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Using fractal dimensions for characterizing intra-urban diversity. The example of Brussels

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Using fractal dimensions for characterizing intra-urban diversity.
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Abstract. The objective of this paper is to compare fractal-based parameters calculated by different fractal methods for urban built-up areas, and to link the observed spatial variations to variables commonly used in urban geography, urban economics or land use planning. Computations are performed on Brussels. Two fractal methods (correlation and dilation) are systematically applied for evaluating the fractal dimension of built-up surfaces; correlation is used to evaluate the fractal dimension of the borders (lines). Analyses show that while fractal dimension is ideal for distinguishing the morphology of Brussels, each estimation technique leads to slightly different results. Interesting associations are to be found between the fractal dimensions and rent, distance, income and planning rules. Despite its limitations, fractal analysis seems to be a promising tool for describing the morphology of the city and for simulating its genesis and planning. The model is robust: it replicates the urban spatial regularities and patterns, and could hence fruitfully be integrated into intra urban simulation processes.

Keywords. Fractal dimension; morphology; urban; Brussels

I. Introduction

Concepts of fractals, scaling and fractal dimension have already widely been used in physical and human geography (see e.g. Mandelbrot 1982; Goodchild and Mark, 1987; Barnsley 1988 ; MacLennan et al., 1991). Let us remind here that **fractals** are objects whose geometric properties include irregularity, scale-dependence and self-similarity. If spatial fractal analyses initially concentrated on "natural" objects, they were later extended to urban forms and urban systems, linking urban hierarchy to fractal geometry, or analyzing the "global" urban physical form and growth processes (see e.g. Arlinghaus 1985; Arlinghaus and Arlinghaus, 1989; White and Engelen 1993; Batty and Longley 1994; Frankhauser 1990 and 1994; Batty and Xie 1996; Wentz 2000; Gomes 2001 or Shen 2002).

This paper aims at **understanding the spatial internal layout of the city** with fractal tools. We know that the city's design emerges on its own in accordance with a locally ordered system (Hillier 1996; Hillier and Hanson 1984). The spatial structure of cities is indeed a disorderly outcome of a long history of small incremental changes that occurred at large scales. The resulting patterns have neither geometrical, nor functional simplicity. The metropolitan feature is here limited to the built-up pattern and hence represented by a lattice of residential sites offering urban amenities and, in the interstices, "green areas" (that is to say areas that are not built up), where consumers enjoy "rural" leisure amenities. These empty spaces are ranked following an inverse hierarchical order. This structure breaks the geometry of the nested and specialized rings of the Thünen City. Hence, in order to formalize the urban area, we need a geometry that enables the nesting of residential and rural areas within a non-homogeneous hierarchically organized pattern, with lacunae and fully occupied cells. Fractal geometry meets these conditions; it is – by construction – a hierarchical organization of nested objects at different scales (see Cavailhès *et al*. 2002 for further discussion).

Batty and Longley (1994, chap. 6) have already tried to link fractality to the morphology of urban land use in the city of Swindon (England). They limited themselves to the type of land parcels (residential, commercial-industrial, educational, transport and open space). It has also already been shown that fractal shapes reduce travel costs to urban sites and green areas, and that peripheries of cities look fractal

(Anas et al., 1998; Frankhauser 1998). This paper aims at further analyzing the structure of the intra-urban built-up areas by (1) taking the built-up structure into account, (2) using different fractal measurement methods, and (3) linking the obtained values to variables commonly used in spatial urban economics, urban geography and land use planning. The experiment is performed on Brussels, a city where residential wards were mainly planned by private property developers. Hence, each urban residential ward has its own morphology. Specifically, this research focuses on whether different intra-urban patterns (historical center, planned urbanization, social housing, etc.) have different and/or unique fractal dimensions. It also attempts to see how far fractal dimension is statistically related to variables such as rent, household income, distance to CBD, etc. In other words, **is fractal dimension a useful index for distinguishing urban wards?** This paper attempts to draw two types of conclusions. On the one hand, it aims at linking the empirical fractal results to geographical and economic theories, to urban planning and land use (geographical conclusions). On the other hand, it aims at testing whether fractal indices enable one to improve the analysis of the structure of the urban wards, their morphology, barriers effects, etc.

The paper is organized as follows. Section II briefly explains what is a fractal), defines the methodology used in this paper as well as the data processing step. Section III discusses the computational results in terms of exploratory data analysis and Section IV in terms of bivariate relationships. Results are interpreted in an urban territorial development context. Conclusions, perspectives and remarks are included in Section V.

II. Methodology

II.a What is fractality?

Most of the currently used measures for describing urban patterns are based on the notion of **density**, that is the ratio between a mass (e.g. the built-up surface in a statistical sector) and the area on which this mass is localized (e.g. the total area of the statistical sector). Density is constant if the mass is proportional to the reference area, whatever the internal structure of the buildings: it gives a global information about the mass distribution in a given reference area.

By definition, fractals are geometric objects in which mass is not distributed homogeneously, but concentrated in clusters at different scales. The particularities of

fractals become obvious, when considering the iterative procedure, which may be used for their generation. An example is given for the **Sierpinski** carpet (Figure 1). An initially given square of base length L is broken down to $N_1 = N = 5$ smaller squares with base length $l_1 = 1/3\ L$, which are arranged like a chessboard within the initial square. This procedure is repeated in a second step for each of the five squares; we get $N_2 = N^2 = 25$ squares of size $l_2 = (1/3)^2 L$. By repeating this operation, the number of squares is multiplied by a constant factor $N = 5$ at each step, whereas their size is reduced at each step by a factor $r = 1/3$. Hence, at a given iteration step n, we find $N_n = N^n$ squares of size $l_n = r^n L$ called built-up or occupied sites. The total surface M_n of the built-up sites can be computed at each step n:

$$M_n = N_n \times (l_n)^2 = L \times (N \times r^2)^n = L\left(\frac{5}{9}\right)^n \qquad [1]$$

Since $5/9 < 1$, M_n tends to vanish for high iteration steps.

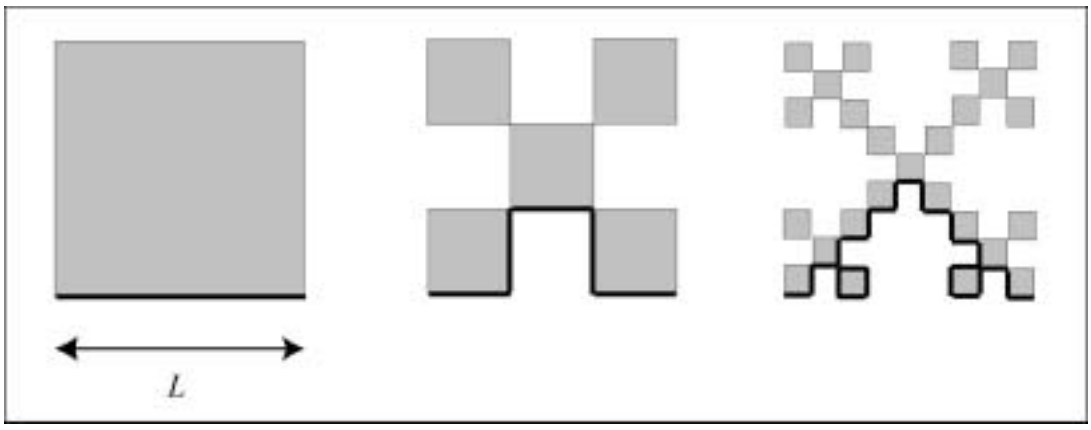


Figure 1: Generating the Sierpinski carpet: the initial square and the two first iteration steps.

In fat: the same steps for generating the border of the Sierpinski carpet

Whereas the built-up squares have at each step the same size l_n, a spatial hierarchy appears in the free spaces (green areas) in the course of the iteration. At the first step, four large lacunae of size l_1 are generated, and the second step adds $\nu_2 = 4 \times N$

= 20 sites of size l_2. Hence, at a given step n, the system of free spaces consists in $N_n = 4 \times N^{n-1}$ lacunae of size $l_n = r^n \times L$, $N_{n-1} = 4 \times N^{n-2}$ lacunae of size $l_{n-1} = r^{n-1} \times L$, etc. This hierarchical distribution of free spaces means that the built-up squares are distributed in a rather non-homogeneous way. A multi-scale cluster structure is observed. Such concentrations also occur in the Fournier dusts (Figure 2 *a*). This example reminds an intra-urban pattern where houses form blocs around courtyards, blocs being grouped around a square.

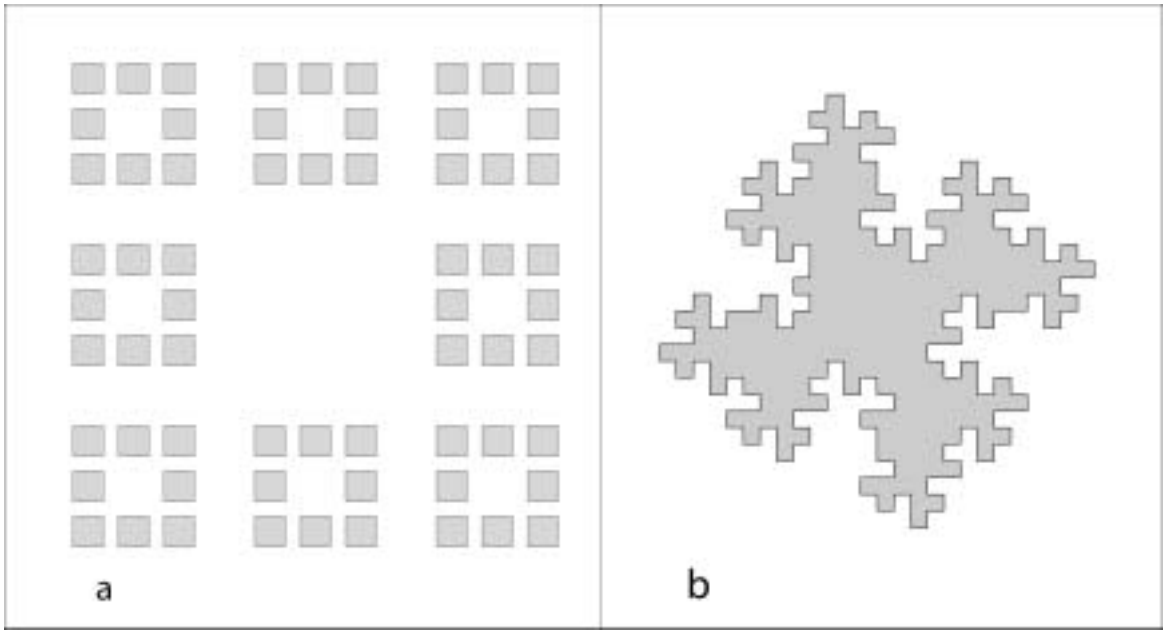


Figure 2: Figure *a* shows a Fournier dust in second iteration (cf. text). Figure 2 *b* shows a teragon: the border is fractal whereas the mass contained is distributed homogeneously all over the iteration steps.

It has been shown that this hierarchical distribution of free spaces corresponds to a Pareto-Zipf distribution (e.g. Mandelbrot, 1982; Goodchild and Mark 1987; Arlinghaus 1985). We can thus expect that exploring their organization across the scales will provide information about their spatial organization.

Fractal geometry is not only useful for describing the spatial distribution of built-up surfaces, but also for describing the filigree structure of the borders of built-up areas. Let us come back to the example of the Sierpinski carpet. As shown in Figure 1 it

is possible to construct its boundary, also by means of an iterative mapping procedure (fat line). We verify that the border length L_n at a given step n is

$$L_n = 4 \times N_n \times l_n = 4 \times L \times (N \times r)^n = 4L\left(\frac{5}{3}\right)^n \qquad [2]$$

and thus tends to infinity although the border remains, topologically, a linear geometric object. Hence, the perimeter of the Sierpinski carpet diverges, while its surface tends to zero. The multi-scale aspect occurs now through the form of the border that consists of a multitude of "loops" of different sizes. First, loops are generated; they form big «bays» of size l_1. Then the line segments of size l_1 are broken by adding smaller loops of size l_2 , etc. Hence, the spatial organization of the border follows a hierarchical principle. Hence, the surface of the Sierpinski carpet tends to zero while its perimeter diverges.

Usual geometric measures like the length of a line or the surface of a two-dimensional object are not appropriate for describing such structures. The measure theory introduces the concept of fractal dimension by means of a relation requiring that there exists a measure $\mathcal{L}$, which doesn't diverge or vanish, but which remains constant all over the iteration steps. The used relationship corresponds to [1] and [2], and introduces a generalized scaling exponent D:

$$\mathcal{L} = N_n \times (l_n)^D \qquad [3]$$

This requirement is fulfilled by the choice of the free parameter D, the **fractal dimension**. Inserting the relation $N_n = N^n$ and $l_n = r^n L$, and setting $\mathcal{L} = L^D$ yields[1] to:

$$D = -\frac{\log N}{\log r}$$

D is directly linked to the iteration parameters N and r. The same concept may be used for describing the distribution of the occupied sites e.g. in a Sierpinski carpet or for describing a linear structure like its border. D describes how the mass is concentrated in a given surface (Frankhauser and Pumain, 2002). A value of D close to

[1] The choice $\mathcal{L} = L^D$ doesn't affect the reasoning, but corresponds just to a particular choice of the unit for the measure.

2.0 describes a rather homogeneous distribution, whereas the lowest the value of D, the more the mass is aggregated at different scales. A dimension close to 0.0 corresponds to a mass concentrated in one point. A value of 1.0 corresponds to a line, but also represents a threshold in fractals: when $D < 1.0$ the structure is necessarily composed of an unconnected set of points. Such a situation corresponds to the Fournier dusts. $D \geq 1.0$ refers to structures that are either Fournier dusts or consist of a unique cluster, highly fragmented, like Sierpinski carpets. For linear fractal object like the border of the Sierpinski carpet, the fractal dimension measures the progressive lengthening of the border when passing from one scale to a smaller one: a value of D close to 1.0 informs us that the borderline doesn't show important deviations from a straight line. Greater values correspond to filigree structures with a multitude of loops of different sizes.

For Sierpinski carpets the set of occupied sites and the border have the same fractal dimension, since the same parameter values $N = 5$ and $r = 1/3$ are used for generating both the geometric objects. This is not necessarily the case as shows the example of the **teragon** in Figure 2 *b*. Here the border is generated by an iteration with $N = 8$ and $r = ¼$, but the inner surface remains constant all over the iteration. Hence, the fractal dimension for the border is equal to 1.5 (D_{Per}) whereas the inner mass distribution remains homogeneous ($D_{Surf} = 2.0$). By analyzing the spatial distribution of the built-up **areas** as well as their **borders**, we obtain fractal dimension values that make it possible to situate the real world patterns with respect to the described types of fractals. These fractals play the role of geometric reference models, like circles or squares in Euclidean geometry. In real-world structures, we do not expect to find well-defined levels like in constructed fractals in all cases. Nevertheless, fractal analysis makes it possible to explore the implicit hierarchy of a spatial pattern: fractal analysis enables one (1) to verify to what extent a spatial pattern[2] is organized in a hierarchical way, and (2) to estimate parameters characterizing this spatial hierarchy.

II.b Measuring fractality

Measuring fractal behavior imitates in some sense the iteration procedure by testing if the same type of spatial organization occurs at each scale. A series of measures ε_n of different size is therefore introduced in analogy to the length l_n of the elements in

[2] In this case, we only consider distributions of elements such as buildings on a two-dimensional surface.

the constructed fractals. For each value ε_n we count then the number $N(\varepsilon_n)$ of elements which are necessary to describe the structure. This means that we neglect all details of the structure whose size is inferior to ε_n. Then an equivalent relation to [3] can be derived (Mandelbrot 1982):

$$N(\varepsilon_n) \times (\varepsilon_n)^D = a \qquad \Rightarrow \qquad N(\varepsilon_n) = a \times \varepsilon_n^{-D} \qquad [4]$$

It can be shown that the so-called form-factor *a* characterizes the general features of the structure which are not linked to fractal geometry (Mandelbrot 1982). E.g. the fact that the Sierpinski carpet looks globally more like a square than an hexagon will affect the value of *a*. In addition, *a* also depends of all relative deviations from the pure fractal law occurring at different scales. Hence, it gives no precise information and may be fixed arbitrarily to $a = 1$ for a given structure, e. g. for a homogenous black surface. The observed values for *a* should vary within a small range around $a = 1$.

$$a(\varepsilon_i) = \frac{N(\varepsilon_i)}{\varepsilon_i^D} = \frac{\text{observed number}}{\text{theoretical number}}$$

In real patterns, we often observe critical values ε_{crit} which separate ranges of ε_n for each of which different fractal laws are observed. By introducing an additional constant *c*, it is possible to fit the fractal laws separately for each range and to estimate correctly their fractal dimensions. Hence:

$$N(\varepsilon_n) = a \times \varepsilon_n^{-D} + c \qquad [5]$$

D is often estimated by using a double logarithmic representation of the power law. If $c = 0$, this law is linear. However, as this corresponds to a nonlinear data transformation, deviations from the law are not treated in the same way all across the estimation range. Hence we prefer to minimize the least square deviations by means of a non-linear regression. Our computer program allows using simplified versions of the relation [5] without *a*, *c* or by keeping either *a* or *c*. Preliminary tests showed that it is wise to maintain a three-parameter version (*a*, *D* and *c*). As pointed out, *a* should remain small; this is usually observed. Otherwise the considered zone is very complex and fractal estimation does not seem useful.

Several methods refer to the discussed logic, each one defining the scales *i* in a particular way. Two methods are used in this paper: the **dilation** and the **correlation** analyses.

In the **dilation** method (noted *Dil*) each built-up site is surrounded by a square whose base length ε_n is incrementally enlarged. Thus the squares referring to built-up sites that are close together intersect first, and in the course of the iteration, more and more clusters appear. $N(\varepsilon_n)$ corresponds to the number of squares of size ε_n necessary for covering the total occupied surface. It is obtained by dividing at each step *n* the occupied surface by the surface ε_n^2 of the squares. Preliminary tests have shown that the method is less reliable for linear structures where the mass of the object is relatively small. Hence this method is exclusively used for investigating built-up areas (surfaces). However, even for such textures, we have observed that the method tends to underestimate the values $N(\varepsilon_n)$, when ε_n values are high. Hence, the estimated *D*-values tend to be lower than those expected by theory. Since this artifact seems to occur in a comparable way for different types of textures, the use of the method remains interesting.

In the **correlation analysis** (noted *Cor*), the texture is not modified: we simply count the number of occupied sites (pixels) that lie within a square[3] of base length ε_n of each occupied site and then compute their mean number $N(\varepsilon_n)$. The procedure is repeated for other values of ε_n. The information obtained is slightly different from the dilation analysis: we get information about the so-called "second-order" effects, i.e. we test the mean neighborhood scaling behavior and not really the cluster distribution. For simple fractal structures, the results tend to be the same, but for more complex structures differences are to be expected. This method turns out to be reliable and can hence be used to analyze surface distributions as well as linear distributions. From a theoretical point-of-view, the dimension obtained by correlation should not exceed that obtained by dilation. However, for real urban patterns as well as for specific constructed patterns, this is not necessarily the case. This may be explained by the above-mentioned artifacts and should not be over-interpreted.

[3] In principle it is possible to choose any shape for the environment, such as a circle, a hexagon, etc. However, since pixels are square-like, the choice of a square helps to avoid rounding errors.

II.c Data processing

Brussels is the capital city of Belgium, located almost in the center of the country. Defining its limits is an objective on its own (GEMACA 1995, Thomas *et al*. 2000, Vanderhaegen et al. 1996): the city sprawls far beyond its original boundaries. In the administrative sense, the Brussels Capital-Region (B.C.R.) is one of the three Regions of the Belgian federal state. Spatially, it corresponds to the enlarged city center (19 communes) and hence excludes recent peripheral wards that mainly extend in the two other administrative and linguistic regions. This paper only refers to the city defined by the limits of the B.C.R. It corresponds to 954,000 inhabitants and 16,138 hectares (in 2001). Ignoring the peripheral communes means better isolating urban from periurban land uses.

The necessary input of our analysis is a **city map** representing built-up areas. The C.I.R.B. (Centre Informatique pour la Région Bruxelloise) has developed a comprehensive geographical information system for the Brussels Capital-Region. This system was developed in the 90's for town planning and administrative uses. The layer corresponding to the built-up areas is the only one taken into consideration in this analysis (Figure 3); each building is delineated. A pixel on the map represents a 2.5×2.5 meters area on the ground. Each built-up information is binary: built-up, not built-up; no information is provided about the function of the building (hospital, plant or residence), nor about the type of green area (garden, public square, etc.). There is no information about roads, parking spaces, rail-tracks, etc.; these spaces are not included in the considered built-up areas.

Fractal dimensions are computed for a set of **26 windows**. The size of a window (2750 × 2250 meters) is defined from the size and shape of the CBD of Brussels: it corresponds to the best fit by a rectangle to the Pentagon (CBD). 10% overlapping of the windows is systematically applied horizontally and vertically in order to optimize the analysis of the spatial structure. A gliding window is then applied from left to right and from top to bottom. Due to the irregular administrative limits of the B.C.R., windows including at least 50% of B.C.R. are the only ones taken into account. Hence, 26 windows are kept in the analysis; each window receives an identification that corresponds to the *x*, *y* location on the grid proposed in Figure 3. Given the characteristics of Brussels and the size of the windows, windows are – by definition – never homogeneous in terms of function or built-up patterns. Ideally, smaller windows

would lead to more homogeneous wards, but then they often reach the limit of the fractal software.

For each of the 26 windows, three fractal dimensions are computed: 2 pertain to the surfaces, one to the borders. Results are first analyzed in a purely descriptive and exploratory way (Section III) and then associated with town planning and geographical variables (Section IV) in order to test whether fractal indices "measure" the urban landscape and give a good idea of the history of the city.

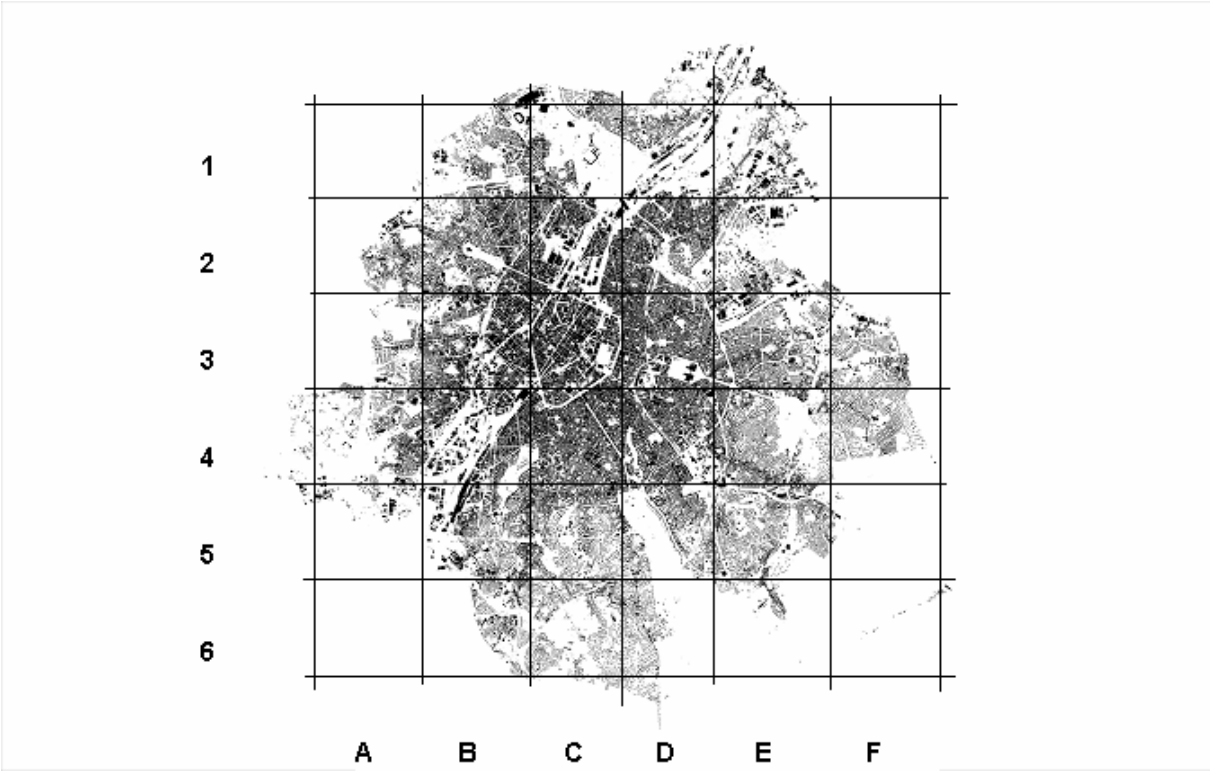


Figure 3: The built-up area of Brussels Capital Region
Source: Brussels Urbis

III. The fractal dimension of Brussels: a descriptive approach

As already mentioned, two methods are applied to built-up surfaces: correlation (*Cor*) and dilation (*Dil*). For the dilation technique, several trials are made in terms of number of iterations; 30 dilation steps seems to be the best fit for most of the studied windows. *Cor* and *Dil* are expected to lead to different results as they refer to different kinds of reasoning. Borders (lines) are also extracted by means of the dilation technique (5 iterations); the fractal dimension of the border is then computed using the correlation

method. In the next paragraphs, $D_{Surf\text{-}Cor}$, $D_{Surf\text{-}Dil}$ and $D_{Per\text{-}Cor}$ are the notations used respectively for fractal dimensions computed on surfaces (*Surf*) or borders (*Per*), with the correlation (*Cor*) or dilation technique (*Dil*).

Whatever the technique, the observed minimum value is 1.24 and the maximum 1.96 (Table 1). As expected, the values of *D* depend upon the estimation technique. Correlation analysis leads to larger values than dilation; this can be explained by the above-mentioned artifact of estimation. In order to have global information about the different samples of windows, we also compute the arithmetic means of the *D* values. We are aware that from a statistical point-of-view, the fractal dimensions are not values that are really observed, since they are estimated by recurring to the values $N(\varepsilon_i)$. We might expect that a mean parameter should refer to these empirical values $N(\varepsilon_i)$. However, since fractal dimensions are obtained by a non-linear estimation procedure over the used ranges of ε_i, we consider them as **independent synthetic measures** of the mass distribution in each window. This justifies the use of the arithmetic means of fractal dimensions as indicator for the order of magnitude of *D*.

As expected, fractal dimensions for a linear topology (*Per*) are smaller than those referring to the built-up surfaces (*Surf*).

Table 1: Descriptive statistics for *D* for different fractal methods

	$D_{Surf\text{-}Cor}$	$D_{Surf\text{-}Dil}$	$D_{Per\text{-}Cor}$
Mean	1.822	1.565	1.719
Minimum	1.338	1.261	1.543
Maximum	1.961	1.793	1.854

On average, whatever the method used, the highest *D* values are observed in the city center, but the variation through the 26 windows differs according to the method used (Table 2). Measures referring to surfaces lead to values positively and significantly associated in the space (+0.589), while the dimension of the border is negatively related to the dimension of the surfaces ($D_{Surf\text{-}Dil}$) or non significant ($D_{Surf\text{-}Cor}$). Let us add at this point that at this scale of analysis, the border of the city has no or little sense. The perimeter is an artifact; it mainly measures the shape of the “non-built up areas” (lacunae). In our case, the denser the built-up areas, the smaller the green areas. All other Pearson’s correlation coefficients are not significant. This suggests that each

fractal estimation method measures a different component of the urban layout, and that the fractal behavior varies within the urban structure.

Table 2: Pearson's correlation coefficients between fractal dimensions

	$D_{Surf\text{-}Cor}$	$D_{Surf\text{-}Dil}$
$D_{Surf\text{-}Dil}$	0.589	1.000
$D_{Per\text{-}Cor}$	n.s.	-0.458

As discussed in Section II.b, we can compare D obtained by correlation on the built-up surfaces ($D_{Surf\text{-}Cor}$) to that obtained on the perimeter ($D_{Per\text{-}Cor}$). We know that (1) if $D_{Surf\text{-}Cor}$ is equal to $D_{Per\text{-}Cor}$, the structure is close to a Sierpinski Carpet (Figure 1), (2) if $D_{Surf\text{-}Cor}$ tends to 2.0 and $D_{Per\text{-}Cor} < 2.0$, the spatial structure tends to a teragon (Figure 2 *b*), and finally, (3) if $D_{Surf\text{-}Cor} < 2.0$ and $D_{Per\text{-}Cor} \leq 2.0$ and different from $D_{Surf\text{-}Cor}$ the structure is mixed. Figure 4 gives the histograms of the $D_{Surf\text{-}Cor}$ and $D_{Per\text{-}Cor}$ values: they are all smaller than 2.0 and greater than 1.3. Figure 5 illustrates the relation (ratio) between D_{Surf} and D_{Per}: most values are smaller than 1.0, that is to say D_{Surf} is often smaller than D_{Per}. In the case of Brussels, the ratio varies between 0.815 and 1.176: the fractal structure of the built-up area is mixed.

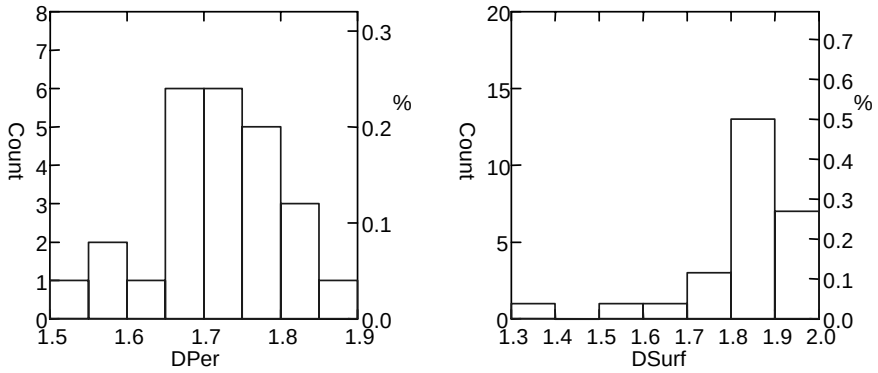


Figure 4: Histogram of $D_{Surf\text{-}Cor}$ and $D_{Per\text{-}Cor}$

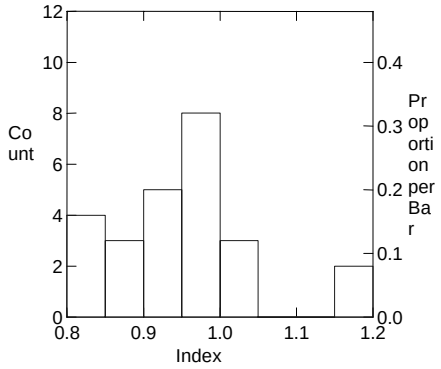


Figure 5: Histogram of the ratio of $D_{Surf\text{-}Cor}$ and $D_{Per\text{-}Cor}$

Let us examine the extent to which the measures of *D* conduct to a coherent urban structure, to a classification of the 26 wards that reflects urbanization and land use planning, that corroborates former geographical analyses. A Ward hierarchical clustering method was applied to the 26 windows on three criterions: $D_{Surf\text{-}Cor}$, $D_{Surf\text{-}Dil}$ and $D_{Per\text{-}Cor}$. Distances were used for measuring resemblance. We can again question whether a classification of the dimensions is justifiable since the real observed values are the $N(\varepsilon)$. However, we adopt the same point-of-view as in the preceding sub-sections and consider that we dispose of a set of independent synthetic parameters *D*. Moreover, we show in Appendix 1 that an estimation of the distances between fractal dimension values *D* is an appropriate measure for comparing scaling laws $N(\varepsilon)$. Table 3 reveals a rather clear center-periphery structure: 3 sub-groups of windows are "urban" and 2 sub-groups of windows belong to the "suburban" part of the Brussels Capital-Region.

The clustering of the windows reflects the history of the city as well as the land-use and socio-economic / demographic characteristics of the city (see e.g. De Keersmaecker 1990; De Keersmaecker and Carton 1992; Mort-Subite 1990; Thomas and Zenou 1999; Goffette-Nagot, Thomas and Zenou, 2000; Vanderhaegen et al. 1996). When Belgium became independent in 1830, Brussels was a market town, limited by ramparts. Its accession to the status of capital city of the country as well as the concomitant Industrial Revolution led to an important increase in population and hence a high pressure on its territory. The city became very dense in the center (behind the former ramparts) in an area called the Pentagon (here noted *C3*), and later extended to the periphery. At the end of the 19th century, large urban public works enabled further urban sprawl and new residential alternatives were offered to upper-class households in

the East (Schaerbeek (*D2*), Woluwé-Saint-Lambert (*E3*) and Ixelles (*D4*)), in the South (Saint-Gilles (*C4*), Forest (*B5*)) and in the North (Koekelberg (*B2*)). All these wards belong to cluster U1 of the classification of the fractal dimensions. During the same period, the "Quartier Léopold" (*D3*) and the "Quartier Nord"(*C2*) developed respectively in the East and North of the city; those wards are centered on a main railway station and have been deeply restructured in the sixties by functionalistic urbanization. This is also the case of some parts of Molenbeek (*B3*), which originally developed around factories during the 19th century; hence, old council flats still characterize this part of the city. These wards belong to cluster U2.

Table 3: Ward classification of 26 windows according to $D_{Surf\text{-}Cor}$, $D_{Surf\text{-}Dil}$ and $D_{Per\text{-}Cor}$

Group	Sub-group	Mean *D*			Content	Description
		Surf-Dil	*Surf-Cor*	*Per-Cor*		
Urban	**U1**	1.70	1.92	1.69	*B2, D2, C3, E3, C4, D4*	City center and mixed wards of the 19th century
	U2	1.75	1.89	1.65	*C2, B3, D3*	CBD with buildings and offices + some residential and old industrial wards
	U3	1.52	1.79	1.73	*C1, D1, B4, E4, B6, B1, E5*	Mixed residential parts of the 20th century and industrial
Suburban	**S1**	1.46	1.82	1.78	*A2, E2, A3, F3, F4, C5, D5, C6*	Urban with gardens, 20th century
	S2	1.42	1.44	1.68	*E1, A4*	Rural, large industrial surfaces and public equipment

The expansion of the city in the Interbellum period (Cluster U3) was characterized by wards forming a first green belt. Several factors were behind this development: new tramway tracks, the Universal Exhibition in 1958 in the North of the city (Heysel – *B1*), the construction of social housing (garden cities such as Floreal in Watermael Boitsfort). After World War II, the city spread further and further away, "diluting" in the countryside; as in many other cities, this was due to the low price of land in the periphery, to the increasing use of the car and to the consumption of rural amenities. This characterizes the detached buildings of cluster S1 and the suburban wards of Uccle and Woluwé. Finally, in the western and northern parts of Brussels,

there are still some rural spaces that are increasingly coveted and settled by industries, schools, hospitals, stores, or other private and public amenities. This characterizes cluster S2. Let us recall that "rehabilitation" is invisible at this level of analysis: it mainly affects the inside part of the buildings and not the relative organization of the buildings (no demolition).

From this descriptive exploratory data analysis, we can conclude that (1) each fractal dimension measures complementary aspects of the structure of the urban built-up area, and (2) the concomitant use of the three dimensions provides a rather accurate reflection of the intra-urban structure of Brussels. The urban structure seems to correspond to a multi-fractal logic OR to the overlay of different fractal patterns.

IV. A binary explanatory exploration of the data

Section III confirms results obtained by models based on land-use characteristics as well as socio-economic and demographic statistics. Let us now test single bi-variate relationships between fractal dimensions and some of those variables.

IV.a Housing characteristics

Bid-rent theory postulates an implicit trade-off in housing decisions between housing space and type, and proximity to central urban functions. Housing type is hence a key variable for defining urban structure: cities are often articulated as spatial patterns with flats near the center, terraced houses occupying the inner suburbs and detached/semi-detached houses in the outer suburbs, each ring reflecting a stage in city growth. However, the date at which the land parcel was integrated into the urban development process also defines the structure of the neighborhood: a specific architectural project can lead to a specific spatial structure that should lead to specific and unique fractal dimensions. Hence, in this first hypothesis, we test the link between the fractal dimension *D* and some morphologic characteristics of the built-up type. A priori, this should be true for Brussels (see Section III), but it is difficult to measure as each window is – by definition – characterized by a mixed urbanization processes. In the case of a city such as Brussels, a homogeneous ward would be far too small to be analyzed by fractal methods. However, if the size of the analyzed areas is too small, the results are no longer reliable: the boundary effects become too important.

For each window, several variables were created in order to quantify the type of housing by available statistics. Most variables are obtained from the 1991 Population Census (I.N.S. 1991). Table 4 gives the relationship between the fractal dimensions and the housing characteristics. Given the density of the built-up areas, the dilation method leads to the most significant results close to those obtained by the correlation method. These are, however, very different from the dimension of the perimeter: this latter relates to the borders of the empty spaces (lacunae). Hence, if the mass is large, the theoretical number of possible arrangements is smaller than if there were only some small black dots to distribute. Hence, the variation of the potential fractal dimensions is smaller for the centers.

Table 4: Pearson's correlation coefficients between fractal dimensions and housing characteristics.

	$D_{Surf\text{-}Cor}$	$D_{Surf\text{-}Dil}$	$D_{Per\text{-}Cor}$
% detached and semi-detached housing	-0.841	-0.713	n.s.
% dense terraced housing	-0.345	n.s.	0.473
% flats /total housing	0.837	0.724	n.s.
% flats in buildings of more than 10 housings	n.s.	0.504	n.s.
Concentration of buildings of more than 10 housings	-0.373	n.s.	0.373
Number of flats/number of houses	0.856	0.571	-0.441
Number of flats in buildings of more than 10 housings/number of detached housings	0.819	0.569	n.s.

Note : n.s. = not significant at $\alpha = 0.05$

Our first conclusion states that each estimation technique (*Dil*, *Cor*) conducts to a value of *D* that is significantly related to the characteristics of the housing. Moreover, land uses that are larger in scope (i.e. residence) have a greater degree of irregularity simply because they are larger in scale and there is hence less effort put into the geometric control on land under development. This second statement confirms Batty and Longley's results (1994, page 226). Last but not least, given the different nature of the fractal estimation methods applied here, differences in the relationships are observed

especially when comparing surfaces to borders; note the opposition in sign between the correlation coefficients computed for surfaces and that for perimeters.

Let us also mention that the history of the development of the city also influences the fractal dimension. Table 5 gives the relationship between the age of the housing and *D*. Pearson's correlation coefficients are never significant for the fractal correlation method computed for surfaces. $D_{Surf\text{-}Dil}$ and $D_{Per\text{-}Cor}$ give more significant results, but the sign of the coefficient is often opposite: $D_{Surf\text{-}Dil}$ refers to the built-up areas (surfaces), $D_{Per\text{-}Cor}$ refers to the sprawling shape of the town border. The older the housing, the more dense the urbanization and the higher $D_{Surf\text{-}Dil}$ but the smoother the town border and hence the smaller $D_{Per\text{-}Cor}$. To the contrary, the more recent the housing, the larger it is (further away from the city center) and hence the sprawl is more important (lengthening of the town border; higher $D_{Per\text{-}Cor}$) and the smaller $D_{Surf\text{-}Dil}$.

Let us recall that Pearson's correlation coefficient between the age of the buildings and $D_{Surf\text{-}Cor}$ (Table 5) is never significant at the 5% level. Their values, however, vary according to the age of the buildings. In fact, dilation of the image ($D_{Surf\text{-}Dil}$) implies dilating the built-up surfaces and hence ignoring the irregularities, the details, the "noises" observed in the built-up areas. Hence, this technique (*Dil*) tends to a generalization, to a better "modeling" of the built-up areas. To the contrary, $D_{Surf\text{-}Cor}$ is computed on built-up surfaces that are not dilated, and hence takes into account more irregularities. They are hence non-significantly related to the average measures of age computed on wards.

Table 5: Pearson's correlation coefficients between fractal dimensions and the age of the housing

	$D_{Surf\text{-}Cor}$	$D_{Surf\text{-}Dil}$	$D_{Per\text{-}Cor}$
% housing built before 1919	0.609	n.s.	-0.655
% housing built before 1945	0.680	n.s.	-0.606
% housing built after 1960	-0.492	n.s.	0.586
% housing built after 1971	-0.481	n.s.	0.507
Mean age of the housing	-0.647	n.s.	0.640
Housing after 1961/housing before 1945	0.500	n.s.	-0.585

Note: n.s.: not significant at $\alpha = 0.05$

IV.b Distance to the CBD

Distance is a key factor for interpreting the internal structure of a city (Anas et al. 1998): on average, the further from the CBD, the more recent the residence, the smaller the rent, the larger the houses and the higher the transportation costs (see e.g. Le Jeannic 1997; Goffette-Nagot 2000; Cavailhès *et al.* 2002). Hence, we expect a significant difference between the fractal dimension in the city center compared to that of the outskirts.

Table 6 gives the Pearson's correlation coefficients between the fractal dimensions D and the crow-fly distance to the city center. Two ways of defining the city center are used: one corresponds to the historical and administrative core of the city (Grand-Place), the other to the location of offices and many national and international administrations (Quartier Léopold). As expected, dimensions computed for surfaces ($D_{Surf\text{-}Cor}$, $D_{Surf\text{-}Dil}$) lead to significant negative coefficients: the center is denser and more homogeneous in terms of built-up areas. Dilation, however, leads to higher values of the correlation coefficients. As expected, dimensions computed for the borders give opposite results: the greater the distance to the center, the higher the value of $D_{Per\text{-}Cor}$: borders penetrate in "green areas" (gardens, etc.) on various scales. This is the case for the windows located in the periphery of the studied area.

Subsection IV.-a showed that fractal dimensions are related to the stage of development of the city (history); this subsection (IV.-b) confirms this fact as it shows that fractal dimensions are significantly related to the distance to the CBD.

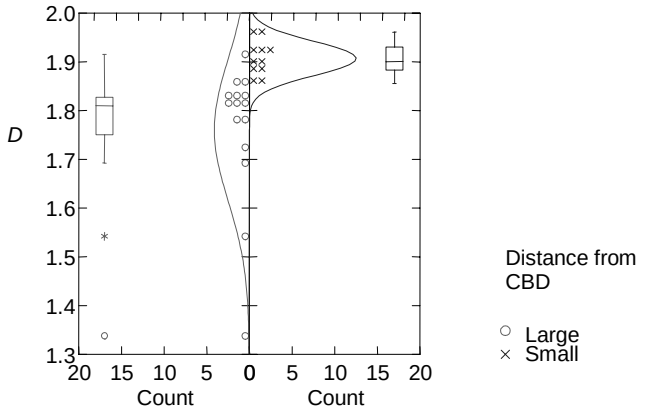
Table 6: Pearson's correlation coefficients between fractal dimensions and distance to city center, for two ways of defining the center

Distance to	$D_{Surf\text{-}Cor}$	$D_{Surf\text{-}Dil}$	$D_{Per\text{-}Cor}$
Grand-Place	-0.623	-0.935	0.525
Quartier Léopold	-0.716	-0.844	n.s.

Note: n.s.: not significant at $\alpha = 0.05$

Figure 6 translates graphically the results of the comparisons of the $D_{Surf\text{-}Cor}$ values for large and small distances from the CBD (larger/lower than 1.800 meters). The absolute values of *D* are greater near the center and their variation is smaller than further away from the city center: centers are more homogeneous and more densely built. More important fluctuations at greater distances from the CBD mean that the spatial organization is weaker at the outskirts and hence that the complexity of the built-up areas is greater: on average, housing in the periphery is not planned and its structure is spatially more heterogeneous. Hence, *D* is a measure of diversity.

Figure 6: Distribution of the $D_{Surf\text{-}Cor}$ values for windows close to the city center (right) and windows further away (left)



IV.c Rent

From spatial economics, we know that a household selects a residential location within a metropolitan area that is made up of several residential sites. The household's fixed income per unit of time is assigned to hiring a residential plot, commuting to the CBD and enjoying urban and rural amenities (see e.g. Cavailhès et al., 2002 and 2003). In this case, we want to test whether this economical mechanism could be translated into the built-up characteristics and hence by fractal dimensions. In Brussels, we know that high rents are associated to periurban locations as far as residence is concerned, and to central locations for offices.

Three rent indicators are considered here. The first is the average monthly rent for housing obtained by an annual survey (De Keersmaecker 1994). The second is the population density, which should give an indication of the intensity of the pressure on real estate. Thirdly, "human density" is considered; it refers to population plus employment density.

Table 7: Pearson's correlation coefficients between fractal dimensions, rent and density.

	$D_{Surf\text{-}Cor}$	$D_{Surf\text{-}Dil}$	$D_{Per\text{-}Cor}$
Average rent for housing	n.s.	-0.680	0.533
Population density	0.723	0.834	-0.405
Population + employment density	0.648	0.869	-0.533

Note: n.s.: not significant at $\alpha = 0.05$

Table 7 reveals quite interesting results: first, it shows that fractal dimensions related to built-up surfaces ($D_{Surf\text{-}Cor}$ and $D_{Surf\text{-}Dil}$) increase when density increases, confirming that large population densities are associated to homogeneous built-up areas (D large) and inversely. This also means that low densities have a different spatial structure of the built-up areas: in terms of planning and land uses, small densities mean less buildings and hence a greater variety in the size of the empty spaces. This confirms the classical North American model: a center with dense, rather small, uncomfortable and low-quality housing, and a periphery with larger housing, less dense and rather comfortable.

Another way of interpreting these results is to refer to former geographical analyses: results are simply to be related to the structure of the housing market. Indeed, former papers (see e.g. De Lannoy and Kesteloot 1985; Vanderhaegen et al. 1996; Kesteloot 1997) have shown that in Brussels the housing market is structured into three concentric circles of decreasing rent. A first set of central wards dating mainly from the 19th century where housing mainly corresponds to old houses divided into flats occupied by tenants; some were renovated but most were not, explaining the low central rents for

housings. The second concentric zone, is – on average – characterized by more recent housing of better quality. In this zone, we also find some wards characterized by social housing; in some of them, the housing has been sold to the private sector. In the third zone, most of the housing is occupied by the owner; housing consists mainly of detached or semi-detached houses or high standard flats.

Whatever the theoretical background for interpreting the results, fractal dimensions are significantly related to the rental housing market.

IV.d Household income

In Brussels, we know that on the average high incomes are located in the periphery and poor people characterize the central wards. Income is a way to approximate the socio-economic characteristics of the inhabitants. Median, average and interquartile income are available in Belgium at the scale of the urban ward, for each ward containing at least 30 households (I.N.S. 1994). A synthetic value is computed for each window.

Table 8: Pearson's correlation coefficients between fractal dimensions and households' income

	$D_{Surf\text{-}Cor}$	$D_{Surf\text{-}Dil}$	$D_{Per\text{-}Cor}$
Average income	n.s.	-0.743	0.496
Median income	n.s.	-0.755	0.569
Interquartile difference in income	n.s.	-0.758	0.457

Note: n.s. = not significant at 0.05 level

D obtained by correlation on the built-up surfaces ($D_{Surf\text{-}Cor}$) is never significantly related to income (Table 8). Dilation applied to surfaces produces *D* values negatively and significantly related to income, whatever the way of measuring the income: high values of $D_{Surf\text{-}Dil}$ are associated to small incomes reflecting the particular structure of Brussels, where rich households are located at the outskirts and poor people occupy the densely built-up areas of the city center (see structure of the housing market in IV.-c). As expected, the correlation method applied to the perimeter leads to opposite results:

the higher the income, the higher $D_{Per\text{-}Cor}$, the more garden areas and other green spaces re linked. Once again, Table 8 cannot be interpreted on its own: a tight link is to be made (1) with the history of the city, its development and the resulting structure of the built-up areas, and (2) with the spatial structure of the housing market and rents.

IV.e A multivariate exploratory attempt

Most variables used in the preceding subsections are now introduced in a multivariate approach. The objective is not here to produce a predictive urban model. In this sense, we can even wonder if the socioeconomic variables "explain" fractal dimension or inversely. Our aim here is simply to see how far the variation of the fractal dimensions are associated to the co-variations of rent, distance, housing structure, etc. commonly used in urban models. A factor analysis followed by a Varimax rotation is applied to the variables used in the preceding sections (Table 9). Correlation between the factors scores and the fractal dimensions are then computed (Table 10).

Table 9 : Rotated loadings

	Factor 1	*Factor 1*	*Factor 3*
% detached housing	0.007	**0.682**	0.455
% terraced housing	0.148	**0.725**	0.023
% buildings > 10 housings	**0.575**	**-0.773**	-0.044
% housing built before 1919	**-0.823**	-0.208	-0.322
% housing built before 1945	**-0.884**	-0.258	-0.270
% housing built after 1960	**0.976**	0.011	0.117
% housing built after 1971	**0.872**	0.056	0.020
Mean age of housing	**0.931**	0.195	0.254
Index of oldness	**0.959**	0.017	0.113
Pop. + employment density	-0.471	**-0.758**	-0.293
Average income	0.280	0.236	**0.880**
Central rent	0.126	0.129	**0.935**
Total rent	0.169	0.200	**0.928**
Distance to center	0.401	**0.762**	0.434

Table 10: Pearson's correlation coefficients

(n.s. = not significant at 0.10)

	$D_{Surf\text{-}Cor}$	$D_{Surf\text{-}Dil}$	$D_{Per\text{-}Cor}$
Factor 1	n.s.	n.s.	0.573
Factor 2	-0.778	-0.630	n.s.
Factor 3	n.s.	-0.543	n.s.

Three factors summarize 86% of the initial information (eigenvalue > 1.0). *Factor 1* mainly translates the characteristics of the age of the housing. *Factor 2* groups the variables measuring density and *Factor 3* is mainly an economic factor grouping variables related to income and rent. Pearson's correlation coefficients reveal the differences between the variations: *D* related to the border ($D_{Per\text{-}Cor}$) is mainly related to the age of the housing (*Factor 1*). In fact, the more recent the housing, the larger the gardens and green spaces and the more detailed the planning, and hence the "best" the border is delineated. Fractal dimensions related to surfaces ($D_{Surf\text{-}Cor}$, $D_{Surf\text{-}Dil}$) are best related to *Factor 2* and *Factor 3*, to respective density measures (*Factor 2*) and to the economic characteristics of the wards (*Factor 3*). This confirms preceding comments: distance to the CBD translates the center-periphery structure of the city not only in terms of the rent of housing and household income, but also in terms of the history of the urbanization procedure. Hence, the fractal dimension is an indicator of the morphology of the urban ward, but each method for estimating this dimension brings complementary information.

V. Conclusions and perspectives

The objective of this paper was to compare fractal-based parameters computed by different fractal methods applied on urban built-up areas in Brussels, and to explain the observed spatial variations by means of variables commonly used in geography,

urban economics and land use planning. Bearing in mind the limited and exploratory nature of the analysis undertaken here, it is possible to state the following conclusions:

(1) With one intra-urban example (Brussels), this paper confirms that with **fractal analysis,** one can quite reasonably describe the spatial dilution of the built-up areas within a metropolitan area. Such analysis describes the structures of the built-up surfaces as well as the limits of the green spaces or lacunae; the fractal descriptors are a priori quite complex but take into account the hierarchical nature of the urbanization process. It is more than a measure of density: density does not take the spatial structure into account.

(2) The paper confirms the **sensitivity** of the results to the fractal method used (Johnsen and Brown, 1994). More particularly, it shows that the structure of the intra-urban built-up areas seems to correspond to a **multi-fractal** logic OR to the overlay of different fractal patterns. The different exploratory data analyses show that the fractal dimensions vary differently and fractal characterization may hence lead to false results when applying only one method. Methods related to the surfaces provide good indicators of the spatial internal structure of the built-up areas; those related to the borderlines give interesting results on the structure and shape of the green areas (e.g. gardens). Methods related to surfaces provide results that co-vary in space but are not equal. Fractal dimensions describe the heterogeneity of the spatial distribution of lines (limits), surfaces or volumes. It gives information on the division of the zone; it measures the structure of the built-up area according to its complexity and its dimensional behavior (tending respectively to a line or to a surface). It gives information on its shape and on its hierarchical structure (nested scales). Fractal dimension is hence a better descriptor than population density: it takes into account the underlying structure of the built-up areas.

(3) Interesting **statistical associations** can be found with the structure of the housing market, the rent, the distance to the city center, the income of the households as well as some planning rules. Given its nature in time and space, housing market and distance to the CBD have a strong center-periphery organization that is to be translated by the computed fractal dimensions. We know that for Brussels, however, the structure is the inverse of Paris: Brussels follows the North American type of urban model. The associations put forward in this paper strongly confirm the interpretation of the urban structure; this is done within a geographical, economical and historical background. Hence, coupled with an adequate model, fractal simulation will definitely improve the

functioning of the city. It “describes” the structure, but is also a powerful instrument for analyzing and planning.

The analyses performed in this paper are suggestive and exploratory rather than definitive. They reveal some basic problems of observation and measurement that are generic to most empirical science. We are aware that our results are only valid in the scope of this application for Brussels. Results are limited by the availability of the data and by the different choices made for defining the built-up areas and the windows. However, the results reported here are relevant and confirm other empirical and modeling approaches used for Brussels. Moreover, they are included in a larger project that will lead to the comparison of several European cities; at this stage of the comparison, results seem to converge.

Despite the limits of the case study as well as the limits of the method itself, fractal analysis seems to be a promising tool for describing the morphology of the city and for understanding its genesis and planning. Fractals are far for being an explanatory tool (and will never be!), but they do seem to be a good tool for reproducing urban inner morphology, for simulating. The model seems to be – on average – robust : it replicates the urban spatial regularities and patterns, and could hence be fruitfully integrated at a later stage in intra-urban simulation processes (saving time and money!).

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Appendix 1

The Ward classification procedure is based on the comparison of "distances" between the observed values (differences). In our case, distances are computed between fractal dimensions *D*. These values are not directly observed but obtained by estimation from the fractal distribution law $N(\varepsilon_i) = \varepsilon_i^D$. Hence, it is of interest to test whether the obtained classification corresponds to that which would be obtained when recurring to the fractal distribution law. If this hypothesis holds, we can affirm that the classification obtained for the dimensions is also valid for the fractal laws.

In order to obtain proof, we show that if the difference between two fractal dimensions D_1 and D_2 is greater than the difference between two others (D_3 *and* D_4), it implies that the same holds for the differences between the corresponding fractal laws $N_1 = \varepsilon^{D1}$, $N_2 = \varepsilon^{D2}$, $N_3 = \varepsilon^{D3}$, $N_1 = \varepsilon^{D1}$.

To this end, we assume that we have classified a set of fractal dimensions D_1, D_2, D_3, D_4 so that:

$$D_1 > D_2 > D_3 > D_4$$

We moreover assume for the distances that

$$D_1 - D_2 > D_3 - D_4$$

This yields

$$(D_1 - D_2) \log \varepsilon > (D_3 - D_4) \log \varepsilon$$

$$\Rightarrow \quad \log \delta_{12} > \log \delta_{34}$$

where $\delta_{12} = \varepsilon^{(D1 - D2)}$ and $\delta_{34} = \varepsilon^{(D3 - D4)}$. Since the logarithm is a monotonous function, this yields:

$$\delta_{12} > \delta_{34} \quad \Rightarrow \quad \varepsilon^{(D1 - D2)} > \varepsilon^{(D3 - D4)}$$

We may rewrite this relation as follows:

$$\frac{\varepsilon^{D_1}}{\varepsilon^{D_2}} > \frac{\varepsilon^{D_3}}{\varepsilon^{D_4}}$$

$$\Rightarrow \quad \frac{\varepsilon^{D_1}}{\varepsilon^{D_2}} - 1 > \frac{\varepsilon^{D_3}}{\varepsilon^{D_4}} - 1$$

$$\frac{\varepsilon^{D_1} - \varepsilon^{D_2}}{\varepsilon^{D_2}} > \frac{\varepsilon^{D_3} - \varepsilon^{D_4}}{\varepsilon^{D_4}}$$

or

$$\varepsilon^{D_1} - \varepsilon^{D_2} > \frac{\varepsilon^{D_2}}{\varepsilon^{D_4}}\left(\varepsilon^{D_3} - \varepsilon^{D_4}\right)$$

Since $D_2 > D_4$ we know that the prefactor on the righthand side is > 1 and thus

$$\frac{\varepsilon^{D_2}}{\varepsilon^{D_4}}\left(\varepsilon^{D_3} - \varepsilon^{D_4}\right) > \left(\varepsilon^{D_3} - \varepsilon^{D_4}\right)$$

so that

$$\varepsilon^{D_1} - \varepsilon^{D_2} > \left(\varepsilon^{D_3} - \varepsilon^{D_4}\right)$$

Thus, if we have obtained a classification of the dimensions, this classification remains valid for the fractal laws.