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## Potential and Spatial Structure of Population

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# Potential and Spatial Structures of Population

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## **Abstract**

The goal of this work is to suggest a mechanism explaining different spatial patterns of residential locations. The basic idea is counterbalance of centripetal and centrifugal forces. This paper complements the previous author's works in this area. It is possible to find the optimal city size assuming some scale economies in production counterbalanced by commuting costs. The rural community of farmers is also considered. Here the average distance to neighbor (as a proxy to market access) is balanced with the benefits from land ownership. Finally, the spatial equilibrium is constructed. It consists of discrete cities of optimal size attracting certain fraction of the population and the continuous farmland between them.

## **1 Introduction**

This article has the objective to further understand the mechanism of emergence of spatial equilibrium in the pattern of residential locations. Like in the majority of economic studies, it is assumed that all agents are identical a priori, and get the same utility in different locations and different occupations.

The goal of this work is to suggest a mechanism explaining different spatial patterns of residential locations. The basic idea is counterbalance of centripetal and centrifugal forces. This paper complements the previous author's works in this area. This article addresses the following questions: a) agglomeration potential, b) optimal city size, c) equilibrium agricultural density, d) influence of agglomeration on land rent.

Why do we observe quite different pattern of residential location? It may be dispersed (both regularly, like Cristaller's hexagons and irregularly) or form clusters. The spatial location of household and business represents a complex and evolving pattern. Both relative location and size distribution of cities and residential patterns in agricultural areas represent interesting objects of study. Here we can divide the overall complexity into several issues:

- a) differences across countries in average population density (studied in Yegorov, 2009b),
- b) structural change between rural and urban density (studied in Yegorov, 2005),
- c) explanation of particular patterns,
- c) share of urban and rural population.

While there are many models explaining agglomeration and asymmetric locations, they typically do not take into account for land as productive factor. Here it is shown that land-intensive production (like agriculture) drives the dispersion of population in space. Spatial equilibrium with relatively low population density is reached through the balance from benefits of having more land per capita and the costs of market access. The necessary assumptions are quite realistic and are based on 2-dimensional continuous Euclidean space, that is decomposed into agricultural land slots. Such an approach is typical for physics, where density determines equilibrium state of condensed matter. The role of density in economics have not been studied until recently (see, for example, Yegorov, 2009b).

The mechanism of balance between attracting and repelling forces is used in many models of formation of urban clusters. Two main forces, centripetal (agglomeration) and centrifugal (congestion) shape this urban pattern. The origin of agglomeration forces is in scale economies, while congestion forces represent a cumulative negative externality from such agglomeration. It is likely that the deterministic component of location pattern can be explained by balance of these forces.

The first article in regional science (von Thünen, 1826) have already used the spatial structure of 2-dimensional continuous space, which was developed in the models of 1930s (by Hotelling, Cristaller, Lösch and others - a good survey is provided in Handbook of Regional Economics) has been largely abandoned in the modern literature, including new economic geography, de-

veloped by Krugman. However, Beckmann and Puu (1991) use continuous space and consider flows of goods in it.

Urban economics (see, for example, Fujita, 1989) present the theory of spatial structure of a city in microeconomic terms. The typical core assumption is in existence of Central Business District (CBD), located in geometrical center of radially symmetric city, with which agents have to commute on daily basis. There are two types of commuting costs (leisure loss, that was in the focus of Henderson, 1985), and transport cost. Interestingly, the interplay between both of them can lead to mapping of agents with different wealth into locations (see Yegorov, 1999), but this type of heterogeneity in wealth will be outside of the models of this article.

**Paper structure.** The paper addresses several related issues and presents a sequence of simple analytical results. Section 2 studies the optimal city size. Section 3 studies agricultural density. Section 4 studies agro-industrial cluster. Section 5 addresses applications. Section 6 concludes.

## 2 Model of Optimal City

There are many factors that influence agglomeration and dispersion of the population in space: pollution, commuting, housing prices, etc. Here we will focus on those forces that are linked to main production activities. Following the stylized facts about different production technologies, it is assumed that agricultural technology creates dispersion force, while industrial technology creates agglomeration force. The basic reason is as follows. Agriculture requires intensive land use. In order to save on commuting between residential location and land to work, farmers have dispersed settlement in space. Still it is not fully dispersed (although we observe isolated farms in some countries), but are often clustered in villages. Village, or rural settlement, represents an optimal balance between saving on commuting with land slots and enjoying some scale effects from residential clustering, like access to market and service centers.

## 2.1 Optimal City

There are many models of optimal city size. The objective of present model is to give a simple presentation, that is consistent with urban economics and also contains the link between optimal city size and some external parameters, like transport cost, that can vary across time and countries.

**Industrial city.** Let  $N$  denotes population of a city, which is assumed to occupy the interior of a circle  $r \in [0, R]$  with CBD in its center. Suppose that all flats are equal in size  $b$ , but standard mechanism of balancing land rent with commuting cost is used to make agents indifferent across locations. It means that every citizen located at distance  $r$  from the CBD has to pay housing services, equal to construction costs  $H$  and land rent  $L(r)$ , plus commuting costs,  $\tau r$ , where  $\tau$  denotes exogenously given unit distance transport cost. Consider an industrial city with scale economies. It is assumed that benefits from living in a city are equal for each citizen independently on location and are proportional to city's population in some power,  $N^\epsilon$ . This benefit can include wage and saving on costs (through cheaper cost per capita on infrastructure). There exist empirical studies showing presence of scale economies in cities. Assuming rent at the city edge equal to zero (typically it equals to agricultural land rent, that can be neglected in this model), we can define potential (that has the meaning similar to utility) through its value on city edge:

$$U = AN^\epsilon - \tau R - H. \quad (1)$$

Since all citizens have identical utility by an assumption, we have the same value of potential for all of them, although its components (land rent and commuting costs) are different for each agent. The equilibrium land rent exactly compensates commuting cost. Considering this potential as parametric function on  $\epsilon$ , we can find an optimal size of city cluster. First we use the geometric link between  $R$  and  $b$  (assuming 1-story buildings with land  $b$  used for living):

$$\pi R^2 = Nb \Rightarrow R = \sqrt{Nb/\pi}. \quad (2)$$

This gives the following expression for potential and optimal cluster size:

$$\begin{aligned} U(N) &= AN^\epsilon - \tau\sqrt{Nb/\pi}, \\ dU/dN = 0 &\Rightarrow N^* = \left(\frac{A\epsilon}{\tau}\sqrt{\pi/b}\right)^{1/(1/2-\epsilon)}. \end{aligned} \quad (3)$$

Since

$$d^2U/dN^2 = A\epsilon(\epsilon - 1)N^{\epsilon-2} + \frac{\tau}{2N^{3/2}}\sqrt{b/\pi},$$

at the point  $N = N^*$ , the second derivative is negative (condition for maximum) only iff  $\epsilon < 1/2$ . Thus, for  $\epsilon \in (0, 1/2)$  we have the finite optimal city size. For  $\epsilon > 1/2$  the critical point is minimum, and the city reaches its optimum for infinite radius.

**Proposition 1** *The optimal city size is finite only for reasonably small scale economies,  $\epsilon \in (0, 1/2)$ .*

**Applications.** This urban model shows that centripetal and centrifugal forces are not always balanced. Only for relatively small scale economies in a city ( $\epsilon < 1/2$ ) its growth is limited. For high scale economies its growth is unbounded, since optimality is reached for infinite size. Since we never can observe infinite population, we see unbounded dynamics of growth for some cities. Such cities as Mexico, Cairo, Bombay, Shanghai already exceed 10 mln. of population, but demographers predicts their growth to 20 mln. or more. In some countries we observe super-giant capital that absorbs high fraction of the total population, even when total urban population share is still modest (the case of developing countries).

**Discussion about country types.** Why it might happen that developing and not developed countries reveal this phenomenon of unbounded city growth? In other words, in what parameters are they different for our model? First, we will exclude the case  $\epsilon > 1/2$  and assume this parameter equal for both cases<sup>1</sup>. Let us denote  $\mu \equiv 1/0.5 - \epsilon$ . For  $\epsilon \in (0, 1/2)$ , we have  $\mu \in (2, +\infty)$ . Consider comparative statics of  $N^*$  with respect to parameters  $b$  and  $\tau$ . Since

$$dN^*/db = -\frac{\mu}{2}\left(\frac{A\epsilon}{\tau}\sqrt{\pi}\right)^\mu b^{-1-\mu/2} < 0, \quad (4)$$

$$dN^*/d\tau = -\mu(A\epsilon\sqrt{\pi/b})^\mu b^{-1-\mu} < 0, \quad (5)$$

the growth of parameters  $b$  and  $\tau$  leads to a decline of optimal city size. Developed countries are richer, and thus people want to have more living

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<sup>1</sup>There are no possible equilibria and these countries are not likely to have higher technology.

space per capita. This leads to higher  $b$ , and thus lower  $N$ . Thus, cities of smaller size are optimal in developed countries. The commuting costs in richer countries are also higher, at least in real terms. This happens because the quality (and price) of both vehicles and roads are higher. This, this factor also works in the same direction, and the model explains the puzzle about faster growth of cities in developing countries.

**Proposition 2** *The optimal city size (population) depends negatively on transport cost,  $\tau$ , and average size of land slot per capita,  $b$ . Technological development in transportation leads to an increase of city size. At the same time, more wealthy economies tend to have smaller cities (through the positive effect of wealth on  $b$ ).*

### 3 Equilibrium Agricultural Density

Here we focus only on the case  $\epsilon < 1/2$ . Assume that space is occupied with agricultural population (density  $\rho_a$ ) and urban clusters. Let  $\gamma$  be the fraction of rural population. Asymptotically neglecting the area occupied by cities, we can write  $\rho_a = \gamma\rho$ .

The simplest way to define agricultural potential is to subtract commuting cost (market access) from the value of land, proportional to its size. However, this will be bad example (see Appendix.2). Good example assumes decreasing returns to scale in the value of land for potential of a farmer.

**Good example.** Consider the case of decreasing returns to land in production :  $V_1 = S^\nu - \tau r$ , for  $\nu < 1$ . We can think about agricultural production function for an individual farmer as Cobb-Douglas for land and labour, where labour unit is supplied inelastically. If we add the cost of going around land, it will be proportional to  $r$ , the distance to neighbour, and thus will add nothing new. Consider the total land size normalized to one, and agricultural population to  $N$ . Since land slot  $S = r^2$ , density  $\rho = N$ , while  $r = \rho^{1/2}$ . Thus,

$$V_1(\rho_a) = S^\nu - \tau r = \rho^{-\nu} - \tau\rho^{-1/2}. \quad (6)$$

Now we also consider the social planner problem (individual cannot rationally consider the problems with variable number of agents), where land is

divided into  $N$  equal slots. Then overall utility of all agricultural population is given by:

$$V_N(\rho) = N(S^\nu - \tau r) = \rho^{1-\nu} - \tau\rho^{1/2}. \quad (7)$$

Differentiation gives the socially optimal population density of farmers:

$$\rho_a^* = \left[ \frac{2(1-\nu)}{\tau} \right]^{1/(\nu-0.5)} \quad (8)$$

Since powers are positive now, we have internal maximum (see Appendix.1) for  $1-\nu < 1/2$ , or taking limitation for  $\nu$  into account, for  $1/2 < \nu < 1$ .

**Proposition 3** *There exists an optimal rural population density for  $\nu \in (0.5, 1)$ , for moderate decreasing returns to scale from land size slot per farmer. In this case farmers keep optimal dispersion (density) in space, that is based on interplay between returns to land slot size in production and cost of market access.*

## 4 Interaction between City and Rural Area

### 4.1 Agro-Industrial Cluster

**Agricultural density.** Let  $S$  denotes land endowment per farmer (all agents involved in agriculture are assumed to be independent farmers),  $r$  - average distance between agents and  $\rho$  population density. Assuming that all land is used for agriculture, we have the following relationships between these variables:

$$\rho = 1/S = 1/r^2. \Rightarrow r = \rho^{-1/2}. \quad (9)$$

**Scale technology in industry.** However, it is more important to explain clustering in urban areas. Consider scale economies, with productivity in industrial (and service production) area growing with the number of agents  $N$  who work there. Let industrial production function be

$$Y = AN^{1+\epsilon}, \quad \epsilon > 0. \quad (10)$$

Then production per capita is  $y = Y/N = AN^\epsilon$ .



**Potential for agro-industrial cluster.** Let commuting cost for agent  $i$  be given by  $\tau r_i$ , with distance  $r_i$  from CBD being different across agents. Consider an agent who has access both to industrial and agricultural technologies. Let us introduce his potential as a proxy for wealth. The potential is defined as value from land endowment and industrial technology minus commuting cost:

$$\Phi = S(r) + AN^\epsilon - \tau r. \quad (11)$$

Here we assumed that agent  $i$  is located at distance  $r$  from CBD, has land endowment  $S(r)$  for farming there and also works in industry located in CBD. Such hypothetical agro-industrial cluster is heterogeneous. To make all agents indifferent across locations (like this is done in urban economic models), potential should be constant:  $\Phi(r) = \text{const}$ . From this condition we can determine  $S(r)$ , and then population density  $\rho(r)$  (persons per sq.km) at distance  $r$ . Let  $\rho_2(r)$  denotes radial population density (number of people living in circle  $[r, r + dr]$ ), and  $\bar{r}$  is the radius of cluster. Then we can close the model by linking total population with the integral from density ( $\pi = 3.14\dots$ ):

$$N = 2\pi \int_0^{\bar{r}} \rho_2(r) dr. \quad (12)$$

## 4.2 Accounting for rural and urban population

In order to make agents indifferent between agricultural and industrial activities, potentials  $U(N^*)$  and  $V_1$  (not  $V_N$ !) are equated:

$$V_1 = U(N^*) \Rightarrow \frac{1}{(\gamma\rho)^\nu} - \frac{\tau}{\sqrt{\gamma\rho}} = AN^{*\epsilon} - \tau\sqrt{N^*b/\pi}, \quad (13)$$

where  $N^*$  is given by the equation (7). Since  $\rho$  is exogenous parameter for a country, fraction of rural population  $\gamma$  can be varied to reach equality. Country will be decomposed into city clusters of optimal size (they depend on scale economies, flat size, commuting cost) and agricultural population with certain density.

This is transcendental equation for  $\gamma$ . Denoting  $U(N^*) \equiv U^*$ , we get it in a form:

$$U^*\gamma^\nu + \frac{\tau}{\rho^{1/2}}\gamma^{\nu-1/2} - \rho^{-\nu} = 0. \quad (14)$$

Consider a special case  $\nu = 3/4$ . Then we can denote new variable  $y \equiv \gamma^{1/3}$ , and get cubic equation for it:

$$U^*y^3 + \frac{\tau}{\rho^{1/2}}y - \rho^{-\nu} = 0. \quad (15)$$

This equation has a unique positive root, which under certain conditions lies in the feasible range  $y \in (0, 1)$ .

Let the total territory is normalized to one, while the total agricultural population is  $N$ . We think about the model of "raisins in a cake", where cities add population to homogeneously spread agricultural population, but occupy negligible territory. This gives  $\rho_a = N$ , and we remember that  $\rho_a = \gamma\rho$ . If we have  $M$  cities of optimal size  $N^*$ , then the average (urban and rural) density  $\rho = MN^* + N$ . Thus, the number of cities is:

$$M = \frac{N(1 - \gamma)}{\gamma N^*}. \quad (16)$$

**Proposition 4** *The equilibrium location structure includes "raisins" of optimal cities in a "cake" of uniformly dispersed agricultural population. For a given total population of a country, population density in cities, transport costs, coefficient of increasing returns in city and decreasing returns on agricultural land, there exists optimal number and size of cities and optimal agricultural density.*

## 5 Applications

### 5.1 Oil Peak Influence

The paper of Robert and Lennert (2010) analyses the impact of oil peaking for Europe. One of their results says that oil peaking will increase prices and reduce transportation, and this will restructure the whole urban system. In particular, sub-urbanization will be reversed and people will move to more compact cities.

How this model of optimal city can be extended? Simple comparative analysis with respect to transport cost will not do all job. It is also important to account for other scarce resources in a city, like limited fresh air,

distance to forest, time in internal commuting, etc. All these factors work for additional congestion.

The model should predict not only unbounded growth of metropolitan areas (observed in the last decade) but a possibility of reverse process: moving to smaller cities if transport cost increase substantially.

## 5.2 Agglomeration and the Dynamics of Land Rent

Suppose that agglomeration force emerges at some moment and leads to growth of a city (see section 2). Urban and housing economics (see CBD model) suggests that the equilibrium price of housing consists of construction price and location rent. However, these theories are essentially static. Urban dynamics is driven by temporal imbalance of centrifugal and centripetal forces. The cause of such disequilibrium may be some social innovation making city more attractive. For example, after 1991 Moscow has gained more attraction in comparison to other Russian cities due to new distribution of financial capital and political power. Huge financial flows to Moscow (including practically all FDI) caused higher wages (via positive jump in coefficient  $A$  in (5)), and thus attraction for migration. This caused a shift from old equilibrium city size  $N^*$  to a larger one,  $N^{**}$ . The new border of the city expanded, and land rent in its center has grown. This caused the price of all existing real estate to rise, and also has created the demand for building new real estate between old and new city borders<sup>2</sup>. First of all, the prices of old housing went up, and the owners were satisfied. Second, there was a substantial construction boom. Third, the process of transition to new equilibrium caused a positive trend in housing price (via the change of land rent) and some speculators took their position. The result was price overshooting with substantial price crash after the world financial crisis of 2008.

## 5.3 Land rent as a collective phenomenon

Price appreciation due to the growth of land rent is a complex collective phenomenon, where different groups of people are affected. The old city dwellers get not only appreciation of their housing, but also utility loss from growing

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<sup>2</sup>Here we ignore the development of some central areas, that give substantial premium in land rent for developer

congestion and higher competition with migrants on labour markets. But the most interesting question is: to whom newly created rent should belong? Note that in this process we have quite complicated accounting. It includes utility loss of other Russian cities (sending migrants to Moscow). Thus, they might have a right for some compensation. The additional land rent created in Moscow city thus belongs not only to citizens but also to other Russian people (via compensation for negative externality effect). And clearly it cannot belong to politicians or construction companies. In an "honest" framework (that only can be created by corresponding laws) construction should take place in a competitive framework, slots for construction should be sold at open auctions, and the revenue should be distributed across citizens, after payment of some tax to state budget (that later can be used for subsidies of regions-lousers).

## 6 Conclusions

1. There are many effects discussed here. First is related to scale economies in cities that act as attracting force to compensate congestion forces in a city of optimal size. The comparative statics with respect to transport cost explains the growth of average city size with technological development.

2. Next, we have a balance between urban and rural population, resulting in equilibrium rural density. The analysis is based on agricultural potential, that involves returns to land in production function of a representative farmer as well as cost of his access to the market.

3. The model of agro-industrial cluster is also developed. It involves a split of the total population into a set of cities of optimal size, surrounded with agricultural area with optimal rural density.

4. External shocks can change optimal city size and lead to temporal disequilibrium processes. In the case of oil peak sub-urbanization can be reversed, people will leave too large cities and move to more compact. In the case of positive shock (like one in Moscow after 1991) there will be a process of further city growth.

5. Combination of city dynamics with the static theory of urban and

housing economics makes it necessary to understand the issue of ownership in land rent. This is a complex collective phenomenon with externalities, and special laws have to be introduced to avoid power abuse and dishonest profits.

## 7 Literature

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## 8 Appendix

### 8.1 About Potentials as Field Differences

The difference of two power potentials  $u(x) = x^a - x^b$  is often used in physics (for example, Lienard-Wiechert) to describe some intermediate stable state. When the forces of attraction and repelling between particles (atoms) depend on some power of distance, there exists equilibrium distance, where both forces are balanced. The origin of these forces may include electromagnetic fields of multipoles, but include also other interactions. If we set  $\phi = x^{-a} - x^{-b}$ ,  $a > b > 0$ , then the field  $\phi$  starts at  $+\infty$  at  $x = 0$ , then declines to minimum  $x^*$  at some finite  $x$ , and later grows to 0 at  $x \rightarrow +\infty$ . In physics, minimum of energy means equilibrium, or stable state.

In economics, we typically seek for some maxima. Thus, we have to use opposite function  $\psi(x) = -\phi(x) = x^{-b} - x^{-a}$ ,  $a > b > 0$ . Such functions of (population density) were used in Yegorov's "Density" model (Chinese Business Review, 2009).

If powers are positive,  $u(x) = x^a - x^b$ , then we get maximum for  $0 < a < b$ . Typical economic example is  $a = 1, b = 2$  (linear-quadratic function).

### 8.2 Bad Example

First, let us introduce the potential for a farmer,  $V$ , as the difference of land slot  $S_a$  and average distance to another farmer, and then use the relationship (1):

$$V(\rho) = S_a - \tau r = \frac{1}{\rho_a} - \frac{\tau}{\sqrt{\rho_a}}. \quad (17)$$

Differentiation of this potential with respect to agricultural density gives the following:

$$\begin{aligned} dV/d\rho_a &= -(\rho_a)^{-2} + (\rho_a)^{-3/2}\tau/2 = 0 \Rightarrow \rho_a^* = 4/\tau^2, \\ d^2V/d\rho_a^2 &= 2\rho^{-3} - \frac{3\tau}{4}\rho^{-5/2}. \end{aligned} \quad (18)$$

A problem emerges here since the second derivative on optimal solution is positive (this is minimum, not maximum). Note that the maximum occurs when the power with sign " + " has lower absolute power.