

Yokoi, Takahisa

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Efficient Maximum Likelihood Estimation of Spatial Autoregressive Models with Normal but Heteroskedastic Disturbances

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Efficient Maximum Likelihood Estimation of Spatial Autoregressive Models with Normal but Heteroskedastic Disturbances*

Takahisa Yokoi, Tohoku University, Japan

Abstract

The likelihood functions for spatial autoregressive models with normal but heteroskedastic disturbances have been derived [Anselin (1988, ch.6)], but there is no implementation of maximum likelihood estimation for these likelihood functions in general cases with heteroskedastic disturbances. This is the reason why less efficient IV-based methods, ‘robust 2-SLS’ estimation for example, must be applied if disturbance terms might be heteroskedastic. In this paper, we develop a new computer program for maximum likelihood estimation and confirm the efficiency of our estimator in cases with heteroskedastic disturbances, using Monte Carlo simulations.

Keywords: *Spatial autoregressive model, Heteroskedasticity*

JEL Classifications: *C13, C21*

1 Introduction

The likelihood functions for spatial autoregressive models with normal but heteroskedastic disturbances have been derived [Anselin (1988, ch.6)], but there is no implementation of maximum likelihood estimation for these likelihood functions in general cases with heteroskedastic disturbances. This is the reason why less efficient IV-based methods, ‘robust 2-SLS’ estimation for example, must be applied if disturbance terms might be heteroskedastic. In this paper, we develop a new computer program for maximum likelihood estimation and confirm the efficiency of our estimator in cases with heteroskedastic disturbances, using Monte Carlo simulations. ¹

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¹ There is literature on the estimation of regression models with heteroskedastic disturbances (Amemiya, 1977; Breusch and Pagan, 1979; Jarque and Bera, 1980; White, 1980; Griffiths, 2003). There are also several discussions of heteroskedastic spatial autoregressive models (Anselin, 1988, 1990; Kelejian and Robinson, 1998; Kelejian and Prucha, 2007; LeSage and Pace, 2009).

The organization of this paper is as follows. In Section 2, we review the general form of spatial autoregressive models including heteroskedasticity. We also discuss estimation methods for such models. In Section 3, we review the formulation of heteroskedasticity, and we define the econometric model which will be analyzed in this paper. In Section 4, we describe Monte Carlo simulations of spatial effects models. We present our conclusions in Section 5.

2 Spatial autoregressive models and heteroskedasticity

In this section, we review the general form of spatial autoregressive models proposed in Anselin (1988, ch.6), and we discuss estimation methods for such models.²

2.1 General form of spatial autoregressive model

Anselin (1988, ch.6) describes the general form of a spatial autoregressive model as follows. \mathbf{y} , \mathbf{X} and $\boldsymbol{\epsilon}$ are a vector of explained variables ($n \times 1$), a matrix of explaining variables ($n \times k$), and a vector of disturbances ($n \times 1$) respectively.³ Two matrices \mathbf{W}_1 and \mathbf{W}_2 are so-called spatial weight matrices ($n \times n$). It represents known spatial structure over the sample space. Scalar variables, ρ and λ , are parameters of spatial auto-correlation among dependent variables and disturbance terms. A vector $\boldsymbol{\mu}$ contains the disturbance terms and its elements obey normal distributions independently. A matrix $\boldsymbol{\Omega}$ is the covariance matrix of $\boldsymbol{\mu}$. h is a function which always takes positive values and determines the variance of each sample. $\boldsymbol{\alpha}$ is a parameter vector of the function h . A vector \mathbf{z}_i represents the variance about the i th sample.

$$\mathbf{y} = \rho \mathbf{W}_1 \mathbf{y} + \mathbf{X} \boldsymbol{\beta} + \boldsymbol{\epsilon} \quad (1)$$

$$\boldsymbol{\epsilon} = \lambda \mathbf{W}_2 \boldsymbol{\epsilon} + \boldsymbol{\mu} \quad (2)$$

$$\boldsymbol{\mu} \sim N(\mathbf{0}, \boldsymbol{\Omega}) \quad (3)$$

$$\Omega_{ii} = h(\boldsymbol{\alpha}, \mathbf{z}_i) > 0, \Omega_{ij} = 0 \text{ for } i \neq j \quad (4)$$

The following useful formulas can be derived. Here, $\boldsymbol{\nu}$ is a vector of random variables which obey the standard normal distribution, $N(0, 1)$, identically and independently.

$$E[\boldsymbol{\mu} \boldsymbol{\mu}'] = \boldsymbol{\Omega}, \boldsymbol{\nu} = \boldsymbol{\Omega}^{-1/2} \boldsymbol{\mu}, \boldsymbol{\mu} = \boldsymbol{\Omega}^{1/2} \boldsymbol{\nu} \quad (5)$$

² The likelihood function, its derivatives, and the information matrix appearing in this section were described in Anselin (1988, ch.6). They are recounted here with some modifications for the sake of clarity.

³ The model has a constant term if every element of the first column of \mathbf{X} is one. The first element of the coefficient vector $\boldsymbol{\beta}$ ($k \times 1$) is the estimator for the constant term in the estimation result.

2.2 Likelihood function

In this section, we adopt the formulation of the log likelihood function for the spatial model.

$$L = -(n/2) \ln \pi - (1/2) \ln |\mathbf{\Omega}| + \ln |\mathbf{A}| + \ln |\mathbf{B}| - (1/2) \boldsymbol{\nu}' \boldsymbol{\nu} \quad (6)$$

$$\boldsymbol{\nu} = \mathbf{\Omega}^{-1/2} \mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}), \quad \boldsymbol{\nu}' = (\mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \mathbf{\Omega}^{-1/2} \quad (7)$$

$$\boldsymbol{\nu}' \boldsymbol{\nu} = (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta})' \mathbf{B}' \mathbf{\Omega}^{-1} \mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}) \quad (8)$$

$$\mathbf{A} = \mathbf{I} - \rho \mathbf{W}_1, \quad \mathbf{B} = \mathbf{I} - \lambda \mathbf{W}_2$$

2.3 First order derivatives

The number of parameters in the optimization is $k + 1 + 1 + p = k + p + 2$. First order conditions on all parameters are necessary for the optimization.

$$\boldsymbol{\theta} = [\boldsymbol{\beta}', \rho, \lambda, \boldsymbol{\alpha}']', \quad \frac{\partial L}{\partial \boldsymbol{\theta}} = \mathbf{0} \quad (9)$$

$$\frac{\partial L}{\partial \boldsymbol{\beta}} = \boldsymbol{\nu}' \mathbf{\Omega}^{-1/2} \mathbf{B} \mathbf{X} (\mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \mathbf{\Omega}^{-1} \mathbf{B} \mathbf{X} \quad (10)$$

$$\frac{\partial L}{\partial \rho} = -\text{tr} (\mathbf{A}^{-1} \mathbf{W}_1) + \boldsymbol{\nu}' \mathbf{\Omega}^{-1/2} \mathbf{B} \mathbf{W}_1 \mathbf{y} \quad (11)$$

$$= -\text{tr} (\mathbf{A}^{-1} \mathbf{W}_1) + (\mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \mathbf{\Omega}^{-1} \mathbf{B} \mathbf{W}_1 \mathbf{y} \quad (12)$$

$$\frac{\partial L}{\partial \lambda} = -\text{tr} (\mathbf{B}^{-1} \mathbf{W}_2) + \boldsymbol{\nu}' \mathbf{\Omega}^{-1/2} \mathbf{W}_2 (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}) \quad (13)$$

$$= -\text{tr} (\mathbf{B}^{-1} \mathbf{W}_2) + (\mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \mathbf{\Omega}^{-1} \mathbf{W}_2 (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}) \quad (14)$$

$$\frac{\partial L}{\partial \alpha_p} = -\frac{1}{2} \text{tr} \mathbf{\Omega}^{-1} \mathbf{H}_p + \frac{1}{2} (\mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \mathbf{\Omega}^{-2} \mathbf{H}_p \mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}) \quad (15)$$

$$= -\frac{1}{2} \text{tr} \mathbf{\Omega}^{-1} \mathbf{H}_p + \frac{1}{2} \boldsymbol{\nu}' \mathbf{\Omega}^{-1} \mathbf{H}_p \boldsymbol{\nu} \quad (16)$$

\mathbf{H}_p is a diagonal matrix and its diagonal elements are $\frac{\partial h}{\partial \alpha_p}$. The solution for $\boldsymbol{\beta}$ is derived from the first order conditions corresponding to Equation (10).

$$\boldsymbol{\beta}^* = (\mathbf{X}' \mathbf{B}' \mathbf{\Omega}^{-1} \mathbf{B} \mathbf{X})^{-1} \mathbf{X}' \mathbf{B}' \mathbf{\Omega}^{-1} \mathbf{B} \mathbf{A} \mathbf{y} \quad (17)$$

2.4 Second order derivatives

$$\frac{\partial^2 L}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'} = -\mathbf{X}' \mathbf{B}' \mathbf{\Omega}^{-1} \mathbf{B} \mathbf{X} = -(\mathbf{B} \mathbf{X})' \mathbf{\Omega}^{-1} \mathbf{B} \mathbf{X} \quad (18)$$

$$\frac{\partial^2 L}{\partial \rho^2} = -\text{tr} (\mathbf{A}^{-1} \mathbf{W}_1 \mathbf{A}^{-1} \mathbf{W}_1) - (\mathbf{B} \mathbf{W}_1 \mathbf{y})' \mathbf{\Omega}^{-1} \mathbf{B} \mathbf{W}_1 \mathbf{y} \quad (19)$$

$$\frac{\partial^2 L}{\partial \lambda^2} = -\text{tr} (\mathbf{B}^{-1} \mathbf{W}_2 \mathbf{B}^{-1} \mathbf{W}_2) - (\mathbf{W}_2 (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \mathbf{\Omega}^{-1} \mathbf{W}_2 (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}) \quad (20)$$

$$\frac{\partial^2 L}{\partial \beta \partial \rho} = \frac{\partial^2 L}{\partial \rho \partial \beta'} = - (BX)' \Omega^{-1} B W_1 y \quad (21)$$

$$\frac{\partial^2 L}{\partial \beta \partial \lambda} = \frac{\partial^2 L}{\partial \lambda \partial \beta'} = - (BX)' \Omega^{-1} W_2 (Ay - X\beta) - (B(Ay - X\beta))' \Omega^{-1} W_2 X \quad (22)$$

$$= - (BX)' \Omega^{-1} W_2 (Ay - X\beta) - \nu' \Omega^{-1/2} W_2 X \quad (23)$$

$$\frac{\partial^2 L}{\partial \beta \partial \alpha_p} = \frac{\partial^2 L}{\partial \alpha_p \partial \beta'} = - (B(Ay - X\beta))' \Omega^{-2} H_p B X = -\nu' \Omega^{-3/2} H_p B X \quad (24)$$

$$\frac{\partial^2 L}{\partial \rho \partial \lambda} = \frac{\partial^2 L}{\partial \lambda \partial \rho} = - (W_2(Ay - X\beta))' \Omega^{-1} B W_1 y - \nu' \Omega^{-1/2} W_2 W_1 y \quad (25)$$

$$= - (W_2(Ay - X\beta))' \Omega^{-1} B W_1 y - (B(Ay - X\beta))' \Omega^{-1} W_2 W_1 y \quad (26)$$

$$\frac{\partial^2 L}{\partial \rho \partial \alpha_p} = \frac{\partial^2 L}{\partial \alpha_p \partial \rho} = - (B(Ay - X\beta))' \Omega^{-2} H_p B W_1 y = -\nu' \Omega^{-2/3} H_p B W_1 y \quad (27)$$

$$\frac{\partial^2 L}{\partial \lambda \partial \alpha_p} = \frac{\partial^2 L}{\partial \alpha_p \partial \lambda} = - (B(Ay - X\beta))' \Omega^{-2} H_p W_2 (Ay - X\beta) \quad (28)$$

$$= - \nu' \Omega^{-3/2} H_p W_2 (Ay - X\beta) \quad (29)$$

H_{pq} is a diagonal matrix. Its diagonal elements are $\frac{\partial}{\partial \alpha_q} \frac{\partial h}{\partial \alpha_p}$.

When a function specifying the heteroskedasticity is not given:

$$\begin{aligned} \frac{\partial^2 L}{\partial \alpha_p \partial \alpha_q} &= (1/2) (\text{tr} (\Omega^{-2} H_p H_q) - \text{tr} (\Omega^{-1} H_{pq})) \\ &\quad + \nu' \{ -\Omega^{-2} H_q H_p + (1/2) \Omega^{-1} H_{pq} \} \nu \end{aligned} \quad (30)$$

$$\begin{aligned} &= (1/2) (\text{tr} (\Omega^{-2} H_p H_q) - \text{tr} (\Omega^{-1} H_{pq})) \\ &\quad + \nu' \{ -\Omega^{-5/2} H_q H_p + (1/2) \Omega^{-3/2} H_{pq} \} B(Ay - X\beta) \end{aligned} \quad (31)$$

$$\begin{aligned} \frac{\partial^2 L}{\partial \alpha_p^2} &= (1/2) (\text{tr} (\Omega^{-2} H_p^2) - \text{tr} (\Omega^{-1} H_{pp})) \\ &\quad + \nu' \{ -\Omega^{-5/2} H_p^2 + (1/2) \Omega^{-3/2} H_{pp} \} B(Ay - X\beta) \end{aligned} \quad (32)$$

When the heteroskedasticity is specified by a linear function:

In this case, we can use $H_{pq} = 0$.

$$\frac{\partial^2 L}{\partial \alpha_p \partial \alpha_q} = (1/2) \text{tr} (\Omega^{-2} H_p H_q) - \nu' \Omega^{-2} H_q H_p \nu \quad (33)$$

$$= (1/2) \text{tr} (\Omega^{-2} H_p H_q) - (B(Ay - X\beta))' \Omega^{-3} H_q H_p B(Ay - X\beta) \quad (34)$$

$$\frac{\partial^2 L}{\partial \alpha_p^2} = (1/2) \text{tr} (\Omega^{-2} H_p^2) - (B(Ay - X\beta))' \Omega^{-3} H_p^2 B(Ay - X\beta) \quad (35)$$

2.5 Information matrix

$$\mathcal{I}_{\beta\beta'} = (BX)' \Omega^{-1} BX \quad (36)$$

$$\mathcal{I}_{\beta\rho} = (BX)' \Omega^{-1} BW_1 A^{-1} X \beta \quad (37)$$

$$\mathcal{I}_{\beta\lambda} = 0 \quad (38)$$

$$\mathcal{I}_{\beta\alpha'} = 0 \quad (39)$$

$$\begin{aligned} \mathcal{I}_{\rho\rho} = & \text{tr} (W_1 A^{-1})^2 + \text{tr} \Omega (BW_1 A^{-1} B^{-1})' \Omega^{-1} (BW_1 A^{-1} B^{-1}) \\ & + (BW_1 A^{-1} X \beta)' \Omega^{-1} (BW_1 A^{-1} X \beta) \end{aligned} \quad (40)$$

$$\mathcal{I}_{\rho\lambda} = \text{tr} (W_2 B^{-1})' \Omega^{-1} BW_1 A^{-1} B^{-1} \Omega + \text{tr} W_2 W_1 A^{-1} B^{-1} \quad (41)$$

$$\mathcal{I}_{\rho\alpha_p} = \text{tr} \Omega^{-1} H_p BW_1 A^{-1} B^{-1} \quad (42)$$

$$\mathcal{I}_{\lambda\lambda} = \text{tr} (W_2 B^{-1})^2 + \text{tr} \Omega (W_2 B^{-1})' \Omega^{-1} W_2 B^{-1} \quad (43)$$

$$\mathcal{I}_{\lambda\alpha_p} = \text{tr} \Omega^{-1} H_p W_2 B^{-1} \quad (44)$$

$$\mathcal{I}_{\alpha_p \alpha_q} = (1/2) \text{tr} \Omega^{-2} H_p H_q \quad (45)$$

2.6 Estimation methods for spatial autoregressive models

2.6.1 Maximum likelihood estimation for homoskedastic models

The maximum likelihood estimation (MLE) is a maximization problem regarding the likelihood function (Equation (6)). With the following assumptions, the model becomes homoskedastic.

$$\text{Assumption : } h = \sigma^2 \quad (46)$$

$$\text{Consequences : } \frac{\partial h}{\partial \sigma^2} = 1, H_1 = H_{\sigma^2} = \frac{\partial \Omega}{\partial \sigma^2} = I, H_{pq} = 0, \quad (47)$$

$$\Omega = \sigma^2 I, \Omega^{-1} = \frac{1}{\sigma^2} I, \Omega^{-2} = \frac{1}{(\sigma^2)^2} I, \quad (48)$$

$$\text{tr} \Omega^{-1} = \frac{n}{\sigma^2}, \text{tr} \Omega^{-2} = \frac{n}{(\sigma^2)^2} \quad (49)$$

MLE is commonly used for this homoskedastic case and is more efficient than other estimation methods. However, the results of MLE may be doubtful if the assumption regarding the disturbance terms is not valid. If the estimated residuals, $\boldsymbol{\nu}$, do not obey the standard normal distribution $N(0, 1)$, the disturbance terms may be heteroskedastic. In such cases, less efficient IV-based methods, ‘robust 2-SLS’ estimation for example, must be applied.

2.6.2 Spatial two-stage least squares method

Weaker assumptions than MLE are necessary in some other estimation methods. For an example, the spatial two-stage least squares method ('S-2SLS') may be applied to the spatial effect model defined below (Equation (62)). In the case of heteroskedastic disturbances, a heteroskedastic-robust version of the spatial two-stage least squares method ('S-2SLS-Robust') may be used instead of S-2SLS.⁴ This method was used in some empirical analyses, including Kim et al. (2003) and Sedgley et al. (2008).

2.6.3 Maximum likelihood estimation of heteroskedastic models

There is no essential difference between the likelihood functions of homoskedastic and heteroskedastic autoregressive models. The problem is that no implementation of maximum likelihood estimation for heteroskedastic models exists yet. Hereafter, we develop a new program for maximum likelihood estimation which facilitates efficient estimators when disturbance terms might be heteroskedastic.⁵

2.7 Modeling heteroskedasticity

Parametric formulations of heteroskedasticity have been categorized (Griffiths, 2003). The following equation is a linear formulation.

$$\Omega_{ii} = \alpha_0 + \alpha_1 z_{1i} + \cdots + \alpha_S z_{Si} \quad (50)$$

Griffiths (2003, p.87) wrote "if (z_{1i}, \dots, z_{Si}) are non-overlapping dummy variables, the specification describes a partition of the sample into $(S+1)$ subsamples, each one with a different error variance". There are alternative specifications which ensure that $\Omega_{ii} > 0$:

$$\Omega_{ii} = (\alpha_0 + \alpha_1 z_{1i} + \cdots + \alpha_S z_{Si})^2 \quad (51)$$

$$\Omega_{ii} = \exp(\alpha_0 + \alpha_1 z_{1i} + \cdots + \alpha_S z_{Si}) \quad (52)$$

⁴ See Anselin and Bera (1998, p.259).

⁵ The authors wrote the program using *Mathematica 5.2*, and will provide it upon request.

3 Specification of heteroskedastic spatial autoregressive model

In this paper, we assume the following linear formulation of heteroskedasticity.

$$\mathbf{y} = \rho \mathbf{W}_1 \mathbf{y} + \mathbf{X} \boldsymbol{\beta} + \boldsymbol{\epsilon} \quad (53)$$

$$\boldsymbol{\epsilon} = \lambda \mathbf{W}_2 \boldsymbol{\epsilon} + \boldsymbol{\mu} \quad (54)$$

$$\boldsymbol{\mu} \sim N(\mathbf{0}, \boldsymbol{\Omega}) \quad (55)$$

$$\Omega_{ii} = \mathbf{z}_i \boldsymbol{\alpha} > 0, \quad \Omega_{ij} = 0 \text{ for } i \neq j \quad (56)$$

$$\frac{\partial h}{\partial \alpha_p} = z_{ip}, \quad (57)$$

$$\mathbf{H}_p = \frac{\partial \boldsymbol{\Omega}}{\partial \alpha_p} = \text{diag}(z_{1p}, \dots, z_{np}) \quad (58)$$

$\boldsymbol{\alpha}$ is a vector of the heteroskedasticity parameters. \mathbf{z}_i is a vector of variables which regulates the variance of the disturbance term for the i th sample. In our experiments we let $\mathbf{z}_i = (1, z_i)$, where the scalar variable z_i is a dummy variable. We assume the values of z_i are known as are the explaining variables in \mathbf{X} . $\text{diag}(a_1, \dots, a_n)$ is a diagonal matrix with diagonal elements a_1, \dots, a_n .

3.1 Details of the optimization in maximum likelihood estimation

$$L(\boldsymbol{\beta}, \rho, \lambda, \boldsymbol{\alpha}) = -(n/2) \ln \pi - (1/2) \ln |\boldsymbol{\Omega}| + \ln |\mathbf{A}| + \ln |\mathbf{B}| - (1/2) \boldsymbol{\nu}' \boldsymbol{\nu} \quad (59)$$

$$\boldsymbol{\nu} = \boldsymbol{\Omega}^{-1/2} \mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}), \quad \mathbf{A} = \mathbf{I} - \rho \mathbf{W}_1, \quad \mathbf{B} = \mathbf{I} - \lambda \mathbf{W}_2$$

1. The solution for $\boldsymbol{\beta}$ is derived from the first order conditions.

$$\boldsymbol{\beta}^*(\rho, \lambda, \boldsymbol{\alpha}) = (\mathbf{X}' \mathbf{B}' \boldsymbol{\Omega}^{-1} \mathbf{B} \mathbf{X})^{-1} \mathbf{X}' \mathbf{B}' \boldsymbol{\Omega}^{-1} \mathbf{B} \mathbf{A} \mathbf{y} \quad (60)$$

2. For given values of ρ, λ , it is possible to solve P first order conditions ($\frac{\partial L}{\partial \alpha_p} = 0$) for α_p , and obtain optimal values $\boldsymbol{\alpha}^*(\rho, \lambda)$.
3. The last step is a maximization problem for the concentrated likelihood function $\bar{L}(\rho, \lambda)$. The initial values are $\rho_0 = \lambda_0 = 0$.

$$\max_{\rho, \lambda} \bar{L}(\rho, \lambda) = L(\boldsymbol{\beta}^*(\rho, \lambda, \boldsymbol{\alpha}^*(\rho, \lambda)), \rho, \lambda, \boldsymbol{\alpha}^*(\rho, \lambda)) \quad (61)$$

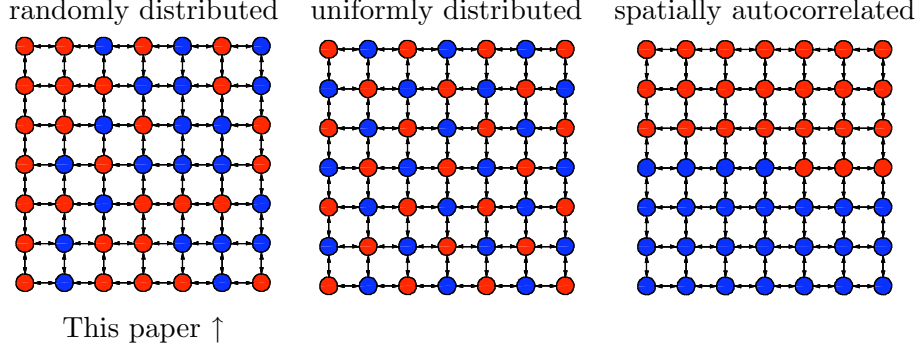


Figure 1: Specification of heteroskedasticity (colors show two subgroups).

4 Monte Carlo Simulations

In this section, we present our experimental design and the results of Monte Carlo simulations intended to investigate the efficiency of maximum likelihood estimation for heteroskedastic spatial autoregressive models. A new program was developed which can compute the maximum likelihood estimation of a general model represented by equations (53) through (56). In this paper, we focus on the spatial effect model (SEM) for which we assume $\lambda = 0$.

4.1 Design

The following assumptions were made in the data generating process (DGP):

- The functional form of the DGP is a spatial autoregressive model with a normal but heteroskedastic disturbance term. As previously mentioned, we assume the functional form of heteroskedasticity is linear. The coefficient vector α is 1×2 . One element is a constant term fixed to unity ($\alpha_0 = 1$). We examine several values of α_1 . The diagonal elements of the covariance matrix are $\Omega_{ii} = (1, z_i) \cdot (\alpha_0, \alpha_1)' = \alpha_0 + \alpha_1 z_i$.

$$\mathbf{y} = \rho \mathbf{W} \mathbf{y} + \mathbf{X} \beta + \epsilon, \quad \epsilon \sim N(\mathbf{0}, \Omega) \quad (62)$$

$$\Omega_{ii} = \alpha_0 + \alpha_1 z_i > 0, \quad \Omega_{ij} = 0 \text{ for } i \neq j \quad (63)$$

In this paper, we consider the case where the disturbance is homoskedastic within subgroups. The following equations and Figure 1 show the ‘groupwise heteroskedasticity of errors’.

$$z_i = 0 \text{ or } 1 \quad (64)$$

$$\therefore \Omega_{ii} = \alpha_0 \text{ or } \alpha_0 + \alpha_1 \quad (65)$$

- The spatial structure is represented by a first-order rook contiguity spatial weight matrix. Two sample sizes are considered: $n = 100$ (10×10) and 400 (20×20).
- There are three explaining variables, $x_{ki} \sim N(0, 1)$ for $k = 1, 2, 3$, and a constant term $x_{0i} = 1$ for the i th observation. Their coefficients are $\beta_k = 1$.
- We performed about 100 replications ($r \approx 100$) for each value of ρ and α_1 . ρ takes values between -0.5 and 0.5 in increments of 0.25. α_1 takes values between 0 and 12 in increments of 4. ⁶When $\alpha_1 = 0$, the DGP is homoskedastic. When $\rho = 0$, there are no spatial effects in the DGP.

Estimations are made by the following four methods.

1. Spatial 2-stage least squares (S-2SLS): The instrumental variables are $[\mathbf{X} \ \mathbf{W}_1 \mathbf{X}]$. We use only linearly independent rows to create the matrix.
2. Heteroskedastic robust version of S-2SLS (S-2SLS-Robust): The same instrumental variables as S-2SLS are used. The residuals of S-2SLS are used to calculate the covariance matrix.
3. Maximum likelihood estimation for homoskedastic model (ML-Homo): We use misspecified disturbance terms, $\epsilon_i \sim N(0, \sigma^2)$. The scalar parameter σ^2 takes the place of α_0 and α_1 .
4. Maximum likelihood estimation for heteroskedastic model (ML-Hetero): The model given by Equations (62) and (63). ⁷ There are four parameters to be estimated, β_k , ρ , α_0 , and α_1 . The total number of parameters is $4 + 1 + 0 + 2 = 7$.

A symbol with an upper-bar ($\bar{\theta}_k$) denotes the true value of a parameter (θ_k).

4.2 Results

4.2.1 Example estimation result

Table 1 shows a typical estimation result for $n = 100$. In this example, there is a positive spatial autocorrelation in the explained variable ($\bar{\rho} = 0.5$) and the variances of half of the samples are larger than the other half ($\Omega_{ii} = \bar{\alpha}_0 + \bar{\alpha}_1 = 13$ or $\Omega_{ii} = \bar{\alpha}_0 = 1$). The row of ‘BS/JB Test’ shows p-values of Bowman and Shenton (1975) (Jarque and Bera (1987)) normality test. In this example, the following points can be observed.

1. For all methods, there is not a large difference between the true value and the estimate.

⁶ The exact number of replications is approximate because some optimizations fail.

⁷ \mathbf{H}_p in Equation (58) may be expressed as follows,

$$\mathbf{H}_0 = \text{diag}(1, \dots, 1) = \mathbf{I}, \text{ and } \mathbf{H}_1 = \text{diag}(z_1, \dots, z_n). \quad (66)$$

Table 1: A typical estimation result ($n = 100, \bar{\rho} = 0.5, \bar{\alpha}_1 = 12$)

Parameter	True value	S-2SLS	S-2SLS- Robust	ML- Homo	ML- Hetero
ρ	0.5	0.429 (0.261)	0.443 (0.202)*	0.476 (0.0975)***	0.573 (0.0506)***
β_0	1	0.988 (0.597)	0.952 (0.464)*	0.892 (0.347)**	1.13 (0.212)***
β_1	1	0.858 (0.297)***	0.866 (0.288)***	0.859 (0.294)***	0.91 (0.13)***
β_2	1	1.46 (0.249)***	1.45 (0.236)***	1.45 (0.241)***	1.01 (0.115)***
β_3	1	0.497 (0.306)	0.516 (0.28)	0.48 (0.29)	0.913 (0.142)***
σ^2				7.35 (1.06)***	
α_0	1				0.778 (0.161)***
α_1	12				13.2 (2.73)***
R^2 , Likelihood		0.456	0.459	-210.	-176.
BS/JB Test		0.00402	0.0028	0.00268	0.583

Notes: Standard errors are in parentheses. *, ** and *** indicate significance levels of 10%, 5% and 1%, respectively.

2. The standard errors of the estimates become smaller if the heteroskedasticity is considered. Please compare S-2SLS-Robust to S-2SLS, or ML-Hetero to ML-Homo.
3. α_1 is the parameter of heteroskedasticity in ML-Hetero. It is significantly non-zero and the null hypothesis $\alpha_1 = 0$ is rejected in the likelihood ratio test between ML-Homo and ML-Hetero.

4.2.2 RMSE of estimators

In this section, we discuss the results of Monte Carlo experiments. The differences in the root mean square error (RMSE) of the estimates of the four methods reveal their efficiencies.⁸ Tables 2 through 41 show the distributions of the parameter estimates.

Each panel in Figure 2 shows important RMSE results. For example, Figure 2(a) shows the relationship between true values of $\bar{\alpha}_1$ and estimates of ρ when $\bar{\rho} = 0.5$. Several points may be observed.

⁸ The definition of the root mean square error (RMSE) we used is

$$\text{RMSE} = \sqrt{\frac{1}{r} \sum_{j=1}^r (\bar{\theta}_k - \tilde{\theta}_{kj})^2} \quad (67)$$

where r is the number of replications, $\bar{\theta}_k$ is the true value of the k th parameter, and $\tilde{\theta}_{kj}$ is an estimate for j th replication.

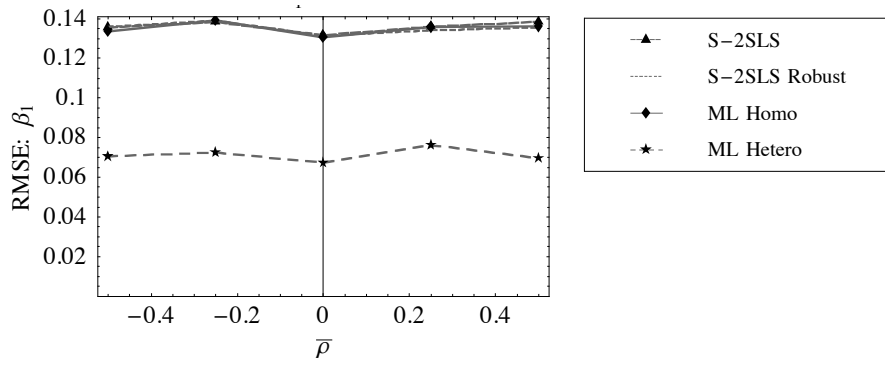
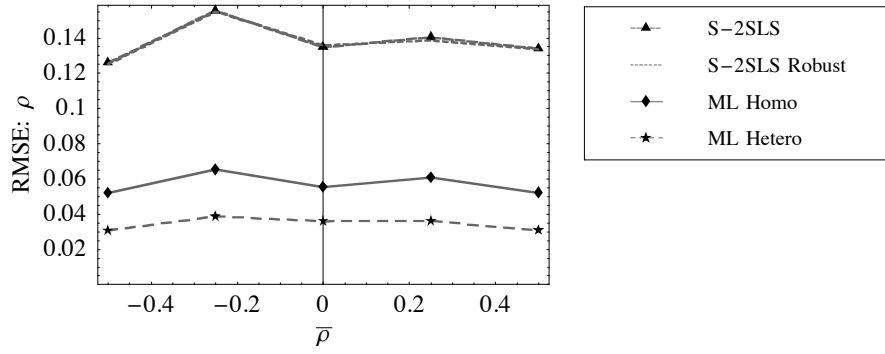
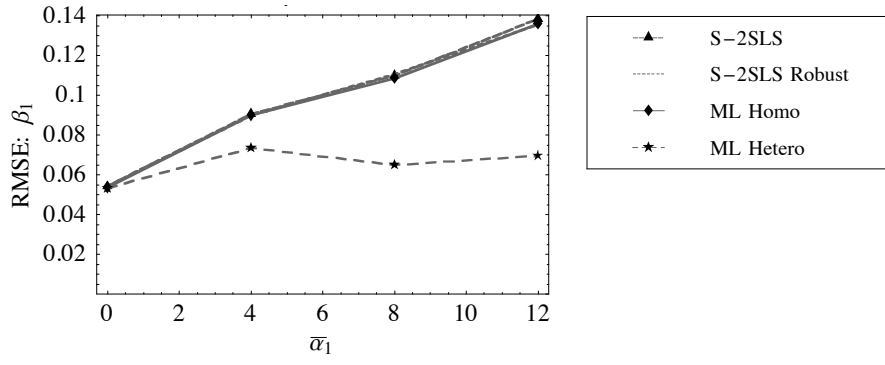
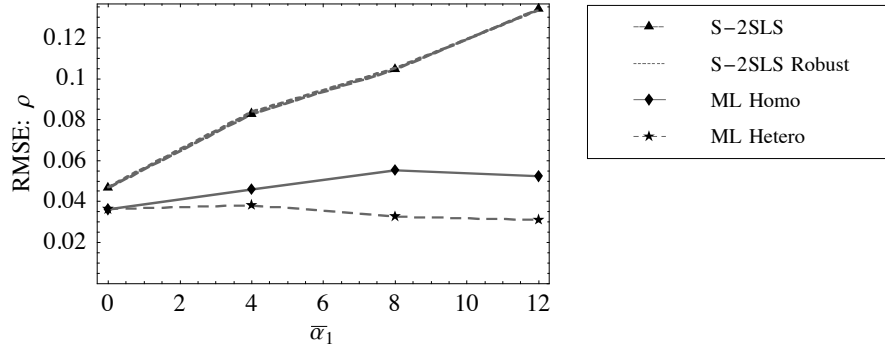


Figure 2: RMSE of estimates for ρ and β_1 ($n = 400$)

1. Estimates of ρ are efficient in two maximum likelihood estimations (Figures 2(a) and 2(c)).
2. It is clear that estimates of β_1 are efficient in ML-Hetero (Figures 2(b) and 2(d)).
3. There is not a big difference between S-2SLS-Robust and S-2SLS.

5 Conclusion

Likelihood functions of spatial autoregressive models with normal but heteroskedastic disturbances have been already derived [Anselin (1988, ch.6)], but there was no implementation of maximum likelihood estimation for these likelihood functions in general with heteroskedastic disturbances. This was the reason why less efficient IV-based methods, ‘robust 2-SLS’ estimation for example, had to be applied when disturbance terms might be heteroskedastic. In this paper, a new computer program for maximum likelihood estimation was presented and the efficiency of this estimator for spatial models with heteroskedastic disturbances was confirmed using Monte Carlo simulations.

We can estimate models with spatially auto-correlated disturbance terms with the program developed in this paper. Additional simulations will show efficiencies in such cases. Experimental comparisons between our method and recently proposed methods (Kelejian and Prucha, 2007, 2010; Lin and Lee, 2010) must also be important.

Table 2: Distribution of estimates: $n = 100, \bar{\rho} = -0.5, \bar{\alpha}_1 = 0$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.482	0.0968	-0.487	0.098	-0.481	0.0992	-0.476	0.101	-0.481	0.0681	-0.484	0.0705	-0.481	0.068	-0.485	0.0702
β_0	0.982	0.132	0.977	0.132	0.98	0.133	0.979	0.134	0.981	0.108	0.968	0.109	0.979	0.11	0.96	0.112
β_1	0.997	0.0911	0.997	0.0907	0.997	0.0923	0.993	0.0919	0.996	0.089	1.	0.0886	0.994	0.0918	0.993	0.0915
β_2	1.02	0.0942	1.02	0.0949	1.02	0.0958	1.02	0.0968	1.01	0.0929	1.02	0.0936	1.01	0.0941	1.01	0.0947
β_3	1.01	0.0992	1.01	0.099	1.01	0.102	1.01	0.102	1.01	0.0978	1.01	0.0975	1.01	0.0997	1.02	0.0995
σ^2									0.963	0.146	0.959	0.15				
α_0													0.953	0.203	0.932	0.207
α_1													0.0233	0.311	-0.0205	0.31
R2,LH	0.808	0.0417	0.817		0.807	0.0422	0.817		-108.	7.69	-109.		-108.	7.67	-108.	
BS/JB	0.569	0.277	0.581		0.563	0.276	0.6		0.573	0.282	0.611		0.561	0.275	0.545	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 3: Distribution of estimates: $n = 100, \bar{\rho} = -0.25, \bar{\alpha}_1 = 0$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.249	0.0985	-0.254	0.098	-0.249	0.0987	-0.251	0.0982	-0.256	0.077	-0.258	0.0768	-0.257	0.0799	-0.257	0.0798
β_0	1.	0.127	0.998	0.126	1.	0.129	1.	0.128	1.01	0.115	1.01	0.115	1.01	0.114	1.01	0.114
β_1	0.997	0.105	1.01	0.105	0.997	0.106	1.01	0.106	0.998	0.105	1.01	0.104	0.998	0.107	1.01	0.107
β_2	1.01	0.112	1.01	0.112	1.01	0.116	1.	0.115	1.01	0.11	1.01	0.11	1.01	0.112	1.02	0.111
β_3	0.989	0.11	0.987	0.11	0.989	0.11	0.991	0.11	0.989	0.11	0.988	0.11	0.99	0.113	0.988	0.113
σ^2									0.96	0.135	0.946	0.14				
α_0													0.939	0.198	0.904	0.206
α_1													0.0466	0.324	0.0591	0.326
R2,LH	0.767	0.0402	0.772		0.767	0.0399	0.773		-106.	6.95	-106.		-105.	6.95	-105.	
BS/JB	0.54	0.286	0.545		0.541	0.285	0.564		0.541	0.277	0.542		0.56	0.273	0.571	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 4: Distribution of estimates: $n = 100, \bar{\rho} = 0, \bar{\alpha}_1 = 0$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.0192	0.105	-0.0207	0.107	-0.0212	0.106	-0.0262	0.108	-0.0145	0.0886	-0.0152	0.0893	-0.0162	0.0876	-0.0157	0.0887
β_0	1.02	0.163	0.991	0.162	1.02	0.168	0.985	0.168	1.01	0.138	0.991	0.138	1.01	0.137	1.01	0.137
β_1	0.993	0.106	0.998	0.106	0.994	0.108	1.01	0.108	0.994	0.107	1.	0.107	0.996	0.106	0.997	0.106
β_2	1.01	0.101	1.01	0.1	1.	0.102	1.01	0.101	1.01	0.1	1.01	0.0998	1.	0.103	1.01	0.102
β_3	0.991	0.0914	0.992	0.0914	0.991	0.0919	0.983	0.0919	0.992	0.0915	0.996	0.0913	0.993	0.0923	0.988	0.0921
σ^2									0.937	0.148	0.929	0.16				
α_0													0.974	0.197	0.958	0.197
α_1									-103.	7.81	-104.		-0.0691	0.277	-0.0535	0.285
R2,LH	0.761	0.0419	0.762		0.761	0.0419	0.762						-103.	7.72	-103.	
BS/JB	0.557	0.269	0.585		0.556	0.275	0.592		0.556	0.266	0.583		0.56	0.234	0.536	

SD: Standard deviation, RMSE: Root mean square error. Results from 99 successfully optimized replications are reported here.

Table 5: Distribution of estimates: $n = 100, \bar{\rho} = 0.25, \bar{\alpha}_1 = 0$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.249	0.112	0.231	0.111	0.249	0.116	0.249	0.115	0.236	0.0773	0.233	0.0782	0.236	0.0774	0.23	0.0783
β_0	1.	0.182	0.992	0.181	0.999	0.186	0.987	0.185	1.02	0.125	1.02	0.126	1.02	0.131	1.02	0.132
β_1	0.997	0.11	1.01	0.11	1.	0.115	1.01	0.114	0.999	0.112	1.02	0.111	1.	0.117	1.02	0.116
β_2	0.988	0.115	0.983	0.115	0.989	0.115	0.987	0.115	0.99	0.112	0.983	0.112	0.99	0.112	0.98	0.112
β_3	1.	0.116	0.995	0.116	1.	0.116	1.	0.116	1.	0.114	0.995	0.114	1.	0.116	0.993	0.116
σ^2									0.976	0.139	0.956	0.141				
α_0													0.954	0.219	0.928	0.222
α_1									-106.	7.18	-106.		0.0477	0.307	0.0479	0.309
R2,LH	0.762	0.0429	0.767		0.761	0.0428	0.766						-106.	7.25	-105.	
BS/JB	0.529	0.282	0.525		0.529	0.284	0.508		0.533	0.289	0.531		0.546	0.279	0.542	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 6: Distribution of estimates: $n = 100, \bar{\rho} = 0.5, \bar{\alpha}_1 = 0$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.518	0.1	0.528	0.101	0.518	0.099	0.529	0.1	0.494	0.0755	0.501	0.0753	0.493	0.0758	0.499	0.0757
β_0	0.969	0.223	0.953	0.224	0.967	0.221	0.948	0.223	1.01	0.169	1.	0.169	1.01	0.171	0.997	0.171
β_1	0.994	0.0958	1.	0.0955	0.992	0.102	0.999	0.101	1.	0.0944	1.01	0.0939	1.	0.0953	1.01	0.0948
β_2	0.984	0.105	0.972	0.105	0.983	0.108	0.981	0.109	0.987	0.104	0.979	0.104	0.987	0.106	0.984	0.106
β_3	0.992	0.106	0.981	0.106	0.996	0.108	0.991	0.107	0.997	0.106	0.984	0.106	0.996	0.108	0.98	0.107
σ^2									0.94	0.131	0.935	0.144				
α_0													0.927	0.181	0.887	0.194
α_1									-108.	6.72	-107.		0.0289	0.278	0.0256	0.278
R2,LH	0.811	0.0376	0.816		0.811	0.0375	0.816						-107.	6.64	-107.	
BS/JB	0.542	0.272	0.564		0.547	0.276	0.562		0.529	0.278	0.545		0.532	0.272	0.526	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 7: Distribution of estimates: $n = 100, \bar{\rho} = -0.5, \bar{\alpha}_1 = 4$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.523	0.162	-0.523	0.163	-0.525	0.159	-0.544	0.16	-0.503	0.0933	-0.512	0.0929	-0.509	0.0734	-0.514	0.0737
β_0	1.05	0.205	1.04	0.209	1.05	0.203	1.04	0.208	1.03	0.166	1.04	0.168	1.01	0.13	1.01	0.129
β_1	0.994	0.179	0.993	0.178	0.991	0.184	0.979	0.183	1.	0.18	1.	0.179	1.	0.129	1.01	0.128
β_2	0.979	0.16	0.994	0.161	0.978	0.154	0.99	0.155	0.99	0.161	0.998	0.161	0.99	0.131	0.999	0.13
β_3	0.986	0.183	0.988	0.183	0.982	0.182	0.966	0.182	0.993	0.181	0.983	0.18	0.995	0.139	0.993	0.139
σ^2									2.81	0.463	2.77	1.86				
α_0													0.927	0.182	0.925	0.195
α_1									-162.	8.04	-162.		3.94	1.05	3.86	1.05
R2,LH	0.628	0.0768	0.637		0.627	0.0761	0.639						-148.	6.32	-147.	
BS/JB	0.3	0.305	0.219		0.294	0.299	0.241		0.293	0.31	0.195		0.573	0.26	0.565	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 8: Distribution of estimates: $n = 100, \bar{\rho} = -0.25, \bar{\alpha}_1 = 4$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.259	0.204	-0.241	0.203	-0.26	0.203	-0.26	0.202	-0.252	0.104	-0.264	0.104	-0.249	0.0836	-0.247	0.0832
β_0	1.03	0.245	1.01	0.245	1.03	0.241	1.01	0.242	1.02	0.178	1.01	0.178	1.	0.132	0.998	0.132
β_1	0.974	0.181	0.997	0.182	0.974	0.178	0.99	0.179	0.982	0.176	1.01	0.176	0.996	0.13	0.989	0.13
β_2	0.98	0.169	0.993	0.17	0.987	0.171	0.994	0.17	0.988	0.164	0.995	0.164	0.986	0.141	0.985	0.141
β_3	1.02	0.162	1.02	0.162	1.02	0.166	1.03	0.166	1.03	0.161	1.03	0.162	1.	0.139	0.991	0.138
σ^2									2.84	0.615	2.77	1.94				
α_0													0.929	0.225	0.907	0.235
α_1									-159.	11.1	-159.		3.94	1.24	3.89	1.23
R2,LH	0.542	0.0786	0.548		0.541	0.0795	0.548						-145.	9.14	-146.	
BS/JB	0.222	0.26	0.105		0.211	0.255	0.108		0.199	0.24	0.0834		0.553	0.292	0.566	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 9: Distribution of estimates: $n = 100, \bar{\rho} = 0., \bar{\alpha}_1 = 4$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.0209	0.194	0.0451	0.194	0.0244	0.2	0.0348	0.201	0.00167	0.107	-0.0143	0.106	0.00257	0.0889	0.00553	0.0885
β_0	0.981	0.261	0.979	0.26	0.984	0.262	0.99	0.261	1.	0.186	0.987	0.185	0.995	0.143	1.	0.142
β_1	0.993	0.17	1.	0.17	0.992	0.171	1.	0.17	0.998	0.169	1.01	0.168	0.997	0.124	1.	0.123
β_2	0.977	0.192	0.975	0.193	0.973	0.197	0.951	0.198	0.98	0.195	0.978	0.195	0.971	0.159	0.965	0.161
β_3	0.978	0.178	0.971	0.179	0.979	0.177	0.975	0.177	0.982	0.179	0.973	0.179	0.991	0.145	0.994	0.145
σ^2									2.75	0.463	2.71	1.81				
α_0													0.93	0.188	0.913	0.2
α_1									-157.	8.43	-157.		3.79	1.01	3.75	1.02
R2,LH	0.516	0.0738	0.527		0.515	0.0749	0.526						-143.	7.23	-143.	
BS/JB	0.256	0.315	0.0936		0.245	0.311	0.0778		0.238	0.309	0.0764		0.579	0.266	0.631	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 10: Distribution of estimates: $n = 100, \bar{\rho} = 0.25, \bar{\alpha}_1 = 4$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.228	0.238	0.238	0.238	0.233	0.24	0.236	0.239	0.228	0.101	0.243	0.103	0.229	0.0916	0.234	0.0935
β_0	1.04	0.388	0.993	0.388	1.04	0.398	0.966	0.398	1.04	0.224	1.04	0.227	1.04	0.175	1.02	0.178
β_1	0.967	0.193	1.	0.195	0.964	0.195	1.01	0.198	0.978	0.182	1.01	0.182	0.974	0.138	0.968	0.14
β_2	0.963	0.204	0.965	0.207	0.964	0.205	0.977	0.207	0.971	0.2	0.966	0.202	0.983	0.144	0.971	0.145
β_3	0.984	0.182	1.01	0.181	0.987	0.186	1.01	0.185	0.993	0.183	1.02	0.182	0.997	0.16	1.01	0.16
σ^2									2.87	0.595	2.82	1.96				
α_0													0.931	0.207	0.904	0.217
α_1	0.512	0.102	0.529		0.512	0.103	0.529		-160.	10.3	-160.		3.93	1.05	3.92	1.05
R2,LH	0.289	0.319	0.166		0.28	0.32	0.132		0.28	0.328	0.11		-145.	9.17	-145.	
BS/JB													0.588	0.266	0.592	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 11: Distribution of estimates: $n = 100, \bar{\rho} = 0.5, \bar{\alpha}_1 = 4$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.489	0.141	0.474	0.141	0.493	0.141	0.486	0.14	0.468	0.0896	0.475	0.0948	0.483	0.0663	0.48	0.068
β_0	1.01	0.349	1.01	0.347	1.01	0.352	1.	0.35	1.06	0.219	1.04	0.226	1.04	0.168	1.02	0.173
β_1	1.	0.167	0.995	0.166	1.	0.175	0.998	0.174	1.01	0.169	1.01	0.169	1.02	0.123	1.02	0.123
β_2	1.02	0.178	1.02	0.178	1.01	0.185	1.02	0.184	1.03	0.172	1.04	0.174	0.997	0.139	1.01	0.138
β_3	0.992	0.181	0.998	0.18	0.993	0.177	1.	0.176	0.999	0.178	0.995	0.177	0.999	0.124	1.01	0.123
σ^2									2.86	0.541	2.82	1.94				
α_0													0.937	0.201	0.926	0.209
α_1	0.616	0.0779	0.626		0.616	0.0774	0.624		-162.	9.5	-163.		3.99	1.12	3.9	1.11
R2,LH	0.204	0.258	0.0633		0.2	0.255	0.0663		0.203	0.259	0.0633		-148.	7.78	-148.	
BS/JB													0.519	0.264	0.538	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 12: Distribution of estimates: $n = 100, \bar{\rho} = -0.5, \bar{\alpha}_1 = 8$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.505	0.241	-0.521	0.24	-0.502	0.24	-0.527	0.239	-0.5	0.085	-0.504	0.0846	-0.503	0.0645	-0.502	0.0643
β_0	0.956	0.273	0.946	0.276	0.953	0.265	0.938	0.268	0.953	0.229	0.957	0.233	0.979	0.14	0.979	0.14
β_1	0.994	0.234	1.02	0.233	0.994	0.221	1.	0.22	1.	0.22	0.996	0.219	1.02	0.135	1.03	0.136
β_2	1.	0.247	1.07	0.246	1.02	0.226	1.05	0.225	1.02	0.233	1.07	0.233	1.03	0.14	1.04	0.143
β_3	0.959	0.217	0.961	0.22	0.963	0.211	0.966	0.214	0.974	0.215	0.965	0.215	1.01	0.153	1.01	0.153
σ^2									4.82	1.01	4.75	3.95				
α_0													0.891	0.179	0.874	0.209
α_1													8.19	1.96	7.97	1.96
R2,LH	0.521	0.105	0.526		0.519	0.106	0.53		-189.	9.87	-189.		-163.	8.72	-163.	
BS/JB	0.114	0.204	0.0188		0.109	0.201	0.011		0.102	0.194	0.00853		0.553	0.224	0.588	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 13: Distribution of estimates: $n = 100, \bar{\rho} = -0.25, \bar{\alpha}_1 = 8$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.226	0.278	-0.238	0.278	-0.23	0.277	-0.249	0.276	-0.244	0.108	-0.26	0.107	-0.25	0.075	-0.249	0.0746
β_0	0.95	0.291	0.933	0.293	0.96	0.288	0.955	0.29	0.962	0.197	0.968	0.199	1.01	0.137	1.01	0.137
β_1	0.942	0.216	0.944	0.223	0.939	0.213	0.941	0.22	0.956	0.215	0.952	0.219	0.985	0.13	0.982	0.131
β_2	0.968	0.226	0.959	0.227	0.973	0.223	0.964	0.223	0.988	0.216	0.97	0.215	1.01	0.137	1.	0.137
β_3	1.04	0.236	1.03	0.239	1.04	0.238	1.04	0.24	1.05	0.235	1.05	0.239	1.02	0.141	1.	0.141
σ^2									4.69	0.887	4.63	3.8				
α_0													0.906	0.208	0.892	0.228
α_1													7.94	1.8	7.82	1.79
R2,LH	0.414	0.089	0.43		0.413	0.0887	0.427		-185.	9.52	-185.		-159.	8.91	-160.	
BS/JB	0.15	0.246	0.0134		0.14	0.234	0.00864		0.126	0.228	0.00647		0.51	0.273	0.5	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 14: Distribution of estimates: $n = 100, \bar{\rho} = 0., \bar{\alpha}_1 = 8.$

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.00435	0.279	0.00622	0.277	-0.00206	0.275	-0.0228	0.273	-0.00786	0.112	-0.0252	0.112	-0.00645	0.09	-0.00332	0.0898
β_0	0.945	0.356	0.908	0.359	0.956	0.355	0.918	0.356	0.955	0.238	0.947	0.241	0.988	0.142	0.985	0.141
β_1	1.02	0.258	1.05	0.257	1.02	0.256	1.06	0.255	1.03	0.254	1.07	0.255	0.995	0.154	1.01	0.154
β_2	0.973	0.231	0.971	0.231	0.979	0.228	0.971	0.227	0.991	0.222	0.975	0.221	0.996	0.157	1.01	0.156
β_3	1.01	0.237	0.983	0.236	1.	0.233	0.994	0.232	1.02	0.235	0.996	0.234	1.01	0.145	1.02	0.145
σ^2									4.88	0.998	4.82	4.01				
α_0													0.906	0.207	0.887	0.226
α_1	0.39	0.0948	0.388		0.39	0.0945	0.389		-186.	10.5	-186.		8.26	2.04	7.87	2.05
R2/LH	0.113	0.21	0.00843		0.109	0.205	0.00674		0.108	0.211	0.00554		-160.	8.6	-159.	
BS/JB													0.554	0.274	0.547	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 15: Distribution of estimates: $n = 100, \bar{\rho} = 0.25, \bar{\alpha}_1 = 8.$

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.243	0.284	0.255	0.282	0.249	0.262	0.271	0.261	0.232	0.112	0.233	0.113	0.24	0.0744	0.243	0.0748
β_0	0.973	0.399	0.97	0.398	0.968	0.359	0.942	0.359	0.993	0.249	0.964	0.248	0.984	0.144	0.958	0.144
β_1	0.998	0.239	0.981	0.238	0.989	0.229	0.977	0.228	1.01	0.237	0.985	0.236	1.01	0.144	1.01	0.144
β_2	0.956	0.228	0.942	0.231	0.961	0.218	0.951	0.22	0.974	0.225	0.951	0.226	1.01	0.149	0.991	0.149
β_3	0.96	0.264	0.946	0.266	0.953	0.261	0.943	0.264	0.981	0.258	0.968	0.258	0.987	0.168	0.979	0.167
σ^2									4.61	0.905	4.49	3.72				
α_0													0.912	0.199	0.902	0.217
α_1													7.83	1.76	7.55	1.76
R2/LH	0.409	0.103	0.418		0.41	0.103	0.414		-184.	9.86	-183.		-158.	9.29	-158.	
BS/JB	0.0822	0.161	0.00408		0.0774	0.153	0.00125		0.0812	0.181	0.00129		0.512	0.287	0.538	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 16: Distribution of estimates: $n = 100, \bar{\rho} = 0.5, \bar{\alpha}_1 = 8$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.484	0.24	0.508	0.239	0.472	0.248	0.495	0.248	0.463	0.108	0.458	0.114	0.478	0.0728	0.483	0.0758
β_0	0.999	0.503	0.934	0.501	1.03	0.53	0.932	0.528	1.05	0.269	1.05	0.273	1.03	0.164	1.03	0.166
β_1	0.979	0.227	0.965	0.227	0.982	0.224	0.964	0.224	0.996	0.221	0.975	0.22	1.01	0.12	1.01	0.119
β_2	0.997	0.209	0.998	0.208	1.	0.207	0.99	0.206	1.01	0.204	1.	0.204	1.02	0.148	1.01	0.148
β_3	0.977	0.28	0.983	0.279	0.986	0.273	0.979	0.272	0.993	0.278	0.987	0.276	1.01	0.144	1.01	0.144
σ^2									4.83	0.965	4.83	3.94				
α_0													0.907	0.203	0.896	0.222
α_1													8.19	1.86	8.15	1.86
R2,LH	0.502	0.113	0.521		0.5	0.115	0.522		-188.	10.5	-189.		-162.	9.97	-162.	
BS/JB	0.142	0.243	0.00887		0.138	0.239	0.00807		0.114	0.196	0.00574		0.574	0.255	0.612	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 17: Distribution of estimates: $n = 100, \bar{\rho} = -0.5, \bar{\alpha}_1 = 12$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.445	0.255	-0.462	0.26	-0.451	0.261	-0.462	0.264	-0.489	0.0946	-0.499	0.0948	-0.493	0.0553	-0.491	0.0554
β_0	0.927	0.315	0.921	0.322	0.925	0.32	0.915	0.327	0.954	0.28	0.964	0.282	0.974	0.135	0.959	0.137
β_1	1.	0.279	0.97	0.278	1.	0.275	0.995	0.274	1.01	0.28	0.987	0.279	1.01	0.16	1.03	0.16
β_2	0.968	0.243	0.936	0.244	0.981	0.242	0.952	0.241	0.978	0.225	0.96	0.225	0.992	0.151	0.988	0.151
β_3	1.01	0.273	0.996	0.272	1.01	0.282	1.01	0.281	1.01	0.265	1.	0.264	0.993	0.128	0.995	0.128
σ^2									6.71	1.43	6.5	5.89				
α_0													0.907	0.191	0.895	0.212
α_1													12.1	2.9	11.6	2.88
R2,LH	0.449	0.114	0.458		0.448	0.114	0.463		-205.	10.7	-205.		-172.	9.14	-172.	
BS/JB	0.0712	0.168	0.00446		0.0687	0.163	0.00277		0.0497	0.111	0.00313		0.565	0.262	0.581	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 18: Distribution of estimates: $n = 100, \bar{\rho} = -0.25, \bar{\alpha}_1 = 12$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.284	0.319	-0.301	0.319	-0.279	0.312	-0.293	0.311	-0.262	0.109	-0.262	0.109	-0.255	0.0778	-0.254	0.0776
β_0	1.05	0.38	1.03	0.382	1.05	0.371	1.02	0.372	1.03	0.27	1.01	0.27	0.984	0.151	0.984	0.151
β_1	0.971	0.26	0.968	0.261	0.972	0.262	0.961	0.262	0.991	0.242	0.971	0.241	0.982	0.148	0.967	0.148
β_2	0.998	0.283	1.04	0.281	1.	0.283	1.02	0.282	1.02	0.274	1.03	0.273	1.02	0.145	1.02	0.146
β_3	0.964	0.309	0.988	0.31	0.962	0.313	0.985	0.313	0.99	0.293	1.04	0.292	1.	0.162	1.01	0.161
σ^2									6.78	1.42	6.66	5.95				
α_0													0.898	0.179	0.885	0.205
α_1	0.349	0.0985	0.352		0.346	0.0986	0.352		-203.	10.9	-203.		12.1	2.83	11.5	2.81
R2,LH	0.0775	0.169	0.00374		0.073	0.161	0.00226		0.0508	0.118	0.000958		-170.	9.98	-171.	
BS/JB													0.609	0.242	0.601	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 19: Distribution of estimates: $n = 100, \bar{\rho} = 0., \bar{\alpha}_1 = 12$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.01	0.269	0.0364	0.268	-0.00897	0.271	0.0169	0.269	-0.0137	0.131	-0.0196	0.131	-0.000636	0.0798	-0.00291	0.0794
β_0	0.998	0.405	0.965	0.403	0.998	0.402	0.955	0.4	0.988	0.278	0.985	0.277	0.964	0.152	0.964	0.155
β_1	0.967	0.299	0.976	0.299	0.966	0.297	0.977	0.297	0.981	0.289	0.972	0.288	0.993	0.151	0.987	0.151
β_2	0.993	0.277	0.987	0.276	0.991	0.275	0.967	0.273	1.	0.265	0.998	0.264	0.988	0.124	0.98	0.124
β_3	0.99	0.251	0.988	0.25	0.984	0.252	0.971	0.251	1.01	0.25	1.03	0.249	0.992	0.13	0.99	0.13
σ^2									6.4	1.3	6.23	5.55				
α_0													0.864	0.216	0.833	0.254
α_1	0.326	0.103	0.324		0.325	0.103	0.32		-199.	10.2	-199.		11.5	2.58	11.3	2.61
R2,LH	0.0812	0.199	0.00108		0.0723	0.187	0.00105		0.0699	0.187	0.000795		-166.	9.82	-165.	
BS/JB													0.56	0.261	0.558	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 20: Distribution of estimates: $n = 100, \bar{\rho} = 0.25, \bar{\alpha}_1 = 12$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.329	0.336	0.31	0.344	0.321	0.311	0.308	0.317	0.225	0.117	0.238	0.119	0.232	0.0779	0.229	0.0796
β_0	0.893	0.497	0.882	0.506	0.903	0.486	0.894	0.493	1.03	0.275	1.01	0.275	1.02	0.151	1.02	0.151
β_1	0.927	0.282	0.939	0.29	0.941	0.285	0.914	0.29	0.959	0.268	0.946	0.27	0.988	0.159	0.986	0.159
β_2	1.01	0.29	1.04	0.289	1.02	0.294	1.01	0.293	1.02	0.28	1.01	0.279	1.02	0.138	1.01	0.139
β_3	0.953	0.282	0.972	0.284	0.955	0.281	0.956	0.283	0.977	0.258	0.982	0.258	0.997	0.152	1.	0.152
σ^2									6.63	1.44	6.5	5.81				
α_0													0.91	0.188	0.912	0.207
α_1									-202.	10.9	-202.		11.9	2.65	11.5	2.64
R2,LH	0.338	0.102	0.353		0.338	0.0992	0.344						-169.	9.95	-170.	
BS/JB	0.087	0.172	0.00432		0.0831	0.17	0.0053		0.0763	0.166	0.0026		0.57	0.268	0.553	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 21: Distribution of estimates: $n = 100, \bar{\rho} = 0.5, \bar{\alpha}_1 = 12$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.458	0.353	0.523	0.354	0.45	0.366	0.52	0.368	0.459	0.0959	0.473	0.104	0.479	0.0692	0.477	0.072
β_0	1.07	0.81	0.915	0.809	1.09	0.835	0.894	0.836	1.08	0.316	1.07	0.324	1.04	0.19	1.03	0.193
β_1	0.965	0.265	0.963	0.266	0.962	0.261	0.962	0.262	0.992	0.265	0.977	0.264	0.991	0.127	0.993	0.126
β_2	0.951	0.247	0.943	0.25	0.963	0.249	0.966	0.25	0.969	0.235	0.973	0.236	1.01	0.144	0.991	0.144
β_3	1.	0.291	1.03	0.29	1.01	0.29	1.02	0.289	1.03	0.291	1.05	0.291	0.996	0.137	0.993	0.136
σ^2									6.79	1.34	6.82	5.95				
α_0													0.896	0.183	0.88	0.209
α_1									-205.	10.3	-207.		12.1	2.42	12.1	2.41
R2,LH	0.434	0.124	0.455		0.43	0.127	0.453						-172.	9.72	-172.	
BS/JB	0.0861	0.169	0.0016		0.0805	0.164	0.00119		0.0709	0.163	0.000561		0.559	0.263	0.568	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 22: Distribution of estimates: $n = 400, \bar{\rho} = -0.5, \bar{\alpha}_1 = 0$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.489	0.0518	-0.485	0.0524	-0.488	0.0532	-0.483	0.054	-0.492	0.037	-0.49	0.0375	-0.493	0.0372	-0.491	0.0376
β_0	0.984	0.0611	0.98	0.0628	0.983	0.0616	0.981	0.0635	0.985	0.0517	0.991	0.0533	0.986	0.0515	0.993	0.053
β_1	1.	0.0497	1.	0.0492	1.	0.0483	1.	0.0479	1.	0.0497	1.	0.0493	1.	0.0505	1.	0.0501
β_2	0.999	0.058	1.	0.0575	1.	0.0577	1.	0.0572	0.999	0.0587	1.01	0.0582	0.999	0.0584	1.01	0.0579
β_3	1.01	0.0437	1.	0.0438	1.01	0.0443	1.	0.0444	1.01	0.0431	1.01	0.0432	1.01	0.0423	1.01	0.0426
σ^2									0.988	0.0573	0.982	0.058				
α_0													1.	0.104	0.999	0.103
α_1									-440.	11.7	-439.		-0.0244	0.181	-0.0342	0.181
R2,LH	0.801	0.0178	0.803		0.801	0.018	0.803						-439.	11.9	-439.	
BS/JB	0.506	0.277	0.486		0.507	0.278	0.488		0.51	0.275	0.462		0.526	0.285	0.49	

SD: Standard deviation, RMSE: Root mean square error. Results from 57 successfully optimized replications are reported here.

Table 23: Distribution of estimates: $n = 400, \bar{\rho} = -0.25, \bar{\alpha}_1 = 0$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.247	0.0535	-0.249	0.0532	-0.247	0.0543	-0.25	0.054	-0.245	0.0407	-0.245	0.0407	-0.245	0.0404	-0.245	0.0405
β_0	0.994	0.0573	0.987	0.0573	0.993	0.0585	0.991	0.0586	0.992	0.0491	0.984	0.0494	0.992	0.0488	0.987	0.0492
β_1	1.	0.0543	1.01	0.0542	1.	0.0538	1.	0.0537	1.01	0.0539	1.01	0.0539	1.01	0.0546	1.01	0.0546
β_2	0.996	0.0493	0.997	0.0492	0.996	0.049	0.996	0.0489	0.996	0.0493	0.998	0.0491	0.997	0.0491	0.997	0.049
β_3	1.	0.0476	1.01	0.0474	1.	0.0479	1.01	0.0478	1.	0.0475	1.01	0.0473	1.	0.0481	1.	0.048
σ^2									0.983	0.0691	0.977	0.0707				
α_0													0.981	0.101	0.963	0.102
α_1									-428.	13.9	-427.		0.00369	0.15	-0.0114	0.149
R2,LH	0.765	0.0204	0.767		0.765	0.0204	0.767						-428.	13.9	-427.	
BS/JB	0.507	0.274	0.479		0.507	0.275	0.469		0.509	0.274	0.447		0.517	0.266	0.471	

SD: Standard deviation, RMSE: Root mean square error. Results from 83 successfully optimized replications are reported here.

Table 26: Distribution of estimates: $n = 400, \bar{\rho} = 0.5, \bar{\alpha}_1 = 0$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.499	0.047	0.499	0.0466	0.499	0.0475	0.496	0.0471	0.487	0.0339	0.493	0.036	0.487	0.0341	0.493	0.0361
β_0	1.	0.107	0.982	0.106	1.	0.109	0.986	0.108	1.03	0.0827	1.02	0.0864	1.03	0.0833	1.02	0.0869
β_1	1.01	0.0541	1.	0.0542	1.01	0.0543	1.01	0.0544	1.01	0.0532	1.01	0.0536	1.01	0.053	1.	0.0533
β_2	1.	0.0479	1.01	0.0475	1.	0.0476	1.01	0.0472	1.	0.0473	1.01	0.047	1.	0.0472	1.	0.0469
β_3	0.998	0.0486	0.99	0.0483	0.999	0.0481	0.99	0.0477	1.	0.0489	0.995	0.0485	1.	0.049	0.994	0.0486
σ^2									0.988	0.0747	0.99	0.0751				
α_0													0.972	0.11	0.974	0.113
α_1									-440.	15.	-440.		0.0306	0.163	0.0711	0.165
R2,LH	0.805	0.0217	0.804		0.804	0.0217	0.803						-439.	15.	-440.	
BS/JB	0.518	0.28	0.544		0.517	0.282	0.546		0.52	0.277	0.531		0.533	0.267	0.574	

SD: Standard deviation, RMSE: Root mean square error. Results from 61 successfully optimized replications are reported here.

Table 27: Distribution of estimates: $n = 400, \bar{\rho} = -0.5, \bar{\alpha}_1 = 4$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.514	0.0736	-0.51	0.0746	-0.514	0.0739	-0.51	0.0748	-0.504	0.0416	-0.508	0.0416	-0.503	0.0392	-0.504	0.0391
β_0	0.999	0.094	1.	0.0935	1.	0.0948	1.	0.0943	0.992	0.0845	0.989	0.0844	0.998	0.0682	0.996	0.0679
β_1	0.996	0.0849	0.989	0.0846	0.996	0.085	0.991	0.0846	0.999	0.085	0.993	0.0845	0.997	0.0608	1.01	0.0606
β_2	1.	0.0989	1.01	0.0984	0.999	0.1	1.01	0.0998	1.	0.0988	1.02	0.0984	1.01	0.0722	1.02	0.0724
β_3	0.995	0.0956	1.01	0.0952	0.995	0.0951	1.01	0.0947	0.998	0.0957	1.	0.0952	0.996	0.0783	0.996	0.078
σ^2									3.02	0.26	3.04	2.04				
α_0													0.985	0.113	0.97	0.113
α_1									-664.	17.7	-667.		4.07	0.467	4.07	0.47
R2,LH	0.607	0.0346	0.61		0.607	0.0348	0.611						-605.	17.8	-604.	
BS/JB	0.00528	0.0213	0		0.00475	0.0185	0		0.00542	0.0227	0		0.535	0.285	0.58	

SD: Standard deviation, RMSE: Root mean square error. Results from 93 successfully optimized replications are reported here.

Table 28: Distribution of estimates: $n = 400, \bar{\rho} = -0.25, \bar{\alpha}_1 = 4$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.254	0.0912	-0.25	0.0908	-0.254	0.0891	-0.249	0.0888	-0.252	0.0523	-0.256	0.052	-0.251	0.0442	-0.256	0.044
β_0	0.999	0.106	0.987	0.106	0.998	0.104	0.991	0.104	0.997	0.0949	0.988	0.0945	0.998	0.0733	1.	0.0729
β_1	1.01	0.085	1.01	0.0848	1.01	0.0859	1.	0.0856	1.01	0.0856	1.01	0.0854	1.01	0.0623	1.01	0.0626
β_2	1.	0.0895	1.	0.089	1.	0.0919	0.998	0.0915	1.	0.0898	1.	0.0893	1.	0.0691	1.	0.0687
β_3	1.01	0.0939	1.01	0.0938	1.01	0.0945	1.01	0.0942	1.01	0.0925	1.01	0.0925	1.01	0.0722	1.	0.0728
σ^2									2.96	0.258	2.95	1.97				
α_0													0.975	0.0887	0.968	0.0918
α_1													4.04	0.464	4.	0.464
R2,LH	0.533	0.0377	0.529		0.533	0.0376	0.529		-649.	17.4	-650.		-589.	16.6	-588.	
BS/JB	0.00625	0.0173	0		0.00623	0.0174	0		0.00603	0.0167	0		0.514	0.267	0.502	

SD: Standard deviation, RMSE: Root mean square error. Results from 98 successfully optimized replications are reported here.

Table 29: Distribution of estimates: $n = 400, \bar{\rho} = 0., \bar{\alpha}_1 = 4$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.0225	0.0961	0.0223	0.0982	0.0208	0.0953	0.0232	0.097	0.00617	0.0582	0.00761	0.0583	0.00208	0.0473	0.00272	0.0471
β_0	0.971	0.125	0.986	0.128	0.972	0.126	0.979	0.128	0.988	0.0943	0.992	0.0946	0.99	0.0754	0.989	0.0756
β_1	0.975	0.0852	0.982	0.0883	0.976	0.0858	0.982	0.0887	0.976	0.0847	0.983	0.0875	0.98	0.0628	0.98	0.0655
β_2	1.01	0.0844	1.01	0.0842	1.01	0.0852	1.	0.0849	1.01	0.086	1.01	0.0859	1.01	0.0635	1.	0.0636
β_3	0.987	0.0838	0.989	0.0843	0.986	0.0843	0.989	0.085	0.989	0.0838	0.994	0.084	0.999	0.0638	1.	0.0635
σ^2									3.01	0.288	3.01	2.03				
α_0													0.987	0.0986	0.984	0.0989
α_1													4.06	0.54	4.05	0.539
R2,LH	0.5	0.0404	0.499		0.5	0.0404	0.499		-649.	19.2	-650.		-590.	16.6	-592.	
BS/JB	0.011	0.044	0		0.0107	0.0429	0		0.00894	0.0343	0		0.547	0.286	0.605	

SD: Standard deviation, RMSE: Root mean square error. Results from 92 successfully optimized replications are reported here.

Table 30: Distribution of estimates: $n = 400, \bar{\rho} = 0.25, \bar{\alpha}_1 = 4$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.26	0.0867	0.263	0.0869	0.26	0.0872	0.259	0.0873	0.245	0.0593	0.251	0.0593	0.25	0.0457	0.253	0.0455
β_0	0.991	0.165	0.991	0.165	0.99	0.166	0.982	0.166	1.01	0.133	1.	0.133	1.01	0.0972	1.	0.097
β_1	0.998	0.089	1.01	0.0886	0.997	0.0906	1.	0.0902	1.	0.088	1.01	0.0875	0.998	0.0647	0.995	0.0644
β_2	1.01	0.089	1.01	0.0887	1.01	0.0903	1.	0.09	1.01	0.0891	1.01	0.089	1.01	0.0632	1.	0.0633
β_3	0.997	0.0832	0.991	0.0828	0.997	0.084	0.986	0.0836	1.	0.0834	0.994	0.083	1.	0.0619	0.998	0.0616
σ^2									2.97	0.28	2.95	1.99				
α_0																
α_1																
R2,LH	0.526	0.0376	0.531		0.526	0.0376	0.531		-650.	18.	-648.		0.979	0.109	0.972	0.111
BS/JB	0.0109	0.0569	0		0.011	0.0568	0		0.0112	0.0565	0		4.	0.521	4.	0.518
													-591.	16.1	-590.	
													0.497	0.265	0.458	

SD: Standard deviation, RMSE: Root mean square error. Results from 99 successfully optimized replications are reported here.

Table 31: Distribution of estimates: $n = 400, \bar{\rho} = 0.5, \bar{\alpha}_1 = 4$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.514	0.082	0.513	0.0827	0.514	0.0829	0.511	0.0837	0.495	0.0459	0.497	0.0459	0.499	0.0383	0.496	0.0381
β_0	0.972	0.179	0.963	0.181	0.97	0.181	0.97	0.182	1.01	0.112	1.	0.112	1.01	0.0881	1.	0.0878
β_1	0.996	0.091	0.988	0.0906	0.996	0.0902	0.988	0.0899	1.	0.0904	0.991	0.0899	1.	0.0738	1.01	0.0735
β_2	1.01	0.0877	1.01	0.0877	1.01	0.0882	1.02	0.0882	1.01	0.0887	1.02	0.0892	1.01	0.0725	1.02	0.0726
β_3	0.993	0.0781	0.996	0.078	0.991	0.0773	0.994	0.0774	0.998	0.0768	1.	0.0765	0.995	0.0597	0.999	0.0596
σ^2									2.94	0.255	2.91	1.95				
α_0																
α_1																
R2,LH	0.61	0.0439	0.618		0.61	0.0438	0.618		-658.	17.7	-659.		0.994	0.0933	0.995	0.0929
BS/JB	0.0171	0.0589	0		0.0166	0.0575	0		0.0173	0.0597	0		3.9	0.424	3.92	0.434
													-601.	17.5	-601.	
													0.494	0.284	0.482	

SD: Standard deviation, RMSE: Root mean square error. Results from 94 successfully optimized replications are reported here.

Table 32: Distribution of estimates: $n = 400, \bar{\rho} = -0.5, \bar{\alpha}_1 = 8$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.505	0.11	-0.5	0.109	-0.505	0.107	-0.498	0.107	-0.494	0.0531	-0.498	0.0532	-0.495	0.0375	-0.497	0.0376
β_0	0.984	0.139	0.985	0.14	0.985	0.138	0.985	0.138	0.977	0.122	0.967	0.123	0.985	0.0759	0.986	0.077
β_1	1.	0.104	1.	0.104	1.	0.107	1.	0.106	1.01	0.102	1.01	0.102	1.	0.0641	1.02	0.0638
β_2	0.986	0.0896	0.985	0.0902	0.989	0.0895	0.987	0.0897	0.992	0.0885	0.99	0.0884	1.01	0.0673	1.	0.0672
β_3	0.995	0.107	0.995	0.107	0.995	0.109	0.996	0.108	1.	0.105	0.998	0.105	1.	0.0623	1.01	0.0619
σ^2									4.88	0.44	4.87	3.91				
α_0													0.982	0.0935	0.989	0.0946
α_1													7.95	0.84	7.88	0.837
R2,LH	0.512	0.0521	0.515		0.512	0.0514	0.516		-760.	17.7	-760.		-658.	17.	-658.	
BS/JB	0.000388	0.00293	0		0.0004	0.00302	0		0.000386	0.00331	0		0.501	0.279	0.511	

SD: Standard deviation, RMSE: Root mean square error. Results from 90 successfully optimized replications are reported here.

Table 33: Distribution of estimates: $n = 400, \bar{\rho} = -0.25, \bar{\alpha}_1 = 8$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.258	0.124	-0.274	0.123	-0.257	0.123	-0.274	0.123	-0.248	0.0552	-0.251	0.0549	-0.248	0.0405	-0.249	0.0403
β_0	0.993	0.147	1.	0.147	0.993	0.148	1.01	0.148	0.985	0.126	0.996	0.126	0.99	0.0766	0.989	0.0768
β_1	0.984	0.105	0.996	0.106	0.987	0.107	0.995	0.107	0.988	0.103	0.996	0.103	0.997	0.0625	1.	0.0623
β_2	0.981	0.121	0.966	0.122	0.981	0.121	0.973	0.122	0.985	0.119	0.976	0.12	0.992	0.0702	0.99	0.0703
β_3	1.	0.115	1.	0.114	1.	0.113	0.997	0.112	1.01	0.112	1.01	0.112	1.01	0.0735	1.01	0.0733
σ^2									4.92	0.464	4.89	3.95				
α_0													0.973	0.1	0.967	0.103
α_1													7.95	0.862	7.9	0.859
R2,LH	0.406	0.0417	0.405		0.405	0.0417	0.404		-750.	19.	-750.		-648.	17.1	-648.	
BS/JB	0	0.000361	0		0	0.000326	0		0	0.00036	0		0.54	0.253	0.514	

SD: Standard deviation, RMSE: Root mean square error. Results from 94 successfully optimized replications are reported here.

Table 34: Distribution of estimates: $n = 400, \bar{\rho} = 0., \bar{\alpha}_1 = 8.$

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	Median	RMSE	Mean	SD	Mean	SD	Median	Mean	SD	Median
ρ	0.0215	0.139	0.0195	0.14	0.0228	0.14	0.0273	0.0533	-0.00632	0.00137	0.0408	-0.00436
β_0	0.965	0.171	0.952	0.173	0.965	0.172	0.992	0.121	0.996	1.01	0.0851	1.
β_1	0.99	0.097	0.99	0.097	0.986	0.1	0.995	0.0952	0.994	0.997	0.0634	0.994
β_2	1.	0.116	1.01	0.115	1.	0.115	1.	0.116	1.01	1.	0.0698	1.
β_3	1.01	0.105	1.01	0.105	1.01	0.105	1.01	0.106	1.01	1.	0.0787	1.01
σ^2							5.04	0.528	5.02			
α_0										0.977	0.09	0.974
α_1							-751.	21.1	-752.	8.2	1.01	8.14
R2,LH	0.374	0.0351	0.373		0.374	0.0353				-648.	16.5	-649.
BS/JB	0.000207	0.0014	0		0.0002	0.00135	0.000179	0.00119	0	0.549	0.277	0.571

SD: Standard deviation, RMSE: Root mean square error. Results from 86 successfully optimized replications are reported here.

Table 35: Distribution of estimates: $n = 400, \bar{\rho} = 0.25, \bar{\alpha}_1 = 8.$

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	Median	RMSE	Mean	SD	Mean	SD	Median	Mean	SD	Median
ρ	0.252	0.105	0.269	0.105	0.253	0.104	0.242	0.0545	0.242	0.246	0.038	0.243
β_0	1.02	0.171	1.02	0.171	1.02	0.169	1.03	0.127	1.02	1.01	0.0878	1.01
β_1	1.	0.113	1.01	0.113	0.998	0.114	1.	0.112	1.01	1.01	0.0762	1.
β_2	1.01	0.113	1.02	0.113	1.01	0.113	1.02	0.112	1.02	1.01	0.0706	1.01
β_3	1.	0.12	1.02	0.12	1.	0.12	1.01	0.122	1.03	0.997	0.0685	0.993
σ^2							5.02	0.552	5.04			
α_0										0.991	0.116	1.
α_1							-754.	22.2	-757.	8.11	1.06	8.17
R2,LH	0.409	0.0423	0.406		0.409	0.0423				-651.	19.8	-654.
BS/JB	0.00075	0.00668	0		0.000756	0.00687	0.000409	0.00369	0	0.524	0.285	0.492

SD: Standard deviation, RMSE: Root mean square error. Results from 98 successfully optimized replications are reported here.

Table 36: Distribution of estimates: $n = 400, \bar{\rho} = 0.5, \bar{\alpha}_1 = 8$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.491	0.105	0.494	0.104	0.493	0.105	0.496	0.105	0.486	0.0537	0.488	0.0552	0.493	0.032	0.491	0.0325
β_0	1.	0.237	0.952	0.236	0.999	0.238	0.967	0.237	1.01	0.16	0.992	0.159	1.02	0.0915	1.01	0.0925
β_1	1.01	0.11	0.997	0.11	1.01	0.111	0.995	0.111	1.01	0.109	1.02	0.109	1.	0.0653	1.	0.0649
β_2	0.992	0.109	0.98	0.108	0.995	0.105	0.977	0.105	0.996	0.106	0.985	0.105	1.	0.0638	1.01	0.0636
β_3	0.99	0.144	0.996	0.143	0.987	0.141	0.999	0.141	0.993	0.144	1.	0.143	1.	0.066	0.992	0.0656
σ^2									4.95	0.5	4.92	3.98				
α_0													0.971	0.0858	0.968	0.09
α_1													7.97	0.862	7.92	0.857
R2,LH	0.506	0.0537	0.504		0.506	0.0536	0.504		-761.	19.6	-761.		-660.	17.1	-661.	
BS/JB	0	0.000218	0		0	0.000222	0		0	0.000191	0		0.547	0.264	0.577	

SD: Standard deviation, RMSE: Root mean square error. Results from 85 successfully optimized replications are reported here.

Table 37: Distribution of estimates: $n = 400, \bar{\rho} = -0.5, \bar{\alpha}_1 = 12$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.504	0.127	-0.508	0.126	-0.5	0.126	-0.509	0.125	-0.496	0.0521	-0.501	0.0519	-0.501	0.0307	-0.504	0.0306
β_0	1.01	0.173	0.992	0.172	1.01	0.171	0.978	0.171	1.01	0.131	0.993	0.13	0.994	0.0655	0.987	0.0654
β_1	0.973	0.134	0.963	0.136	0.975	0.134	0.967	0.136	0.979	0.133	0.958	0.134	1.	0.0709	0.991	0.0705
β_2	0.989	0.131	0.983	0.131	0.99	0.13	0.989	0.13	0.993	0.127	1.	0.126	1.	0.0649	1.	0.0645
β_3	0.983	0.135	0.987	0.135	0.984	0.136	0.99	0.136	0.988	0.135	0.993	0.135	0.989	0.0654	0.995	0.0659
σ^2									6.81	0.796	6.82	5.87				
α_0													0.991	0.084	0.997	0.084
α_1													11.8	1.38	11.9	1.39
R2,LH	0.447	0.0583	0.447		0.446	0.0589	0.446		-826.	22.2	-827.		-696.	19.4	-695.	
BS/JB	0	0	0		0	0	0		0	0	0		0.502	0.305	0.513	

SD: Standard deviation, RMSE: Root mean square error. Results from 83 successfully optimized replications are reported here.

Table 38: Distribution of estimates: $n = 400, \bar{\rho} = -0.25, \bar{\alpha}_1 = 12$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	-0.243	0.156	-0.252	0.155	-0.247	0.156	-0.261	0.155	-0.236	0.0641	-0.239	0.0653	-0.24	0.0375	-0.236	0.0386
β_0	1.	0.19	0.978	0.188	1.	0.19	0.982	0.189	0.996	0.147	0.989	0.146	0.99	0.0749	0.99	0.0752
β_1	1.01	0.139	1.	0.138	1.01	0.139	1.01	0.139	1.02	0.139	1.01	0.139	1.02	0.0689	1.04	0.0725
β_2	0.995	0.133	0.996	0.133	0.993	0.131	0.996	0.13	1.	0.131	0.997	0.13	0.993	0.0696	0.995	0.0696
β_3	1.01	0.119	1.01	0.119	1.01	0.116	1.02	0.115	1.01	0.118	1.02	0.118	1.	0.0691	1.	0.0687
σ^2																
α_0									6.97	0.829	6.92	6.03				
α_1													0.968	0.104	0.961	0.108
R2,LH	0.337	0.0521	0.341		0.337	0.0525	0.341		-819.	23.7	-820.		12.2	1.38	12.2	1.38
BS/JB	0	0	0		0	0	0		0	0	0		-684.	21.9	-685.	
													0.564	0.268	0.569	

SD: Standard deviation, RMSE: Root mean square error. Results from 92 successfully optimized replications are reported here.

Table 39: Distribution of estimates: $n = 400, \bar{\rho} = 0., \bar{\alpha}_1 = 12$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.0188	0.134	0.0234	0.135	0.0181	0.135	0.0103	0.136	-0.0047	0.0555	-0.00807	0.0554	0.00121	0.0361	0.00625	0.0359
β_0	1.	0.161	0.993	0.16	1.	0.162	0.996	0.161	1.02	0.125	1.03	0.127	1.	0.077	1.	0.0765
β_1	1.	0.132	0.988	0.132	0.999	0.132	0.993	0.131	1.	0.132	0.989	0.131	0.996	0.0674	0.989	0.0671
β_2	0.974	0.121	0.96	0.123	0.975	0.122	0.963	0.124	0.978	0.12	0.967	0.122	0.982	0.0549	0.987	0.0574
β_3	1.	0.147	1.01	0.146	1.01	0.146	1.02	0.146	1.01	0.147	1.02	0.146	0.984	0.0618	0.99	0.0636
σ^2																
α_0									6.9	0.742	6.95	5.95				
α_1													0.976	0.103	0.969	0.105
R2,LH	0.303	0.0464	0.304		0.303	0.0465	0.304		-814.	21.3	-817.		12.	1.31	12.	1.3
BS/JB	0	0.000115	0		0	0.000125	0		0	0.000167	0		-682.	19.9	-679.	
													0.538	0.263	0.519	

SD: Standard deviation, RMSE: Root mean square error. Results from 81 successfully optimized replications are reported here.

Table 40: Distribution of estimates: $n = 400, \bar{\rho} = 0.25, \bar{\alpha}_1 = 12$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.258	0.141	0.273	0.14	0.253	0.139	0.261	0.139	0.246	0.061	0.251	0.0608	0.248	0.0362	0.246	0.036
β_0	0.951	0.227	0.941	0.231	0.959	0.224	0.947	0.227	0.966	0.163	0.984	0.166	1.	0.0707	0.988	0.0704
β_1	1.01	0.137	0.993	0.136	1.01	0.135	0.995	0.134	1.01	0.136	0.994	0.136	1.	0.0766	1.	0.0762
β_2	0.999	0.135	1.01	0.134	0.997	0.135	1.	0.134	1.	0.134	1.01	0.134	1.01	0.0701	1.	0.0701
β_3	0.975	0.138	0.98	0.139	0.975	0.137	0.98	0.139	0.981	0.139	0.979	0.139	1.	0.0707	1.	0.0704
σ^2									6.89	0.69	6.8	5.93				
α_0													0.978	0.0952	0.978	0.0972
α_1									12.	1.29	12.	1.29	12.	1.29	12.	1.29
R2,LH	0.337	0.0466	0.331		0.337	0.0466	0.332		-818.	19.9	-816.		-684.	18.5	-685.	
BS/JB	0	0	0		0	0	0		0	0	0		0.533	0.294	0.565	

SD: Standard deviation, RMSE: Root mean square error. Results from 92 successfully optimized replications are reported here.

Table 41: Distribution of estimates: $n = 400, \bar{\rho} = 0.5, \bar{\alpha}_1 = 12$.

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
ρ	0.495	0.135	0.515	0.134	0.497	0.134	0.513	0.134	0.489	0.0512	0.497	0.0522	0.488	0.0287	0.49	0.0308
β_0	1.02	0.313	1.01	0.312	1.01	0.307	1.01	0.306	1.03	0.176	1.	0.178	1.03	0.0916	1.03	0.0964
β_1	1.	0.139	0.991	0.138	1.	0.136	0.993	0.136	1.	0.137	0.999	0.136	1.	0.07	0.988	0.0696
β_2	0.992	0.126	0.972	0.125	0.992	0.128	0.978	0.127	0.999	0.123	0.979	0.123	0.992	0.0604	0.988	0.0606
β_3	1.01	0.123	0.991	0.122	1.	0.119	0.991	0.119	1.01	0.12	1.01	0.12	1.	0.0635	0.996	0.0631
σ^2									6.96	0.804	6.89	6.01				
α_0													0.988	0.108	0.996	0.108
α_1									12.	1.38	12.1	1.37	12.	1.38	12.1	1.37
R2,LH	0.444	0.052	0.453		0.445	0.052	0.454		-830.	22.5	-829.		-697.	22.	-694.	
BS/JB	0	0	0		0	0	0		0	0	0		0.518	0.272	0.56	

SD: Standard deviation, RMSE: Root mean square error. Results from 86 successfully optimized replications are reported here.

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