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## Emergence and Evolution of Heterogeneous Spatial Patterns

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# Emergence and Evolution of Heterogeneous Spatial Patterns

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6 June 2006

## Abstract

We live in a quite heterogeneous space. There are cities and rural areas, and population density varies a lot across space. People migrate and commute to the places of their work. The goal of this article is to clarify the mechanism of commuting as an equilibrium in heterogeneous space with different technologies. It is well known that agricultural production requires substantial amount of land per unit of labour, while most industrial production and services require much lower land input. We assume that all industrial production and service sector is located in urban areas, while all agriculture is in rural area. Historically, the share of labour in agriculture was declining due to more rapid growth of productivity there in comparison to service sector. At the same time, people change the location of their residence much slower. That is why at some point in time we face the situation, when rural area has excessive labour (not enough work for all in agriculture), while urban areas create an increasing number of jobs. A relatively simple mathematical model is proposed to explain the emergence of spatial pattern with heterogeneous density and phase transition between urban and rural areas. There are three types of agents: workers who live in a city, farmers who live in a rural area and workers commuters from rural area to city. In an equilibrium they are indifferent between occupation and residence. An indifference across

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locations for a priori identical agents implies the shape of land rent. If some parameters of the model change, they imply the change of the whole spatial pattern. In particular, split of rural residents into commuters and farmers depends on road infrastructure development through transport cost. Two types of shocks (decline in commuting transport cost by construction of fast roads and the relative decline in agricultural price) can perturb agricultural zone. Some former farmers start commuting to city while keeping residence in rural area. This is how a functional area of a city with integrated labour market emerges.

JEL Classification: D5, R0.

KEYWORDS: continuous space, equilibrium, rural and urban areas, commuting.

## 1 Introduction

We live in a quite heterogeneous space. There are cities and rural areas, and population density varies a lot across space. People migrate and commute to the places of their work. The goal of this article is to clarify the mechanism of commuting as an equilibrium in heterogeneous space with different technologies.

It is well known that agricultural production requires substantial amount of land per unit of labour, while most industrial production and services require much lower land input. We assume that all industrial production and service sector is located in urban areas, while all agriculture is in rural area. Historically, the share of labour in agriculture was declining due to more rapid growth of productivity there in comparison to service sector [1]. At the same time, people change the location of their residence much slower. That is why at some point in time we face the situation, when rural area has excessive labour (not enough work for all in agriculture), while urban areas create an increasing number of jobs.

The goal of this article is to present two models of endogeneous population split across workers and farmers. This split is caused by existence of sectors that have different land intensity with respect to unit of labour. The focus is on emerging heterogeneity of spatial population density despite the fact that agents are indifferent across locations.

The first, simple model, generalizes agglomeration model from Yegorov [3] and optimal city model from Mascarilla Miro, Yegorov [2]. In paper [3] the split of population between point-like city and rural area with variable and finite population density was described. The assumption about point-like city with infinite population density is a reasonable approximation but is not fully realistic. The paper [2] presents a simple model of city structure, based on CBD type models that have been developed by Alonso [8] and Fujita [7]. It also relates the wage and price indices of a city with its spatial size. The idea about optimal city size was developed by Henderson [9]. While the equilibrium population split across two cities is studied in [2], the present article focuses on such a split between a city and rural area that supplies it with food.

The developed model tries to combine two ideas from regional science and urban economics literature. Von Thünen [5] was the first to describe heterogeneous rural area around a city. Urban economics literature [7,8] was focusing on land use in a city or around it for residential purpose, but did not study agricultural use of land. There are some extensions of von Thünen model in [6], but the general approach of these authors to modelling agricultural production is far from being realistic. Typically it is modelled as a basic sector with decreasing returns to scale, but the assumption about zero transport cost for agricultural product is beyond any criticism. It is well known that one unit of weight of agricultural products is typically cheaper than for industrial products. One kg of potatoes or milk is much cheaper than 1 kg of textile, not mentioning computers or mobile phones.

Another model is more sophisticated. It captures the phenomenon of commuting and tries to explain the phenomenon of periurbanization as the reaction on globalization and local shocks for agricultural production. In equilibrium, workers, farmers and commuters coexist in space.

Section 2 deals with general formulation of the model and some intermediate derivations. Section 3 presents a simple model. Section 4 discusses potential problems. Section 5 describes the model with 3 types of agents, including commuters. Paper ends with conclusions.

## 2 The Model Structure

In this simple model there are only two types of agents (workers and farmers), and in equilibrium they are indifferent between locations and occupations.

While regional science deals with the issue of agricultural land use, there is no unique standard about modelling agricultural production function. That is why here agricultural land is assumed to belong to self-employed farmer. He inelastically supplied one unit of labour, and the return of land to labour is decreasing in scale. This captures the fact of existence of a set of possibilities to use land for different activities (mentioned already by von Thünen

### 2.1 Assumptions

1. The space is assumed to have linear one-dimensional structure, formally represented by an interval  $[0, 2]$ .<sup>1</sup> Two CBDs are located at points  $x = 0$  and  $x = 2$ . they correspond to business centers of cities, each with population  $N_u$ . The total population on interval  $[0, 2]$  equals to labor force  $M$  and is split between this cities and rural area between them. Labor is freely mobile and can choose its location on an interval. There are  $N_u$  of urban residents,  $N_a$  of farmers and  $N_c$  of commuters.

2. There exist two technologies: land-consuming (agricultural) and land-free. Land free technology can exist in multiple forms; it covers all industrial activity and service sector. We assume, that each city has a firm with Leontief type of technology  $Q = \min\{K, L\}$  and fixed cost  $F$  per firm. Because capital is more mobile than labour<sup>2</sup>, its return  $r$  is equalized across cities, while labour has to share fixed cost. Hence, the urban wages are  $w_u = p_i - r - F/N$ , where  $p$  is product price and  $N$  is the number of workers in a city. Considering radially symmetric model of a city with constant population density and linear transport cost with price  $t$  per unit of distance, we arrive to price index in a city as the function of its size, given by the formula:  $P_u = a + b\sqrt{N}$  [2].

3. Agricultural technology is self-employment of farmers. They pay location- dependent land rent price  $R(x)$  per unit of used land and Samuelson

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<sup>1</sup>We can also postulate periodic structure with period 2 on it. This allows not only to cover all geographical space, but also to replicate the economy by having many identical cities - a useful tool to introduce competition.

<sup>2</sup>Capital mobility has cost close to zero, while for labour it is finite.

transport cost to deliver their goods to the market in a city. It is assumed that production does not require labour<sup>3</sup>, but only land. The wage of farmers is formally given by expression  $w_a = S(x)^\beta(p_a - t_a x) - R(x)S(x)$ ;  $0 < \beta < 1$ .

4. All agents have to live in houses, which for simplicity (and because we have a priori identical agents) have equal size, but different location-specific value. The utility of agents is the difference between their wage and housing price. (to write more about housing price formation)

5. There are initially two types of agents: workers living in cities and farmers. The initial split of population between urban and rural residents is defined through the condition to be indifferent across occupations.

6. At the second stage we have population and technological growth, which leads to the following situation: cities reach their optimal size, and further population growth there is suboptimal, while there is not enough work in agriculture for all rural residents. At the same time, the development of transportation technologies makes commuting a viable option, and some fraction of rural population commutes to city for the job.

## 2.2 Intermediate Derivations

We consider only an interval  $[0, 1]$ . The subinterval  $[0, r]$  is occupied by the city, while the rest is rural area with heterogeneous population density  $\rho(x)$ . We can consider fully radially symmetric model (like von Thunen's isolated city), but for simplicity we will treat rural area as a line.<sup>4</sup>

There exists three types of agents: worker (w), agricultural farmer (a) and commuter (c). Let  $H$  be construction price of a house, converted into equivalent flow of rental prices. It is assumed that agricultural land in location  $x$  has unit price  $R(x)$  (to be determined endogeneously) and can be used both for agricultural and residential use.<sup>5</sup> The unit distance transport

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<sup>3</sup>In fact, we can assume that opportunity cost of leisure is zero for farmers and the time constraint is not binding at any farm.

<sup>4</sup>This topology may correspond to a valley of constant width between two cities.

<sup>5</sup>In some countries there exist no free choice of land use by owner, but we abstract from this for simplicity.

cost is  $t$  for both agents and unit of agricultural good.

Since atomic measure of agents live in cities, markets are also located there. Agricultural good is traded for generalized industrial good (that includes services) there. For the moment, we do not consider the preferences, but take agricultural and industrial prices  $p_a, p_i$  as given.

The utilities of all types of agents are given by the formulae

$$U_w = w_u - P_u, \quad U_a(x) = w_a(x) - P_a(x), \quad U_c = w_u - P_a(x) - tx, \quad (1)$$

$$x^* < x < 1, \quad x^* \equiv \sqrt{N_u/\rho_u\pi},$$

where

$$w_u = p_i - r - F/N, \quad P_u = R(x^*) + H + t\sqrt{N_u/\pi\rho_u}, \quad (2)$$

$$w_a(x) = S(x)^\beta(p_a - tx) - R(x)S(x), \quad P_a(x) = R(x) + H. \quad (3)$$

(should we use  $hR(x)$  instead of  $R(x)$  here?)

Note that  $N = N_u + N_c$ . The urban population density  $\rho_u$  is assumed to be constant. Utilities of different agents should be equalized in equilibrium, i.e.  $U_w = U_a = U_c$ . The border of city  $x^*$ , rural population density  $\rho(x)$  and the share of commuters  $s(x)$  in each location is to be determined endogeneously.

Let us write expressions for utilities as the function of endogenous and exogenous variables:

$$U_w = p_i - r - F/(N_u + N_c) - R(x^*)h - H - t\sqrt{N_u/\pi\rho_u}, \quad (4)$$

$$U_a = S^\beta(x)(p_a - t_ax) - R(x)S(x) - R(x)h - H, \quad (5)$$

$$U_c = p_i - r - F/(N_u + N_c) - R(x) - H - t_ax. \quad (6)$$

Also, we have (need to change some of these equations):

$$N_c = \int_{x^*}^1 \rho_c(x)dx, \quad (7)$$

$$\rho_a(x) + \rho_c(x) = \rho(x), \quad (8)$$

$$N_u + \int_{x^*}^1 \rho(x)dx = M, \quad (9)$$

$$N_u = \pi\rho_u(x^*)^2. \quad (10)$$

We can distinguish between different costs of commuting, assuming that  $t_a \neq t$ . Most likely that  $t_a < t$ ; this will correspond to congested transport in city. Putting cost  $t_a$  for all commuting distance is an implicit assumption of an existence of fast road from rural area to the CBD. We have neglected the share of land used for housing in rural area assuming it to be much smaller than those for agricultural use.<sup>6</sup>

**Spatial structure.** We consider radially symmetric model. The city is already radially symmetric, and the formula (2) already takes it into account (for derivation see [2] and/or Appendix). Radial symmetry assumes that the measure of land at the distance interval  $[x, x+dx]$  is  $2\pi x dx$ , i.e. grows linearly with distance. As for spatial density of agents, we introduce linear densities of farmers,  $\rho_a(x)$ , and commuters,  $\rho_c(x)$ , so that the number of agents of corresponding type in a small ring  $[x, x+dx]$  (for all angles  $0 < \phi < 2\pi$ ) equals to  $dN_a = \rho_a(x)dx$  and  $dN_c = \rho_c(x)dx$  correspondingly.

**Market for land.** Each farmer located at distance  $x$  from the closest CBD<sup>7</sup> would optimally choose land quantity  $S(x)$ , taking into account equilibrium land rent price  $R(x)$ . A standard rural housing requires  $h$  units of land, and this residential land will be demanded by both farmers and commuters. In every small ring  $[x, x+dx]$  the total demand for land would be ( $D(x)$  is demand density)

$$D(x)dx = [\rho_a(x)S(x) + (\rho_a(x) + \rho_c(x))h]dx. \quad (11)$$

and its supply is  $2\pi x dx$ . In the limit  $dx \rightarrow 0$  we get the exact equality between demand and supply densities:

$$[\rho_a(x)S(x) + (\rho_a(x) + \rho_c(x))h] = 2\pi x. \quad (12)$$

This land market clearing equation gives one relation between 3 endogeneous functions  $\rho_a(x)$ ,  $\rho_c(x)$  and  $S(x)$ .

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<sup>6</sup>Now the situation changes with an increase of commuter shares, and explicit accounting for rural land split for residential and agricultural use can be the next extension of the model.

<sup>7</sup>Clearly, we will have symmetric location in space in the case of many cities located with symmetrical pattern (we do not move into details of hexagon structure) when we move away from our one-city model a la von Thunen.



**Indifference condition for commuters across locations.** Since we assume transport cost linear in distance and no explicit preferences for leisure, in the environment of fixed size of housing, equilibrium land rent price should be linearly decreasing with distance. This follows from  $U_c(x) = w_i - H - R(x) - t_a x = \text{const}$ . If unit distance transport cost out of the city is different from internal transport cost ( $t_a \neq t$ , the land rent price function would be piece-wise linear, with a kink at the edge of the city, i.e.

$$R(x) = R(x^*) - t_a(x - x^*), \quad x^* < x < 1, \quad (13)$$

$$R(x) = R(x^*) + t(x^* - x), \quad 0 < x < x^*. \quad (14)$$

**Optimal city size.** As it was shown in [2], in the absence of commuting the optimal city population is  $N^* = (2F/b)^{2/3}$ , where  $b \equiv t/\sqrt{\pi\rho_u}$ . Then the city radius is  $x^* = \sqrt{N^*/\pi\rho_u}$ . Since  $t, F, \rho_u$  are exogeneous parameters,  $x^*$  depends only on urban population  $N^*$ . In the presence of commuting this population is no longer optimal. We will return to this issue later.

**Optimal decision of a farmer.** Each farmer is an optimizer not only across locations, but also with respect to land size. Formally, his income (profit) is maximized with respect to land size  $S$ :

$$\frac{dw_a}{dS} = \beta S^{\beta-1}(p_a - t_a x) - R(x) = 0, \quad (15)$$

$$S(x) = \left[ \frac{R(x)}{\beta(p_a - t_a x)} \right]^{1/(\beta-1)}. \quad (16)$$

**Indifference of farmers across locations.** We should set farmer's utility to constant value:

$$U_a(x) = S^\beta(x)(p_a - t_a x) - R(x)(S(x) + h) - H = \text{const}. \quad (17)$$

The function  $R(x)$  is already found (from indifference of commuters and we know  $S(x)$  for given  $R(x)$  from the farmer optimization.

(to re-think this paragraph!)...This equation defines implicitly function  $S(x)$  as the function of  $R(x)$ . Explicit solution is possible only for some particular cases. For example, for  $\beta = 1/2$  we get a quadratic equation for

$y(x) \equiv \sqrt{S(x)}$  and can solve it for each  $x$ . We get the following equation and result:

$$[(y^2(x) + h)R(x) - y(x)[p_a - t_a x] + H + hR(x) - U_a = 0, \quad (18)$$

$$y_{1,2}(x) = \frac{p_a - t_a x \pm \sqrt{(p_a - t_a x)^2 - 4R(x)(H + hR(x) - U_a)}}{2R(x)}. \quad (19)$$

Positive solution are only available for  $x < p_a/t_a$ . If  $U_a > H + hR(x)$ , we have two positive solutions, and only one in the opposite case.  $U_a = U_w$ .

**How to find the densities?** Now we know  $R(x)$  and  $S(x)$ . Consider land market equations (12) and overall population equations (7-9). From (12), we can express commuter density via density of farmers and already known  $S(x)$ :

$$\rho_c(x) = \frac{2\pi x - \rho_a(x)[S(x) + h]}{h}. \quad (20)$$

Let us exploit the equality  $U_w = U_c$ . Since income is the same, we have  $P_u = P_a + t_a x$ , and this equality gives us:

$$R(x^*) + t/\sqrt{\pi\rho_u} = R(x) + t_a x, \\ x^* = \frac{t}{t_a\sqrt{\pi\rho_u}}. \quad (21)$$

Thus, we have defines the raduis of the city. In the presence of commuting the edge of the city and its population is no longer defined by an internal city optimum, but through its interaction with its neighbourhood.

**Proposition 1** *In the presence of commuting labour and housing market of the city and its neighbourhood become integrated, and form functional area, with endogeneously determined size. The city can no longer an in internal optimal condition, and its population and labour force are now determined through the interaction with its neighbourhood.*

**Closing the model.** We have not used two two equations:  $U_a = U_w$  and

$$N_u + \int_{x^*}^1 [\rho_a(x) + \rho_c(x)]dx = M, \quad (22)$$

following from (7-9). Note that 1 is the upper border only if there exists an internal solution, i.e. farmers can access city markets. If agricultural price is not too small, and transport cost is not too high, this is always the case. At least, this is the typical case for developed economies, and we will concentrate on it. Since we know already  $x^*$ , we know city population. But its utility  $U_w$  depends positively on the number of commuters, since  $N = N_u + N_c$  in formula (2). Now we substitute  $S(x)(U_w)$  into farmers' utility and equate it to  $U_w$ . We can do it in any point  $x$ ; this is algebraic and not functional equation. Solving it, we find the equilibrium value of  $U_w$ . Through the expressions (1,2) for  $U_w$  we find its link with  $N_c$ . The last step is to find unknown densities.

### 3 Simple Model: No Commuters

Consider the case  $\rho_c = 0$ . Then we have the city of optimal size surrounded by rural area. It resembles von Thunen model. We can find rent bid curve (last formula), where  $U_a = U_w$ , to make agents indifferent. The agricultural circle is finite, because the upper border cannot exceed  $x = p_a/t_a$ . However, few rural areas in developed countries are located at higher distances from the closest cities.

We have the following equations. The utility of workers, city optimal size and city radius are given by the formulae:

$$U_w = p_i - r - F/(N_u + N_c) - R(x^*)h - H - t\sqrt{N_u}/\sqrt{\pi\rho_u}, \quad (23)$$

$$N_u = \left(\frac{4F^2\pi\rho_u}{t^2}\right)^{1/3}, \quad x^* = \sqrt{N_u/\pi\rho_u}. \quad (24)$$

The optimization of land use by farmer gives optimal land slots  $S(x)$ , utility of farmers and rent bid curve  $R(x)$ , making them indifferent across locations:

$$S(x) = \frac{\beta^2(p_a - t_ax)^2}{R^2(x)}, \quad (25)$$

$$U_a(x) = \beta(1 - \beta)\frac{(p_a - t_ax)^2}{R(x)} - R(x)h - H, \quad (26)$$

$$R(x) = \frac{-H - U_a + \sqrt{(H + U_a)^2 + 4h\beta(1 - \beta)(p_a - t_ax)^2}}{2h}. \quad (27)$$

The land market equilibrium in 2-dimensional symmetric space is:

$$\rho_a(x)S(x) = 2\pi x, \quad x > x^*. \quad (28)$$

The indifference between being worker and farmer,  $U_w = U_a$ , gives the land rent at the city border<sup>8</sup>:

$$R(x^*) = \left[ p_i - r - \frac{F}{N_u} - \frac{t\sqrt{N_u}}{\sqrt{\pi\rho_u}} \right]^{-1} \frac{1}{\beta(1-\beta)(p_a - t_ax^*)^2}. \quad (29)$$

Now we can uniquely reconstruct the land rent function:

$$R(x) = \frac{-H - U_a + \sqrt{(H + U_a)^2 + 4h\beta(1-\beta)(p_a - t_ax)^2}}{2h}, \quad x > x^*, \quad (30)$$

$$R(x) = R(x^*) + t(x^* - x), \quad 0 < x < x^*. \quad (31)$$

Further we reconstruct agricultural population density  $\rho_a(x)$  (this is linear and not area density) and total agricultural population  $N_a$  (here we assume that periodic spatial pattern expands until  $x = 1$ ):

$$\rho_a(x) = \frac{2\pi x}{S(x)}, \quad (32)$$

$$N_a = \int_{x^*}^1 \rho(x) dx. \quad (33)$$

This solution represents a phase transition between urban zone with high population density and rural zone with substantially lower density. If we fix the number of economic agents  $M$  (in a certain territory), then a fraction of them will form a city of optimal size, while the rest will be located in rural area around it. If we have too small population, there will be some vacant land. If we have too much population, all space will be densely occupied, and we may have lower utility.

This is a kind of open-economy macro model, because agents take world prices of industrial and agricultural goods as given. They also do not have a chance to select the size of housing (for the sake of model simplicity) and take standard housing as given. But in fact, we do not always observe all demanded sizes of housing at the market. Some laws may prohibit construction of too small houses, and poorer agents may find themselves excluded from some attractive cities or districts.

<sup>8</sup>In the model for equilibrium across cities [2] it was taken as exogeneous

### 3.1 Reaction to Shocks

Consider the spatial dynamics of equilibrium, associated with technological and population growth. In fact, we need to study how the spatial pattern of residence and occupation reacts on change in total population (demographic change), transport cost (transport innovations), industrial and agricultural innovations. The latter will be modelled simply as an exogeneous decline in  $p_i$  or  $p_a$ .

We start with the initial allocation of city and rural area. The city has initially optimal population and occupies territory  $0 < x < x^*$  (see model derivation in [2]).

Then structural change happens.

Consider two types of shocks, that perturb agricultural zone. The first is a decline in commuting transport cost (construction of fast road). It changes spatial coordinates, and leads not only to the extension of the external border of agricultural ring, but also to shift of some agents to commuting. The labor market in a city due to scale economies is interested in more workers, but congestion do not allow for its further growth. These workers now come as commuters. We have an emergence of suburban area (to calculate exactly). To isolate commuting effect, we can introduce distinction between commuting cost along fast road and agricultural transport cost, that stay the same, since fast roads cannot cover densely all agricultural productive area.

Another type of shock is the decline in agricultural price, that makes farmers worse off, unless some of them shift to work in city and make other farmers more productive.

**Shift of rent bid curves.** Rent bid curves  $R_a(x)$ ,  $R_c(x)$  represent such rental prices for land, that make agents of certain group (agricultural farmers, commuters) indifferent across locations. In the absence of commuters only farmers were located beyond city border. They have their own  $R_a(x) = R_a(x)$ . In other words, if only one type of agents is located in certain point, their rent bid curve defines equilibrium rent price.

If rural transport cost is lower than urban, it gives agents in suburban area some advantage in transportation to the CBD. Then some farmers will

find it useful to become commuters. Lower transport costs from just above the city border (due to fast road to CBD and no local congestion) push bid rent above its level on city border, and we can have rent discontinuity there. The rent bid curve of commuters have higher slope (absolute value of gradient) and higher value at the city border. At a certain distance it will be lower than rent bid curve of farmers (see Fig.1 [EXCEL]). Thus, this area will be occupied by commuters, while farmers would locate further.

## 4 Potential Problems

### 4.1 Limit Case- Does not Help

In order to arrive to more explicit solution, consider the limit case  $h \rightarrow 0$ . Formally this corresponds to the case when almost all rural land is used for agriculture, and only small fraction for housing. Then the land market equilibrium condition simplifies to:

$$\rho_a(x)S(x) = 2\pi x. \quad (34)$$

Since we know  $S(x)$ , we can reconstruct agricultural population density in all locations  $x$ . This also gives us the total agricultural population:

$$N_a = \int_{x^*}^1 \frac{2\pi x}{S(x)} dx. \quad (35)$$

Then, from the balance population equation, we get:  $N_c = M - N_u - N_a$ . Now we need to find its distribution in space, which is governed by land rent price and transport cost.

### 4.2 Does Unique Rent Bid Curve Exist for Two Types of Agents?

Typically we have segregation in space, where particular location is occupied by agents paying the highest bid. We have already found that commuters are indifferent across locations if the slope of rent bid curve coincides with agricultural transport cost (to compensate higher rents by lower transportation costs, like in a city).

Consider the special case  $\beta = 1/2$ , for simplicity. In this case for given rent curve  $R(x)$ , farmers in location  $x$  optimally choose land size

$$S(x) = \frac{\beta^2(p_a - t_a x)^2}{R^2(x)}. \quad (36)$$

If we substitute this into utility of farmers, we get:

$$U_a(x) = \beta(1 - \beta) \frac{(p_a - t_a x)^2}{R(x)} - R(x)h - H. \quad (37)$$

For any functional form of  $R(x)$  different from the solution  $U_a(x) = \text{const}$ ,

$$R(x) = \frac{-H - U_a + \sqrt{(H + U_a)^2 + 4h\beta(1 - \beta)(p_a - t_a x)^2}}{2h}, \quad (38)$$

we get different utility in different locations. Thus, either farmers choose one (the best) location, or they should not be able to choose optimal land parcels.

In this situation we typically have spatial segregation. We can expect that commuters occupy all the land between the city edge and new agricultural border, where land rent gradient will correspond to transport cost, while farmers would locate further and have rent function given by the last formula of this subsection.

### 4.3 Mixed type solution?

Is it possible to have mixed type solution, where at each distance there is a certain share of commuters and the rest are farmers? Maybe if we have heterogeneous wealth and/or the possibility to choose house size. But this question requires further elaboration. In the present assumptions it is not likely to occur because mixed population will produce some intermediate equilibrium rent curve between rent bid curve for farmers and commuters, and each member will be no longer indifferent across locations. If rent bid curve  $R_i(x)$  for group  $i = a, c$  has higher slope than equilibrium rent curve  $R(x)$ , then at lower distances the bid of this group will exceed equilibrium bid, and they will be better off by relocating there. Thus, we do not get equilibrium in this case.

## 5 Extension: Workers, Farmers and Commuters

This section is devoted to elaborated model, that includes 3 types of agents: workers, farmers and commuters.

We can consider fully microeconomic specification with preferences for both size of housing and the rest of the good. For example, utility may be Cobb-Douglas in housing quantity and the rest of goods  $C$ . The housing of size  $q$  at location  $x$  costs  $q(H + R(x)h)$ , where  $h$  is occupied land and  $H$  is construction cost.

### 5.1 Income and Preferences

Like in the core model, the income of worker (who is also urban resident) is:

$$w_i = p_i - r - \frac{F}{N_u + N_c}, \quad (39)$$

where  $N_u$  is number of urban residents, while  $N_c$  is the number of commuters. The income of farmer depends on his land slot in location  $S(x)$ , price of agricultural good  $p_a$  and agricultural unit transport cost  $t_a$ , which can differ from unit distance travel cost  $t$ :

$$w_a = S(x)^\beta (p_a - t_a x). \quad (40)$$

It is assumed that workers and farmers have standard (or minimal) housing, that requires  $h$  units of land and construction cost  $H$ , spread over time horizon. Their utilities come from net income which is their gross income minus cost of housing and transportation:

$$U_w = w_i - R(x^*)h - H - tx^*, \quad (41)$$

$$U_a = w_a - R(x)(S(x) + h) - H. \quad (42)$$

Commuters form a separate group. Like workers, they have income from industrial or service activity in a city. But their residences are located outside of city border  $x^*$ . Utility of commuters is derived from both extra housing space and net consumption. To capture this fact, their utility is:

$$U_c = (w_c - q(H + R(x)h) - tx)^a (q - 1)^{1-a}, \quad (43)$$



where  $q$  is housing size coefficient. It is possible to assume also that their income,  $w_c$ , is higher due to skill difference:  $w_c = \mu w_i, \mu \geq 1$ . If we keep  $\mu = 1$ , all agents are still indifferent a priori, while for  $\mu > 1$  we have two classes of agents with different income, and thus utility. In the first case all agents are indifferent across all occupations and locations, while in the second case commuters and farmers are not indifferent across occupations, even if they live at the same location  $x$ .<sup>9</sup>

## 5.2 Choosing Housing Size

The formulae (1-6) will be modified now. It is assumed that  $H$  denotes construction cost of minimal basic housing, while  $h$  is minimal land for it. The idea of such minimal consumption bundle was presented in the papers of C.Koulovatianos et al [4], however in a different context. This is especially important for the countries where substantial share of population lives near subsistence level and is less relevant (but not negligible) in rich countries.

Further it is assumed that  $B \equiv H + hR(x)$  is the cost of such minimal housing bundle that does not bring utility. Before (see (1-6)) the utility was defined as a difference between income  $w$  and basic cost,  $P(x) + tx$ . Now the utility will be defined as Cobb-Douglas function of non-basic consumption  $C$  and housing:

$$U = C^a(q - 1)^{1-a}. \quad (44)$$

What is left in  $C$ ? Housing and transport expenditures should be subtracted from income. Hence,  $C = w - tx - qB(x)$ . Thus, a particular commuter should maximize the following expression with respect to choice parameter  $q$  (housing size) conditionally on  $q \geq 1$ :

$$\max_q [w - tx - qB(x)]^a (q - 1)^{1-a}, \quad B(x) \equiv H + hR(x). \quad (45)$$

Optimization gives:

$$\frac{w - qB(x) - tx}{q - 1} = \frac{aB(x)}{1 - a}, \quad (46)$$

$$q(x) = a + \frac{(1 - a)(w - tx)}{B(x)}. \quad (47)$$

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<sup>9</sup>Location with respect to city distance; there are plenty of them due to radial symmetry.

We need to have  $q \geq 1$ . Now, after adding possibility to choose housing size, at each location out of city farmers and commuters will make their land bids and land rent will have such profile, that they are indifferent across locations. (Hope that we can get the declining share of commuters with distance).

### 5.3 Optimization Problems for Different Agents. Rent Bid Curves

All workers in a city are already indifferent across locations. As was shown earlier, with identical housing size, the rent difference is exactly compensated with the difference in transport costs.

**Optimal decision of a farmer.** Each farmer is an optimizer not only across locations, but also with respect to land size. Formally, his income (profit) is maximized with respect to land size  $S$ :

$$\frac{dw_a}{dS} = \beta S^{\beta-1}(p_a - t_a x) - R(x) = 0, \quad (48)$$

$$S(x) = \left[ \frac{R(x)}{\beta(p_a - t_a x)} \right]^{1/(\beta-1)}. \quad (49)$$

If  $\beta = (n - 1)/n$  ( $n \geq 1$ ), we get:

$$S(x) = \left[ \frac{\beta(p_a - t_a x)}{R(x)} \right]^n. \quad (50)$$

Since  $n > 1$ , it immediately follows that:  $\partial S(x)/\partial p_a > 0$ ,  $\partial S(x)/\partial t_a < 0$ ,  $\partial S(x)/\partial R(x) < 0$ . Thus,

**Lemma 1** *The optimal land slot chosen by a farmer depends positively on agricultural price and negatively on agricultural transport cost and equilibrium land rent.*

**Indifference of farmers across locations.** We should set farmer's utility to constant value:

$$U_a(x) = S^\beta(x)(p_a - t_a x) - R(x)(S(x) + h) - H = \text{const}. \quad (51)$$

Substitution of optimal land size into the last formula gives indirect utility:

$$U_a(x) = (1 - \beta)\beta^{n-1} \frac{(p_a - t_a x)^n}{R(x)^{n-1}} - R(x)h - H. \quad (52)$$

For an arbitrary natural  $n$  this is a polynomial equation w.r.t.  $R(x)$ . In particular case  $n = 2$  ( $\beta = 0.5$ ), it reduces to quadratic equation, and its unique positive solution  $R_a(x)$  can be expressed explicitly:

$$R_a(x) = \frac{-H - U_a + \sqrt{(H + U_a)^2 + 4h\beta(1 - \beta)(p_a - t_a x)^2}}{2h}. \quad (53)$$

It is easy to see that  $dR(x)/dx < 0$ . In the general case it is also possible to prove the existence and uniqueness of farmer's rent bid curve  $R_a(x)$ .

**Lemma 2** *There exists a unique continuous curve  $R_a(x)$ , such that farmers are indifferent across locations. Moreover,  $R'(x) < 0$ .*

**Proof:**

The equation  $U_x = \text{const}$  can be written as  $f(R) = g(R)$ , where  $f(R) = cR^{1-n}$ ,  $g(R) = b + hR$  and  $b = \text{const} + H$ . We have:  $f'(R) < 0$ ,  $g'(R) > 0$  and  $f(0) = +\infty$ ,  $g(0) = b > 0$ ,  $f(+\infty) = 0$ ,  $g(+\infty) = +\infty$ . Hence, there exists a unique point  $R = R^* \in (0, +\infty)$ , where both curves intersect.

Further, both  $f$  and  $g$  are continuous functions of  $R$  and all parameters, including  $x$ , for  $0 < R < +\infty$ . By the theorem of implicit function, there exists an implicit function  $R_a(x)$ , continuous in  $x$ .

Consider function  $F \equiv f(R(x), x) - g(R(x), x)$ . The implicit equation  $F = 0$  is equivalent to  $f(R; x) = g(R, x)$ . The derivative is:  $dR_a(x)/dx = -F'_R/F'_x$ . Since  $\forall n > 1$  and  $p_a > t_a x$ ,  $f'_R < 0$ ,  $g'_R > 0$ ,  $f'_x < 0$  and  $g'_x = 0$ , we have  $F'_R = f'_R - g'_R < 0$  and  $F'_x < 0$ . Hence,  $R'(x) < 0$ .

**Commuters: indirect utility and rent bid curve.** After substitution of the formula  $q(x)$  into  $U_c$  we get indirect utility of commuters:

$$U_c(x) = \frac{a^a(1 - a)^{1-a}[w_c - tx - H - hR(x)]}{(H + hR(x))^{1-a}}. \quad (54)$$

Consider the condition of indifference of commuters across locations:  $U_c(x) = c = \text{const}$ . It is linear in  $x$  and transcendental in  $R$ . Hence, we can consider

it as implicit equation  $F(x, R(x)) = 0$ , where  $F \equiv a^a(1 - a)^{1-a}[w_c - tx - H - hR(x) - c[H + hR(x)]^{1-a}]$ , and then solve for  $x = x(R)$ , obtaining the inverse function of rent bid curve for commuters.

## 5.4 Equilibrium Location

Here we shall consider local land market equilibria and all rent bid curves.

**Land market clearing.** Consider radially symmetric model. The measure of land (supply) at the small distance interval  $[x, x + dx]$  is  $2\pi x dx$ . The demand for land outside the city border  $x > x^*$  comes from 3 sources: a) each farmer needs  $S(x)$  for agricultural use, b) each farmer needs  $h$  for housing, c) each commuter needs  $q(x)h$  for housing. Let  $\rho_a(x)$  be the density of farmers. This means that the quantity of farmers in the interval  $[x, x + dx]$  is  $\rho(x)dx$ . Their total number is  $\int_{x^*}^{\hat{x}} \rho_a(x)dx$ , where  $\hat{x} \equiv p_a/t_a$  is endogeneous border for farming activity oriented at the market in our city. Similarly, we can introduce the density of commuters,  $\rho_c(x)$ . The total number of commuters, that enters the equation for  $w_i$ , is:

$$N_c = \int_{x^*}^{x^{**}} \rho_c(x)dx, \quad (55)$$

where  $x^{**}$  is the upper distance for commuters settlement (to be defined later).

**Survival areas for commuters and farmers.** The largest distance  $x^{**}$ , where commuter can locate, is one where he gets zero utility due to  $q = 1$  and where he spends all income on basic housing and commuting:  $x^{**} = (\mu w_i - H - hR(x^{**}))/t$ . Farmers can locate no further than  $\hat{x} = p_a/t_a$ . If this distance is smaller, equilibrium land rent may be very close to zero, and they can always choose land slot to get the income sufficient to cover  $H$ . In principle, it may happen that agricultural price is so low that  $x^{**} > \hat{x}$ , and then commuters can fully outbid farmers. This is more typical for small open economy, which takes world prices as given. Although there is no endogeneity of agricultural and industrial prices in this model, it is possible to introduce Leontief preferences, for example, and produce the same quantity of agricultural and industrial good (see [3]). Scarcity of agricultural good

would push the price for it, and the areas of agricultural activity would expand beyond the upper border for commuters.

**Equilibrium land rent.** Different possibilities exist. Consider first the case when rent bid curves for farmers and commuters coincide just in one point. Typically, we can expect rent bid curve for commuters to be higher in city neighbourhood and lower at large distances.<sup>10</sup> Let there exist a point  $x^* < \bar{x} < x^{**}$ , so that  $R_a(\bar{x}) = R_c(\bar{x})$ , and  $R_c(x) > R_a(x)$ , for  $x < \bar{x}$ . Then all commuters would settle in the interval  $[x^*, \bar{x}]$ , and farmers - further. The equilibrium land rent would then have the following structure:

$$R(x) = R(x^* + t(x^* - x)), \quad 0 < x < x^*, \quad (56)$$

$$R(x) = R_c(x), \quad x^* < x < \bar{x}, \quad (57)$$

$$R(x) = R_a(x), \quad \bar{x} < x < \tilde{x} \leq \hat{x}, \quad (58)$$

$$R(x) = 0, \quad x > \tilde{x}. \quad (59)$$

In a general equilibrium framework, there exists total population, which is split between workers and farmers. There also exists some exogeneous number of commuters. Depending on their number and income premium factor  $\mu$ , the width of commuting ring can vary. This is a generalization of von Thünen's "Isolated City", but with more tiny structure of a city itself, based on CBD model.

**Phase transition in density.** Under our assumptions, the population density in a city itself is high and constant.<sup>11</sup> At the city border, commuters start to settle, and form the further ring on 1-story housing with some gardens. The size of these gardens is growing with distance. For  $\mu > 1$ , we can have discontinuity of population density, while land rent is still continuous. This ring ends at the distance  $\bar{r}$ , where agricultural activity starts to take over settlement of commuters. It may happen that population density at this point declines sharply (second phase transition), while land rent is still continuous. In the simple model with no commuters the phase transition in density at the edge of the city is more sharp.

<sup>10</sup>It would be interesting to find explicit conditions for parameters for this to hold. It can be done, for example, for  $a = 1/2$  and  $\beta = 1/2$ , where rent bid curves can be found explicitly.

<sup>11</sup>In reality, it may be higher in city centre, but we need to introduce then buildings of variable height.

## 6 Conclusions

1. The present paper combines the ideas of von Thünen [5] extends the previous work of author [3], where he describes the split of space into point cities and continuous agricultural area by introducing two technologies: industry and farming. It also expands the model [2], where city microstructure is added, by introducing a category of commuters who can also choose the size of land slot (like it is often done in urban economic literature).

2. In a first, simple model, the solution for a city surrounded by agricultural area is constructed explicitly. We have continuously declining land rent and discontinuity in population density (phase transition) at the edge of the city. All agents are indifferent across locations and occupations in this model.

3. The second, elaborated, model includes also commuters. They live outside city border in villas with gardens and work in city. The city can grow further, because commuters do not add to congestion effects but only to scale economies. They can choose the size of their housing. Typically, commuters form a second ring around the city with declining population density, and push agricultural ring further away from the city.

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