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draft

Optimal distances between metro stations in centre dominated metropolitan areas

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Abstract

When the city centre is the major destination of passengers, public transport vehicles will get fuller as they approach it. The disadvantage of time loss due to stops on the way to the centre is thus experienced by an increasing number of travellers. We demonstrate that optimal stop distances increase as vehicles get closer to the centre. This is at variance with the usual assumption of constant stop distances employed in this type of models. A countervailing force is that urban densities increase as one approaches the centre. We demonstrate that there exist combinations of the various cost and density gradient parameters that result in constant stop distances as an optimal outcome. However, this is found for rather steep density gradients, so that the overall conclusion is that there are good reasons to let stop distances increase as one approaches the city centre.

Keywords: stop distance, public transport, density gradient, public transport access

1. Introduction.

Most cities depend on a mixture of public and private transport to satisfy travel demand. Acces and speed are essential elements of a well-functioning public transport system. It should preferably be close to the places where people live, work or do other activities, and it should also preferably be fast. This leads to various trade-offs, one of them related to the density of the network, and more in particular the optimal stop distances. Small stop distances allow high accessibility of the network, but lead to low speeds because of the many stops the vehicles have to make.

In the literature (Spasovic et al., 1994, Van Nes, 2002) optimal stop distances in public transport networks are usually modelled to be constant. This seems to be a plausible approach when the density of travellers is uniform along public transport lines. In the present paper we will focus on an aspect that calls for variable stop distances, even when the density of travellers is uniform. The background relates to the point that when all travellers go to the same destination, public transport vehicles get fuller and fuller as one approaches the city centre implying that more and more passengers experience the disadvantage of the waiting times.

In section 2 we give a simple formulation of a model to determine optimal stop distances. Minimisation of generalised costs can be demonstrated to imply shorter stop distances near the city fringe compared to stops closer to the centre. Some variants of the model are discussed, such as the case where the centre is not the sole destination of trips. In section 3 we present a case where population densities increase as one approaches the centre. We demonstrate that with exponential density gradients there are certain combinations of cost and density parameters that would lead to constant stop distances throughout the urban area, but such combinations are implausible because very steep density gradients would be implied. A numerical illustration is given for transport networks in Dutch cities. Section 4 concludes

2. Optimal distances between metro stops, basic formulation.

Consider a line that runs from the city edge O to D where D is the centre of the metropolitan area. We assume that all passengers use the metro to travel between their residence and the centre and vice versa. Thus, in the morning the metro gets fuller and fuller when it approaches the city centre, and in the afternoon the opposite occurs. Passengers are assumed to be uniformly distributed along the line with a density a per unit distance. Passengers are assumed to go to the closest stop[1]. Demand for metro services is inelastic: it does not depend on service levels and location of stops.

Consider the costs as far as they are associated with the location of the stops. The present paper does not address the optimal total number of stations N, but only, once N is known the optimal distances between them. From the perspective of the traveller we consider two components of generalised costs: costs of in-vehicle travel time and costs of walking to the nearest stop.

We use L_1, L_2, …, L_N to represent the location of the stops between edge O and city centre D. O is located left from L_1, and D right from L_N. Thus L_n denotes the distance between the edge O and stop n.

In-vehicle time as a function of the number of metro stops.

We start with computing the expected number of times people sitting in the metro experience a stop. For the stop at the city edge O (n=0) our assumption that travellers go to the nearest stop implies for the expected number of passengers entering the metro:

$$P_0 = 0.5(L_1)a \qquad (1)$$

Before arriving in the city centre D, these travellers will experience N stops.

The expected number of passengers entering at stop n is:

[1] This assumption has as an advantage that it keeps the formulas simple since the border of a market area between two stops is just in the middle. An alternative approach would be to have the border between two stops shifted a little towards the edge of the city since a traveller living just between two stops would gain a little by walking into the direction of the centre because this would save in-vehicle time. This case is shortly discussed in Appendix 1. It appears that the way of separating market areas of stops does not affect the outcomes of our analysis in terms of stop distances.

$$P_n = 0.5(L_n - L_{n-1})a + 0.5(L_{n+1} - L_n)a = 0.5(L_{n+1} - L_{n-1})a \qquad n=1,2,\ldots,N \qquad (2)$$

These passengers will experience N-n stops while sitting in the metro before arriving in D. Note that in the case of n=1, P_1 equals $0.5(L_2)a$ because L_0 equals 0.

Thus, the mean number of stops E(S) the collective of travellers experiences depends on the location of the stops in the following way:

$$E(S) = L_1\,[(0.5)a(N) - (0.5)a(N-2)] + L_2\,[(0.5)a(N-1) - (0.5)a(N-3)] + \ldots\ldots =$$

$$(L_1 + L_2 + \ldots)a \qquad (3)$$

The loss of time due to a stop is denoted as T_S. Note that this is more than the time the vehicle stands still at the platform, since it includes time loss related to accelerating and decelerating. Assume that there is no congestion at the metro stations: T_S does not depend on the number of passengers entering the metro. Then total time loss due to entering passengers equals $E(S).T_S$. Assuming that congestion is absent, it also appears that as long as the number of stops is given, from the operator's viewpoint, the choice of location of stops does not have an impact on operator's costs. Therefore it is sufficient for our analysis of optimal stop distances to focus on passenger related costs.

Let v be the time costs involved of a stop per seated passenger. This is the product of time loss T_S times the value of time related to a stop. Then the related costs of in-vehicle travel time are:

$$E(S).v = (L_1 + L_2 + \ldots).a.v \qquad (4)$$

It follows from (4) that the marginal in-vehicle time cost of shifting a stop one unit of distance towards the centre equals a.v: moving a stop location towards the centre implies that more passengers are will board at the preceding stop and these passengers experience a longer in-vehicle time because of an extra stop.

Expected walking distance as a function of the number of metro stops.
The average walking distance to stop O is $L_1/4$. The expected number of people experiencing this is P_0 as given in (1). Similarly, the average walking distance of passengers using stop n and arriving from the left side is $(L_n-L_{n-1})/4$, whereas for those arriving from the right side it is $(L_{n+1}-L_n)/4$. The expected numbers of passengers involved are $0.5(L_n-L_{n-1})a$ and $0.5(L_{n+1}-L_n)a$.

Hence the expected walking distance of travellers E(d) is:

$$E(d) = a\,[(L_1)^2 + (L_2-L_1)^2 + (L_3-L_2)^2 + \ldots.]/4 \qquad (5)$$

This quadratic formula makes clear that when travellers are uniformly distributed, increasing the space between two stops has two reinforcing effects on walking distances: the average distance to be walked increases, but also the number of passengers involved is larger.

The effect of a marginal shift in the location of stop n towards the centre implies that at the left side of the stop walking distances increase, and at the right side they decrease. Minimisation of expected walking distance as a function of the location of stop n results in $[(L_n-L_{n-1})-(L_{n+1}-L_n)]=0$, and hence

$$L_n = (L_{n+1}+L_{n-1})/2$$

This implies that stop n will be in the middle of stops of n-1 and n+1. Thus, in order to minimise expected walking distances, stops should be equally spaced, as long as the spatial distribution of passengers is uniform. We now turn to an integrated treatment of total costs as a function of both access costs and in-vehicle time costs.

Minimisation of social costs.
We assume that the transport company minimises social costs of transport, i.e., the sum of costs of operations and the generalised costs of passengers. For our purpose it is sufficient to specify only those costs that depend on the location of stops. Since, as already outlined above operator costs do not depend on the location of stops under the given assumptions, we focus on traveller costs. Let w denote the time costs involved

in walking a unit distance to a stop per passenger. Then the total costs as far as they depend on the location of stops are:

$$C = v.a\ (L_1+L_2+\ldots) \quad + \quad w.a\ [(L_1)^2 + (L_2-L_1)^2 + (L_3-L_2)^2 + \ldots.]/4 \qquad (6)$$

The first order condition for cost minimisation is:

$$\partial C/\partial L_n = v.a\ + w.a\ [(L_n-L_{n-1})-(L_{n+1}-L_n)]/2 = 0 \qquad n=1,2,\ldots,N \qquad (7)$$

This implies that as one comes closer to the city centre the intervals should become larger:

$$(L_{n+1}-L_n) = (L_n-L_{n-1}) + 2.v/w,$$

hence

$$L_n = (L_{n+1}+L_{n-1})/2 - v/w,$$

so that each stop is not in the middle of the two neighbouring stops, but shifted a distance v/w away from the centre. When we assume that the value of time of in-vehicle transport and walking would be equal, v/w equals the ratio of stop time and walking time per unit distance. This is equal to the distance a pedestrian walks during the time loss of the metro vehicle due to a stop. This implies equality of marginal costs of shifting a stop away from the centre in terms of longer access time and marginal benefits in terms of a smaller number of passengers that experience the delay due to an extra stop.

Consider the following numerical example. Total time loss related to a metro stop is about 0.5 minutes. The access speed will be about 5 kms per hour. When the value of time would be the same for walking and sitting in-vehicle, the optimal increase in the distance 2.v/w between neighbour stations as one approaches the centre is about 83 meters. This is two times the distance a pedestrian would walk during the time loss due to a stop, as outlined above. A similar result is found for the increase in distances between bus stops, because time loss due to stops is of a similar magnitude.

Consider for example a line with 10 stops and a distance between the origin L_0 and the first stop L_1 of say 600 m, then the distance between the last two stops L_{10} and L_{11} would be 600 + 10.(83) = 1430 m. Thus, in this numerical example the stop distance substantially increases as one moves from the fringe to the city centre.

For trains we would have larger increases in stop distances since time loss due to stops is higher. If we would count a loss of 2 minutes per stop for a train, the increase in stop distances as one approaches the centre would be about 333 meters.

The above calculations are based on the assumption that the value of time is equal for in-vehicle time or access time. A general result of the literature is that access-time is valued higher than in-vehicle time, at least with a factor 2 (see for example Wardman, 2002). This would imply that the above values suggested for the spacing of stops should be halved. On the other hand, it may well be that passengers have a certain dislike against stop-and-go of public transport vehicles and hence value in-vehicle time at stops higher than in-vehicle time when vehicles are moving. This would move the optimal stop distances again into the direction of the figures mentioned above.

The result illustrated in Table 1 implies that the metro gradually changes in its function along the line. At the edge of the metropolitan areas it starts as a feeder with the function of *collecting* passengers near their residences. It gradually moves into a transport mode with a *connecting* function where high speed is important. The shifting balance from collecting (priority on short access times) to connecting (priority on high speeds) is reflected by the increasing number of travellers in the vehicles.

In more refined network settings, collecting and connecting functions are usually carried out by different vehicles. For example, busses may collect passengers near their homes and bring them to metro stations. But also in such cases the principles outlined here will appear to be relevant. First, busses collecting passengers to bring them to metro stations are typical examples of public transport where demand builds up as one approaches the station, so that stopping distances should preferably increase. And second, irrespective of the access mode used by metro travellers, the basic trade-off between access time and in-vehicle time remains of relevance,

implying that the basic results also remain valid for the metro part of the transport system.

Extensions

There are several empirical reasons why this result must be used with care when putting it in practice. First, not all passengers will use the metro until the end. If all passengers would enter and leave at random places the first term in the total cost function would vanish and we would simply arrive at equal stop distances to be optimal. When for example only a share b of the passengers would use the metro to travel to the end, and the other half would leave at random places, total costs would be modified in two ways. First, the alighting costs would only put a burden on the share b, and the walking costs would now also be relevant for the share (1-b) on its way to the city centre. Thus, the modified cost function C(b) becomes:

$$C(b) = b.v.a\ (L_1+L_2+\ldots) \quad + \quad (2\text{-}b)\ w.a\ [(L_1)^2 + (L_2\text{-}L_1)^2 + (L_3\text{-}L_2)^2 + \ldots.]/4 \qquad (8)$$

The first order condition for cost minimisation implies:

$$(L_{n+1}\text{-}L_n) = (L_n\text{-}L_{n-1}) \ + 2.b.v/[(2\text{-}b).w] \qquad (9)$$

It is immediately clear that for b=1 we arrive at the original model formulation. When b=.5 the result is that optimal stop distances no longer increase with 83 meter, but at the much more moderate pace of about 30 meter (83/3).

The second reason why the model does not always comply with reality is that a metro line may indeed have an end station D being the dominant destination of most passengers, but that in reality it may well be that near D there are several important destinations. City centres are not just points, but in stead they are often rather spread out. As a result, when many travellers would benefit from a stop *near* D instead of *in* D, the optimal stop distance formula has to be modified for stops near D so that with in the central area the stop distances may again be smaller

3. Stop distances and density gradients.

Another point that deserves our special attention is that in urban areas the assumption of constant density is unrealistic. Near the centre densities tend to be higher and therefore there is a countervailing force: when the number of entering passengers increases rapidly as one approaches the city centre, one might even expect decreasing rather than increasing stop distances. Consider the case that the density gradient is exponential, a result that is often found in empirical urban research (Mills and Hamilton 1984):

$$\text{Density}_n = \exp(\gamma L_n) \text{ where } \gamma \geq 0 \qquad (10)$$

By dropping the assumption of constant density, the formula becomes more complex. First, we have to determine the expected number of passengers entering the metro at point O:

$$P_O = \int_0^{0.5L_1} \exp(\gamma x)dx = \frac{1}{\gamma}\exp(0.5\gamma L_1) - \frac{1}{\gamma},$$

as well as the expected number of passengers entering at an arbritrary point n:

$$P_n = \int_{0.5(L_{n-1}+L_n)}^{0.5(L_n+L_{n+1})} \exp(\gamma x)dx = \frac{1}{\gamma}\exp[0.5\gamma(L_n + L_{n+1})] - \frac{1}{\gamma}\exp[0.5\gamma(L_{n-1} + L_n)]$$

In this way the collective of travellers experiences the following number of stops:

$$\sum_{i=0}^{N-1}(N-i)P_i = \sum_{i=0}^{N-1}\{\frac{1}{\gamma}\exp[0.5\gamma(L_i + L_{i+1})] - \frac{1}{\gamma}\} \qquad (11)$$

This is the first part of the cost function, the second part is the expected walking distance of the travellers. The walking distance can be divided into two components:

- The distance contributed by travellers coming from the right of the stop:

$$\sum_{i=0}^{N-1}\int_{L_i}^{0.5(L_i+L_{i+1})}(x - L_i)\exp(\gamma x)dx$$

- The distance contributed by travellers coming from the left of the stop:

$$\sum_{i=1}^{N} \int_{0.5(L_{i-1}+L_i)}^{L_i} (L_i - x)\exp(\gamma x)dx$$

Thus the expected walking distance of the travellers equals:

$$\sum_{i=0}^{N-1} \int_{L_i}^{0.5(L_i+L_{i+1})} (x - L_i)\exp(\gamma x)dx + \sum_{i=1}^{N} \int_{0.5(L_{i-1}+L_i)}^{L_i} (L_i - x)\exp(\gamma x)dx =$$

$$\sum_{i=0}^{N-1} \{\frac{1}{\gamma^2}\exp(\gamma L_i) + \frac{1}{\gamma^2}\exp(\gamma L_{i+1}) - \frac{2}{\gamma^2}\exp[0.5\gamma(L_i + L_{i+1})]\} \qquad (12)$$

Combining equations (11) and (12) gives the cost function C(γ):

$$C(\gamma) = \sum_{i=0}^{N-1} v\{\frac{1}{\gamma}\exp[0.5\gamma(L_i + L_{i+1})] - \frac{1}{\gamma}\} +$$

$$\sum_{i=0}^{N-1} w\{\frac{1}{\gamma^2}\exp(\gamma L_i) + \frac{1}{\gamma^2}\exp(\gamma L_{i+1}) - \frac{2}{\gamma^2}\exp[0.5\gamma(L_i + L_{i+1})]\}$$

The first order condition for cost minimisation implies for n=1,…,N-1:

$$\frac{\partial C}{\partial L_n} = [\frac{v}{2} - \frac{w}{\gamma}]\exp[0.5\gamma(L_n + L_{n+1})] + [\frac{v}{2} - \frac{w}{\gamma}]\exp[0.5\gamma(L_{n-1} + L_n)] + \frac{2w}{\gamma}\exp(\gamma L_n) = 0 \qquad (13)$$

It appears that when γ approaches the value 0, implying a uniform distribution of population, this formula again coincides with the old formulation. The effect of the introduction of increasing densities is that a countervailing force starts to operate so that stop distances may both increase and decrease as one approaches the centre. Various patterns might be possible such as increasing stop distances; decreasing stop distances; decreasing stop distances near the city edge followed by increasing stop distances as one approaches the centre, etc. Figure 1 gives results for various values of the gradient parameter γ given values for v and w as used above. The figure shows

that for low values of γ stop distances increase monotonically, and for high values of γ stop distances are decreasing as one approaches the centre. An interesting finding is that there are certain combinations of v, w, and γ that yield constant stop distances.[2]

This result of constant stop distances is interesting because it would imply that the constraint usually adopted in this type of modelling may follow as a result of a more complex underlying model with varying spatial densities and focus of traffic flows on one destination. However, it is important to note that the parameter combination for which this result is found is rather extreme. A density gradient with γ=0.377 must be considered as very high. It would imply that the population density near the centre is about 43 times higher than at a distance 10 kms away. In reality one expects much flatter density gradients along public transport lines. This implies that the result of constant stop distances as an optimal outcome in a non-uniform density city is implausible. On the other hand, incorporation of other model elements such as the one in equation (8) indicating that part of the metro travellers does not have the centre as their destination would make the curves in Figure 1 flatter and this would lead to equally spaced stops for lower values of γ.

Figure 1. Distance between stops for various values of the density gradient parameter γ.

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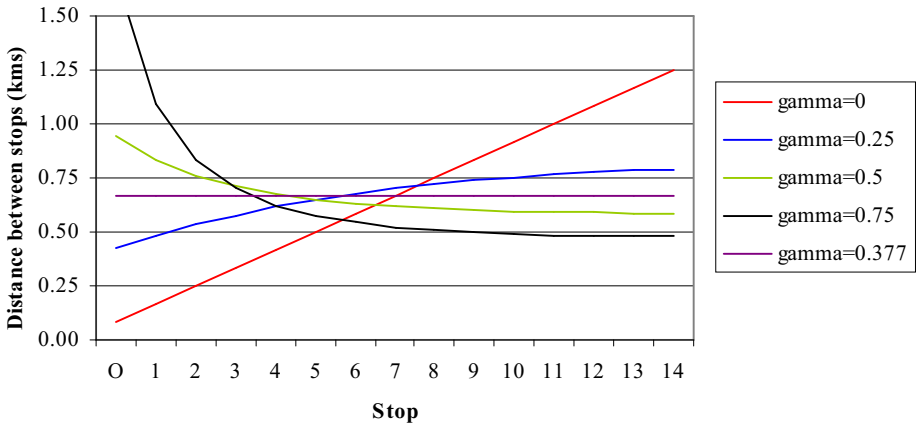
[2] To under stand why this result follows, rewrite L_n+L_{n+1} as $2L_n+x_n$, and $L_{n-1}+L_n$ as $2L_{n-1}-x_{n-1}$ so that x_n denotes stop distance between stops n and n+1. Then, by dividing both sides of the equation (13) by $\exp(L_n)$ we arrive at

$$[\frac{v}{2}-\frac{w}{\gamma}]\exp[0.5\gamma(x_n)]+[\frac{v}{2}-\frac{w}{\gamma}]\exp[0.5\gamma(-x_{n-1})]+\frac{2w}{\gamma}=0$$

Note that this equation holds for all n=1,…,N. Thus, when a combination of v,w,γ exists such that x_n equals x_{n-1} for a certain n, then this holds for all n, implying equal stop distances. The existence of such a point where x_n equals x_{n-1} follows from Brouwer's fixed point theorem.



Finally we note that the form of the stop distance curves in Figure 1 depend on the specification of the density function. In appendix 2 we carry out a parallel analysis for a power function, and it appears that here other results are obtained: decreasing stop distances at the beginning of the line, followed by increasing distances as one gets closer to the centre. Thus, the actual spatial density pattern does have a definite impact on optimal stop distances.

As an empirical test we collected data on public transport networks in the three largest metropolitan areas in The Netherlands (Amsterdam, Rotterdam and The Hague). For the railways we compared distances between stations at the edge of the city and near the city centre. We find a slight tendency that distances between railway stations are smaller near the edge (see Table 2). The background is partly that during the last decades as cities expanded new railway stations have been built to serve the newly developed residential areas and that station distances were chosen to be relatively low. For metro we did a similar exercise; here densities near the centre are similar to those near the edge. An interesting observation is that a more recently constructed light metro line in the Amsterdam region that uses the same infrastructure as the ordinary metro near the centre indeed has low stop distances at the city edge. This light metro line thus provides a good illustration of the principle discussed in this paper, the rail case is a weak illustration, and the ordinary metro does not comply with the principle. We note in passing that there may still be another reason to have higher stop density at the edge. The fixed costs of constructing a station tend to be lower there, not only

because land is cheaper, but also because near the centre the stations tend to be underground and at the edge they may well be above the ground.

Table 2. Stop distances in metropolitan areas in The Netherlands: train versus metro[3].

Mode	Average distance between stops near city edge (metres)	Average distance between stops near city centre (metres)	Ratio of stop distance at edge and near centre
Train	1500	1700	0.88
Metro	880	850	1.03
Light metro	490	850	0.58

4. Concluding remarks.

In our analysis we focussed on central city oriented public transport systems. When the centre is the major destination, vehicles will get fuller as they approach the centre and as a consequence stopping times will be experienced by an increasing number of travellers. We demonstrate that optimal stop distances increase as vehicles get closer to the centre. A countervailing force is that urban densities increase as one approaches the centre. We demonstrate that there exist combinations of the various cost and density gradient parameters that result in constant stop distances as an optimal outcome. However, this is found for very steep density gradients, so that the overall conclusion is that there are good reasons to let stop distances increase as one approaches the city centre. This tendency will be stronger, the larger the dominance of the centre as an origin or destination of passenger flows.

Confrontation of these outcomes with public transport systems in Dutch metropolitan areas only gives a partial confirmation that this principle is used. For light rail stop distances are indeed smaller near the fringe. But for metro systems no such result could be obtained.

[3] The figures in the table are based on the following railway lines: Berkel-Hofplein, Vlaardingen-Rotterdam CS, Barendrecht-Rotterdam Zuid, Mariahoeve- The Hague CS, Voorburg-The Hague CS,

Of course the present analysis can be extended by a more refined treatment of travel demand such as dropping the assumption of inelastic demand. This will make the analysis more complex, but there is little reason to expect that the results will be different in a qualitative sense.

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For metro the figures are based on lines 53 and 54 in Amsterdam. For light metro the figures relate to line 51.

Appendix 1. Model formulation when market areas of stations are based on minimisation of generalised travel costs.

Consider the case that travellers go to the station implying the lowest generalised travel costs. Consider the locations L_n and L_{n+1} of two subsequent stops. Let x_n denote the border between the market areas of both stops. Then the resident who lives at x_n will be indifferent between travelling via stop n and stop n+1. Hence: $(x_n-L_n)w+v+(L_{n+1}-L_n)z=(L_{n+1}-x_n)w$ where z is the time cost per unit distance so that $x_n= ½[(L_{n-1}+L_n)-(z/w)(L_{n+1}-L_n)-(v/w)]$. This shows that the market area border between the two stops is shifted away from the centre. Total costs are:

$$
\begin{aligned}
\mathbf{TC} = \; & ½\, a\, x_0^2\, w \\
& ½\, a\, (L_1-x_0)^2\, w + ½\, a\, (L_1-x_1)^2\, w + a\, (x_1-x_0)(-L_1 z) + \\
& ½\, a\, (L_2-x_1)^2\, w + ½\, a\, (L_2-x_2)^2\, w + a\, (x_2-x_1)(-L_2 z) + \ldots \\
& \ldots \\
& \ldots + \\
& ½\, a\, (L_{N-1}-x_{N-2})^2\, w + ½\, a\, (L_{N-1}-x_{N-1})^2\, w + a\, (x_{N-1}-x_{N-2})(-L_{N-1} z) + \\
& ½\, a\, (L_N-x_{N-1})^2\, w + \\
& a\, x_{N-1}\, L_N\, z + a.v(x_0+x_1+\ldots+x_{N-1})
\end{aligned}
$$

and where:

$x_0 = ½\,[L_1-(z/w)L_1-(v/w)]$
$x_1 = ½\,[L_1+L_2-(z/w)(L_2-L_1)-(v/w)]$
$x_2 = ½\,[L_2+L_3-(z/w)(L_3-L_2)-(v/w)]$
...
$x_{N-1} = ½\,[L_{N-1}+L_N-(z/w)(L_N-L_{N-1})-(v/w)]$

Minimisation of total costs leads to a linear system of equations in terms of $L_1,..,L_N$. Applying Cramer's rule leads to the conclusion that for the differences between stops the same results are found as in section 2.

Appendix 2. Stop distances given a power function density gradient.

$\text{Density}_n = L_n^{\,c}$ where $c \geq 0$

Thus, as distance from the centre increases with 1%, the number of customers decreases with c%.

The expected number of passengers entering the metro at point O:

$$P_O = \int_0^{0.5L_1} x^c dx = \frac{(0.5)^{c+1}}{c+1} L_1^{c+1}$$

The expected number of passengers entering the metro at point n:

$$P_n = \int_{0.5(L_{n-1}+L_n)}^{0.5(L_n+L_{n+1})} x^c dx = \frac{(0.5)^{c+1}}{c+1}(L_n + L_{n+1})^{c+1} - \frac{(0.5)^{c+1}}{c+1}(L_{n-1} + L_n)^{c+1}$$

The expected number of stops the collective of travellers experiences:

$$\sum_{i=0}^{N-1}(N-i)P_i = \sum_{i=0}^{N-1}\frac{(0.5)^{c+1}}{c+1}(L_i + L_{i+1})^{c+1}$$

The expected walking distance of travellers:

Two components

- Travellers coming from the right:

  $$\sum_{i=0}^{N-1}\int_{L_i}^{0.5(L_i+L_{i+1})}(x - L_i)x^c dx$$

- Travellers coming from the left:

  $$\sum_{i=1}^{N}\int_{0.5(L_{i-1}+L_i)}^{L_i}(L_i - x)x^c dx$$

Thus,

$$\sum_{i=0}^{N-1}\int_{L_i}^{0.5(L_i+L_{i+1})}(x-L_i)x^c dx+\sum_{i=1}^{N}\int_{0.5(L_{i-1}+L_i)}^{L_i}(L_i-x)x^c dx=$$

$$\sum_{i=0}^{N-1}\{[\frac{1}{c+1}-\frac{1}{c+2}]L_i^{c+2}+[\frac{1}{c+1}-\frac{1}{c+2}]L_{i+1}^{c+2}-2[\frac{1}{c+1}-\frac{1}{c+2}](0.5)^{c+2}(L_i+L_{i+1})^{c+2}\}$$

The total costs to be minimised are:

$$C=\sum_{i=0}^{N-1}v\frac{(0.5)^{c+1}}{c+1}(L_i+L_{i+1})^{c+1}+$$

$$\sum_{i=0}^{N-1}w\{[\frac{1}{c+1}-\frac{1}{c+2}]L_i^{c+2}+[\frac{1}{c+1}-\frac{1}{c+2}]L_{i+1}^{c+2}-2[\frac{1}{c+1}-\frac{1}{c+2}](0.5)^{c+2}(L_i+L_{i+1})^{c+2}\}$$

The first order condition for cost minimisation is:

$$\frac{\partial C}{\partial L_n}=[v-\frac{w(L_n+L_{n+1})}{c+1}](0.5)^{c+1}(L_n+L_{n+1})^c+[v-\frac{w(L_{n-1}+L_n)}{c+1}](0.5)^{c+1}(L_{n-1}+L_n)^c$$

$$+\frac{2w}{c+1}L_n^{c+1}$$

Figure 2.