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## Efficient Maximum Likelihood Estimation of Spatial Autoregressive Models with Normal but Heteroskedastic Disturbances

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# Efficient Maximum Likelihood Estimation of Spatial Autoregressive Models with Normal but Heteroskedastic Disturbances\*

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## Abstract

The likelihood functions for spatial autoregressive models with normal but heteroskedastic disturbances have been derived [Anselin (1988, ch.6)], but there is no implementation of maximum likelihood estimation for these likelihood functions in general cases with heteroskedastic disturbances. This is the reason why less efficient IV-based methods, ‘robust 2-SLS’ estimation for example, must be applied if disturbance terms might be heteroskedastic. In this paper, we develop a new computer program for maximum likelihood estimation and confirm the efficiency of our estimator in cases with heteroskedastic disturbances, using Monte Carlo simulations.

**Keywords:** *Spatial autoregressive model, Heteroskedasticity*

**JEL Classifications:** *C13, C21*

## 1 Introduction

The likelihood functions for spatial autoregressive models with normal but heteroskedastic disturbances have been derived [Anselin (1988, ch.6)], but there is no implementation of maximum likelihood estimation for these likelihood functions in general cases with heteroskedastic disturbances. This is the reason why less efficient IV-based methods, ‘robust 2-SLS’ estimation for example, must be applied if disturbance terms might be heteroskedastic. In this paper, we develop a new computer program for maximum likelihood estimation and confirm the efficiency of our estimator in cases with heteroskedastic disturbances, using Monte Carlo simulations. <sup>1</sup>

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<sup>1</sup> There is literature on the estimation of regression models with heteroskedastic disturbances (Amemiya, 1977; Breusch and Pagan, 1979; Jarque and Bera, 1980; White, 1980; Griffiths, 2003). There are also several discussions of heteroskedastic spatial autoregressive models (Anselin, 1988, 1990; Kelejian and Robinson, 1998; Kelejian and Prucha, 2007; LeSage and Pace, 2009).

The organization of this paper is as follows. In Section 2, we review the general form of spatial autoregressive models including heteroskedasticity. We also discuss estimation methods for such models. In Section 3, we review the formulation of heteroskedasticity, and we define the econometric model which will be analyzed in this paper. In Section 4, we describe Monte Carlo simulations of spatial effects models. We present our conclusions in Section 5.

## 2 Spatial autoregressive models and heteroskedasticity

In this section, we review the general form of spatial autoregressive models proposed in Anselin (1988, ch.6), and we discuss estimation methods for such models.<sup>2</sup>

### 2.1 General form of spatial autoregressive model

Anselin (1988, ch.6) describes the general form of a spatial autoregressive model as follows.  $\mathbf{y}$ ,  $\mathbf{X}$  and  $\boldsymbol{\epsilon}$  are a vector of explained variables ( $n \times 1$ ), a matrix of explaining variables ( $n \times k$ ), and a vector of disturbances ( $n \times 1$ ) respectively.<sup>3</sup> Two matrices  $\mathbf{W}_1$  and  $\mathbf{W}_2$  are so-called spatial weight matrices ( $n \times n$ ). It represents known spatial structure over the sample space. Scalar variables,  $\rho$  and  $\lambda$ , are parameters of spatial auto-correlation among dependent variables and disturbance terms. A vector  $\boldsymbol{\mu}$  contains the disturbance terms and its elements obey normal distributions independently. A matrix  $\boldsymbol{\Omega}$  is the covariance matrix of  $\boldsymbol{\mu}$ .  $h$  is a function which always takes positive values and determines the variance of each sample.  $\boldsymbol{\alpha}$  is a parameter vector of the function  $h$ . A vector  $\mathbf{z}_i$  represents the variance about the  $i$ th sample.

$$\mathbf{y} = \rho \mathbf{W}_1 \mathbf{y} + \mathbf{X} \boldsymbol{\beta} + \boldsymbol{\epsilon} \quad (1)$$

$$\boldsymbol{\epsilon} = \lambda \mathbf{W}_2 \boldsymbol{\epsilon} + \boldsymbol{\mu} \quad (2)$$

$$\boldsymbol{\mu} \sim N(\mathbf{0}, \boldsymbol{\Omega}) \quad (3)$$

$$\Omega_{ii} = h(\boldsymbol{\alpha}, \mathbf{z}_i) > 0, \quad \Omega_{ij} = 0 \text{ for } i \neq j \quad (4)$$

The following useful formulas can be derived. Here,  $\boldsymbol{\nu}$  is a vector of random variables which obey the standard normal distribution,  $N(0, 1)$ , identically and independently.

$$E[\boldsymbol{\mu} \boldsymbol{\mu}'] = \boldsymbol{\Omega}, \quad \boldsymbol{\nu} = \boldsymbol{\Omega}^{-1/2} \boldsymbol{\mu}, \quad \boldsymbol{\mu} = \boldsymbol{\Omega}^{1/2} \boldsymbol{\nu} \quad (5)$$

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<sup>2</sup> The likelihood function, its derivatives, and the information matrix appearing in this section were described in Anselin (1988, ch.6). They are recounted here with some modifications for the sake of clarity.

<sup>3</sup> The model has a constant term if every element of the first column of  $\mathbf{X}$  is one. The first element of the coefficient vector  $\boldsymbol{\beta}$  ( $k \times 1$ ) is the estimator for the constant term in the estimation result.

## 2.2 Likelihood function

In this section, we adopt the formulation of the log likelihood function for the spatial model.

$$L = -(n/2) \ln \pi - (1/2) \ln |\boldsymbol{\Omega}| + \ln |\mathbf{A}| + \ln |\mathbf{B}| - (1/2) \boldsymbol{\nu}' \boldsymbol{\nu} \quad (6)$$

$$\boldsymbol{\nu} = \boldsymbol{\Omega}^{-1/2} \mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}), \quad \boldsymbol{\nu}' = (\mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \boldsymbol{\Omega}^{-1/2} \quad (7)$$

$$\boldsymbol{\nu}' \boldsymbol{\nu} = (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta})' \mathbf{B}' \boldsymbol{\Omega}^{-1} \mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}) \quad (8)$$

$$\mathbf{A} = \mathbf{I} - \rho \mathbf{W}_1, \quad \mathbf{B} = \mathbf{I} - \lambda \mathbf{W}_2$$

## 2.3 First order derivatives

The number of parameters in the optimization is  $k + 1 + 1 + p = k + p + 2$ . First order conditions on all parameters are necessary for the optimization.

$$\boldsymbol{\theta} = [\boldsymbol{\beta}', \rho, \lambda, \boldsymbol{\alpha}']', \quad \frac{\partial L}{\partial \boldsymbol{\theta}} = \mathbf{0} \quad (9)$$

$$\frac{\partial L}{\partial \boldsymbol{\beta}} = \boldsymbol{\nu}' \boldsymbol{\Omega}^{-1/2} \mathbf{B} \mathbf{X} (\mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \boldsymbol{\Omega}^{-1} \mathbf{B} \mathbf{X} \quad (10)$$

$$\frac{\partial L}{\partial \rho} = -\text{tr} (\mathbf{A}^{-1} \mathbf{W}_1) + \boldsymbol{\nu}' \boldsymbol{\Omega}^{-1/2} \mathbf{B} \mathbf{W}_1 \mathbf{y} \quad (11)$$

$$= -\text{tr} (\mathbf{A}^{-1} \mathbf{W}_1) + (\mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \boldsymbol{\Omega}^{-1} \mathbf{B} \mathbf{W}_1 \mathbf{y} \quad (12)$$

$$\frac{\partial L}{\partial \lambda} = -\text{tr} (\mathbf{B}^{-1} \mathbf{W}_2) + \boldsymbol{\nu}' \boldsymbol{\Omega}^{-1/2} \mathbf{W}_2 (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}) \quad (13)$$

$$= -\text{tr} (\mathbf{B}^{-1} \mathbf{W}_2) + (\mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \boldsymbol{\Omega}^{-1} \mathbf{W}_2 (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}) \quad (14)$$

$$\frac{\partial L}{\partial \alpha_p} = -\frac{1}{2} \text{tr} \boldsymbol{\Omega}^{-1} \mathbf{H}_p + \frac{1}{2} (\mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \boldsymbol{\Omega}^{-2} \mathbf{H}_p \mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}) \quad (15)$$

$$= -\frac{1}{2} \text{tr} \boldsymbol{\Omega}^{-1} \mathbf{H}_p + \frac{1}{2} \boldsymbol{\nu}' \boldsymbol{\Omega}^{-1} \mathbf{H}_p \boldsymbol{\nu} \quad (16)$$

$\mathbf{H}_p$  is a diagonal matrix and its diagonal elements are  $\frac{\partial h}{\partial \alpha_p}$ . The solution for  $\boldsymbol{\beta}$  is derived from the first order conditions corresponding to Equation (10).

$$\boldsymbol{\beta}^* = (\mathbf{X}' \mathbf{B}' \boldsymbol{\Omega}^{-1} \mathbf{B} \mathbf{X})^{-1} \mathbf{X}' \mathbf{B}' \boldsymbol{\Omega}^{-1} \mathbf{B} \mathbf{A} \mathbf{y} \quad (17)$$

## 2.4 Second order derivatives

$$\frac{\partial^2 L}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'} = -\mathbf{X}' \mathbf{B}' \boldsymbol{\Omega}^{-1} \mathbf{B} \mathbf{X} = -(\mathbf{B} \mathbf{X})' \boldsymbol{\Omega}^{-1} \mathbf{B} \mathbf{X} \quad (18)$$

$$\frac{\partial^2 L}{\partial \rho^2} = -\text{tr} (\mathbf{A}^{-1} \mathbf{W}_1 \mathbf{A}^{-1} \mathbf{W}_1) - (\mathbf{B} \mathbf{W}_1 \mathbf{y})' \boldsymbol{\Omega}^{-1} \mathbf{B} \mathbf{W}_1 \mathbf{y} \quad (19)$$

$$\frac{\partial^2 L}{\partial \lambda^2} = -\text{tr} (\mathbf{B}^{-1} \mathbf{W}_2 \mathbf{B}^{-1} \mathbf{W}_2) - (\mathbf{W}_2 (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}))' \boldsymbol{\Omega}^{-1} \mathbf{W}_2 (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}) \quad (20)$$

$$\frac{\partial^2 L}{\partial \beta \partial \rho} = \frac{\partial^2 L}{\partial \rho \partial \beta'} = -(\mathbf{B}\mathbf{X})' \Omega^{-1} \mathbf{B}\mathbf{W}_1 \mathbf{y} \quad (21)$$

$$\frac{\partial^2 L}{\partial \beta \partial \lambda} = \frac{\partial^2 L}{\partial \lambda \partial \beta'} = -(\mathbf{B}\mathbf{X})' \Omega^{-1} \mathbf{W}_2 (\mathbf{A}\mathbf{y} - \mathbf{X}\beta) - (\mathbf{B}(\mathbf{A}\mathbf{y} - \mathbf{X}\beta))' \Omega^{-1} \mathbf{W}_2 \mathbf{X} \quad (22)$$

$$= -(\mathbf{B}\mathbf{X})' \Omega^{-1} \mathbf{W}_2 (\mathbf{A}\mathbf{y} - \mathbf{X}\beta) - \nu' \Omega^{-1/2} \mathbf{W}_2 \mathbf{X} \quad (23)$$

$$\frac{\partial^2 L}{\partial \beta \partial \alpha_p} = \frac{\partial^2 L}{\partial \alpha_p \partial \beta'} = -(\mathbf{B}(\mathbf{A}\mathbf{y} - \mathbf{X}\beta))' \Omega^{-2} \mathbf{H}_p \mathbf{B}\mathbf{X} = -\nu' \Omega^{-3/2} \mathbf{H}_p \mathbf{B}\mathbf{X} \quad (24)$$

$$\frac{\partial^2 L}{\partial \rho \partial \lambda} = \frac{\partial^2 L}{\partial \lambda \partial \rho} = -(\mathbf{W}_2 (\mathbf{A}\mathbf{y} - \mathbf{X}\beta))' \Omega^{-1} \mathbf{B}\mathbf{W}_1 \mathbf{y} - \nu' \Omega^{-1/2} \mathbf{W}_2 \mathbf{W}_1 \mathbf{y} \quad (25)$$

$$= -(\mathbf{W}_2 (\mathbf{A}\mathbf{y} - \mathbf{X}\beta))' \Omega^{-1} \mathbf{B}\mathbf{W}_1 \mathbf{y} - (\mathbf{B}(\mathbf{A}\mathbf{y} - \mathbf{X}\beta))' \Omega^{-1} \mathbf{W}_2 \mathbf{W}_1 \mathbf{y} \quad (26)$$

$$\frac{\partial^2 L}{\partial \rho \partial \alpha_p} = \frac{\partial^2 L}{\partial \alpha_p \partial \rho} = -(\mathbf{B}(\mathbf{A}\mathbf{y} - \mathbf{X}\beta))' \Omega^{-2} \mathbf{H}_p \mathbf{B}\mathbf{W}_1 \mathbf{y} = -\nu' \Omega^{-2/3} \mathbf{H}_p \mathbf{B}\mathbf{W}_1 \mathbf{y} \quad (27)$$

$$\frac{\partial^2 L}{\partial \lambda \partial \alpha_p} = \frac{\partial^2 L}{\partial \alpha_p \partial \lambda} = -(\mathbf{B}(\mathbf{A}\mathbf{y} - \mathbf{X}\beta))' \Omega^{-2} \mathbf{H}_p \mathbf{W}_2 (\mathbf{A}\mathbf{y} - \mathbf{X}\beta) \quad (28)$$

$$= -\nu' \Omega^{-3/2} \mathbf{H}_p \mathbf{W}_2 (\mathbf{A}\mathbf{y} - \mathbf{X}\beta) \quad (29)$$

$\mathbf{H}_{pq}$  is a diagonal matrix. Its diagonal elements are  $\frac{\partial}{\partial \alpha_q} \frac{\partial h}{\partial \alpha_p}$ .

**When a function specifying the heteroskedasticity is not given:**

$$\begin{aligned} \frac{\partial^2 L}{\partial \alpha_p \partial \alpha_q} &= (1/2) (\text{tr} (\Omega^{-2} \mathbf{H}_p \mathbf{H}_q) - \text{tr} (\Omega^{-1} \mathbf{H}_{pq})) \\ &\quad + \nu' \{ -\Omega^{-2} \mathbf{H}_q \mathbf{H}_p + (1/2) \Omega^{-1} \mathbf{H}_{pq} \} \nu \end{aligned} \quad (30)$$

$$\begin{aligned} &= (1/2) (\text{tr} (\Omega^{-2} \mathbf{H}_p \mathbf{H}_q) - \text{tr} (\Omega^{-1} \mathbf{H}_{pq})) \\ &\quad + \nu' \{ -\Omega^{-5/2} \mathbf{H}_q \mathbf{H}_p + (1/2) \Omega^{-3/2} \mathbf{H}_{pq} \} \mathbf{B} (\mathbf{A}\mathbf{y} - \mathbf{X}\beta) \end{aligned} \quad (31)$$

$$\begin{aligned} \frac{\partial^2 L}{\partial \alpha_p^2} &= (1/2) (\text{tr} (\Omega^{-2} \mathbf{H}_p^2) - \text{tr} (\Omega^{-1} \mathbf{H}_{pp})) \\ &\quad + \nu' \{ -\Omega^{-5/2} \mathbf{H}_p^2 + (1/2) \Omega^{-3/2} \mathbf{H}_{pp} \} \mathbf{B} (\mathbf{A}\mathbf{y} - \mathbf{X}\beta) \end{aligned} \quad (32)$$

**When the heteroskedasticity is specified by a linear function:**

In this case, we can use  $\mathbf{H}_{pq} = \mathbf{0}$ .

$$\frac{\partial^2 L}{\partial \alpha_p \partial \alpha_q} = (1/2) \text{tr} (\Omega^{-2} \mathbf{H}_p \mathbf{H}_q) - \nu' \Omega^{-2} \mathbf{H}_q \mathbf{H}_p \nu \quad (33)$$

$$= (1/2) \text{tr} (\Omega^{-2} \mathbf{H}_p \mathbf{H}_q) - (\mathbf{B}(\mathbf{A}\mathbf{y} - \mathbf{X}\beta))' \Omega^{-3} \mathbf{H}_q \mathbf{H}_p \mathbf{B} (\mathbf{A}\mathbf{y} - \mathbf{X}\beta) \quad (34)$$

$$\frac{\partial^2 L}{\partial \alpha_p^2} = (1/2) \text{tr} (\Omega^{-2} \mathbf{H}_p^2) - (\mathbf{B}(\mathbf{A}\mathbf{y} - \mathbf{X}\beta))' \Omega^{-3} \mathbf{H}_p^2 \mathbf{B} (\mathbf{A}\mathbf{y} - \mathbf{X}\beta) \quad (35)$$

## 2.5 Information matrix

$$\mathcal{I}_{\beta\beta'} = (\mathbf{B}\mathbf{X})' \boldsymbol{\Omega}^{-1} \mathbf{B}\mathbf{X} \quad (36)$$

$$\mathcal{I}_{\beta\rho} = (\mathbf{B}\mathbf{X})' \boldsymbol{\Omega}^{-1} \mathbf{B}\mathbf{W}_1 \mathbf{A}^{-1} \mathbf{X}\beta \quad (37)$$

$$\mathcal{I}_{\beta\lambda} = \mathbf{0} \quad (38)$$

$$\mathcal{I}_{\beta\alpha'} = \mathbf{0} \quad (39)$$

$$\begin{aligned} \mathcal{I}_{\rho\rho} = & \text{tr} (\mathbf{W}_1 \mathbf{A}^{-1})^2 + \text{tr} \boldsymbol{\Omega} (\mathbf{B}\mathbf{W}_1 \mathbf{A}^{-1} \mathbf{B}^{-1})' \boldsymbol{\Omega}^{-1} (\mathbf{B}\mathbf{W}_1 \mathbf{A}^{-1} \mathbf{B}^{-1}) \\ & + (\mathbf{B}\mathbf{W}_1 \mathbf{A}^{-1} \mathbf{X}\beta)' \boldsymbol{\Omega}^{-1} (\mathbf{B}\mathbf{W}_1 \mathbf{A}^{-1} \mathbf{X}\beta) \end{aligned} \quad (40)$$

$$\mathcal{I}_{\rho\lambda} = \text{tr} (\mathbf{W}_2 \mathbf{B}^{-1})' \boldsymbol{\Omega}^{-1} \mathbf{B}\mathbf{W}_1 \mathbf{A}^{-1} \mathbf{B}^{-1} \boldsymbol{\Omega} + \text{tr} \mathbf{W}_2 \mathbf{W}_1 \mathbf{A}^{-1} \mathbf{B}^{-1} \quad (41)$$

$$\mathcal{I}_{\rho\alpha_p} = \text{tr} \boldsymbol{\Omega}^{-1} \mathbf{H}_p \mathbf{B}\mathbf{W}_1 \mathbf{A}^{-1} \mathbf{B}^{-1} \quad (42)$$

$$\mathcal{I}_{\lambda\lambda} = \text{tr} (\mathbf{W}_2 \mathbf{B}^{-1})^2 + \text{tr} \boldsymbol{\Omega} (\mathbf{W}_2 \mathbf{B}^{-1})' \boldsymbol{\Omega}^{-1} \mathbf{W}_2 \mathbf{B}^{-1} \quad (43)$$

$$\mathcal{I}_{\lambda\alpha_p} = \text{tr} \boldsymbol{\Omega}^{-1} \mathbf{H}_p \mathbf{W}_2 \mathbf{B}^{-1} \quad (44)$$

$$\mathcal{I}_{\alpha_p \alpha_q} = (1/2) \text{tr} \boldsymbol{\Omega}^{-2} \mathbf{H}_p \mathbf{H}_q \quad (45)$$

## 2.6 Estimation methods for spatial autoregressive models

### 2.6.1 Maximum likelihood estimation for homoskedastic models

The maximum likelihood estimation (MLE) is a maximization problem regarding the likelihood function (Equation (6)). With the following assumptions, the model becomes homoskedastic.

$$\text{Assumption : } h = \sigma^2 \quad (46)$$

$$\text{Consequences : } \frac{\partial h}{\partial \sigma^2} = 1, \mathbf{H}_1 = \mathbf{H}_{\sigma^2} = \frac{\partial \boldsymbol{\Omega}}{\partial \sigma^2} = \mathbf{I}, \mathbf{H}_{pq} = \mathbf{0}, \quad (47)$$

$$\boldsymbol{\Omega} = \sigma^2 \mathbf{I}, \boldsymbol{\Omega}^{-1} = \frac{1}{\sigma^2} \mathbf{I}, \boldsymbol{\Omega}^{-2} = \frac{1}{(\sigma^2)^2} \mathbf{I}, \quad (48)$$

$$\text{tr} \boldsymbol{\Omega}^{-1} = \frac{n}{\sigma^2}, \text{tr} \boldsymbol{\Omega}^{-2} = \frac{n}{(\sigma^2)^2} \quad (49)$$

MLE is commonly used for this homoskedastic case and is more efficient than other estimation methods. However, the results of MLE may be doubtful if the assumption regarding the disturbance terms is not valid. If the estimated residuals,  $\boldsymbol{\nu}$ , do not obey the standard normal distribution  $N(0, 1)$ , the disturbance terms may be heteroskedastic. In such cases, less efficient IV-based methods, 'robust 2-SLS' estimation for example, must be applied.

## 2.6.2 Spatial two-stage least squares method

Weaker assumptions than MLE are necessary in some other estimation methods. For an example, the spatial two-stage least squares method ('S-2SLS') may be applied to the spatial effect model defined below (Equation (62)). In the case of heteroskedastic disturbances, a heteroskedastic-robust version of the spatial two-stage least squares method ('S-2SLS-Robust') may be used instead of S-2SLS. <sup>4</sup> This method was used in some empirical analyses, including Kim et al. (2003) and Sedgley et al. (2008).

## 2.6.3 Maximum likelihood estimation of heteroskedastic models

There is no essential difference between the likelihood functions of homoskedastic and heteroskedastic autoregressive models. The problem is that no implementation of maximum likelihood estimation for heteroskedastic models exists yet. Hereafter, we develop a new program for maximum likelihood estimation which facilitates efficient estimators when disturbance terms might be heteroskedastic. <sup>5</sup>

## 2.7 Modeling heteroskedasticity

Parametric formulations of heteroskedasticity have been categorized (Griffiths, 2003). The following equation is a linear formulation.

$$\Omega_{ii} = \alpha_0 + \alpha_1 z_{1i} + \cdots + \alpha_S z_{Si} \quad (50)$$

Griffiths (2003, p.87) wrote "if  $(z_{1i}, \dots, z_{Si})$  are non-overlapping dummy variables, the specification describes a partition of the sample into  $(S+1)$  subsamples, each one with a different error variance". There are alternative specifications which ensure that  $\Omega_{ii} > 0$ :

$$\Omega_{ii} = (\alpha_0 + \alpha_1 z_{1i} + \cdots + \alpha_S z_{Si})^2 \quad (51)$$

$$\Omega_{ii} = \exp(\alpha_0 + \alpha_1 z_{1i} + \cdots + \alpha_S z_{Si}) \quad (52)$$

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<sup>4</sup> See Anselin and Bera (1998, p.259).

<sup>5</sup> The authors wrote the program using *Mathematica 5.2*, and will provide it upon request.

### 3 Specification of heteroskedastic spatial autoregressive model

In this paper, we assume the following linear formulation of heteroskedasticity.

$$\mathbf{y} = \rho \mathbf{W}_1 \mathbf{y} + \mathbf{X} \boldsymbol{\beta} + \boldsymbol{\epsilon} \quad (53)$$

$$\boldsymbol{\epsilon} = \lambda \mathbf{W}_2 \boldsymbol{\epsilon} + \boldsymbol{\mu} \quad (54)$$

$$\boldsymbol{\mu} \sim N(\mathbf{0}, \boldsymbol{\Omega}) \quad (55)$$

$$\Omega_{ii} = \mathbf{z}_i \boldsymbol{\alpha} > 0, \quad \Omega_{ij} = 0 \text{ for } i \neq j \quad (56)$$

$$\frac{\partial h}{\partial \alpha_p} = z_{ip}, \quad (57)$$

$$\mathbf{H}_p = \frac{\partial \boldsymbol{\Omega}}{\partial \alpha_p} = \text{diag}(z_{1p}, \dots, z_{np}) \quad (58)$$

$\boldsymbol{\alpha}$  is a vector of the heteroskedasticity parameters.  $\mathbf{z}_i$  is a vector of variables which regulates the variance of the disturbance term for the  $i$ th sample. In our experiments we let  $\mathbf{z}_i = (1, z_i)$ , where the scalar variable  $z_i$  is a dummy variable. We assume the values of  $z_i$  are known as are the explaining variables in  $\mathbf{X}$ .  $\text{diag}(a_1, \dots, a_n)$  is a diagonal matrix with diagonal elements  $a_1, \dots, a_n$ .

#### 3.1 Details of the optimization in maximum likelihood estimation

$$L(\boldsymbol{\beta}, \rho, \lambda, \boldsymbol{\alpha}) = -(n/2) \ln \pi - (1/2) \ln |\boldsymbol{\Omega}| + \ln |\mathbf{A}| + \ln |\mathbf{B}| - (1/2) \boldsymbol{\nu}' \boldsymbol{\nu} \quad (59)$$

$$\boldsymbol{\nu} = \boldsymbol{\Omega}^{-1/2} \mathbf{B} (\mathbf{A} \mathbf{y} - \mathbf{X} \boldsymbol{\beta}), \quad \mathbf{A} = \mathbf{I} - \rho \mathbf{W}_1, \quad \mathbf{B} = \mathbf{I} - \lambda \mathbf{W}_2$$

1. The solution for  $\boldsymbol{\beta}$  is derived from the first order conditions.

$$\boldsymbol{\beta}^*(\rho, \lambda, \boldsymbol{\alpha}) = (\mathbf{X}' \mathbf{B}' \boldsymbol{\Omega}^{-1} \mathbf{B} \mathbf{X})^{-1} \mathbf{X}' \mathbf{B}' \boldsymbol{\Omega}^{-1} \mathbf{B} \mathbf{A} \mathbf{y} \quad (60)$$

2. For given values of  $\rho, \lambda$ , it is possible to solve  $P$  first order conditions ( $\frac{\partial L}{\partial \alpha_p} = 0$ ) for  $\alpha_p$ , and obtain optimal values  $\boldsymbol{\alpha}^*(\rho, \lambda)$ .
3. The last step is a maximization problem for the concentrated likelihood function  $\bar{L}(\rho, \lambda)$ . The initial values are  $\rho_0 = \lambda_0 = 0$ .

$$\max_{\rho, \lambda} \bar{L}(\rho, \lambda) = L(\boldsymbol{\beta}^*(\rho, \lambda, \boldsymbol{\alpha}^*(\rho, \lambda)), \rho, \lambda, \boldsymbol{\alpha}^*(\rho, \lambda)) \quad (61)$$



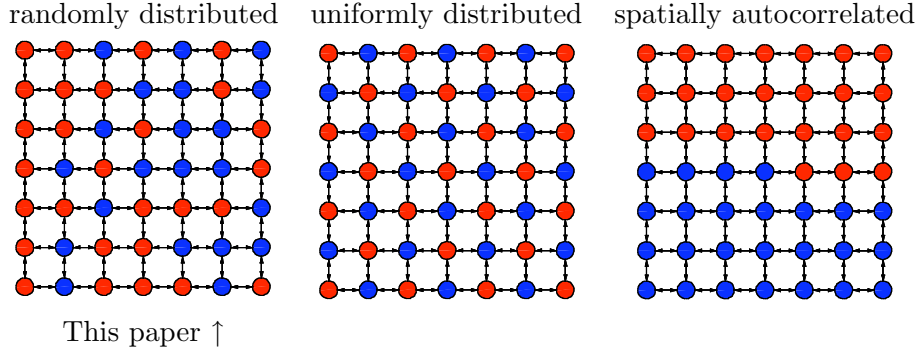


Figure 1: Specification of heteroskedasticity (colors show two subgroups).

## 4 Monte Carlo Simulations

In this section, we present our experimental design and the results of Monte Carlo simulations intended to investigate the efficiency of maximum likelihood estimation for heteroskedastic spatial autoregressive models. A new program was developed which can compute the maximum likelihood estimation of a general model represented by equations (53) through (56). In this paper, we focus on the spatial effect model (SEM) for which we assume  $\lambda = 0$ .

### 4.1 Design

The following assumptions were made in the data generating process (DGP):

- The functional form of the DGP is a spatial autoregressive model with a normal but heteroskedastic disturbance term. As previously mentioned, we assume the functional form of heteroskedasticity is linear. The coefficient vector  $\alpha$  is  $1 \times 2$ . One element is a constant term fixed to unity ( $\alpha_0 = 1$ ). We examine several values of  $\alpha_1$ . The diagonal elements of the covariance matrix are  $\Omega_{ii} = (1, z_i) \cdot (\alpha_0, \alpha_1)' = \alpha_0 + \alpha_1 z_i$ .

$$\mathbf{y} = \rho \mathbf{W} \mathbf{y} + \mathbf{X} \boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \boldsymbol{\Omega}) \quad (62)$$

$$\Omega_{ii} = \alpha_0 + \alpha_1 z_i > 0, \quad \Omega_{ij} = 0 \text{ for } i \neq j \quad (63)$$

In this paper, we consider the case where the disturbance is homoskedastic within subgroups. The following equations and Figure 1 show the ‘groupwise heteroskedasticity of errors’.

$$z_i = 0 \text{ or } 1 \quad (64)$$

$$\therefore \Omega_{ii} = \alpha_0 \text{ or } \alpha_0 + \alpha_1 \quad (65)$$

- The spatial structure is represented by a first-order rook contiguity spatial weight matrix. Two sample sizes are considered:  $n = 100$  ( $10 \times 10$ ) and  $400$  ( $20 \times 20$ ).
- There are three explaining variables,  $x_{ki} \sim N(0, 1)$  for  $k = 1, 2, 3$ , and a constant term  $x_{0i} = 1$  for the  $i$ th observation. Their coefficients are  $\beta_k = 1$ .
- We performed about 100 replications ( $r \approx 100$ ) for each value of  $\rho$  and  $\alpha_1$ .  $\rho$  takes values between -0.5 and 0.5 in increments of 0.25.  $\alpha_1$  takes values between 0 and 12 in increments of 4. <sup>6</sup>When  $\alpha_1 = 0$ , the DGP is homoskedastic. When  $\rho = 0$ , there are no spatial effects in the DGP.

Estimations are made by the following four methods.

1. Spatial 2-stage least squares (S-2SLS): The instrumental variables are  $[\mathbf{X} \ \mathbf{W}_1 \mathbf{X}]$ . We use only linearly independent rows to create the matrix.
2. Heteroskedastic robust version of S-2SLS (S-2SLS-Robust): The same instrumental variables as S-2SLS are used. The residuals of S-2SLS are used to calculate the covariance matrix.
3. Maximum likelihood estimation for homoskedastic model (ML-Homo): We use misspecified disturbance terms,  $\epsilon_i \sim N(0, \sigma^2)$ . The scalar parameter  $\sigma^2$  takes the place of  $\alpha_0$  and  $\alpha_1$ .
4. Maximum likelihood estimation for heteroskedastic model (ML-Hetero): The model given by Equations (62) and (63). <sup>7</sup> There are four parameters to be estimated,  $\beta_k$ ,  $\rho$ ,  $\alpha_0$ , and  $\alpha_1$ . The total number of parameters is  $4 + 1 + 0 + 2 = 7$ .

A symbol with an upper-bar ( $\bar{\theta}_k$ ) denotes the true value of a parameter ( $\theta_k$ ).

## 4.2 Results

### 4.2.1 Example estimation result

Table 1 shows a typical estimation result for  $n = 100$ . In this example, there is a positive spatial autocorrelation in the explained variable ( $\bar{\rho} = 0.5$ ) and the variances of half of the samples are larger than the other half ( $\Omega_{ii} = \bar{\alpha}_0 + \bar{\alpha}_1 = 13$  or  $\Omega_{ii} = \bar{\alpha}_0 = 1$ ). The row of ‘BS/JB Test’ shows p-values of Bowman and Shenton (1975) (Jarque and Bera (1987)) normality test. In this example, the following points can be observed.

1. For all methods, there is not a large difference between the true value and the estimate.

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<sup>6</sup> The exact number of replications is approximate because some optimizations fail.

<sup>7</sup>  $\mathbf{H}_p$  in Equation (58) may be expressed as follows,

$$\mathbf{H}_0 = \text{diag}(1, \dots, 1) = \mathbf{I}, \text{ and } \mathbf{H}_1 = \text{diag}(z_1, \dots, z_n). \quad (66)$$

Table 1: A typical estimation result ( $n = 100, \bar{\rho} = 0.5, \bar{\alpha}_1 = 12$ )

Parameter	True value	S-2SLS	S-2SLS- Robust	ML- Homo	ML- Hetero
$\rho$	0.5	0.429 (0.261)	0.443 (0.202)*	0.476 (0.0975)***	0.573 (0.0506)***
$\beta_0$	1	0.988 (0.597)	0.952 (0.464)*	0.892 (0.347)**	1.13 (0.212)***
$\beta_1$	1	0.858 (0.297)***	0.866 (0.288)***	0.859 (0.294)***	0.91 (0.13)***
$\beta_2$	1	1.46 (0.249)***	1.45 (0.236)***	1.45 (0.241)***	1.01 (0.115)***
$\beta_3$	1	0.497 (0.306)	0.516 (0.28)	0.48 (0.29)	0.913 (0.142)***
$\sigma^2$				7.35 (1.06)***	
$\alpha_0$	1				0.778 (0.161)***
$\alpha_1$	12				13.2 (2.73)***
$R^2$ , Likelihood		0.456	0.459	-210.	-176.
BS/JB Test		0.00402	0.0028	0.00268	0.583

Notes: Standard errors are in parentheses. \*, \*\* and \*\*\* indicate significance levels of 10%, 5% and 1%, respectively.

2. The standard errors of the estimates become smaller if the heteroskedasticity is considered. Please compare S-2SLS-Robust to S-2SLS, or ML-Hetero to ML-Homo.
3.  $\alpha_1$  is the parameter of heteroskedasticity in ML-Hetero. It is significantly non-zero and the null hypothesis  $\alpha_1 = 0$  is rejected in the likelihood ratio test between ML-Homo and ML-Hetero.

#### 4.2.2 RMSE of estimators

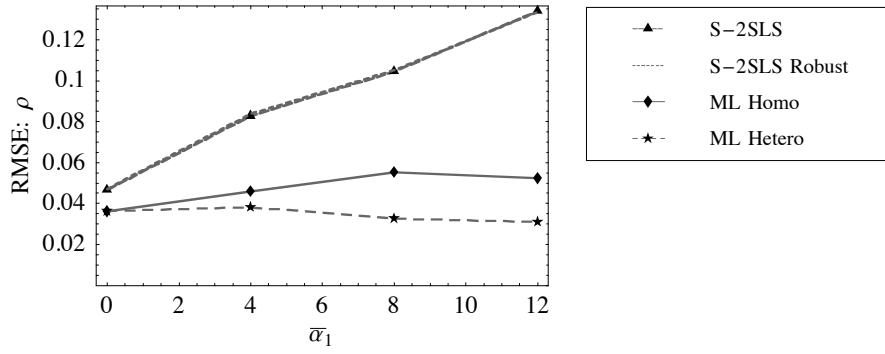
In this section, we discuss the results of Monte Carlo experiments. The differences in the root mean square error (RMSE) of the estimates of the four methods reveal their efficiencies.<sup>8</sup> Tables 2 through 41 show the distributions of the parameter estimates.

Each panel in Figure 2 shows important RMSE results. For example, Figure 2(a) shows the relationship between true values of  $\bar{\alpha}_1$  and estimates of  $\rho$  when  $\bar{\rho} = 0.5$ . Several points may be observed.

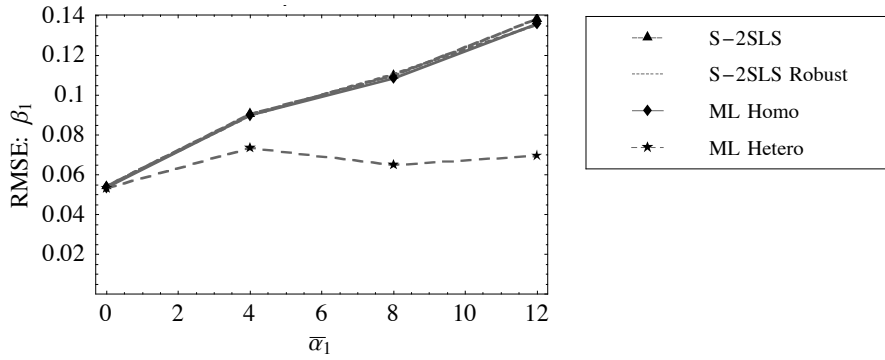
<sup>8</sup> The definition of the root mean square error (RMSE) we used is

$$\text{RMSE} = \sqrt{\frac{1}{r} \sum_{j=1}^r (\bar{\theta}_k - \tilde{\theta}_{kj})^2} \quad (67)$$

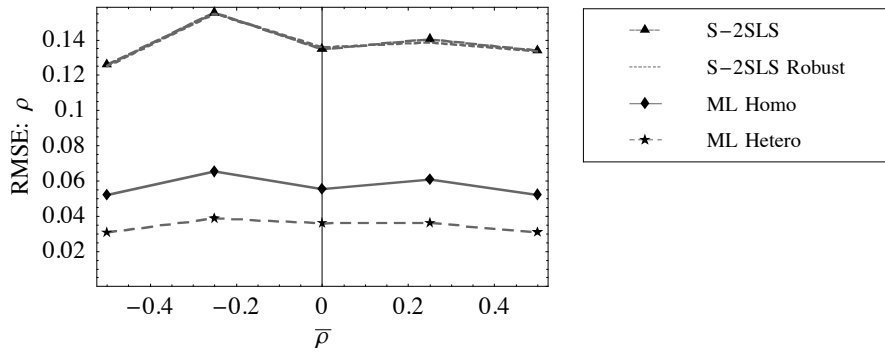
where  $r$  is the number of replications,  $\bar{\theta}_k$  is the true value of the  $k$ th parameter, and  $\tilde{\theta}_{kj}$  is an estimate for  $j$ th replication.



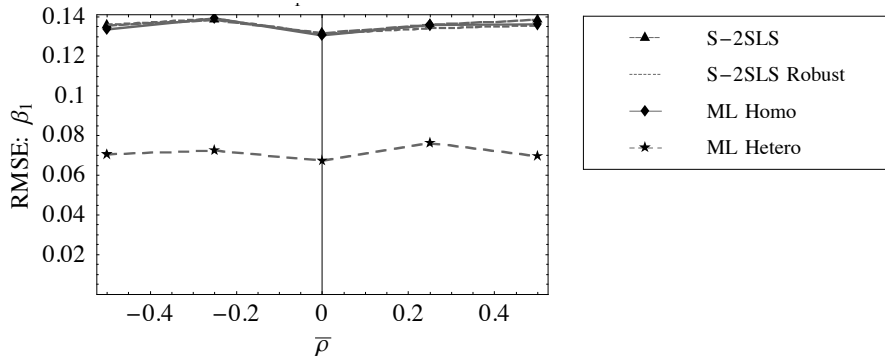
(a)  $\bar{\rho} = 0.5$



(b)  $\bar{\rho} = 0.5$



(c)  $\bar{\alpha}_1 = 12$



(d)  $\bar{\alpha}_1 = 12$

Figure 2: RMSE of estimates for  $\rho$  and  $\beta_1$  ( $n = 400$ )

1. Estimates of  $\rho$  are efficient in two maximum likelihood estimations (Figures 2(a) and 2(c)).
2. It is clear that estimates of  $\beta_1$  are efficient in ML-Hetero (Figures 2(b) and 2(d)).
3. There is not a big difference between S-2SLS-Robust and S-2SLS.

## 5 Conclusion

Likelihood functions of spatial autoregressive models with normal but heteroskedastic disturbances have been already derived [Anselin (1988, ch.6)], but there was no implementation of maximum likelihood estimation for these likelihood functions in general with heteroskedastic disturbances. This was the reason why less efficient IV-based methods, ‘robust 2-SLS’ estimation for example, had to be applied when disturbance terms might be heteroskedastic. In this paper, a new computer program for maximum likelihood estimation was presented and the efficiency of this estimator for spatial models with heteroskedastic disturbances was confirmed using Monte Carlo simulations.

We can estimate models with spatially auto-correlated disturbance terms with the program developed in this paper. Additional simulations will show efficiencies in such cases. Experimental comparisons between our method and recently proposed methods (Kelejian and Prucha, 2007, 2010; Lin and Lee, 2010) must also be important.

Table 2: Distribution of estimates:  $n = 100, \bar{\rho} = -0.5, \bar{\alpha}_1 = 0$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero					
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE			
$\rho$	-0.482	0.0968	-0.487	-0.098	0.098	-0.487	-0.481	0.0992	-0.476	0.101	0.101	-0.481	0.068	-0.485	0.0702
$\beta_0$	0.982	0.132	0.977	0.132	0.132	0.977	0.98	0.133	0.979	0.134	0.134	0.981	0.108	0.96	0.112
$\beta_1$	0.997	0.0911	0.997	0.0907	0.0907	0.997	0.997	0.0923	0.993	0.0919	0.0919	0.996	0.089	1.	0.0886
$\beta_2$	1.02	0.0942	1.02	0.0949	0.0949	1.02	1.02	0.0958	1.02	0.0968	0.0968	1.01	0.0929	1.02	0.0936
$\beta_3$	1.01	0.0992	1.01	0.099	0.099	1.01	1.01	0.102	1.01	0.102	0.102	1.01	0.0978	1.01	0.0975
$\sigma^2$							0.963	0.146	0.959	0.15	0.15				
$\alpha_0$												0.953	0.203	0.932	0.207
$\alpha_1$												0.0233	0.311	-0.0205	0.31
R2,LH	0.808	0.0417	0.817				0.807	0.0422	0.817			-108.	7.69	-109.	
BS/JB	0.569	0.277	0.581				0.563	0.276	0.6			0.573	0.282	0.611	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 3: Distribution of estimates:  $n = 100, \bar{\rho} = -0.25, \bar{\alpha}_1 = 0$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero					
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE			
$\rho$	-0.249	0.0985	-0.254	0.098	0.098	-0.251	-0.249	0.0987	-0.251	0.0982	0.0982	-0.257	0.0799	-0.257	0.0798
$\beta_0$	1.	0.127	0.998	0.126	0.126	1.	1.	0.129	1.	0.128	0.128	1.01	0.114	1.01	0.114
$\beta_1$	0.997	0.105	1.01	0.105	0.105	1.01	0.997	0.106	1.01	0.106	0.106	0.998	0.107	1.01	0.107
$\beta_2$	1.01	0.112	1.01	0.112	0.112	1.	1.01	0.116	1.	0.115	0.115	1.01	0.112	1.02	0.111
$\beta_3$	0.989	0.11	0.987	0.11	0.11	0.991	0.989	0.11	0.991	0.11	0.11	0.99	0.113	0.988	0.113
$\sigma^2$							0.96	0.135	0.946	0.14	0.14				
$\alpha_0$												0.939	0.198	0.904	0.206
$\alpha_1$												0.0466	0.324	0.0591	0.326
R2,LH	0.767	0.0402	0.772				0.767	0.0399	0.773			-105.	6.95	-105.	
BS/JB	0.54	0.286	0.545				0.541	0.285	0.564			0.56	0.273	0.571	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 4: Distribution of estimates:  $n = 100, \bar{\rho} = 0., \bar{\alpha}_1 = 0.$

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero						
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	-0.0192	0.105	-0.0207	0.107	-0.0212	0.106	-0.0262	0.108	-0.0145	0.0886	-0.0152	0.0893	-0.0162	0.0876	-0.0157	0.0887
$\beta_0$	1.02	0.163	0.991	0.162	1.02	0.168	0.985	0.168	1.01	0.138	0.991	0.138	1.01	0.137	1.01	0.137
$\beta_1$	0.993	0.106	0.998	0.106	0.994	0.108	1.01	0.108	0.994	0.107	1.	0.107	0.996	0.106	0.997	0.106
$\beta_2$	1.01	0.101	1.01	0.1	1.	0.102	1.01	0.101	1.01	0.1	1.01	0.0998	1.	0.103	1.01	0.102
$\beta_3$	0.991	0.0914	0.992	0.0914	0.991	0.0919	0.983	0.0919	0.992	0.0915	0.996	0.0913	0.993	0.0923	0.988	0.0921
$\sigma^2$									0.937	0.148	0.929	0.16				
$\alpha_0$													0.974	0.197	0.958	0.197
$\alpha_1$									-103.	7.81	-104.		-0.0691	0.277	-0.0535	0.285
R2,LH	0.761	0.0419	0.762		0.761	0.0419	0.762									
BS/JB	0.557	0.269	0.585		0.556	0.275	0.592		0.556	0.266	0.583		0.56	0.234	0.536	

SD: Standard deviation, RMSE: Root mean square error. Results from 99 successfully optimized replications are reported here.

Table 5: Distribution of estimates:  $n = 100, \bar{\rho} = 0.25, \bar{\alpha}_1 = 0.$

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero						
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.249	0.112	0.231	0.111	0.249	0.116	0.249	0.115	0.236	0.0773	0.233	0.0782	0.236	0.0774	0.23	0.0783
$\beta_0$	1.	0.182	0.992	0.181	0.999	0.186	0.987	0.185	1.02	0.125	1.02	0.126	1.02	0.131	1.02	0.132
$\beta_1$	0.997	0.11	1.01	0.11	1.	0.115	1.01	0.114	0.999	0.112	1.02	0.111	1.	0.117	1.02	0.116
$\beta_2$	0.988	0.115	0.983	0.115	0.989	0.115	0.987	0.115	0.99	0.112	0.983	0.112	0.99	0.112	0.98	0.112
$\beta_3$	1.	0.116	0.995	0.116	1.	0.116	1.	0.116	1.	0.114	0.995	0.114	1.	0.116	0.993	0.116
$\sigma^2$									0.976	0.139	0.956	0.141				
$\alpha_0$													0.954	0.219	0.928	0.222
$\alpha_1$									-106.	7.18	-106.		0.0477	0.307	0.0479	0.309
R2,LH	0.762	0.0429	0.767		0.761	0.0428	0.766									
BS/JB	0.529	0.282	0.525		0.529	0.284	0.508		0.533	0.289	0.531		-106.	7.25	-105.	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 6: Distribution of estimates:  $n = 100, \bar{\rho} = 0.5, \bar{\alpha}_1 = 0$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	0.518	0.1	0.528	0.518	0.099	0.1	0.494	0.0755	0.501	0.493	0.0758	0.499
$\beta_0$	0.969	0.223	0.953	0.967	0.221	0.948	1.01	0.169	1.	1.01	0.171	0.997
$\beta_1$	0.994	0.0958	1.	0.992	0.102	0.999	1.	0.0944	1.01	1.	0.0953	1.01
$\beta_2$	0.984	0.105	0.972	0.983	0.108	0.981	0.987	0.104	0.979	0.987	0.106	0.984
$\beta_3$	0.992	0.106	0.981	0.996	0.108	0.991	0.997	0.106	0.984	0.996	0.108	0.98
$\sigma^2$							0.94	0.131	0.935			
$\alpha_0$										0.927	0.181	0.887
$\alpha_1$	0.811	0.0376	0.816	0.811	0.0375	0.816	-108.	6.72	-107.	0.0289	0.278	0.0256
R2,LH	0.542	0.272	0.564	0.547	0.276	0.562	0.529	0.278	0.545	-107.	6.64	-107.
BS/JB										0.532	0.272	0.526

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 7: Distribution of estimates:  $n = 100, \bar{\rho} = -0.5, \bar{\alpha}_1 = 4$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	-0.523	0.162	-0.523	-0.525	0.159	-0.544	-0.503	0.0933	-0.512	-0.509	0.0734	-0.514
$\beta_0$	1.05	0.205	1.04	1.05	0.203	1.04	1.03	0.166	1.04	1.01	0.13	1.01
$\beta_1$	0.994	0.179	0.993	0.991	0.184	0.979	1.	0.18	1.	1.	0.129	1.01
$\beta_2$	0.979	0.16	0.994	0.978	0.154	0.99	0.99	0.161	0.998	0.99	0.131	0.999
$\beta_3$	0.986	0.183	0.988	0.982	0.182	0.966	0.993	0.181	0.983	0.995	0.139	0.993
$\sigma^2$							2.81	0.463	2.77			
$\alpha_0$										0.927	0.182	0.925
$\alpha_1$	0.628	0.0768	0.637	0.627	0.0761	0.639	-162.	8.04	-162.	3.94	1.05	3.86
R2,LH	0.3	0.305	0.219	0.294	0.299	0.241	0.293	0.31	0.195	-148.	6.32	-147.
BS/JB										0.573	0.26	0.565

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.



Table 8: Distribution of estimates:  $n = 100, \bar{\rho} = -0.25, \bar{\alpha}_1 = 4$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	-0.259	0.204	-0.241	0.203	-0.26	0.203	-0.26	0.202	-0.252	0.104	-0.264	0.104	-0.249	0.0836	-0.247	0.0832
$\beta_0$	1.03	0.245	1.01	0.245	1.03	0.241	1.01	0.242	1.02	0.178	1.01	0.178	1.	0.132	0.998	0.132
$\beta_1$	0.974	0.181	0.997	0.182	0.974	0.178	0.99	0.179	0.982	0.176	1.01	0.176	0.996	0.13	0.989	0.13
$\beta_2$	0.98	0.169	0.993	0.17	0.987	0.171	0.994	0.17	0.988	0.164	0.995	0.164	0.986	0.141	0.985	0.141
$\beta_3$	1.02	0.162	1.02	0.162	1.02	0.166	1.03	0.166	1.03	0.161	1.03	0.162	1.	0.139	0.991	0.138
$\sigma^2$									2.84	0.615	2.77	1.94				
$\alpha_0$													0.929	0.225	0.907	0.235
$\alpha_1$									3.94	1.24	3.89	1.23				
R2,LH	0.542	0.0786	0.548		0.541	0.0795	0.548		-159.	11.1	-159.					
BS/JB	0.222	0.26	0.105		0.211	0.255	0.108		0.199	0.24	0.0834					

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 9: Distribution of estimates:  $n = 100, \bar{\rho} = 0, \bar{\alpha}_1 = 4$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.0209	0.194	0.0451	0.194	0.0244	0.2	0.0348	0.201	0.00167	0.107	-0.0143	0.106	0.00257	0.0889	0.00553	0.0885
$\beta_0$	0.981	0.261	0.979	0.26	0.984	0.262	0.99	0.261	1.	0.186	0.987	0.185	0.995	0.143	1.	0.142
$\beta_1$	0.993	0.17	1.	0.17	0.992	0.171	1.	0.17	0.998	0.169	1.01	0.168	0.997	0.124	1.	0.123
$\beta_2$	0.977	0.192	0.975	0.193	0.973	0.197	0.951	0.198	0.98	0.195	0.978	0.195	0.971	0.159	0.965	0.161
$\beta_3$	0.978	0.178	0.971	0.179	0.979	0.177	0.975	0.177	0.982	0.179	0.973	0.179	0.991	0.145	0.994	0.145
$\sigma^2$									2.75	0.463	2.71	1.81				
$\alpha_0$													0.93	0.188	0.913	0.2
$\alpha_1$													3.79	1.01	3.75	1.02
R2,LH	0.516	0.0738	0.527		0.515	0.0749	0.526		-157.	8.43	-157.					
BS/JB	0.256	0.315	0.0936		0.245	0.311	0.0778		0.238	0.309	0.0764					

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 10: Distribution of estimates:  $n = 100, \bar{\rho} = 0.25, \bar{\alpha}_1 = 4$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.228	0.238	0.238	0.238	0.233	0.24	0.236	0.239	0.228	0.101	0.243	0.103	0.229	0.0916	0.234	0.0935
$\beta_0$	1.04	0.388	0.993	0.388	1.04	0.398	0.966	0.398	1.04	0.224	1.04	0.227	1.04	0.175	1.02	0.178
$\beta_1$	0.967	0.193	1.	0.195	0.964	0.195	1.01	0.198	0.978	0.182	1.01	0.182	0.974	0.138	0.968	0.14
$\beta_2$	0.963	0.204	0.965	0.207	0.964	0.205	0.977	0.207	0.971	0.2	0.966	0.202	0.983	0.144	0.971	0.145
$\beta_3$	0.984	0.182	1.01	0.181	0.987	0.186	1.01	0.185	0.993	0.183	1.02	0.182	0.997	0.16	1.01	0.16
$\sigma^2$									2.87	0.595	2.82	1.96				
$\alpha_0$													0.931	0.207	0.904	0.217
$\alpha_1$									3.93	1.05	3.92	1.05	3.93	1.05	3.92	1.05
R2,LH	0.512	0.102	0.529		0.512	0.103	0.529		-160.	10.3	-160.		-145.	9.17	-145.	
BS/JB	0.289	0.319	0.166		0.28	0.32	0.132		0.28	0.328	0.11		0.588	0.266	0.592	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 11: Distribution of estimates:  $n = 100, \bar{\rho} = 0.5, \bar{\alpha}_1 = 4$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.489	0.141	0.474	0.141	0.493	0.141	0.486	0.14	0.468	0.0896	0.475	0.0948	0.483	0.0663	0.48	0.068
$\beta_0$	1.01	0.349	1.01	0.347	1.01	0.352	1.	0.35	1.06	0.219	1.04	0.226	1.04	0.168	1.02	0.173
$\beta_1$	1.	0.167	0.995	0.166	1.	0.175	0.998	0.174	1.01	0.169	1.01	0.169	1.02	0.123	1.02	0.123
$\beta_2$	1.02	0.178	1.02	0.178	1.01	0.185	1.02	0.184	1.03	0.172	1.04	0.174	0.997	0.139	1.01	0.138
$\beta_3$	0.992	0.181	0.998	0.18	0.993	0.177	1.	0.176	0.999	0.178	0.995	0.177	0.999	0.124	1.01	0.123
$\sigma^2$									2.86	0.541	2.82	1.94				
$\alpha_0$													0.937	0.201	0.926	0.209
$\alpha_1$									3.99	1.12	3.9	1.11	3.99	1.12	3.9	1.11
R2,LH	0.616	0.0779	0.626		0.616	0.0774	0.624		-162.	9.5	-163.		-148.	7.78	-148.	
BS/JB	0.204	0.258	0.0633		0.2	0.255	0.0663		0.203	0.259	0.0633		0.519	0.264	0.538	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 12: Distribution of estimates:  $n = 100, \bar{\rho} = -0.5, \bar{\alpha}_1 = 8$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	-0.505	0.241	0.24	-0.502	0.24	0.239	-0.5	0.085	0.0846	-0.503	0.0645	0.0643
$\beta_0$	0.956	0.273	0.276	0.953	0.265	0.268	0.953	0.229	0.233	0.979	0.14	0.14
$\beta_1$	0.994	0.234	0.233	0.994	0.221	0.22	1.	0.22	0.219	1.02	0.135	0.136
$\beta_2$	1.	0.247	0.246	1.02	0.226	0.225	1.02	0.233	0.233	1.03	0.14	0.143
$\beta_3$	0.959	0.217	0.22	0.963	0.211	0.214	0.974	0.215	0.215	1.01	0.153	0.153
$\sigma^2$							4.82	1.01	3.95			
$\alpha_0$										0.891	0.179	0.209
$\alpha_1$										8.19	1.96	1.96
R2,LH	0.521	0.105	0.526	0.519	0.106	0.53	-189.	9.87	-189.	-163.	8.72	-163.
BS/JB	0.114	0.204	0.0188	0.109	0.201	0.011	0.102	0.194	0.00853	0.553	0.224	0.588

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 13: Distribution of estimates:  $n = 100, \bar{\rho} = -0.25, \bar{\alpha}_1 = 8$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	-0.226	0.278	0.278	-0.23	0.277	0.276	-0.244	0.108	0.107	-0.25	0.075	0.0746
$\beta_0$	0.95	0.291	0.293	0.96	0.288	0.29	0.962	0.197	0.199	1.01	0.137	0.137
$\beta_1$	0.942	0.216	0.223	0.939	0.213	0.22	0.956	0.215	0.219	0.985	0.13	0.131
$\beta_2$	0.968	0.226	0.227	0.973	0.223	0.223	0.988	0.216	0.215	1.01	0.137	0.137
$\beta_3$	1.04	0.236	0.239	1.04	0.238	0.24	1.05	0.235	0.239	1.02	0.141	0.141
$\sigma^2$							4.69	0.887	3.8			
$\alpha_0$										0.906	0.208	0.228
$\alpha_1$										7.94	1.8	1.79
R2,LH	0.414	0.089	0.43	0.413	0.0887	0.427	-185.	9.52	-185.	-159.	8.91	-160.
BS/JB	0.15	0.246	0.0134	0.14	0.234	0.00864	0.126	0.228	0.00647	0.51	0.273	0.5

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 14: Distribution of estimates:  $n = 100, \bar{\rho} = 0, \bar{\alpha}_1 = 8$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero							
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE					
$\rho$	0.00435	0.279	0.00622	0.277	0.00622	0.277	-0.00206	0.275	0.273	-0.00786	0.112	-0.0252	0.112	-0.00645	0.09	-0.00332	0.0898
$\beta_0$	0.945	0.356	0.908	0.359	0.908	0.359	0.956	0.355	0.356	0.955	0.238	0.947	0.241	0.988	0.142	0.985	0.141
$\beta_1$	1.02	0.258	1.05	0.257	1.06	0.255	1.02	0.256	0.255	1.03	0.254	1.07	0.255	0.995	0.154	1.01	0.154
$\beta_2$	0.973	0.231	0.971	0.231	0.979	0.227	0.979	0.228	0.227	0.991	0.222	0.975	0.221	0.996	0.157	1.01	0.156
$\beta_3$	1.01	0.237	0.983	0.236	1.	0.232	1.	0.233	0.232	1.02	0.235	0.996	0.234	1.01	0.145	1.02	0.145
$\sigma^2$										4.88	0.998	4.82	4.01				
$\alpha_0$														0.906	0.207	0.887	0.226
$\alpha_1$	0.39	0.0948	0.388		0.39	0.0945	0.389			-186.	10.5	-186.		8.26	2.04	7.87	2.05
R2,LH	0.113	0.21	0.00843		0.109	0.205	0.00674			0.108	0.211	0.00554		-160.	8.6	-159.	
BS/JB														0.554	0.274	0.547	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 15: Distribution of estimates:  $n = 100, \bar{\rho} = 0.25, \bar{\alpha}_1 = 8$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero							
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE					
$\rho$	0.243	0.284	0.255	0.282	0.249	0.262	0.271	0.261	0.261	0.232	0.112	0.233	0.113	0.24	0.0744	0.243	0.0748
$\beta_0$	0.973	0.399	0.97	0.398	0.968	0.359	0.942	0.359	0.359	0.993	0.249	0.964	0.248	0.984	0.144	0.958	0.144
$\beta_1$	0.998	0.239	0.981	0.238	0.989	0.229	0.977	0.228	0.228	1.01	0.237	0.985	0.236	1.01	0.144	1.01	0.144
$\beta_2$	0.956	0.228	0.942	0.231	0.961	0.218	0.951	0.22	0.22	0.974	0.225	0.951	0.226	1.01	0.149	0.991	0.149
$\beta_3$	0.96	0.264	0.946	0.266	0.953	0.261	0.943	0.264	0.264	0.981	0.258	0.968	0.258	0.987	0.168	0.979	0.167
$\sigma^2$										4.61	0.905	4.49	3.72				
$\alpha_0$														0.912	0.199	0.902	0.217
$\alpha_1$	0.409	0.103	0.418		0.41	0.103	0.414			-184.	9.86	-183.		7.83	1.76	7.55	1.76
R2,LH	0.0822	0.161	0.00408		0.0774	0.153	0.00125			0.0812	0.181	0.00129		-158.	9.29	-158.	
BS/JB														0.512	0.287	0.538	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 16: Distribution of estimates:  $n = 100, \bar{\rho} = 0.5, \bar{\alpha}_1 = 8$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.484	0.24	0.508	0.239	0.472	0.248	0.495	0.248	0.463	0.108	0.458	0.114	0.478	0.0728	0.483	0.0758
$\beta_0$	0.999	0.503	0.934	0.501	1.03	0.53	0.932	0.528	1.05	0.269	1.05	0.273	1.03	0.164	1.03	0.166
$\beta_1$	0.979	0.227	0.965	0.227	0.982	0.224	0.964	0.224	0.996	0.221	0.975	0.22	1.01	0.12	1.01	0.119
$\beta_2$	0.997	0.209	0.998	0.208	1.	0.207	0.99	0.206	1.01	0.204	1.	0.204	1.02	0.148	1.01	0.148
$\beta_3$	0.977	0.28	0.983	0.279	0.986	0.273	0.979	0.272	0.993	0.278	0.987	0.276	1.01	0.144	1.01	0.144
$\sigma^2$									4.83	0.965	4.83	3.94				
$\alpha_0$													0.907	0.203	0.896	0.222
$\alpha_1$	0.502	0.113	0.521		0.5	0.115	0.522		-188.	10.5	-189.		8.19	1.86	8.15	1.86
R2,LH	0.142	0.243	0.00887		0.138	0.239	0.00807		0.114	0.196	0.00574		-162.	9.97	-162.	
BS/JB													0.574	0.255	0.612	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 17: Distribution of estimates:  $n = 100, \bar{\rho} = -0.5, \bar{\alpha}_1 = 12$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	-0.445	0.255	-0.462	0.26	-0.451	0.261	-0.462	0.264	-0.489	0.0946	-0.499	0.0948	-0.493	0.0553	-0.491	0.0554
$\beta_0$	0.927	0.315	0.921	0.322	0.925	0.32	0.915	0.327	0.954	0.28	0.964	0.282	0.974	0.135	0.959	0.137
$\beta_1$	1.	0.279	0.97	0.278	1.	0.275	0.995	0.274	1.01	0.28	0.987	0.279	1.01	0.16	1.03	0.16
$\beta_2$	0.968	0.243	0.936	0.244	0.981	0.242	0.952	0.241	0.978	0.225	0.96	0.225	0.992	0.151	0.988	0.151
$\beta_3$	1.01	0.273	0.996	0.272	1.01	0.282	1.01	0.281	1.01	0.265	1.	0.264	0.993	0.128	0.995	0.128
$\sigma^2$									6.71	1.43	6.5	5.89				
$\alpha_0$													0.907	0.191	0.895	0.212
$\alpha_1$	0.449	0.114	0.458		0.448	0.114	0.463		-205.	10.7	-205.		12.1	2.9	11.6	2.88
R2,LH	0.0712	0.168	0.00446		0.0687	0.163	0.00277		0.0497	0.111	0.00313		-172.	9.14	-172.	
BS/JB													0.565	0.262	0.581	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 18: Distribution of estimates:  $n = 100, \bar{\rho} = -0.25, \bar{\alpha}_1 = 12$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	-0.284	0.319	0.319	-0.279	0.312	0.311	-0.262	0.109	0.109	-0.255	0.0778	0.0776
$\beta_0$	1.05	0.38	0.382	1.05	0.371	0.372	1.03	0.27	0.27	0.984	0.151	0.151
$\beta_1$	0.971	0.26	0.261	0.972	0.262	0.262	0.991	0.242	0.241	0.982	0.148	0.148
$\beta_2$	0.998	0.283	0.281	1.	0.283	0.282	1.02	0.274	0.273	1.02	0.145	0.146
$\beta_3$	0.964	0.309	0.31	0.962	0.313	0.313	0.99	0.293	0.292	1.	0.162	0.161
$\sigma^2$							6.78	1.42	5.95			
$\alpha_0$										0.898	0.179	0.205
$\alpha_1$							-203.	10.9	-203.	12.1	2.83	2.81
R2,LH	0.349	0.0985	0.352	0.346	0.0986	0.352	0.0508	0.118	0.000958	-170.	9.98	-171.
BS/JB	0.0775	0.169	0.00374	0.073	0.161	0.00226				0.609	0.242	0.601

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 19: Distribution of estimates:  $n = 100, \bar{\rho} = 0., \bar{\alpha}_1 = 12$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	-0.01	0.269	0.268	-0.00897	0.271	0.269	-0.0137	0.131	0.131	-0.000636	0.0798	0.0794
$\beta_0$	0.998	0.405	0.403	0.998	0.402	0.4	0.988	0.278	0.277	0.964	0.152	0.155
$\beta_1$	0.967	0.299	0.299	0.966	0.297	0.297	0.981	0.289	0.288	0.993	0.151	0.151
$\beta_2$	0.993	0.277	0.276	0.991	0.275	0.273	1.	0.265	0.264	0.988	0.124	0.124
$\beta_3$	0.99	0.251	0.25	0.984	0.252	0.251	1.01	0.25	0.249	0.992	0.13	0.13
$\sigma^2$							6.4	1.3	5.55			
$\alpha_0$										0.864	0.216	0.254
$\alpha_1$							-199.	10.2	-199.	11.5	2.58	2.61
R2,LH	0.326	0.103	0.324	0.325	0.103	0.32	0.0699	0.187	0.000795	-166.	9.82	-165.
BS/JB	0.0812	0.199	0.00108	0.0723	0.187	0.00105				0.56	0.261	0.558

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 20: Distribution of estimates:  $n = 100, \bar{\rho} = 0.25, \bar{\alpha}_1 = 12$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.329	0.336	0.31	0.344	0.321	0.311	0.308	0.317	0.225	0.117	0.238	0.119	0.232	0.0779	0.229	0.0796
$\beta_0$	0.893	0.497	0.882	0.506	0.903	0.486	0.894	0.493	1.03	0.275	1.01	0.275	1.02	0.151	1.02	0.151
$\beta_1$	0.927	0.282	0.939	0.29	0.941	0.285	0.914	0.29	0.959	0.268	0.946	0.27	0.988	0.159	0.986	0.159
$\beta_2$	1.01	0.29	1.04	0.289	1.02	0.294	1.01	0.293	1.02	0.28	1.01	0.279	1.02	0.138	1.01	0.139
$\beta_3$	0.953	0.282	0.972	0.284	0.955	0.281	0.956	0.283	0.977	0.258	0.982	0.258	0.997	0.152	1.	0.152
$\sigma^2$									6.63	1.44	6.5	5.81				
$\alpha_0$													0.91	0.188	0.912	0.207
$\alpha_1$													11.9	2.65	11.5	2.64
R2,LH	0.338	0.102	0.353		0.338	0.0992	0.344		-202.	10.9	-202.		-169.	9.95	-170.	
BS/JB	0.087	0.172	0.00432		0.0831	0.17	0.0053		0.0763	0.166	0.0026		0.57	0.268	0.553	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 21: Distribution of estimates:  $n = 100, \bar{\rho} = 0.5, \bar{\alpha}_1 = 12$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.458	0.353	0.523	0.354	0.45	0.366	0.52	0.368	0.459	0.0959	0.473	0.104	0.479	0.0692	0.477	0.072
$\beta_0$	1.07	0.81	0.915	0.809	1.09	0.835	0.894	0.836	1.08	0.316	1.07	0.324	1.04	0.19	1.03	0.193
$\beta_1$	0.965	0.265	0.963	0.266	0.962	0.261	0.962	0.262	0.992	0.265	0.977	0.264	0.991	0.127	0.993	0.126
$\beta_2$	0.951	0.247	0.943	0.25	0.963	0.249	0.966	0.25	0.969	0.235	0.973	0.236	1.01	0.144	0.991	0.144
$\beta_3$	1.	0.291	1.03	0.29	1.01	0.29	1.02	0.289	1.03	0.291	1.05	0.291	0.996	0.137	0.993	0.136
$\sigma^2$									6.79	1.34	6.82	5.95				
$\alpha_0$													0.896	0.183	0.88	0.209
$\alpha_1$													12.1	2.42	12.1	2.41
R2,LH	0.434	0.124	0.455		0.43	0.127	0.453		-205.	10.3	-207.		-172.	9.72	-172.	
BS/JB	0.0861	0.169	0.0016		0.0805	0.164	0.00119		0.0709	0.163	0.000561		0.559	0.263	0.568	

SD: Standard deviation, RMSE: Root mean square error. Results from 100 successfully optimized replications are reported here.

Table 22: Distribution of estimates:  $n = 400, \bar{\rho} = -0.5, \bar{\alpha}_1 = 0$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero						
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	-0.489	0.0518	-0.485	0.0524	-0.488	0.0532	-0.483	0.054	-0.492	0.037	-0.49	0.0375	-0.493	0.0372	-0.491	0.0376
$\beta_0$	0.984	0.0611	0.98	0.0628	0.983	0.0616	0.981	0.0635	0.985	0.0517	0.991	0.0533	0.986	0.0515	0.993	0.053
$\beta_1$	1.	0.0497	1.	0.0492	1.	0.0483	1.	0.0479	1.	0.0497	1.	0.0493	1.	0.0505	1.	0.0501
$\beta_2$	0.999	0.058	1.	0.0575	1.	0.0577	1.	0.0572	0.999	0.0587	1.01	0.0582	0.999	0.0584	1.01	0.0579
$\beta_3$	1.01	0.0437	1.	0.0438	1.01	0.0443	1.	0.0444	1.01	0.0431	1.01	0.0432	1.01	0.0423	1.01	0.0426
$\sigma^2$									0.988	0.0573	0.982	0.058				
$\alpha_0$													1.	0.104	0.999	0.103
$\alpha_1$									-440.	11.7	-439.		-0.0244	0.181	-0.0342	0.181
R2,LH	0.801	0.0178	0.803		0.801	0.018	0.803									
BS/JB	0.506	0.277	0.486		0.507	0.278	0.488		0.51	0.275	0.462		0.526	0.285	0.49	

SD: Standard deviation, RMSE: Root mean square error. Results from 57 successfully optimized replications are reported here.

Table 23: Distribution of estimates:  $n = 400, \bar{\rho} = -0.25, \bar{\alpha}_1 = 0$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero						
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	-0.247	0.0535	-0.249	0.0532	-0.247	0.0543	-0.25	0.054	-0.245	0.0407	-0.245	0.0407	-0.245	0.0404	-0.245	0.0405
$\beta_0$	0.994	0.0573	0.987	0.0573	0.993	0.0585	0.991	0.0586	0.992	0.0491	0.984	0.0494	0.992	0.0488	0.987	0.0492
$\beta_1$	1.	0.0543	1.01	0.0542	1.	0.0538	1.	0.0537	1.01	0.0539	1.01	0.0539	1.01	0.0546	1.01	0.0546
$\beta_2$	0.996	0.0493	0.997	0.0492	0.996	0.049	0.996	0.0489	0.996	0.0493	0.998	0.0491	0.997	0.0491	0.997	0.049
$\beta_3$	1.	0.0476	1.01	0.0474	1.	0.0479	1.01	0.0478	1.	0.0475	1.01	0.0473	1.	0.0481	1.	0.048
$\sigma^2$									0.983	0.0691	0.977	0.0707				
$\alpha_0$													0.981	0.101	0.963	0.102
$\alpha_1$									-428.	13.9	-427.		0.00369	0.15	-0.0114	0.149
R2,LH	0.765	0.0204	0.767		0.765	0.0204	0.767									
BS/JB	0.507	0.274	0.479		0.507	0.275	0.469		0.509	0.274	0.447		0.517	0.266	0.471	

SD: Standard deviation, RMSE: Root mean square error. Results from 83 successfully optimized replications are reported here.



Table 24: Distribution of estimates:  $n = 400, \bar{\rho} = 0, \bar{\alpha}_1 = 0$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	-0.000109	0.0513	0.000703	0.051	0.0677	0.0514	-0.000813	0.0376	0.0374	-0.000858	0.0379	0.0377
$\beta_0$	1.	0.0681	1.	0.0677	1.	0.0689	1.	0.061	0.0607	1.	0.0612	0.0609
$\beta_1$	1.01	0.0445	1.	0.0445	1.	0.0451	1.01	0.0442	0.0443	1.01	0.0451	0.0452
$\beta_2$	1.01	0.0498	1.01	0.05	1.	0.0507	1.01	0.0498	0.05	1.01	0.0501	0.0503
$\beta_3$	0.998	0.045	0.998	0.0448	0.998	0.045	0.999	0.0452	0.0449	0.999	0.0457	0.0455
$\sigma^2$							0.992	0.0699	0.0699			
$\alpha_0$										0.99	0.0944	0.0943
$\alpha_1$										0.00458	0.132	0.131
R2,LH	0.751	0.0204	0.75	0.749			-427.	14.1	-427.			
BS/JB	0.498	0.291	0.528	0.524			0.502	0.294	0.527			

SD: Standard deviation, RMSE: Root mean square error. Results from 90 successfully optimized replications are reported here.

Table 25: Distribution of estimates:  $n = 400, \bar{\rho} = 0.25, \bar{\alpha}_1 = 0$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	0.247	0.055	0.249	0.0547	0.248	0.0559	0.252	0.0556	0.244	0.0467	0.246	0.0468
$\beta_0$	0.994	0.0943	0.984	0.094	0.993	0.0956	0.99	0.0953	0.997	0.0796	0.994	0.0792
$\beta_1$	1.01	0.0459	1.01	0.0462	1.01	0.047	1.01	0.0472	1.01	0.0455	1.01	0.0459
$\beta_2$	1.	0.0478	1.	0.0476	1.	0.048	0.999	0.0478	1.	0.0478	1.	0.0477
$\beta_3$	0.991	0.0529	0.996	0.0534	0.991	0.0525	1.	0.053	0.991	0.053	0.996	0.0534
$\sigma^2$									0.993	0.0631	0.994	0.0631
$\alpha_0$												
$\alpha_1$												
R2,LH	0.763	0.0219	0.765	0.765	0.763	0.0219	0.765	-431.	12.8	-431.		
BS/JB	0.558	0.273	0.584	0.582	0.558	0.272	0.582	0.555	0.274	0.574		

SD: Standard deviation, RMSE: Root mean square error. Results from 83 successfully optimized replications are reported here.

Table 26: Distribution of estimates:  $n = 400, \bar{\rho} = 0.5, \bar{\alpha}_1 = 0$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	0.499	0.047	0.0466	0.499	0.0475	0.0471	0.487	0.0339	0.036	0.487	0.0341	0.0361
$\beta_0$	1.	0.107	0.106	1.	0.109	0.108	1.03	0.0827	0.0864	1.03	0.0833	0.0869
$\beta_1$	1.01	0.0541	0.0542	1.01	0.0543	0.0544	1.01	0.0532	0.0536	1.01	0.053	0.0533
$\beta_2$	1.	0.0479	0.0475	1.	0.0476	0.0472	1.	0.0473	0.047	1.	0.0472	0.0469
$\beta_3$	0.998	0.0486	0.0483	0.999	0.0481	0.0477	1.	0.0489	0.0485	1.	0.049	0.0486
$\sigma^2$							0.988	0.0747	0.0751			
$\alpha_0$										0.972	0.11	0.113
$\alpha_1$							-440.	15.	-440.	0.0306	0.163	0.165
R2,LH	0.805	0.0217	0.804	0.804	0.0217	0.803				-439.	15.	-440.
BS/JB	0.518	0.28	0.544	0.517	0.282	0.546	0.52	0.277	0.531	0.533	0.267	0.574

SD: Standard deviation, RMSE: Root mean square error. Results from 61 successfully optimized replications are reported here.

Table 27: Distribution of estimates:  $n = 400, \bar{\rho} = -0.5, \bar{\alpha}_1 = 4$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	-0.514	0.0736	0.0746	-0.514	0.0739	0.0748	-0.504	0.0416	0.0416	-0.503	0.0392	0.0391
$\beta_0$	0.999	0.094	0.0935	1.	0.0948	0.0943	0.992	0.0845	0.0844	0.998	0.0682	0.0679
$\beta_1$	0.996	0.0849	0.0846	0.996	0.085	0.0846	0.999	0.085	0.0845	0.997	0.0608	0.0606
$\beta_2$	1.	0.0989	0.0984	0.999	0.1	0.0998	1.	0.0988	1.02	1.01	0.0722	1.02
$\beta_3$	0.995	0.0956	0.0952	0.995	0.0951	0.0947	0.998	0.0957	1.	0.996	0.0783	0.078
$\sigma^2$							3.02	0.26	2.04			
$\alpha_0$										0.985	0.113	0.113
$\alpha_1$							-664.	17.7	-667.	4.07	0.467	4.07
R2,LH	0.607	0.0346	0.61	0.607	0.0348	0.611				-605.	17.8	-604.
BS/JB	0.00528	0.0213	0	0.00475	0.0185	0	0.00542	0.0227	0	0.535	0.285	0.58

SD: Standard deviation, RMSE: Root mean square error. Results from 93 successfully optimized replications are reported here.

Table 28: Distribution of estimates:  $n = 400, \bar{\rho} = -0.25, \bar{\alpha}_1 = 4$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	-0.254	0.0912	0.0908	-0.254	0.0891	0.0888	-0.252	0.0523	0.052	-0.251	0.0442	0.044
$\beta_0$	0.999	0.106	0.106	0.998	0.104	0.104	0.997	0.0949	0.0945	0.998	0.0733	0.0729
$\beta_1$	1.01	0.085	0.0848	1.01	0.0859	0.0856	1.01	0.0856	0.0854	1.01	0.0623	0.0626
$\beta_2$	1.	0.0895	0.089	1.	0.0919	0.0915	1.	0.0898	0.0893	1.	0.0691	0.0687
$\beta_3$	1.01	0.0939	0.0938	1.01	0.0945	0.0942	1.01	0.0925	0.0925	1.01	0.0722	0.0728
$\sigma^2$							2.96	0.258	1.97			
$\alpha_0$										0.975	0.0887	0.0918
$\alpha_1$										4.04	0.464	4.
R2,LH	0.533	0.0377	0.529	0.533	0.0376	0.529	-649.	17.4	-650.	-589.	16.6	-588.
BS/JB	0.00625	0.0173	0	0.00623	0.0174	0	0.00603	0.0167	0	0.514	0.267	0.502

SD: Standard deviation, RMSE: Root mean square error. Results from 98 successfully optimized replications are reported here.

Table 29: Distribution of estimates:  $n = 400, \bar{\rho} = 0., \bar{\alpha}_1 = 4$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	0.0225	0.0961	0.0223	0.0208	0.0953	0.0232	0.00617	0.0582	0.00761	0.00208	0.0473	0.00272
$\beta_0$	0.971	0.125	0.986	0.972	0.126	0.979	0.988	0.0943	0.992	0.99	0.0754	0.989
$\beta_1$	0.975	0.0852	0.982	0.976	0.0858	0.982	0.976	0.0847	0.983	0.98	0.0628	0.98
$\beta_2$	1.01	0.0844	1.01	1.01	0.0852	1.	1.01	0.086	1.01	1.01	0.0635	1.
$\beta_3$	0.987	0.0838	0.989	0.986	0.0843	0.989	0.989	0.0838	0.994	0.999	0.0638	1.
$\sigma^2$							3.01	0.288	3.01			
$\alpha_0$										0.987	0.0986	0.984
$\alpha_1$										4.06	0.54	4.05
R2,LH	0.5	0.0404	0.499	0.5	0.0404	0.499	-649.	19.2	-650.	-590.	16.6	-592.
BS/JB	0.011	0.044	0	0.0107	0.0429	0	0.00894	0.0343	0	0.547	0.286	0.605

SD: Standard deviation, RMSE: Root mean square error. Results from 92 successfully optimized replications are reported here.

Table 30: Distribution of estimates:  $n = 400, \bar{\rho} = 0.25, \bar{\alpha}_1 = 4$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.26	0.0867	0.263	0.0869	0.26	0.0872	0.259	0.0873	0.245	0.0593	0.251	0.0593	0.25	0.0457	0.253	0.0455
$\beta_0$	0.991	0.165	0.991	0.165	0.99	0.166	0.982	0.166	1.01	0.133	1.	0.133	1.01	0.0972	1.	0.097
$\beta_1$	0.998	0.089	1.01	0.0886	0.997	0.0906	1.	0.0902	1.	0.088	1.01	0.0875	0.998	0.0647	0.995	0.0644
$\beta_2$	1.01	0.089	1.01	0.0887	1.01	0.0903	1.	0.09	1.01	0.0891	1.01	0.089	1.01	0.0632	1.	0.0633
$\beta_3$	0.997	0.0832	0.991	0.0828	0.997	0.084	0.986	0.0836	1.	0.0834	0.994	0.083	1.	0.0619	0.998	0.0616
$\sigma^2$									2.97	0.28	2.95	1.99				
$\alpha_0$													0.979	0.109	0.972	0.111
$\alpha_1$									4.	0.521	4.	0.518				
R2,LH	0.526	0.0376	0.531		0.526	0.0376	0.531		-650.	18.	-648.					
BS/JB	0.0109	0.0569	0		0.011	0.0568	0		0.0112	0.0565	0		0.497	0.265	0.458	

SD: Standard deviation, RMSE: Root mean square error. Results from 99 successfully optimized replications are reported here.

Table 31: Distribution of estimates:  $n = 400, \bar{\rho} = 0.5, \bar{\alpha}_1 = 4$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.514	0.082	0.513	0.0827	0.514	0.0829	0.511	0.0837	0.495	0.0459	0.497	0.0459	0.499	0.0383	0.496	0.0381
$\beta_0$	0.972	0.179	0.963	0.181	0.97	0.181	0.97	0.182	1.01	0.112	1.	0.112	1.01	0.0881	1.	0.0878
$\beta_1$	0.996	0.091	0.988	0.0906	0.996	0.0902	0.988	0.0899	1.	0.0904	0.991	0.0899	1.	0.0738	1.01	0.0735
$\beta_2$	1.01	0.0877	1.01	0.0877	1.01	0.0882	1.02	0.0882	1.01	0.0887	1.02	0.0892	1.01	0.0725	1.02	0.0726
$\beta_3$	0.993	0.0781	0.996	0.078	0.991	0.0773	0.994	0.0774	0.998	0.0768	1.	0.0765	0.995	0.0597	0.999	0.0596
$\sigma^2$									2.94	0.255	2.91	1.95				
$\alpha_0$													0.994	0.0933	0.995	0.0929
$\alpha_1$									3.9	0.424	3.92	0.434				
R2,LH	0.61	0.0439	0.618		0.61	0.0438	0.618		-658.	17.7	-659.					
BS/JB	0.0171	0.0589	0		0.0166	0.0575	0		0.0173	0.0597	0		0.494	0.284	0.482	

SD: Standard deviation, RMSE: Root mean square error. Results from 94 successfully optimized replications are reported here.

Table 32: Distribution of estimates:  $n = 400, \bar{\rho} = -0.5, \bar{\alpha}_1 = 8$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero						
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	-0.505	0.11	-0.5	0.109	-0.505	0.107	-0.498	0.107	-0.494	0.0531	-0.498	0.0532	-0.495	0.0375	-0.497	0.0376
$\beta_0$	0.984	0.139	0.985	0.14	0.985	0.138	0.985	0.138	0.977	0.122	0.967	0.123	0.985	0.0759	0.986	0.077
$\beta_1$	1.	0.104	1.	0.104	1.	0.107	1.	0.106	1.01	0.102	1.01	0.102	1.	0.0641	1.02	0.0638
$\beta_2$	0.986	0.0896	0.985	0.0902	0.989	0.0895	0.987	0.0897	0.992	0.0885	0.99	0.0884	1.01	0.0673	1.	0.0672
$\beta_3$	0.995	0.107	0.995	0.107	0.995	0.109	0.996	0.108	1.	0.105	0.998	0.105	1.	0.0623	1.01	0.0619
$\sigma^2$									4.88	0.44	4.87	3.91				
$\alpha_0$													0.982	0.0935	0.989	0.0946
$\alpha_1$									-760.	17.7	-760.		7.95	0.84	7.88	0.837
R2,LH	0.512	0.0521	0.515		0.512	0.0514	0.516						-658.	17.	-658.	
BS/JB	0.000388	0.00293	0		0.0004	0.00302	0		0.000386	0.00331	0		0.501	0.279	0.511	

SD: Standard deviation, RMSE: Root mean square error. Results from 90 successfully optimized replications are reported here.

Table 33: Distribution of estimates:  $n = 400, \bar{\rho} = -0.25, \bar{\alpha}_1 = 8$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero						
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	-0.258	0.124	-0.274	0.123	-0.257	0.123	-0.274	0.123	-0.248	0.0552	-0.251	0.0549	-0.248	0.0405	-0.249	0.0403
$\beta_0$	0.993	0.147	1.	0.147	0.993	0.148	1.01	0.148	0.985	0.126	0.996	0.126	0.99	0.0766	0.989	0.0768
$\beta_1$	0.984	0.105	0.996	0.106	0.987	0.107	0.995	0.107	0.988	0.103	0.996	0.103	0.997	0.0625	1.	0.0623
$\beta_2$	0.981	0.121	0.966	0.122	0.981	0.121	0.973	0.122	0.985	0.119	0.976	0.12	0.992	0.0702	0.99	0.0703
$\beta_3$	1.	0.115	1.	0.114	1.	0.113	0.997	0.112	1.01	0.112	1.01	0.112	1.01	0.0735	1.01	0.0733
$\sigma^2$									4.92	0.464	4.89	3.95				
$\alpha_0$													0.973	0.1	0.967	0.103
$\alpha_1$									-750.	19.	-750.		7.95	0.862	7.9	0.859
R2,LH	0.406	0.0417	0.405		0.405	0.0417	0.404						-648.	17.1	-648.	
BS/JB	0	0.000361	0		0	0.000326	0		0	0.00036	0		0.54	0.253	0.514	

SD: Standard deviation, RMSE: Root mean square error. Results from 94 successfully optimized replications are reported here.

Table 34: Distribution of estimates:  $n = 400, \bar{\rho} = 0., \bar{\alpha}_1 = 8.$

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero						
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.0215	0.139	0.0195	0.14	0.0228	0.14	0.0235	0.141	-0.00273	0.0533	-0.00632	0.053	0.00137	0.0408	-0.00436	0.0406
$\beta_0$	0.965	0.171	0.952	0.173	0.965	0.172	0.953	0.174	0.992	0.121	0.996	0.12	1.01	0.0851	1.	0.0848
$\beta_1$	0.99	0.097	0.99	0.097	0.986	0.1	0.986	0.101	0.995	0.0952	0.994	0.0948	0.997	0.0634	0.994	0.0631
$\beta_2$	1.	0.116	1.01	0.115	1.	0.115	1.01	0.114	1.	0.116	1.01	0.116	1.	0.0698	1.	0.0694
$\beta_3$	1.01	0.105	1.01	0.105	1.01	0.105	1.01	0.105	1.01	0.106	1.01	0.106	1.	0.0787	1.01	0.0782
$\sigma^2$									5.04	0.528	5.02	4.07				
$\alpha_0$													0.977	0.09	0.974	0.0923
$\alpha_1$									-751.	21.1	-752.		8.2	1.01	8.14	1.03
R2,LH	0.374	0.0351	0.373		0.374	0.0353	0.373						-648.	16.5	-649.	
BS/JB	0.000207	0.0014	0		0.0002	0.00135	0		0.000179	0.00119	0		0.549	0.277	0.571	

SD: Standard deviation, RMSE: Root mean square error. Results from 86 successfully optimized replications are reported here.

Table 35: Distribution of estimates:  $n = 400, \bar{\rho} = 0.25, \bar{\alpha}_1 = 8.$

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero						
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.252	0.105	0.269	0.105	0.253	0.104	0.262	0.104	0.242	0.0545	0.242	0.0548	0.246	0.038	0.243	0.0379
$\beta_0$	1.02	0.171	1.02	0.171	1.02	0.169	1.03	0.169	1.03	0.127	1.02	0.13	1.01	0.0878	1.01	0.0883
$\beta_1$	1.	0.113	1.01	0.113	0.998	0.114	1.	0.113	1.	0.112	1.01	0.112	1.01	0.0762	1.	0.0765
$\beta_2$	1.01	0.113	1.02	0.113	1.01	0.113	1.03	0.114	1.02	0.112	1.02	0.113	1.01	0.0706	1.01	0.0705
$\beta_3$	1.	0.12	1.02	0.12	1.	0.12	1.01	0.119	1.01	0.122	1.03	0.122	0.997	0.0685	0.993	0.0682
$\sigma^2$									5.02	0.552	5.04	4.06				
$\alpha_0$													0.991	0.116	1.	0.116
$\alpha_1$									-754.	22.2	-757.		8.11	1.06	8.17	1.06
R2,LH	0.409	0.0423	0.406		0.409	0.0423	0.406						-651.	19.8	-654.	
BS/JB	0.00075	0.00668	0		0.000756	0.00687	0		0.000409	0.00369	0		0.524	0.285	0.492	

SD: Standard deviation, RMSE: Root mean square error. Results from 98 successfully optimized replications are reported here.

Table 36: Distribution of estimates:  $n = 400, \bar{\rho} = 0.5, \bar{\alpha}_1 = 8$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	0.491	0.105	0.104	0.493	0.105	0.105	0.486	0.0537	0.488	0.493	0.032	0.0325
$\beta_0$	1.	0.237	0.236	0.999	0.238	0.237	1.01	0.16	0.992	1.02	0.0915	0.0925
$\beta_1$	1.01	0.11	0.997	1.01	0.111	0.111	1.01	0.109	1.02	1.	0.0653	1. 0.0649
$\beta_2$	0.992	0.109	0.108	0.995	0.105	0.105	0.996	0.106	0.985	1.	0.0638	1.01 0.0636
$\beta_3$	0.99	0.144	0.143	0.987	0.141	0.141	0.993	0.144	1.	1.	0.066	0.992 0.0656
$\sigma^2$							4.95	0.5	4.92			3.98
$\alpha_0$										0.971	0.0858	0.968 0.09
$\alpha_1$										7.97	0.862	7.92 0.857
R2,LH	0.506	0.0537	0.504	0.506	0.0536	0.504	-761.	19.6	-761.	-660.	17.1	-661.
BS/JB	0	0.000218	0	0	0.000222	0	0	0.000191	0	0.547	0.264	0.577

SD: Standard deviation, RMSE: Root mean square error. Results from 85 successfully optimized replications are reported here.

Table 37: Distribution of estimates:  $n = 400, \bar{\rho} = -0.5, \bar{\alpha}_1 = 12$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	-0.504	0.127	-0.508	-0.5	0.126	-0.509	-0.496	0.0521	-0.501	-0.501	0.0307	-0.504 0.0306
$\beta_0$	1.01	0.173	0.992	1.01	0.171	0.978	1.01	0.131	0.993	0.994	0.0655	0.987 0.0654
$\beta_1$	0.973	0.134	0.963	0.975	0.134	0.967	0.979	0.133	0.958	1.	0.0709	0.991 0.0705
$\beta_2$	0.989	0.131	0.983	0.99	0.13	0.989	0.993	0.127	1.	1.	0.0649	1. 0.0645
$\beta_3$	0.983	0.135	0.987	0.984	0.136	0.99	0.988	0.135	0.993	0.989	0.0654	0.995 0.0659
$\sigma^2$							6.81	0.796	6.82			5.87
$\alpha_0$										0.991	0.084	0.997 0.084
$\alpha_1$										11.8	1.38	11.9 1.39
R2,LH	0.447	0.0583	0.447	0.446	0.0589	0.446	-826.	22.2	-827.	-696.	19.4	-695.
BS/JB	0	0	0	0	0	0	0	0	0	0.502	0.305	0.513

SD: Standard deviation, RMSE: Root mean square error. Results from 83 successfully optimized replications are reported here.

Table 38: Distribution of estimates:  $n = 400, \bar{\rho} = -0.25, \bar{\alpha}_1 = 12$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	-0.243	0.156	0.155	-0.247	0.156	0.155	-0.236	0.0641	0.0653	-0.24	0.0375	0.0386
$\beta_0$	1.	0.19	0.188	1.	0.19	0.189	0.996	0.147	0.146	0.99	0.0749	0.0752
$\beta_1$	1.01	0.139	0.138	1.01	0.139	0.139	1.02	0.139	0.139	1.02	0.0689	0.0725
$\beta_2$	0.995	0.133	0.133	0.993	0.131	0.13	1.	0.131	0.13	0.993	0.0696	0.0696
$\beta_3$	1.01	0.119	0.119	1.01	0.116	0.115	1.01	0.118	0.118	1.	0.0691	0.0687
$\sigma^2$							6.97	0.829	6.03			
$\alpha_0$										0.968	0.104	0.108
$\alpha_1$										12.2	1.38	1.38
R2,LH	0.337	0.0521	0.341	0.337	0.0525	0.341	-819.	23.7	-820.	-684.	21.9	-685.
BS/JB	0	0	0	0	0	0	0	0	0	0.564	0.268	0.569

SD: Standard deviation, RMSE: Root mean square error. Results from 92 successfully optimized replications are reported here.

Table 39: Distribution of estimates:  $n = 400, \bar{\rho} = 0., \bar{\alpha}_1 = 12$ .

	S-2SLS			S-2SLS-Robust			ML-Homo			ML-Hetero		
	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE	Mean	SD	RMSE
$\rho$	0.0188	0.134	0.135	0.0181	0.135	0.136	-0.0047	0.0555	0.0554	0.00121	0.0361	0.0359
$\beta_0$	1.	0.161	0.16	1.	0.162	0.161	1.02	0.125	0.127	1.	0.077	0.0765
$\beta_1$	1.	0.132	0.132	0.999	0.132	0.131	1.	0.132	0.131	0.996	0.0674	0.0671
$\beta_2$	0.974	0.121	0.123	0.975	0.122	0.124	0.978	0.12	0.122	0.982	0.0549	0.0574
$\beta_3$	1.	0.147	0.146	1.01	0.146	0.146	1.01	0.147	0.146	0.984	0.0618	0.0636
$\sigma^2$							6.9	0.742	5.95			
$\alpha_0$										0.976	0.103	0.969
$\alpha_1$										12.	1.31	12.
R2,LH	0.303	0.0464	0.304	0.303	0.0465	0.304	-814.	21.3	-817.	-682.	19.9	-679.
BS/JB	0	0.000115	0	0	0.000125	0	0	0.000167	0	0.538	0.263	0.519

SD: Standard deviation, RMSE: Root mean square error. Results from 81 successfully optimized replications are reported here.



Table 40: Distribution of estimates:  $n = 400, \bar{\rho} = 0.25, \bar{\alpha}_1 = 12$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.258	0.141	0.273	0.14	0.253	0.139	0.261	0.139	0.246	0.061	0.251	0.0608	0.248	0.0362	0.246	0.036
$\beta_0$	0.951	0.227	0.941	0.231	0.959	0.224	0.947	0.227	0.966	0.163	0.984	0.166	1.	0.0707	0.988	0.0704
$\beta_1$	1.01	0.137	0.993	0.136	1.01	0.135	0.995	0.134	1.01	0.136	0.994	0.136	1.	0.0766	1.	0.0762
$\beta_2$	0.999	0.135	1.01	0.134	0.997	0.135	1.	0.134	1.	0.134	1.01	0.134	1.01	0.0701	1.	0.0701
$\beta_3$	0.975	0.138	0.98	0.139	0.975	0.137	0.98	0.139	0.981	0.139	0.979	0.139	1.	0.0707	1.	0.0704
$\sigma^2$									6.89	0.69	6.8	5.93				
$\alpha_0$													0.978	0.0952	0.978	0.0972
$\alpha_1$									12.	1.29	12.	1.29	12.	1.29	12.	1.29
R2,LH	0.337	0.0466	0.331		0.337	0.0466	0.332		-818.	19.9	-816.		-684.	18.5	-685.	
BS/JB	0	0	0		0	0	0		0	0	0		0.533	0.294	0.565	

SD: Standard deviation, RMSE: Root mean square error. Results from 92 successfully optimized replications are reported here.

Table 41: Distribution of estimates:  $n = 400, \bar{\rho} = 0.5, \bar{\alpha}_1 = 12$ .

	S-2SLS				S-2SLS-Robust				ML-Homo				ML-Hetero			
	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE	Mean	SD	Median	RMSE
$\rho$	0.495	0.135	0.515	0.134	0.497	0.134	0.513	0.134	0.489	0.0512	0.497	0.0522	0.488	0.0287	0.49	0.0308
$\beta_0$	1.02	0.313	1.01	0.312	1.01	0.307	1.01	0.306	1.03	0.176	1.	0.178	1.03	0.0916	1.03	0.0964
$\beta_1$	1.	0.139	0.991	0.138	1.	0.136	0.993	0.136	1.	0.137	0.999	0.136	1.	0.07	0.988	0.0696
$\beta_2$	0.992	0.126	0.972	0.125	0.992	0.128	0.978	0.127	0.999	0.123	0.979	0.123	0.992	0.0604	0.988	0.0606
$\beta_3$	1.01	0.123	0.991	0.122	1.	0.119	0.991	0.119	1.01	0.12	1.01	0.12	1.	0.0635	0.996	0.0631
$\sigma^2$									6.96	0.804	6.89	6.01				
$\alpha_0$													0.988	0.108	0.996	0.108
$\alpha_1$									12.	1.38	12.1	1.37	12.	1.38	12.1	1.37
R2,LH	0.444	0.052	0.453		0.445	0.052	0.454		-830.	22.5	-829.		-697.	22.	-694.	
BS/JB	0	0	0		0	0	0		0	0	0		0.518	0.272	0.56	

SD: Standard deviation, RMSE: Root mean square error. Results from 86 successfully optimized replications are reported here.

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