beta^m}(t_i^m, \mathbf{t}_{-(i,j)}) \mid \forall k \notin \{i, j\} [\mathbf{t}_k \leq z_k^m - \epsilon_m] \right] - 2\eta - (\underline{b}_i^m + \eta) \right) \\ & > 5\eta - 2\eta - \eta = 2\eta. \end{aligned} \tag{119}$$

Substep 3.c: Bidder j 's opportunity cost of winning:

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbb{E} \left[L_{ji}(\mathbf{t}_{-j} \mid z_j^m, \beta^m) \mid b_j^m > \max_{k \neq j} \beta_k^m(\mathbf{t}_k) \right] \\ & \stackrel{(115)}{\leq} \lim_{m \rightarrow \infty} \mathbb{E} \left[L_{ji}(\mathbf{t}_{-j} \mid z_j^m, \beta^m) \mid b_j^m > \max_{k \neq j} \beta_k^m(\mathbf{t}_k); \underline{b}_i^m > \max_{k \notin \{i,j\}} \beta_k^m(\mathbf{t}_k) \right] + \eta \\ & = \lim_{m \rightarrow \infty} \mathbb{E} \left[L_{ji}(\mathbf{t}_{-j} \mid z_j^m, \beta^m) \mid b_j^m > \max_{k \neq j} \beta_k^m(\mathbf{t}_k); \forall k \notin \{i, j\} [\mathbf{t}_k \leq z_k^m - \epsilon_m] \right] + \eta \\ & = \lim_{m \rightarrow \infty} \mathbb{E} \left[z_j^m - p_{j,i,\beta_i^m(t_i),\beta^m}(t_i, \mathbf{t}_{-(i,j)}) \mid b_j^m > \beta_i^m(\mathbf{t}_i); \forall k \notin \{i, j\} [\mathbf{t}_k \leq z_k^m - \epsilon_m] \right] + \eta \\ & \stackrel{(119)}{<} \lim_{m \rightarrow \infty} (z_j^m - b_j^m) - \eta; \end{aligned}$$

here the first equality is because the difference between the events $\underline{b}_i^m > \max_{k \notin \{i,j\}} \beta_k^m(\mathbf{t}_k)$ and $\forall k \notin \{i, j\} [\mathbf{t}_k \leq z_k^m - \epsilon_m]$ vanishes as $m \rightarrow \infty$; the second equality is due to $t_k \leq z_k^m - \epsilon_m \leq z_j^m$ for all $k \notin \{i, j\}$ and $t_i < z_j^m$ for large m (due to Ineq. (114) and $\eta < z_j$).

Ineq. (115) also implies that for any $k \notin \{i, j\}$

$$\lim_{m \rightarrow \infty} \mathbb{E} \left[\mathbf{1} \left[\beta_k^m(\mathbf{t}_k) > \max_{l \notin \{j,k\}} \beta_l^m(\mathbf{t}_l) \right] L_{jk}(\mathbf{t}_{-j} \mid z_j^m, \beta^m) \mid b_j^m > \max_{k \neq j} \beta_k^m(\mathbf{t}_k) \right] < \eta.$$

Combining the two inequalities displayed above with Eq. (9), we have

$$\lim_{m \rightarrow \infty} \mathbb{E} \left[L_j(\mathbf{t}_{-j} \mid z_j^m, \beta^m) \mid b_j^m > \max_{k \neq j} \beta_k^m(\mathbf{t}_k) \right] < \lim_{m \rightarrow \infty} (z_j^m - b_j^m).$$

Therefore, by Eq. (116) and the fact $W_j(\mathbf{t}_{-j} \mid b_j^m, z_j^m, \beta^m) \geq z_j^m$,

$$\lim_{m \rightarrow \infty} \bar{\Pi}_j^m(z_j^m) \geq \lim_{m \rightarrow \infty} \left(z_j^m - b_j^m - \mathbb{E} \left[L_j(\mathbf{t}_{-j} \mid z_j^m, \beta^m) \mid b_j^m > \max_{k \neq j} \beta_k^m(\mathbf{t}_k) \right] \right) > 0,$$

as desired.

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Index

$(i, b_i) \succ (b_k)_{k \in J}, (i, b_i) \not\succeq (b_k)_{k \in J}, 12$

$(i, b_i), 10$

$B_i, 8$

$B_i^m, 18$

$F_k(\cdot \mid i, b_i, \beta), 10$

$I, 8$

$L_i(t, \beta), 13$

$L_{ij}(t_{-i} \mid t_i, \beta), 13$

$M_i(b_i, t_i, \beta), 11$

$Q_i(b_i, t, \beta), 15$

$R(t_{i_*}, t_{-i_*}, \zeta), 34$

$T, T_i, T_{-i}, 9$

$U_i(b_i, t_i, \beta), 14$

$V_{k, b_i, \beta}(t_k), V_k(t_k \mid b_i, \beta), 10$

$V_{k, b_i, \beta}^{-1}, 12$

$W_i(t_{-i} \mid b_i, t_i, \beta), 13$

$\Delta U_i^n(t_i^n), 20$

$\Delta W_i^n(t_i^n), 21$

$\Omega_i^n, 24$

$\bar{\beta}^n, 20$

$\beta_i, 9$

$\beta_{-i}, 10$

$\beta_{i, \inf}^{-1}(b), 9$

$\beta_{i, \sup}^{-1}(b), 9$

$\mathbf{1}[E], 9$

$\mathbf{t}, t, 9$

$\mathbf{t}_i, t_i, 9$

$\mathbf{t}_{-(i, j)}, 9$

$\mathbf{t}_{-i}, t_{-i}, 9$

$\delta_n, 45$

$\mathbb{E}[g(\mathbf{x})], \mathbb{E}[g(\mathbf{x}) \mid E], 9$

$\mathbb{M}(\zeta), 34$

$\mathcal{V}_{k, x}(t_k), \mathcal{V}_k, 23, 37$

$\bar{L}_i(b'_i, b_i, t_i, \beta), 46$

$\bar{L}_i^n(t_i^n), 21$

$\bar{Q}_i(b_i, t, \beta), 15$

$\bar{R}(\zeta), 34$

$\bar{V}_{k, i, b_i, \beta}(t_k), 10$

$\bar{W}_i(b_i, t_i, \beta), 15$

$\bar{\varphi}_i, 38, 57$

$\underline{b}_i^*, 59$

$\varphi_i(t_i, t_{i_*}), 37$

$\zeta, 34$

$b_i \succ \mathbf{t}_{-i}, b_i \not\succeq \mathbf{t}_{-i}, 17, 46$

$c_i^n, c_j^n, 21, 59$

$f_k(\cdot \mid i, b_i, \beta), 10$

$l, 8$

m -approximation game, 18

m -equilibrium, 18

$p_{k, i, b_i, \beta}(t_{-k}), 11, 12$

$q_i(t, \beta), 15$

$q_{ij}(t, \beta), 15$

$r, 8$

$t_j^n, 47$

$v_k^n(t_{-k}), 53$

$z_i, 29$

$z_j, 60$

$z_k^m, 59$

approximation equilibrium, 2, 18

atom, 10

consequential, 19

dominant bidder, 6

inconsequential, 7

monotone profile, 9

payoff security, 5

revenue effect, 6

serious bid, 8, 13

winner's curse, 2, 4, 21, 26