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## Slow and steady wins the race: Approximating Nash equilibria in nonlinear quadratic tracking games

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# JENA ECONOMIC RESEARCH PAPERS



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## **Slow and steady wins the race: approximating Nash equilibria in nonlinear quadratic tracking games**

by

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# Slow and steady wins the race: approximating Nash equilibria in nonlinear quadratic tracking games

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## Abstract

We propose a meta-heuristic approach for solving nonlinear dynamic tracking games. In contrast to more ‘traditional’ methods based on linear-quadratic (LQ) techniques, this derivative-free method is very flexible (e.g. to introduce inequality constraints). The meta-heuristic is applied to a three-player dynamic game and tested versus derivative-dependent method in approximating Nash solution in different game specifications. We demonstrate the superiority of the proposed approach in identifying Nash equilibria, where LQ methods are not applicable.

**Keywords:** Dynamic games; Nash equilibrium; Differential Evolution

**JEL Classification:** C61, C63, C72, C73, E61.

## 1 Introduction

We consider infinite dynamic games in discrete time, where payoffs are defined analytically penalizing differences between outcomes of a nonlinear dynamic system and the corresponding predefined target paths. The standard game-theoretic solution methods rely on the linear-quadratic (LQ) optimization techniques deriving analytical response functions. The OPTGAME algorithm (Blueschke *et al.* (2013a)) is an example for such solvers and is used here as a benchmark for approximating Nash and cooperative Pareto-efficient equilibria.

Over the last few decades a large number of studies tried to extend or rather to replace classical methods, which impose strong restrictions on the model at hand, by using different simulation techniques. One of the methods that becomes increasingly popular is heuristic optimization. It imposes no additional restrictions at the price of being more

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computationally demanding.<sup>1</sup> The proposed approach makes use of those strengths of heuristic methods ensuring convergence to an approximate Nash equilibrium.

To test the proposed approach, we use a three-player dynamic game between fiscal and monetary policy in a monetary union. In the first step we check the quality of the approximate Nash solution by comparing the results to the ones of the OPTGAME algorithm. After that, we solve a ‘non-standard’ extension of this game (in particular, with an inequality constraint), where the traditional methods are not applicable.

## 2 Problem Description

We consider infinite dynamic games in discrete time given in tracking form. The players aim at minimizing quadratic deviations of the equilibrium values from given target values.<sup>2</sup> Each player minimizes an objective function (a loss function)  $J^i$ :

$$\min_{u_1^i, \dots, u_T^i} J^i = \min_{u_1^i, \dots, u_T^i} \sum_{t=1}^T L_t^i(x_t, u_t^1, \dots, u_t^N), \quad i = 1, \dots, N, \quad (1)$$

with

$$L_t^i(x_t, u_t^1, \dots, u_t^N) = \frac{1}{2}[X_t - \tilde{X}_t^i]'\Omega_t^i[X_t - \tilde{X}_t^i]. \quad (2)$$

The parameter  $N$  denotes the number of players (decision makers).  $T$  is the terminal period of the planning horizon.  $X_t$  is an aggregated vector

$$X_t = [x_t \ u_t^1 \ u_t^2 \ \dots \ u_t^N]', \quad (3)$$

consisting of an  $(n_x \times 1)$  vector of state variables and  $N$   $(n_i \times 1)$  vectors of control variables. The game is infinite due to the action sets of the players, which comprise an infinite number (continuous space) of alternatives.

The desired levels of the state and the control variables enter (1)-(2) via the terms

$$\tilde{X}_t^i = [\tilde{x}_t^i \ \tilde{u}_t^{i1} \ \tilde{u}_t^{i2} \ \dots \ \tilde{u}_t^{iN}]. \quad (4)$$

Finally, (2) contains a penalty matrix  $\Omega_t^i$  weighting the deviations of states and controls from their desired levels at any period  $t$ .

The dynamic system constraining the choices of the decision makers is given in state-space form by a first-order system of nonlinear difference equations:

$$x_t = f(x_{t-1}, x_t, u_t^1, \dots, u_t^N, z_t), \quad x_0 = \bar{x}_0. \quad (5)$$

$\bar{x}_0$  contains the initial values of the states,  $z_t$  contains non-controlled exogenous variables.

Equations (1), (2) and (5) define a nonlinear dynamic tracking game problem, which equilibrium solutions we try to find representing them by  $N$  trajectories of control variables  $u_t^i$  minimizing the postulated objective functions subject to the dynamic system.

<sup>1</sup>For a discussion of the matter see Gilli and Schumann (2014).

<sup>2</sup>A quadratic form prevails as a standard approach (Blueschke and Savin 2015).

### 3 Optimization Algorithms

A standard way to solve nonlinear dynamic tracking games is the OPTGAME algorithm.<sup>3</sup> As an alternative we propose a ‘meta-heuristic’ approach in finding the equilibrium solutions. To this end, two linked procedures are applied: a heuristic approach (the Differential Evolution algorithm) for finding individual optimal strategies, and an approach for finding an approximate game equilibrium.

#### 3.1 Differential Evolution

Differential Evolution (DE) is a population-based optimization technique designed to tackle complex problems and detect global optima of various objective functions (eligible for certain constraints).<sup>4</sup> We consider dynamic games, where each player optimizes its own objective function given by (1) - (2), and apply DE for that purpose. A detailed description of how DE deals with an optimal control problem for a single decision maker is described in Blueschke *et al.* (2013b). In short, starting with an initial population of random solutions (line 2 in Algorithm 1), DE updates this population by linear combination (line 7: with  $F$  being the shrinkage rate) and crossover (line 9: with  $CR$  – the crossover rate) of four different solutions into one selecting the fittest solutions among the original and the updated population. This continues until some stopping criterion is met.

For all  $N$  players we apply the same DE procedure with the only difference of the individual number of control variables  $n_i$ . Each member of the population (each candidate solution) contains the control variables of player  $i$  for all time periods.

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**Algorithm 1** Pseudocode for Differential Evolution (individual optimization)

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- 1: initialize parameters  $m, T, p, F$  and  $CR$
  - 2: randomly initialize  $P_{j,t,k}^{(1)}, j = 1, \dots, n_i; t = 1, \dots, T; k = 1, \dots, p$
  - 3: **while** the stopping criterion is not met **do**
  - 4:    $P^{(0)} = P^{(1)}$
  - 5:   **for**  $k = 1$  to  $p$  **do**
  - 6:     generate  $r_1, r_2, r_3 \in 1, \dots, p, r_1 \neq r_2 \neq r_3 \neq k$
  - 7:     compute  $P_{\dots,k}^{(v)} = P_{\dots,r_1}^{(0)} + F \times (P_{\dots,r_2}^{(0)} - P_{\dots,r_3}^{(0)})$
  - 8:     **for**  $j = 1$  to  $n_i$  and  $t = 1$  to  $T$  **do**
  - 9:       **if**  $u < CR$  **then**  $P_{j,t,k}^{(n)} = P_{j,t,k}^{(v)}$  **else**  $P_{j,t,k}^{(n)} = P_{j,t,k}^{(0)}$
  - 10:     **end for**
  - 11:     **if**  $J(P_{\dots,k}^{(n)}) < J(P_{\dots,k}^{(0)})$  **then**  $P_{\dots,k}^{(1)} = P_{\dots,k}^{(n)}$  **else**  $P_{\dots,k}^{(1)} = P_{\dots,k}^{(0)}$
  - 12:   **end for**
  - 13: **end while**
- 

#### 3.2 Stochastic approximation of a Nash equilibrium

The dynamic games are characterized by interacting agents, whose strategies determine the outcome. Here we consider two kinds of game strategies, a cooperative (Pareto

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<sup>3</sup>See Blueschke *et al.* (2013a) for details on the OPTGAME algorithm.

<sup>4</sup>For an overview of the heuristic methods see Gilli and Winker (2009).

optimal) and a non-cooperative (Nash equilibrium) game strategy. In the former case, players cooperate and act as one player. To this end, a joint objective function is created,

$$J = \sum_{t=1}^T \sum_{i=1}^N \mu^i L_t^i(x_t, u_t^1, \dots, u_t^N), \quad \sum_{i=1}^N \mu^i = 1, \quad (6)$$

according to the individual weights of the players ( $\mu^i$ ).<sup>5</sup> The resulting problem can be considered as a single-player optimal control problem and solved using Algorithm 1.

To find a Nash solution is a more demanding task. The Nash equilibrium is characterized by the fact that no player can improve her performance by changing only own strategy. To approximate the equilibrium, we use an iterative method where the players successively find individual best strategies using the available information about other players' choices. The pseudocode of this procedure is given in Algorithm 2.

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**Algorithm 2** Pseudocode for finding approximate Nash equilibrium

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- 1: set criteria for convergence, set tentative  $\overset{\circ}{u}$
  - 2: **while** the convergence criterion is not met **do**
  - 3:   **for** each player  $i = 1 : N$  **do**
  - 4:      $\min_{u^i} J^i = f(x, u^{1*}, \dots, u^i, \dots, \overset{\circ}{u}^N)$  to get  $u^{i*}$  (see Algorithm 1)
  - 5:   **end for**
  - 6:   check convergence, if no convergence set  $\overset{\circ}{u} = u^*$
  - 7: **end while**
- 

As the convergence criterion we ask the control variables of all players to be the same as in the previous iteration.<sup>6</sup> If the algorithm converges, it means no player can improve its individual performance by a one-sided deviation from the optimal strategy.<sup>7</sup> As a result, we get a stochastic approximation of the Nash equilibrium.

## 4 Simulation Results

To test our approach, we use a dynamic macroeconomic model of a monetary union consisting of two countries (or two blocs of countries) with a common central bank, which is called MUMOD1.<sup>8</sup> The model is calibrated to deal with the problem of public debt targeting in a situation that resembles the one currently prevailing in the European Union. MUMOD1 is formulated in terms of deviations from a long-run growth path and includes following state variables: output ( $y$ ), real interest rate ( $r$ ), nominal interest rate ( $I$ ), inflation ( $\pi$ ), union-wide inflation and output ( $\pi_E, y_E$ ), public debt ( $D$ ) and interest rate on bonds ( $BI$ ). Furthermore, the model includes three decision-makers: the common central bank decides on the prime rate  $R_{Et}$  (a nominal rate of interest); the national governments decide on fiscal policy:  $g_{it}$  denotes country  $i$ 's ( $i = 1, 2$ ) real fiscal surplus (or, if negative, deficit), measured in relation to real GDP.

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<sup>5</sup> $L_t^i$  is defined by (2). The parameters  $\mu^i$  reflect player  $i$ 's 'power' in the joint objective function.

<sup>6</sup>This means that we require them to lie inside an  $\epsilon$ -tube with  $\epsilon = 1 \times 10^{-5}$ .

<sup>7</sup>We are aware that this convergence is not guaranteed in general. For the future research we plan to consider a Particle Swarm Optimization (PSO)-based approach increasing the robustness of the meta-heuristic search.

<sup>8</sup>For details see Neck and Blueschke (2014).

### 4.1 Baseline

First we present results of the baseline scenario of MUMOD1 to show the ability of our approach to achieve the same solution as calculated by OPTGAME. In the left panel of Figure 1 we plot simultaneously Pareto and Nash solutions for both methods: OPTGAME (denoted by ‘Pareto’ and ‘Nash’, respectively) and the proposed meta-heuristic (‘Pareto\_DE’ and ‘Nash\_DE’, respectively).

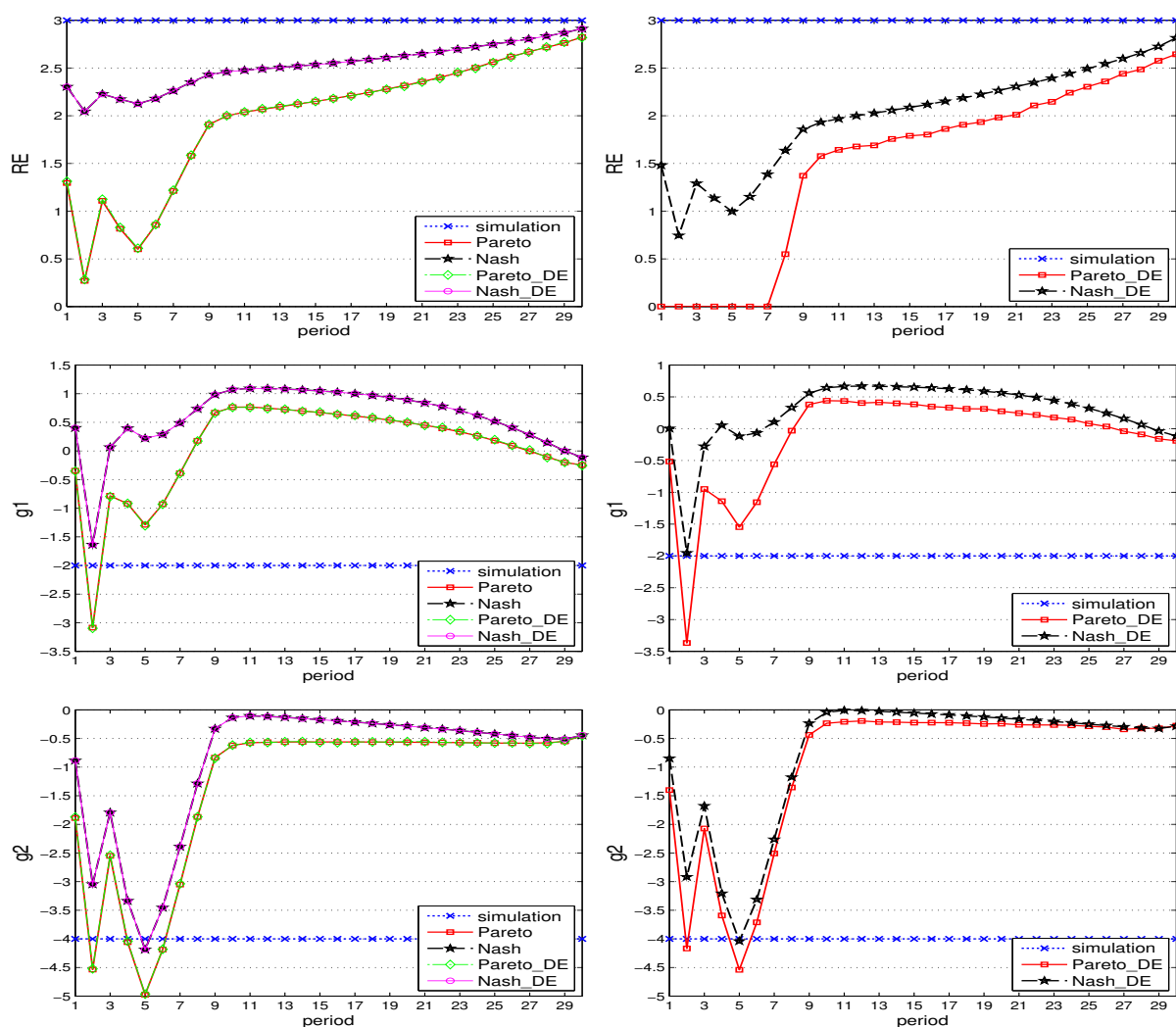


Figure 1: Optimal paths for the control variables  $RE$  and  $g_i$ . The left panel reports baseline solution and the right panel: inequality constraint scenario.

As the graphical results nearly coincide, Table 1 gives more details on comparison between the two alternatives. The meta-heuristic approximates the analytical solutions fairly well. In the case of the Nash solution, the new approach gives the same results for all individual objective functions. In the case of the Pareto solution, it even slightly outperforming the OPTGAME in respect to  $\sum J_i$ . However, the proposed method is extremely computationally demanding exceeding OPTGAME by factors 100 - 1500. This clearly indicates that for a ‘normal’ case (where an analytical solution is available) the meta-heuristic approach is not useful. Its usage is justified only for situations where an analytical solution is not available (or reliable) as presented in the next section.

Table 1: Results for the MUMOD1 dynamic game

	baseline				inequality constraint	
	OPTGAME		Meta-heuristic		Pareto	Nash
	Pareto	Nash	Pareto	Nash		
$J_1$	79.72	76.73	79.34	76.73	53.93	56.82
$J_2$	38.75	72.14	39.11	72.14	28.62	44.59
$J_3$	118.33	152.36	118.34	152.36	97.05	122.24
$\sum J_i$	236.80	301.23	236.79	301.23	179.6	223.64
cpu	2.1s	2.1s	281s	3459s	321s	3597s

## 4.2 Inequality constraint

The main reason of applying the proposed method is its flexibility allowing to solve certain scenarios of dynamic tracking games, which cannot be addressed (or extremely difficult to address) by ‘traditional’ methods. To illustrate such a ‘useful’ application, we consider a modification of the MUMOD1 model. In the baseline scenario, the control variable of the central bank is assigned with an importance (weight) by factor three in comparison to the controls of national governments. The main justification for it is the fact that this is the only way to get a certain analytical solution. Figure 2 presents results for a scenario where the prime rate has the same weight as other controls, namely one.

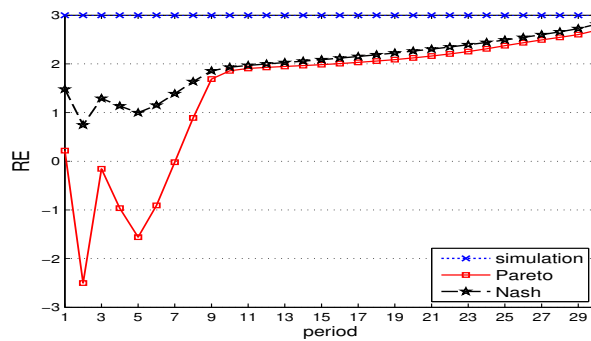


Figure 2: prime rate  $R_{Et}$  controlled by the central bank ( $W(R_E) = 1$ )

The central bank is required to set  $R_{Et}$  negative in periods 2-6 (Figure 2). Since 2014, the ECB uses negative interest rates, mainly as a discount window, while the prime rate is still positive (0.05%). There is a discussion whether a negative prime rate is a reasonable instrument. However, a prime rate equaling -2.5% is clearly inappropriate. The standard way to get a proper analytical solution is to assign a higher weight to the concerned variable. However, without a theoretic justification this changes the conditions of the game and can produce misleading results.

In contrast, by exploiting the flexibility of the meta-heuristic we can easily restrict the prime rate from being negative without changing its weight. The introduction of an inequality constraint ( $R_E \geq 0$ ) is one way to do so. Thus, the prime rate has a significantly higher weight (equal to 100) only for its negative values, and 1 otherwise.<sup>9</sup>

<sup>9</sup>Taking inequality constraints into account is only one example, where the meta-heuristic is useful.



Thus, we force (in a smooth way) the prime rate to stay non-negative without the risk of uncontrolled jumps of this variable. In contrast, to introduce inequality constraints into the OPTGAME algorithm is very difficult.

Right panel of Figure 1 shows the paths of the control variables. The central bank is forced to hold its interest level close to zero in the first seven periods due to the inequality constraint. Comparing the results to the ones given in the left panel, one observes a significant change regarding the Nash equilibrium, namely, a more expansionary monetary policy is required. For brevity reasons, we skip the interpretation of the results. Table 1 summarizes the objective function values and the computational time for the case with the inequality constraint.

## 5 Conclusion

We propose a meta-heuristic approach to solve an infinite dynamic tracking game in discrete time and identify a cooperative (Pareto optimal) and a non-cooperative (Nash equilibrium) game strategies. We both demonstrate that the meta-heuristic indeed allows us to obtain a fairly well approximation of the analytically calculated Nash equilibrium, but also show that thanks to its flexibility the new approach can be easily extended to games that are extremely difficult to solve by ‘traditional’ algorithms. As an example, we consider an extension of the MUMOD1 model including inequality constraints. For further research we leave application of the proposed method to different dynamic games and extension to more robust algorithms (e.g., PSO-based) to ensure better convergence.

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Another example is to allow for an asymmetric objective function as described in Blueschke *et al.* (2013b).