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# A Migration Model Of Capitalists And Residents

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## Abstract

The unipole concentration phenomenon is well-known in Japan. Inside Tokyo metropolitan area, there are many capitalists who provide jobs for the residents therefore the residents enjoy high incomes. However, the congestion lowers the residents' real utility. It is rational to let some capitalists and residents move to local regions. By assuming the full mobility of capitalists and residents, this paper examines their migration. The existence, stability of an equilibrium and the comparative statics are analyzed. Finally, this paper forms a model of optimal tax policy to settle the unipole concentration problem.

*Keywords:* Migration; Resident; Capitalist; Tax; Optimal

*JEL classification:* R23, R13, R38

## 1 Introduction

It is well-known that in Japan, the population in Tokyo metropolitan area is very large (the so-called unipole concentration phenomenon). Inside Tokyo metropolitan area, there are many capitalists (firms) and residents. The firms provide jobs for the residents therefore the residents enjoy a high income. However, the heavy congestion in the area lowers the residents utility level. To induce some residents to move to local areas, through a migration model of all residents in a country, Chapter 4 of Kaiyama (1993) and Kanemoto (1995) provide an analysis for optimal tax policy. Their results suggest the central government levy high tax on the residents in crowded regions, and subsidize the residents in sparse regions. After the possible migration from crowded regions to sparse regions, all residents enjoy a higher utility finally.

However, another important factor in showing the prosperity of a region, the number of firms, is not considered in Kaiyama (1993) and Kanemoto (1995), therefore it is difficult for their models to explain where the residents' incomes come from. In real life, a national government collects taxes from both its residents and its firms, the capitalists and residents are close related. For a resident, the possible income amount strongly depends on the number of firms in his/her residential area. On the other hand, for a capitalist (firm), the number of potential workers, which strongly depends on the number of residents, is also an important factor in location. However, those two kinds of people should be treated differently. For example, capitalists and residents have different attitudes toward congestion. In a crowded area, residents feel uncomfortable because of the heavy congestion but firms are happy because the labor cost should be low due to the rivalry among residents.

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Suppose that both capitalists and residents are fully mobile and move to regions which best satisfy them, this paper gives an equilibrium analysis, taking account of their migration simultaneously. Although the idea of “voting with one’s feet” is from Tiebout (1956), our model is different from Tiebout model in three aspects.

- There are several types of “consumers” in Tiebout model and the “consumers” segregate themselves into homogeneous communities in an equilibrium. In contrast, our model only involves two kinds of people: residents and capitalists. They are related to each other and therefore do not segregate themselves into different regions in an equilibrium.
- Each resident works at a firm in his/her residential region and live on the wage. The income amount depends on the numbers of capitalists and residents of the residential region. In contrast, “consumers” in Tiebout model live on dividend income, which is the same in all regions.
- We are interested in optimal tax policy of the whole country, instead of local public goods.

In Section 2, we illustrate the basic model. To simplify the analysis, here we assume that each capitalist owns a firm (i.e., each capitalist is the only stockholder of a firm), therefore the profit of a firm is the income of its capitalist. The government owns all the land. Each resident is allowed to use an average amount of land free in his/her residential region for living, and a firm does not need any land. Therefore the land amount in a region is related to the utility level of residents directly but it is not directly related to the utility level of capitalists. All the capitalists are homogeneous, in the sense that their preferences, their amounts of money are the same. All the residents are also homogeneous, in the sense that their preferences, their working technologies are the same. When residents and firms determine their location, the pure income after subtracting taxes is an important factor. In Section 3, we show that there exists at least one migration equilibrium for any tax policy, where all capitalists and residents in any region do not wish to move to another region. Our result is not implied by the existence of Tiebout equilibrium, because the utilities of a capitalist and a resident in a region are related to each other. In Section 4, we analyze the stability of a migration equilibrium. Some more assumptions are imposed there and we show that the assumptions assure the Hicks’ stability. The result generalizes the stability result of Kaiyama (1993). To show the rationality of the assumptions, we give an example satisfying them. In Section 5, we give some comparative statics analyses for an equilibrium. We conclude that when we increase the tax amount on the capitalists (residents) in a region, the number of capitalists (residents) in the region decreases, the number of residents (capitalists) in the region does not increase, the utility level of capitalists (residents) decreases but the utility level of residents (capitalists) is possibly decreasing or increasing. Since tax policy is directly related to the net income amounts of capitalists and residents, a government can control the migration of capitalists and residents by a suitable tax policy. Based on the results of Sections 3–5, Section 6 forms a model of optimal tax policy. Finally, Section 7 concludes this paper and provides some topics for future research.

## 2 Basic model

We consider a country which consists of  $m$  regions. Let  $n_c^i$  and  $n_r^i$  respectively denote the number of capitalists and residents in regions  $i = 1, \dots, m$ , where subscript  $c$  stands for capitalist and subscript  $r$  stands for resident. Here we suppose that each capitalist owns a firm, and we identify a capitalist with his/her firm. Therefore the number of firms in region  $i$  is also  $n_c^i$ . All

the capitalists are homogeneous, in the sense that they hold the same preferences, the same amount of capital. All the residents are also homogeneous, in the sense that they hold the same preferences, the same working technology and others. Hence there are  $n_r^i/n_c^i$  workers in each firm of region  $i$ . For convenience, we allow all the numbers to be ordinary real numbers (they may not be integers).

Land is an important factor in the study of regional economics. In our model, the congestion of region  $i$  is represented by population density, which depends partially on the land square  $g^i$  of this region. We suppose that the government owns all the land, and each resident is allowed to use an average amount of land free in his/her residential region for living. However, for simplicity, our model supposes that firms do not use any land, and the only variable input of firms is labor. Therefore congestion depends on the number of residents, but does not depend on the number of firms.

As in Kaiyama (1993), we use a variable  $Q^i$  to denote the uncontrollable region specific factor (URSF) of region  $i$  such as its amenities, its public service scale and level. Here  $Q^i$  is independent of the numbers  $n_c^i$ ,  $n_r^i$ , and the land square  $g^i$  of region  $i$ . A larger  $Q^i$  represents a better equipped region.

The government determines a suitable tax policy. A tax policy is represented by the amount  $s_c^i$  of tax imposed on each capitalist in region  $i$ , and the amount  $s_r^i$  of tax imposed on each resident in region  $i$ , where a negative value means a subsidy.

Each resident in region  $i$  works in a firm of region  $i$  and gets a gross income  $\gamma_r^i$ . By subtracting tax  $s_r^i$ , the resident obtains a net income  $\eta_r^i = \gamma_r^i - s_r^i$ . The residents use their incomes to buy composite good, which includes all consumer goods. We specify the utility function of a resident in region  $i$  as  $u_r(\eta_r^i, d^i, Q^i)$ , where  $d^i$  is the population density in region  $i$ . Let  $\gamma_c^i$  be the remainder of the profit of a firm in region  $i$  subtracted by workers' wages. By further subtracting tax, a capitalist in region  $i$  obtains net income  $\eta_c^i = \gamma_c^i - s_c^i$ . The capitalists also use their incomes to buy composite consumer good. We specify the utility function of a capitalist in region  $i$  as  $u_c(\eta_c^i, Q^i)$ . Since  $\gamma_c^i$  and  $\gamma_r^i$  depend on  $n_c^i$  and  $n_r^i$ ,  $d^i$  depends on  $n_r^i$  and  $g^i$ , we introduce the following two functions  $\Phi_c$  and  $\Phi_r$ :

$$\begin{aligned}\Phi_c(n_c^i, n_r^i, s_c^i, Q^i) &= u_c(\eta_c^i, Q^i) = u_c(\gamma_c^i - s_c^i, Q^i), \\ \Phi_r(n_c^i, n_r^i, g^i, s_r^i, Q^i) &= u_r(\eta_r^i, d^i, Q^i) = u_r(\gamma_r^i - s_r^i, d^i, Q^i).\end{aligned}\tag{2.1}$$

Summing up the numbers of capitalists, residents and the tax amounts, we introduce vectors  $\mathbf{n}_c = (n_c^1, \dots, n_c^m)$ ,  $\mathbf{n}_r = (n_r^1, \dots, n_r^m)$ ,  $\mathbf{s}_c = (s_c^1, \dots, s_c^m)$ ,  $\mathbf{s}_r = (s_r^1, \dots, s_r^m)$ . As in Tiebout model, we suppose that all capitalists and residents can move among the regions without any cost. Hence under tax policy  $(\mathbf{s}_c, \mathbf{s}_r)$ , an equilibrium state (denoted by  $\langle \mathbf{n}_c, \mathbf{n}_r, \mathbf{s}_c, \mathbf{s}_r \rangle$ ) appears in which the capitalists in all regions share the same utility  $U_c(\mathbf{s}_c, \mathbf{s}_r)$ , and the residents in all regions also share the same utility  $U_r(\mathbf{s}_c, \mathbf{s}_r)$ . In mathematics, an equilibrium  $\langle \mathbf{n}_c, \mathbf{n}_r, \mathbf{s}_c, \mathbf{s}_r \rangle$  can be described by the following relations:

$$\left\{ \begin{array}{ll} \Phi_c(n_c^i, n_r^i, s_c^i, Q^i) = U_c(\mathbf{s}_c, \mathbf{s}_r), & \forall i \text{ such that } n_c^i > 0, \\ \Phi_c(n_c^i, n_r^i, s_c^i, Q^i) \leq U_c(\mathbf{s}_c, \mathbf{s}_r), & \forall i \text{ such that } n_c^i = 0, \\ \Phi_r(n_c^i, n_r^i, g^i, s_r^i, Q^i) = U_r(\mathbf{s}_c, \mathbf{s}_r), & \forall i \text{ such that } n_r^i > 0, \\ \Phi_r(n_c^i, n_r^i, g^i, s_r^i, Q^i) \leq U_r(\mathbf{s}_c, \mathbf{s}_r), & \forall i \text{ such that } n_r^i = 0, \\ \sum_{i=1}^m n_c^i = N_c, & \\ \sum_{i=1}^m n_r^i = N_r. & \end{array} \right.\tag{2.2}$$

where  $N_c$  is the number of total capitalists in the country and  $N_r$  is the total number of total residents in the country.

### 3 The existence of an equilibrium

There are some results concerning the existence of a Tiebout equilibrium (for example, Bewley (1981)). We mentioned in Section 1 that our model is different from Tiebout model because capitalists and residents are related to each other. We have to provide a proof for the existence of an equilibrium in our new model.

**Theorem 3.1** *If utility functions  $\Phi_c$  and  $\Phi_r$  are continuous with respect to  $n_c^i$  and  $n_r^i$  for all  $i$ , then for any tax policy  $\langle \mathbf{s}_c, \mathbf{s}_r \rangle$ , there exists at least one equilibrium  $\langle \mathbf{n}_c^*, \mathbf{n}_r^*, \mathbf{s}_c, \mathbf{s}_r \rangle$ .*

**Proof:** Let  $p_c^i = n_c^i/N_c$ ,  $p_r^i = n_r^i/N_r$ ,  $i = 1, \dots, m$ . Then

$$\mathbf{p}_c = (p_c^1, \dots, p_c^m), \mathbf{p}_r = (p_r^1, \dots, p_r^m) \in S_m = \left\{ (p^1, \dots, p^m) \mid \sum_{j=1}^m p^j = 1, p^j \geq 0 \right\}.$$

For convenience we denote, for  $i = 1, \dots, m$ ,

$$\Phi_c^i = \Phi_c(n_c^i, n_r^i, s_c^i, Q^i) = \Phi_c(p_c^i N_c, p_r^i N_r, s_c^i, Q^i),$$

$$\Phi_r^i = \Phi_r(n_c^i, n_r^i, g^i, s_r^i, Q^i) = \Phi_r(p_c^i N_c, p_r^i N_r, g^i, s_r^i, Q^i).$$

Now define the following mapping  $T : S_m \times S_m \rightarrow S_m \times S_m$ ,

$$T(\mathbf{p}_c, \mathbf{p}_r) = (T_c^1(\mathbf{p}_c, \mathbf{p}_r), \dots, T_c^m(\mathbf{p}_c, \mathbf{p}_r), T_r^1(\mathbf{p}_c, \mathbf{p}_r), \dots, T_r^m(\mathbf{p}_c, \mathbf{p}_r)),$$

where

$$T_c^i(\mathbf{p}_c, \mathbf{p}_r) = \lambda_c \left( p_c^i + \max \left\{ 0, \Phi_c^i - \min_{\substack{j=1, \dots, m \\ p_c^j > 0}} \Phi_c^j \right\} \right), \quad i = 1, \dots, m,$$

$$T_r^i(\mathbf{p}_c, \mathbf{p}_r) = \lambda_r \left( p_r^i + \max \left\{ 0, \Phi_r^i - \min_{\substack{j=1, \dots, m \\ p_r^j > 0}} \Phi_r^j \right\} \right), \quad i = 1, \dots, m,$$

$$\lambda_c = \frac{1}{1 + \sum_{k=1}^m \max \left\{ 0, \Phi_c^k - \min_{\substack{j=1, \dots, m \\ p_c^j > 0}} \Phi_c^j \right\}},$$

$$\lambda_r = \frac{1}{1 + \sum_{k=1}^m \max \left\{ 0, \Phi_r^k - \min_{\substack{j=1, \dots, m \\ p_r^j > 0}} \Phi_r^j \right\}}.$$

Since  $\Phi_c$  and  $\Phi_r$  are continuous functions of  $n_c^i$  and  $n_r^i$ , we know that  $T$  is a continuous mapping. By the following lemma, we know that  $(\mathbf{p}_c^*, \mathbf{p}_r^*)$  is a fixed point of the mapping  $T$  if and only if  $\mathbf{n}_c^* = (p_c^{*1} N_c, \dots, p_c^{*m} N_c)$ ,  $\mathbf{n}_r^* = (p_r^{*1} N_r, \dots, p_r^{*m} N_r)$ ,  $\mathbf{s}_c = (s_c^1, \dots, s_c^m)$ ,  $\mathbf{s}_r = (s_r^1, \dots, s_r^m)$  form an equilibrium  $\langle \mathbf{n}_c^*, \mathbf{n}_r^*, \mathbf{s}_c, \mathbf{s}_r \rangle$ . Since  $S_m \times S_m$  is a nonempty, compact and convex set, by Brouwer's fixed-point theorem (Theorem 2.E.2 of Takayama, 1985), such a fixed point always exists, which concludes our proof.  $\square$

**Lemma 3.1** *Point  $(\mathbf{p}_c^*, \mathbf{p}_r^*)$  is a fixed point of mapping  $T$  if and only if*

$$\Phi_c^{i*} (= \Phi_c(n_c^{i*}, n_r^{i*}, s_c^i, Q^i)) = \max_{j=1, \dots, m} \Phi_c^{j*}, \quad \forall i \text{ such that } p_c^{i*} > 0, \quad (3.1)$$

$$\Phi_r^{i*} (= \Phi_r(n_c^{i*}, n_r^{i*}, g^i, s_r^i, Q^i)) = \max_{j=1, \dots, m} \Phi_r^{j*}, \quad \forall i \text{ such that } p_r^{i*} > 0. \quad (3.2)$$

**Proof:** We first show that

$$T_c^i(\mathbf{p}_c^*, \mathbf{p}_r^*) = p_c^{i*}, \quad \forall i = 1, \dots, m \quad (3.3)$$

if and only if (3.1) holds.

*Sufficiency.* If (3.1) holds, then for any  $i, j$  such that  $p_c^{i*} > 0, p_c^{j*} > 0$ , we have  $\Phi_c^{i*} = \Phi_c^{j*}$ . Hence

$$\Phi_c^{i*} = \min_{\substack{j=1, \dots, m \\ p_c^{j*} > 0}} \Phi_c^{j*}, \quad \forall i \text{ such that } p_c^{i*} > 0,$$

and

$$\Phi_c^{i*} \leq \min_{\substack{j=1, \dots, m \\ p_c^{j*} > 0}} \Phi_c^{j*}, \quad \forall i \text{ such that } p_c^{i*} = 0.$$

Therefore (3.3) holds for any  $i$  from the definition of  $T$ .

*Necessity.* From the definition of  $T$ , we know that (3.3) implies

$$(1 - \lambda_c)p_c^{i*} = \lambda_c \max \left\{ 0, \Phi_c^{i*} - \min_{\substack{j=1, \dots, m \\ p_c^{j*} > 0}} \Phi_c^{j*} \right\}, \quad \forall i = 1, \dots, m. \quad (3.4)$$

Furthermore, by the definition of  $\lambda_c$ , (3.4) leads to

$$\sum_{k=1}^m \max \left\{ 0, \Phi_c^{k*} - \min_{\substack{j=1, \dots, m \\ p_c^{j*} > 0}} \Phi_c^{j*} \right\} p_c^{i*} = \max \left\{ 0, \Phi_c^{i*} - \min_{\substack{j=1, \dots, m \\ p_c^{j*} > 0}} \Phi_c^{j*} \right\}, \quad (3.5)$$

$$\forall i = 1, \dots, m.$$

Since  $\sum_{i=1}^m p_c^{i*} = 1$ , we know that  $p_c^{j*} > 0$  holds for some  $j$ . Let  $l$  be a region satisfying

$$\Phi_c^{l*} = \min_{\substack{j=1, \dots, m \\ p_c^{j*} > 0}} \Phi_c^{j*}.$$

Then for  $i = l$ , the right hand side of (3.5) should be zero. Since  $p_c^{l*} > 0$  by the definition of  $l$ , we know that the coefficient of  $p_c^{l*}$  in (3.5) should be zero. Therefore

$$\Phi_c^{k*} \leq \Phi_c^{l*}, \quad \forall k = 1, \dots, m. \quad (3.6)$$

For  $i$  such that  $p_c^{i*} > 0$  we have  $\Phi_c^{i*} \geq \Phi_c^{l*}$  by the definition of  $l$ . Combining with (3.6) we know that for those  $i, \Phi_c^{i*} = \Phi_c^{l*}$  holds. Furthermore, since (3.6) holds for all  $k$ , we obtain (3.1).

Similarly, we can show that  $T_r^i(\mathbf{p}_c^*, \mathbf{p}_r^*) = p_r^{i*}$  holds for all  $i$  if and only if relation (3.2) holds, which concludes the proof.  $\square$

## 4 The stability analysis of an equilibrium

In this section, we consider the stability of equilibrium  $(\mathbf{n}_c, \mathbf{n}_r, \mathbf{s}_c, \mathbf{s}_r)$ . The stability concept for such kind of equilibrium is first described for a simple case that residents move between 2 regions in Boadway and Flatters (1982), without strict definition. Kaiyama (1993) then considers the case that residents move among  $m$  regions and gives a sufficient condition for the stability (P. 77), which generalizes the stability conclusion of Boadway and Flatters (1982) (P. 619). Here we consider the stability of our new model, which is more general than Kaiyama (1993).

## 4.1 More assumptions

Section 3 shows that, to assure the existence of an equilibrium,  $\Phi_c$  and  $\Phi_r$  are only required to be continuous with respect to  $n_c^i$  and  $n_r^i$ . In order to examine the stability and comparative statics analysis, in the rest part of this paper, we give more technical assumptions. At first, the utility functions  $\Phi_c$  and  $\Phi_r$  are now supposed to be continuously differentiable with respect to all variables. The following notations will be used:

$$w_i = \frac{\partial \Phi_c(n_c^i, n_r^i, s_c^i, Q^i)}{\partial n_c^i}, \quad x_i = \frac{\partial \Phi_c(n_c^i, n_r^i, s_c^i, Q^i)}{\partial n_r^i},$$

$$y_i = \frac{\partial \Phi_r(n_c^i, n_r^i, g^i, s_r^i, Q^i)}{\partial n_c^i}, \quad z_i = \frac{\partial \Phi_r(n_c^i, n_r^i, g^i, s_r^i, Q^i)}{\partial n_r^i},$$

where  $i = 1, \dots, m$ , the differentials are valued at the considered equilibrium  $\langle \mathbf{n}_c, \mathbf{n}_r, \mathbf{s}_c, \mathbf{s}_r \rangle$ . If  $n_c^i = 0$  ( $n_c^i = N_c$ ) for some  $i$ , then  $w_i$  and  $y_i$  are the right (left) derivatives. If  $n_r^i = 0$  ( $n_r^i = N_r$ ), then  $x_i$  and  $z_i$  are the right (left) derivatives.

If  $x_i = y_i = 0$  holds constantly for all  $i$ , the utilities of capitalists and residents are independent of each other. Those two kinds of people can be treated as two types of customers in Tiebout model. If furthermore  $N_c = 0$ , then our model degenerates into Kaiyama's model.

The following assumption (4.1) is popularly used in literature, which says that each people will be more happy with a larger net income, and each resident will be happy by mitigation of congestion. (4.1) needs no extra explanation.

$$\frac{\partial u_c}{\partial \eta_c^i} > 0, \quad \frac{\partial u_r}{\partial \eta_r^i} > 0, \quad \frac{\partial u_r}{\partial d^i} \leq 0. \quad (4.1)$$

By (4.1) and (2.1), we have

$$\frac{\partial \Phi_c}{\partial s_c^i} = -\frac{\partial u_c}{\partial \eta_c^i} < 0, \quad \frac{\partial \Phi_r}{\partial s_r^i} = -\frac{\partial u_r}{\partial \eta_r^i} < 0. \quad (4.2)$$

Relation (4.2) says that the utilities of a capitalist and a resident decrease if their tax amounts increase.

The following two assumptions need more explanations.

$$w_i < 0, x_i \geq 0, y_i \geq 0, z_i < 0, \quad \forall i = 1, 2, \dots, m, \quad (4.3)$$

$$w_i z_j - x_i y_j \geq 0, \quad \text{and} \quad w_i z_i - x_i y_i > 0, \quad \forall i, j = 1, \dots, m. \quad (4.4)$$

Assumption (4.3) says that when the number  $n_c^i$  of capitalists in region  $i$  increases, the utility level of capitalists in region  $i$  decreases but the utility level of residents in region  $i$  does not decrease. Similarly, when the number  $n_r^i$  of residents in region  $i$  increases, the utility level of residents in region  $i$  decreases but the utility level of capitalists in region  $i$  does not decrease. As an intuitive explanation of this assumption, when the number of residents increases, a competition in the worker market decreases the labor cost of firms and capitalists enjoy a higher utility. On the other hand, when the number of capitalists increases, it becomes easier for a resident to find a good job and hence residents enjoy a higher utility. Now we use Figure 1 to explain the condition (4.4). Figure 1 illustrates the situation that a resident, named  $R$ , moves from region  $i$  to region  $j$ . Since  $n_r^i$  decreases by 1, the utility level  $U_r^i$  of residents remaining in region  $i$  increases by  $|z_i|$  directly. At the same time, the utility level of capitalists in region  $i$  decreases by  $|x_i|$ , which can be recovered by letting  $|x_i/w_i|$  capitalists of region  $i$  emigrate. Due to the decrease of  $n_c^i$ ,  $U_r^i$  decreases by  $|y_i||x_i/w_i|$ . The latter half of condition (4.4) says that the

total increase of  $U_r^i$  is  $|z_i| - |y_i||x_i/w_i| > 0$ . Hence, the residents remaining in region  $i$  do not prefer moving after  $R$ . On the other hand, let us speculate how much will  $R$  benefit from the move.  $R$ 's move directly decreases the utility level of residents in region  $j$  by  $|x_j|$ . The  $|x_i/w_i|$  capitalists emigrating from region  $i$  move to regions other than  $i$ . In the case best to  $R$ , all the  $|x_i/w_i|$  capitalists move to region  $j$  and  $U_r^j$  increases by  $|y_j||x_i/w_i|$ . The former part of (4.4) says that the total increase of  $U_r^j$  is  $|y_j||x_i/w_i| - |z_j| \leq 0$ . Therefore  $R$  does not benefit from the move. Figure 1 intuitively explains that no residents prefer moving therefore the equilibrium is stable. In this section we theoretically show that the equilibrium is actually stable in the Hicks sense.

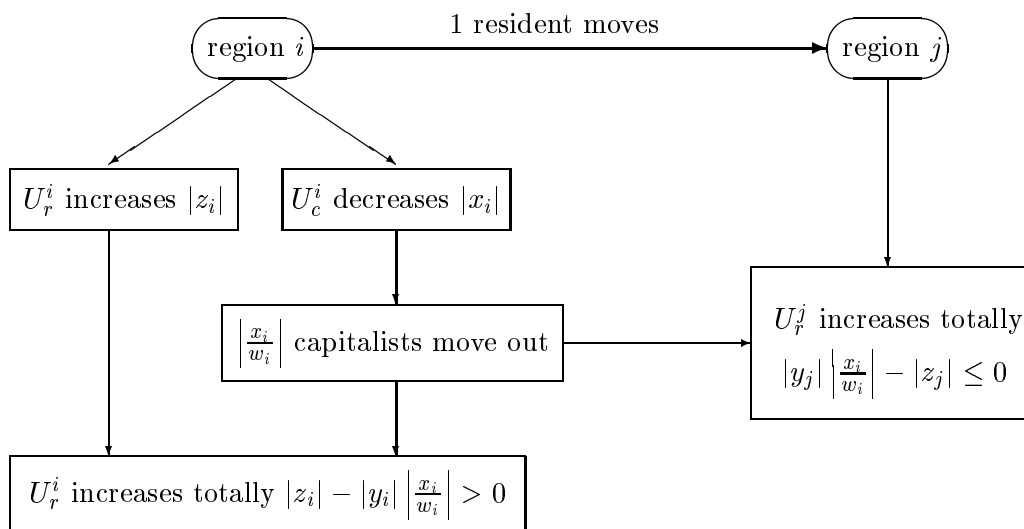


Figure 1: Explanation of (4.4)

We give a comment on assumptions (4.3) and (4.4). The relations are only required to be valid at the considered equilibrium, instead of being valid constantly. Therefore, for example, our model does not preclude the case that  $\Phi_c$  is a concave function of  $n_c^i$ , hence  $w_i$  is positive when  $n_c^i$  is small but it becomes negative when  $n_c^i$  is large.

## 4.2 Stability

Here we give another mathematical description for the migration of capitalists and residents. In each region, a capitalist (resident) compares the current utility level with the average utility



level of all regions. If the current utility level is lower, the capitalist (resident) moves to another region with higher utility level. Therefore, we revise (2.2) as follows.

$$\left\{ \begin{array}{l} 0 = \Phi_c(n_c^1, n_r^1, s_c^1, Q^1) - \frac{1}{m} \sum_{i=1}^m \Phi_c(n_c^i, n_r^i, s_c^i, Q^i) \\ 0 = \Phi_r(n_c^1, n_r^1, g^1, s_r^1, Q^1) - \frac{1}{m} \sum_{i=1}^m \Phi_r(n_c^i, n_r^i, g^i, s_r^i, Q^i) \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ 0 = \Phi_c(n_c^k, n_r^k, s_c^k, Q^k) - \frac{1}{m} \sum_{i=1}^m \Phi_c(n_c^i, n_r^i, s_c^i, Q^i) \\ 0 = \Phi_r(n_c^k, n_r^k, g^k, s_r^k, Q^k) - \frac{1}{m} \sum_{i=1}^m \Phi_r(n_c^i, n_r^i, g^i, s_r^i, Q^i) \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ 0 = \Phi_c(n_c^{m-1}, n_r^{m-1}, s_c^{m-1}, Q^{m-1}) - \frac{1}{m} \sum_{i=1}^m \Phi_c(n_c^i, n_r^i, s_c^i, Q^i) \\ 0 = \Phi_r(n_c^{m-1}, n_r^{m-1}, g^{m-1}, s_r^{m-1}, Q^{m-1}) - \frac{1}{m} \sum_{i=1}^m \Phi_r(n_c^i, n_r^i, g^i, s_r^i, Q^i), \end{array} \right. \quad (4.5)$$

where  $n_c^m$  and  $n_r^m$  in  $\Phi_c(n_c^m, n_r^m, s_c^m, Q^m)$  and  $\Phi_r(n_c^m, n_r^m, g^m, s_r^m, Q^m)$  are not independent variables. They satisfy

$$n_c^m = N_c - \sum_{i=1}^{m-1} n_c^i, \quad n_r^m = N_r - \sum_{i=1}^{m-1} n_r^i. \quad (4.6)$$

For convenience, we denote the right hand side of (4.5) as  $\Psi_c^k(n_c, n_r)$  and  $\Psi_r^k(n_c, n_r)$  respectively. That is,

$$\begin{aligned} \Psi_c^k(n_c, n_r) &= \Phi_c(n_c^k, n_r^k, s_c^k, Q^k) - \frac{1}{m} \sum_{i=1}^m \Phi_c(n_c^i, n_r^i, s_c^i, Q^i) \\ \Psi_r^k(n_c, n_r) &= \Phi_r(n_c^k, n_r^k, g^k, s_r^k, Q^k) - \frac{1}{m} \sum_{i=1}^m \Phi_r(n_c^i, n_r^i, g^i, s_r^i, Q^i) \end{aligned} \quad k = 1, \dots, m-1. \quad (4.7)$$

Furthermore, for  $i, j = 1, \dots, m-1$ ,  $i \neq j$ , we denote

$$\begin{aligned} a_{ii} &= \frac{\partial \Psi_c^i}{\partial n_c^i} = \frac{(m-1)w_i + w_m}{m}, & a_{ij} &= \frac{\partial \Psi_c^i}{\partial n_c^j} = \frac{-w_j + w_m}{m}, \\ b_{ii} &= \frac{\partial \Psi_c^i}{\partial n_r^i} = \frac{(m-1)x_i + x_m}{m}, & b_{ij} &= \frac{\partial \Psi_c^i}{\partial n_r^j} = \frac{-x_j + x_m}{m}, \\ c_{ii} &= \frac{\partial \Psi_r^i}{\partial n_c^i} = \frac{(m-1)y_i + y_m}{m}, & c_{ij} &= \frac{\partial \Psi_r^i}{\partial n_c^j} = \frac{-y_j + y_m}{m}, \\ d_{ii} &= \frac{\partial \Psi_r^i}{\partial n_r^i} = \frac{(m-1)z_i + z_m}{m}, & d_{ij} &= \frac{\partial \Psi_r^i}{\partial n_r^j} = \frac{-z_j + z_m}{m}, \end{aligned} \quad (4.8)$$

where  $i, j = 1, \dots, m-1$ ,  $i \neq j$ . The relations are from (4.6) and (4.7).

Hicks (1946) gives the following specification of stability on exchange economy (see also Negishi, 1962). A rise of the price of any commodity above the equilibrium must be accompanied by an excess supply of that commodity, and a fall below the equilibrium by an excess demand, so that a force is generated to bring the changed price back to equilibrium. This behavior must hold regardless of the state of other markets, i.e., whether or not other prices are unchanged or adjusted so as to maintain equilibrium in the relevant markets. For our migration problem, if we take the numbers of capitalists and residents as prices of goods, the stability can be defined in a similar way. That is, an equilibrium is *stable* if the equilibrium is not destroyed when a capitalist or a resident migrates from one region to another. Specifically, when a capitalist (resident) in region  $i$  migrates to another region, and either

1. the numbers of capitalists or residents in other regions do not change,

2. one number of capitalists or residents in a region is adjusted to maintain the equilibrium,
3. two numbers of capitalists or residents in one or more regions are adjusted to maintain the equilibrium,
4. similarly to the above cases, several numbers of capitalists or residents in several regions are adjusted to maintain the equilibrium,

then some capitalists (residents) are willing to migrate to region  $i$ . As in Hicks (1946), to investigate the stability of equilibrium (4.5), we use the following  $2(m-1) \times 2(m-1)$  Jacobian matrix.

$$\begin{pmatrix} a_{11} & b_{11} & \cdots & a_{1k} & b_{1k} & \cdots & a_{1,m-1} & b_{1,m-1} \\ c_{11} & d_{11} & \cdots & c_{1k} & d_{1k} & \cdots & c_{1,m-1} & d_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k1} & b_{k1} & \cdots & a_{kk} & b_{kk} & \cdots & a_{k,m-1} & b_{k,m-1} \\ c_{k1} & d_{k1} & \cdots & c_{kk} & d_{kk} & \cdots & c_{k,m-1} & d_{k,m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-1,1} & b_{m-1,1} & \cdots & a_{m-1,k} & b_{m-1,k} & \cdots & a_{m-1,m-1} & b_{m-1,m-1} \\ c_{m-1,1} & d_{m-1,1} & \cdots & c_{m-1,k} & d_{m-1,k} & \cdots & c_{m-1,m-1} & d_{m-1,m-1} \end{pmatrix}, \quad (4.9)$$

where  $a_{ij}$ ,  $b_{ij}$ ,  $c_{ij}$  and  $d_{ij}$  are defined by (4.8).

According to Hicks (1946), the equilibrium is stable if and only if the signs of the principal minors of (4.9) are alternatively negative and positive.

The next theorem says that assumptions (4.3) and (4.4) assure the stability.

**Theorem 4.1** *If (4.3) and (4.4) hold at an equilibrium  $\langle n_c, n_r, s_c, s_r \rangle$ , then this equilibrium is stable.*

**Proof:** The conclusion holds according to Lemmas A.4 and A.5 in the Appendix.  $\square$

We give two comments on Theorem 4.1. First, as stated in Section 4.1, our model is a generalization of Kaiyama's model. When  $N_c = 0$ ,  $x_i = 0$  and  $y_i = 0$  for all  $i$ , (4.3) and (4.4) degenerate to  $z_i < 0$ . Therefore Theorem 4.1 generalizes the stability conclusion of Kaiyama (1993) (P. 77), which itself generalizes the stability result of Boadway and Flatters (1982) (P. 619). Second, if (4.3) and (4.4) are valid constantly, then Theorems 3.1 and 4.1 ensure the existence of a stable equilibrium.

Now we explain why Hicks' definition of stability is used here. Historically, Hicks' stability is the first one and it has been replaced by Samuelson (1941, 1942), who argues that the stability problem should be specified with a dynamic adjustment process. However, very few useful propositions are derived from Samuelson's criteria (P. 11 of Arrow and Hahn, 1971), and Hicksian condition has been proved to be useful in comparative statics. Furthermore, the Hicksian condition is necessary if the dynamic process is stable regardless of the value of speeds of adjustment (Metzler, 1945).

### 4.3 An example

We here give an example which satisfies all the conditions of (4.3) and (4.4).

**Example:** Consider the case of a country consisting of  $m$  symmetric regions, in the sense that  $g^i = g$  and  $Q^i = Q$  for all  $i = 1, \dots, m$ . Let  $f(n, Q)$  be the production function of each firm in a region, where  $n$  is the number of workers in this firm. Hence the gross profit of a firm in region  $i$  is  $f(n_r^i/n_c^i, Q)$ . Suppose that the production function is twice differentiable with respect to  $n$ , and satisfies

$$f_{11} = \frac{\partial^2 f(n, Q)}{\partial n^2} < 0. \quad (4.10)$$

Therefore the marginal productivity  $f_1 = \partial f(n, Q)/\partial n$  decreases. All residents get incomes by working. Each resident's wage  $\gamma_r^i$  is the marginal productivity  $f_1$ , and the net income is

$$\eta_r^i = \gamma_r^i - s_r^i = f_1\left(\frac{n_r^i}{n_c^i}, Q\right) - s_r^i.$$

Notice that the land square in region  $i$  is  $g^i$ , the population density is  $d^i = n_r^i/g^i$ . Here, since a capitalist in region  $i$  may not be a resident of region  $i$ , and we assume that firms do not use any land, therefore we do not count the number of firms (capitalists) when we calculate the population density. In this way, the utility of a resident in region  $i$  is

$$\Phi_r(n_c^i, n_r^i, g^i, s_r^i, Q) = u_r(\eta_r^i, d^i, Q) = u_r\left(f_1\left(\frac{n_r^i}{n_c^i}, Q\right) - s_r^i, \frac{n_r^i}{g^i}, Q\right). \quad (4.11)$$

The net income of a capitalist in region  $i$  is the gross profit subtracted by the amount of wages and tax:

$$\eta_c^i = \gamma_c^i - s_c^i = f\left(\frac{n_r^i}{n_c^i}, Q\right) - \frac{n_r^i}{n_c^i}\gamma_r^i - s_c^i = f\left(\frac{n_r^i}{n_c^i}, Q\right) - \frac{n_r^i}{n_c^i}f_1\left(\frac{n_r^i}{n_c^i}, Q\right) - s_c^i.$$

The utility level of a capitalist in region  $i$  becomes

$$\Phi_c(n_c^i, n_r^i, s_c^i, Q) = u_c(\eta_c^i, Q) = u_c\left(f\left(\frac{n_r^i}{n_c^i}, Q\right) - \frac{n_r^i}{n_c^i}f_1\left(\frac{n_r^i}{n_c^i}, Q\right) - s_c^i, Q\right) \quad (4.12)$$

Since all regions are symmetric, we know that there is a symmetric equilibrium in which  $n_r^i = N_r/m$ ,  $n_c^i = N_c/m$ ,  $s_c^i = s_c^j$  and  $s_r^i = s_r^j$  for all  $i, j = 1, \dots, m$ . We show that (4.3) and the latter half of (4.4) hold constantly and the former part of (4.3) holds at this equilibrium therefore this symmetric equilibrium is stable by Theorem 4.1. At first, from (4.12), we have the following relations for all region  $i$ .

$$\begin{aligned} x_i &= \frac{\partial \Phi_c}{\partial n_r^i} = -\frac{n_r^i}{(n_c^i)^2} f_{11}\left(\frac{n_r^i}{n_c^i}, Q\right) \frac{\partial u_c(\eta_c^i, Q)}{\partial \eta_c^i} > 0, \\ w_i &= \frac{\partial \Phi_c}{\partial n_c^i} = \frac{(n_r^i)^2}{(n_c^i)^3} f_{11}\left(\frac{n_r^i}{n_c^i}, Q\right) \frac{\partial u_c(\eta_c^i, Q)}{\partial \eta_c^i} \\ &= -\frac{n_r^i}{n_c^i} \left( -\frac{n_r^i}{(n_c^i)^2} f_{11}\left(\frac{n_r^i}{n_c^i}, Q\right) \frac{\partial u_c(\eta_c^i, Q)}{\partial \eta_c^i} \right) \\ &= -\frac{n_r^i}{n_c^i} x_i < 0, \end{aligned} \quad (4.13)$$

where the inequalities are from (4.1) and (4.10). Furthermore, by (4.11),

$$z_i = \frac{\partial \Phi_r}{\partial n_r^i} = \frac{1}{n_c^i} f_{11}\left(\frac{n_r^i}{n_c^i}, Q\right) \frac{\partial u_r(\eta_r^i, d^i, Q)}{\partial \eta_r^i} + \frac{1}{g^i} \frac{\partial u_r(\eta_r^i, d^i, Q)}{\partial d^i}$$

$$\begin{aligned}
&< \frac{1}{n_c^i} f_{11} \left( \frac{n_r^i}{n_c^i}, Q \right) \frac{\partial u_r(\eta_r^i, d^i, Q)}{\partial \eta_r^i}, & i = 1, \dots, m, \\
y_i &= \frac{\partial \Phi_r}{\partial n_c^i} = -\frac{n_r^i}{(n_c^i)^2} f_{11} \left( \frac{n_r^i}{n_c^i}, Q \right) \frac{\partial u_r(\eta_r^i, d^i, Q)}{\partial \eta_r^i} \\
&= -\frac{n_r^i}{n_c^i} \left( \frac{1}{n_c^i} f_{11} \left( \frac{n_r^i}{n_c^i}, Q \right) \frac{\partial u_r(\eta_r^i, d^i, Q)}{\partial \eta_r^i} \right) < -\frac{n_r^i}{n_c^i} z_i, & i = 1, \dots, m
\end{aligned}$$

hold, where the inequalities are from (4.1) and (4.10). Since  $f_{11} < 0$ ,  $\partial u_r / \partial \eta_r^i > 0$ , we know  $y_i > 0$  by the last equality. From the last inequality, we have

$$z_i < -\frac{n_c^i}{n_r^i} y_i < 0, \quad \forall i = 1, \dots, m. \quad (4.14)$$

By combing (4.13) and (4.14) we obtain

$$w_i z_i > x_i y_i \quad \forall i = 1, \dots, m. \quad (4.15)$$

By (4.13) again, we know that relations  $w_i/x_i = -n_r^i/n_c^i = -n_r^j/n_c^j = w_j/x_j$  hold for any two regions  $i$  and  $j$  at the symmetric equilibrium. Therefore

$$w_i z_j - x_i y_j = x_i z_j \left( \frac{w_i}{x_i} - \frac{y_j}{z_j} \right) = x_i z_j \left( \frac{w_j}{x_j} - \frac{y_j}{z_j} \right) = \frac{x_i}{x_j} (w_j z_j - x_j y_j) > 0,$$

where the last inequality is because of (4.15).

## 5 Comparative statics analysis

The previous two sections consider the existence and stability of an equilibrium. This section examines how an equilibrium changes when the numbers of total capitalists and residents and the tax policy change.

We suppose that all the assumptions in Section 4 hold, therefore the considered equilibrium is stable. Furthermore, for convenience, here we assume that  $n_c^i > 0$  and  $n_r^i > 0$  for all region  $i$ , and that the size of land, the URSF of each region  $i$  do not change. Therefore only four equalities remains in (2.2). By calculating the total differentiation of the four equations, we obtain (5.1).

$$\left\{ \begin{array}{l} w_i dn_c^i + x_i dn_r^i - dU_c = -\frac{\partial \Phi_c}{\partial s_c^i} ds_c^i, & i = 1, \dots, m, \\ y_i dn_c^i + z_i dn_r^i - dU_r = -\frac{\partial \Phi_r}{\partial s_r^i} ds_r^i, & i = 1, \dots, m, \\ \sum_{i=1}^m dn_c^i = dN_c, \\ \sum_{i=1}^m dn_r^i = dN_r. \end{array} \right. \quad (5.1)$$

We treat the above  $2m + 2$  formulas as equations of  $2m + 2$  variables  $dn_c^i, dn_r^i, dU_c, dU_r$ .

For convenience, we introduce the following notation for the coefficients in (5.1).

$$\Delta(w_1, \dots, w_m; x_1, \dots, x_m; y_1, \dots, y_m; z_1, \dots, z_m)$$

$$\begin{aligned}
&= \begin{vmatrix} w_1 & \cdots & 0 & x_1 & \cdots & 0 & -1 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & w_m & 0 & \cdots & x_m & -1 & 0 \\ y_1 & \cdots & 0 & z_1 & \cdots & 0 & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & y_m & 0 & \cdots & z_m & 0 & -1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 \end{vmatrix} \\
&= \prod_{l=1}^m (w_l z_l - x_l y_l) \sum_{i=1}^m \sum_{j=1}^m \frac{w_i z_j - x_i y_j}{(w_i z_i - x_i y_i)(w_j z_j - x_j y_j)} > 0,
\end{aligned}$$

where the inequality is from (4.4). Therefore (5.1) has a unique solution for any  $ds_c^i$ ,  $ds_r^i$ ,  $dN_c$  and  $dN_r$ ,  $i = 1, \dots, m$ .

### 5.1 The impact of $N_c$ and $N_r$ on $U_c$ and $U_r$

**Theorem 5.1** *By increasing the number  $N_c$  of total capitalists, the utility level  $U_c$  of capitalists decreases but the utility level  $U_r$  of residents does not decrease. Similarly, by increasing the number  $N_r$  of total residents, the utility  $U_r$  of residents decreases but the utility  $U_c$  of capitalists does not decrease.*

**Proof:** Let  $ds_c^i = ds_r^i = dN_r = 0$  in (5.1). By Cramer's formula, we have

$$dU_c = \frac{\begin{vmatrix} w_1 & \cdots & 0 & x_1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & w_m & 0 & \cdots & x_m & 0 & 0 \\ y_1 & \cdots & 0 & z_1 & \cdots & 0 & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & y_m & 0 & \cdots & z_m & 0 & -1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & dN_c & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 \end{vmatrix}}{\Delta(w_1, \dots, w_m; x_1, \dots, x_m; y_1, \dots, y_m; z_1, \dots, z_m)}.$$

Therefore

$$\frac{\partial U_c}{\partial N_c} = \frac{\sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l}}{\begin{vmatrix} \sum_{i=1}^m \frac{z_i}{w_i z_i - x_i y_i} & -\sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ -\sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} & \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{vmatrix}} < 0,$$

where the inequality is from (4.3)-(4.4).

Similarly,

$$\frac{\partial U_r}{\partial N_r} = \frac{\sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l}}{\begin{vmatrix} \sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l} & -\sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ -\sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} & \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{vmatrix}} < 0,$$

$$\frac{\partial U_r}{\partial N_c} = \frac{\sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l}}{\begin{vmatrix} \sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l} & -\sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ -\sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} & \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{vmatrix}} \geq 0,$$

$$\frac{\partial U_c}{\partial N_r} = \frac{\sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l}}{\begin{vmatrix} \sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l} & -\sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ -\sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} & \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{vmatrix}} \geq 0.$$

□

Many developing countries exert themselves to the utmost to attract foreign capitalists. The above result shows that this may increase the utility of citizens in the countries.

## 5.2 The Impact of tax policy on $n_c^i$ and $n_r^i$

In China, as a step of economic reform, the Shenzhen city in Guangdong province had a special tax policy (reduction or exemption tax policy in several years for several kinds of firms in the city). As a result, many capitalists and residents have been moving to Shenzhen city and the population of this city increases rapidly. Now we give a theoretical support for this phenomenon.

**Theorem 5.2** *By increasing tax  $s_c^i$  on the capitalists in region  $i$ , number  $n_c^i$  decreases and number  $n_r^i$  does not increase. Number  $n_c^k$  of another region  $k$  increases if and only if*

$$\begin{vmatrix} z_i & \sum_{l=1}^m \frac{x_k z_l - x_l z_k}{w_l z_l - x_l y_l} \\ -y_i & \sum_{l=1}^m \frac{w_l z_k - x_k y_l}{w_l z_l - x_l y_l} \end{vmatrix} < 0, \quad (5.2)$$

and number  $n_r^k$  of region  $k$  increases if and only if

$$\begin{vmatrix} z_i & \sum_{l=1}^m \frac{w_k z_l - x_l y_k}{w_l z_l - x_l y_l} \\ -y_i & \sum_{l=1}^m \frac{y_k w_l - y_l w_k}{w_l z_l - x_l y_l} \end{vmatrix} > 0. \quad (5.3)$$

Similarly, by increasing tax  $s_r^i$  on residents in region  $i$ , number  $n_r^i$  decreases and number  $n_c^i$  does not increase. Number  $n_c^k$  of another region  $k$  increases if and only if

$$\begin{vmatrix} x_i & \sum_{l=1}^m \frac{x_k z_l - x_l z_k}{w_l z_l - x_l y_l} \\ -w_i & \sum_{l=1}^m \frac{w_l z_k - x_k y_l}{w_l z_l - x_l y_l} \end{vmatrix} > 0, \quad (5.4)$$

and number  $n_r^k$  of region  $k$  increases if and only if

$$\begin{vmatrix} x_i & \sum_{l=1}^m \frac{w_k z_l - x_l y_k}{w_l z_l - x_l y_l} \\ -w_i & \sum_{l=1}^m \frac{y_k w_l - y_l w_k}{w_l z_l - x_l y_l} \end{vmatrix} < 0. \quad (5.5)$$

**Proof:** Here we only show the case of  $i = 1, k = m$  and that tax  $s_c^1$  increases. (Other cases can be proved in a similar way.) To examine the change of number  $n_c^1$  of capitalists in region 1, we have the following expressions from (5.1).

$$\frac{\partial n_c^1}{\partial s_c^1} = \frac{\begin{vmatrix} -\partial \Phi_c / \partial s_c^1 & 0 & \cdots & 0 & x_1 & 0 & \cdots & 0 & -1 & 0 \\ 0 & w_2 & \cdots & 0 & 0 & x_2 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & w_m & 0 & 0 & \cdots & x_m & -1 & 0 \\ 0 & 0 & \cdots & 0 & z_1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & y_2 & \cdots & 0 & 0 & z_2 & \cdots & 0 & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & y_m & 0 & 0 & \cdots & z_m & 0 & -1 \\ 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & 0 \end{vmatrix}}{\Delta(w_1, \dots, w_m; x_1, \dots, x_m; y_1, \dots, y_m; z_1, \dots, z_m)}$$

$$\begin{aligned}
&= -\frac{\frac{\partial \Phi_c}{\partial s_c^1} z_1 \left( \frac{1}{z_1} \sum_{l=2}^m \frac{z_l}{w_l z_l - x_l y_l} + \sum_{l_1=2}^m \sum_{l_2=2}^m \frac{w_{l_1} z_{l_2} - x_{l_1} y_{l_2}}{(w_{l_1} z_{l_1} - x_{l_1} y_{l_1})(w_{l_2} z_{l_2} - x_{l_2} y_{l_2})} \right)}{(w_1 z_1 - x_1 y_1) \left| \begin{array}{cc} \sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l} & - \sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ - \sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} & \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{array} \right|} \\
&< 0,
\end{aligned}$$

where the inequality is from (4.2)–(4.4). Therefore number  $n_c^1$  decreases. To examine the change of number  $n_r^1$  of residents in region 1, from (5.1), we have

$$\begin{aligned}
\frac{\partial n_r^1}{\partial s_c^1} &= \frac{\begin{vmatrix} w_1 & 0 & \cdots & 0 & -\partial \Phi_c / \partial s_c^1 & 0 & \cdots & 0 & -1 & 0 \\ 0 & w_2 & \cdots & 0 & 0 & x_2 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & w_m & 0 & 0 & \cdots & x_m & -1 & 0 \\ y_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & y_2 & \cdots & 0 & 0 & z_2 & \cdots & 0 & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & y_m & 0 & 0 & \cdots & z_m & 0 & -1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & 0 \end{vmatrix}}{\Delta(w_1, \dots, w_m; x_1, \dots, x_m; y_1, \dots, y_m; z_1, \dots, z_m)} \\
&= \frac{\frac{\partial \Phi_c}{\partial s_c^1} y_1 \left( \sum_{l_1=2}^m \sum_{l_2=2}^m \frac{w_{l_1} z_{l_2} - x_{l_1} y_{l_2}}{(w_{l_1} z_{l_1} - x_{l_1} y_{l_1})(w_{l_2} z_{l_2} - x_{l_2} y_{l_2})} + \frac{1}{y_1} \sum_{l=2}^m \frac{y_l}{w_l z_l - x_l y_l} \right)}{(w_1 z_1 - x_1 y_1) \left| \begin{array}{cc} \sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l} & - \sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ - \sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} & \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{array} \right|} \\
&\leq 0,
\end{aligned}$$

where the inequality is from (4.2)–(4.4). Hence the number of residents in region 1 does not increase.

To consider the change of number  $n_c^m$  of region  $k = m$ , we use the following expression from (5.1):

$$\begin{aligned}
\frac{\partial n_c^m}{\partial s_c^1} &= \frac{\begin{vmatrix} w_1 & \cdots & 0 & -\partial \Phi_c / \partial s_c^1 & x_1 & \cdots & 0 & 0 & -1 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & w_{m-1} & 0 & 0 & \cdots & x_{m-1} & 0 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & x_m & -1 & 0 \\ y_1 & \cdots & 0 & 0 & z_1 & \cdots & 0 & 0 & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & y_{m-1} & 0 & 0 & \cdots & z_{m-1} & 0 & 0 & -1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & z_m & 0 & -1 \\ 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 & 0 & 0 \end{vmatrix}}{\Delta(w_1, \dots, w_m; x_1, \dots, x_m; y_1, \dots, y_m; z_1, \dots, z_m)} \\
&= \frac{\frac{\partial \Phi_c}{\partial s_c^1} \left| \begin{array}{cc} \frac{z_1}{w_1 z_1 - x_1 y_1} & \sum_{l=1}^m \frac{x_m z_l - x_l z_m}{w_l z_l - x_l y_l} \\ - \frac{y_1}{w_1 z_1 - x_1 y_1} & \sum_{l=1}^m \frac{w_l z_m - x_m y_l}{w_l z_l - x_l y_l} \end{array} \right|}{(w_m z_m - x_m y_m) \left| \begin{array}{cc} \sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l} & - \sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ - \sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} & \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{array} \right|}. \tag{5.6}
\end{aligned}$$

From (4.2)–(4.4), (5.6) is positive if and only if (5.2) holds. Finally, to examine the change of number  $n_r^m$  of region  $k = m$ , we use the following expression from (5.1).

$$\begin{aligned}
\frac{\partial n_r^m}{\partial s_c^1} &= \frac{\begin{vmatrix} w_1 & \cdots & 0 & 0 & x_1 & \cdots & 0 & -\partial\Phi_c/\partial s_c^1 & -1 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & w_{m-1} & 0 & 0 & \cdots & x_{m-1} & 0 & -1 & 0 \\ 0 & \cdots & 0 & w_m & 0 & \cdots & 0 & 0 & -1 & 0 \\ y_1 & \cdots & 0 & 0 & z_1 & \cdots & 0 & 0 & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & y_{m-1} & 0 & 0 & \cdots & z_{m-1} & 0 & 0 & -1 \\ 0 & \cdots & 0 & y_m & 0 & \cdots & 0 & 0 & 0 & -1 \\ 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & 0 & 0 \end{vmatrix}}{\Delta(w_1, \dots, w_m; x_1, \dots, x_m; y_1, \dots, y_m; z_1, \dots, z_m)} \\
&= -\frac{\frac{\partial\Phi_c}{\partial s_c^1} \left| \begin{array}{c} \frac{z_1}{w_1 z_1 - x_1 y_1} \quad \sum_{l=1}^m \frac{w_m z_l - x_l y_m}{w_l z_l - x_l y_l} \\ -\frac{y_1}{w_1 z_1 - x_1 y_1} \quad \sum_{l=1}^m \frac{y_m w_l - y_l w_m}{w_l z_l - x_l y_l} \end{array} \right|}{(w_m z_m - x_m y_m) \left| \begin{array}{c} \sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l} \quad -\sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ -\sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} \quad \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{array} \right|}. \tag{5.7}
\end{aligned}$$

By (4.2)–(4.4), (5.7) is positive if and only if (5.3) holds.  $\square$

Theorem 5.2 tells us that, if  $s_c^i$  ( $s_r^i$ ) is increased, some capitalists (residents) of region  $i$  move out. However, it does not mean that the capitalists (residents) in all other regions increase. Now we give an explanation of condition (5.4), which can be rewritten as follows.

$$x_i + x_k + x_k \sum_{\substack{l=1, \dots, m \\ l \neq i, l \neq k}} \frac{w_i z_l - x_i y_l}{w_l z_l - x_l y_l} + z_k \sum_{\substack{l=1, \dots, m \\ l \neq i, l \neq k}} \frac{w_l x_i - w_i x_l}{w_l z_l - x_l y_l} > 0, \tag{5.8}$$

The preceding three items of (5.8) are nonnegative hence it is quite possible that (5.8) is true. However, in the case that  $m > 2$  and

$$\left| \frac{x_i}{w_i} \right| \ll \left| \frac{x_l}{w_l} \right| \quad \forall l = 1, \dots, m, l \neq i, k, \tag{5.9}$$

(5.8) may fail to hold. From Figure 1, we know that  $|x_i/w_i|$  is the number of capitalists emigrate from region  $i$  to recover the original capitalists' utility level when one resident emigrates from region  $i$ . Intuitively, first some residents emigrate from region  $i$  to region  $l$  because  $s_r^i$  increases for some  $l$ . Since (5.9) says that  $|x_i/w_i|$  is small, few capitalists emigrate from region  $i$ . However, since  $|x_l/w_l|$  is large by (5.9), many capitalists immigrate to region  $l$ . Therefore it happens that the number of capitalists in region  $k$  decreases. Other conditions (5.2), (5.3) and (5.5) can be explained in a similar way.

### 5.3 The impact of tax policy on $U_c$ and $U_r$

From the conclusions of Section 5.2, we know that when tax  $s_c^i$  on capitalists in region  $i$  increases, number  $n_c^i$  decreases and number  $n_r^i$  does not increase. Now we show that  $U_c$  decreases but  $U_r$  possibly decreases and also possibly increases. Similarly, when we increase tax  $s_r^i$  on residents in region  $i$ , the utility level  $U_c$  possibly decreases and also possibly increases.



**Theorem 5.3** *By increasing tax  $s_c^i$  on capitalists of region  $i$ , the utility level  $U_c$  of capitalists decreases. Furthermore, the utility level  $U_r$  of residents increases if and only if*

$$\sum_{l=1}^m \frac{z_i y_l - y_i z_l}{w_l z_l - x_l y_l} < 0.$$

*By increasing tax  $s_r^i$  on residents in region  $i$ , the utility level  $U_r$  of residents decreases. Furthermore, the utility level  $U_c$  of capitalists increases if and only if*

$$\sum_{l=1}^m \frac{w_i x_l - x_i w_l}{w_l z_l - x_l y_l} < 0.$$

**Proof:** Let  $ds_r^j = dQ^j = dN_c = dN_r = 0$  for all  $j$ , and furthermore  $ds_c^j = 0$  for all  $j \neq i$ . Then from (5.1) we have

$$dU_c = \frac{\begin{vmatrix} w_1 & \cdots & 0 & \cdots & 0 & x_1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & w_i & \cdots & 0 & 0 & \cdots & x_i & \cdots & 0 & -(\partial\Phi_c/\partial s_c^i)ds_c^i & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & w_m & 0 & \cdots & 0 & \cdots & x_m & 0 & 0 \\ y_1 & \cdots & 0 & \cdots & 0 & z_1 & \cdots & 0 & \cdots & 0 & 0 & -1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & y_i & \cdots & 0 & 0 & \cdots & z_i & \cdots & 0 & 0 & -1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & y_m & 0 & \cdots & 0 & \cdots & z_m & 0 & -1 \\ 1 & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 1 & 0 & 0 \end{vmatrix}}{\Delta(w_1, \dots, w_m; x_1, \dots, x_m; y_1, \dots, y_m; z_1, \dots, z_m)}.$$

Therefore

$$\frac{\partial U_c}{\partial s_c^i} = \frac{\frac{\partial\Phi_c}{\partial s_c^i} \sum_{l=1}^m \frac{w_l z_l - x_l y_l}{w_l z_l - x_l y_l}}{(w_i z_i - x_i y_i) \left| \begin{array}{cc} \sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l} & - \sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ - \sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} & \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{array} \right|} < 0,$$

where the inequality is from (4.2)–(4.4).

Similarly, we have

$$\frac{\partial U_r}{\partial s_c^i} = \frac{\frac{\partial\Phi_c}{\partial s_c^i} \sum_{l=1}^m \frac{z_i y_l - y_i z_l}{w_l z_l - x_l y_l}}{(w_i z_i - x_i y_i) \left| \begin{array}{cc} \sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l} & - \sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ - \sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} & \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{array} \right|}, \quad (5.10)$$

$$\frac{\partial U_c}{\partial s_r^i} = \frac{\frac{\partial\Phi_r}{\partial s_r^i} \sum_{l=1}^m \frac{w_i x_l - x_i w_l}{w_l z_l - x_l y_l}}{(w_i z_l - x_i y_l) \left| \begin{array}{cc} \sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l} & - \sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ - \sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} & \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{array} \right|}, \quad (5.11)$$

$$\frac{\partial U_r}{\partial s_r^i} = \frac{\frac{\partial\Phi_r}{\partial s_r^i} \sum_{l=1}^m \frac{w_i z_l - x_i y_l}{w_l z_l - x_l y_l}}{(w_i z_i - x_i y_i) \left| \begin{array}{cc} \sum_{l=1}^m \frac{z_l}{w_l z_l - x_l y_l} & - \sum_{l=1}^m \frac{x_l}{w_l z_l - x_l y_l} \\ - \sum_{l=1}^m \frac{y_l}{w_l z_l - x_l y_l} & \sum_{l=1}^m \frac{w_l}{w_l z_l - x_l y_l} \end{array} \right|} < 0.$$

From the above expressions, the conclusions of this theorem hold evidently.  $\square$

The following two corollaries explain the conditions in Theorem 5.3.

**Corollary 5.1** *There exists at least one region  $i$ , such that when  $s_c^i$  increases,  $U_c$  decreases and  $U_r$  does not increase; There exists at least one region  $j$ , such that when  $s_r^j$  increases,  $U_r$  decreases and  $U_c$  does not increase.*

**Proof:** Let  $i$  and  $j$  be the region satisfying

$$\left| \frac{y_i}{z_i} \right| = \max_{l=1, \dots, m} \left| \frac{y_l}{z_l} \right|, \quad \left| \frac{x_j}{w_j} \right| = \max_{l=1, \dots, m} \left| \frac{x_l}{w_l} \right|.$$

Then it is easy to check that  $\partial U_r / \partial s_c^i \leq 0$  and  $\partial U_c / \partial s_r^j \leq 0$  by (5.10) and (5.11).  $\square$

**Corollary 5.2** *By increasing  $s_c^i$  in region  $i$  satisfying (5.12),  $U_c$  decreases but  $U_r$  increases; By increasing  $s_r^j$  in region  $j$  satisfying (5.13),  $U_r$  decreases but  $U_c$  increases.*

$$\left| \frac{y_i}{z_i} \right| = \min_{l=1, \dots, m} \left| \frac{y_l}{z_l} \right| < \max_{l=1, \dots, m} \left| \frac{y_l}{z_l} \right|, \quad (5.12)$$

$$\left| \frac{x_j}{w_j} \right| = \min_{l=1, \dots, m} \left| \frac{x_l}{w_l} \right| < \max_{l=1, \dots, m} \left| \frac{x_l}{w_l} \right|. \quad (5.13)$$

The above two corollaries tell us that if the government needs more tax, increasing taxes  $s_c^i$  and  $s_r^j$  is better than increasing other taxes until

$$\min_l \left| \frac{y_l}{z_l} \right| = \max_l \left| \frac{y_l}{z_l} \right|, \quad \min_l \left| \frac{x_l}{w_l} \right| = \max_l \left| \frac{x_l}{w_l} \right|. \quad (5.14)$$

To explain (5.14), we use Figure 1 again. The ratio  $|x_l/w_l|$  can be explained as the sensitivity of capitalists' migration to residents' migration. Therefore the latter part of (5.14) says that the sensitivity in any region is the same. The former part of (5.14) can be explained in a similar way.

## 6 A model for optimal tax policy

The net income, which is the gross income subtracted by tax, is an important factor for capitalists and residents to locate. Therefore the government can control the migration by a suitable tax policy. We suppose that the total numbers  $N_c$  and  $N_r$  of capitalists and residents in the country are fixed and the government seeks to maximize the welfare of its own residents and capitalists. By introducing weights  $\alpha$  and  $\beta$ , we model the objective function of the government as follows.

$$U(\mathbf{s}_c, \mathbf{s}_r) = \alpha N_c U_c(\mathbf{s}_c, \mathbf{s}_r) + \beta N_r U_r(\mathbf{s}_c, \mathbf{s}_r).$$

The weights  $\alpha$ ,  $\beta$  are nonnegative and  $\alpha + \beta = 1$ . A tax policy which levies a person more tax than his/her income is considered to be infeasible. Therefore we suppose that  $s_c^i \leq \gamma_c^i$  and  $s_r^i \leq \gamma_r^i$  for all region  $i$ . By tax policy  $\langle \mathbf{s}_c, \mathbf{s}_r \rangle$ , the government should obtain total tax  $T$ , which is equal to the total expenditure of the whole country, including the national defense fee, the education fee, etc. The optimal tax policy can be described as a solution to the following optimal problem.

$$\begin{aligned} \max \quad & \alpha N_c U_c(\mathbf{s}_c, \mathbf{s}_r) + \beta N_r U_r(\mathbf{s}_c, \mathbf{s}_r) \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^m (n_c^i s_c^i + n_r^i s_r^i) = T \\ s_c^i \leq \gamma_c^i \\ s_r^i \leq \gamma_r^i. \end{cases} \end{aligned} \quad (6.1)$$

To find the optimal solution of (6.1), we now introduce the following Lagrange function.

$$\begin{aligned} L(\mathbf{s}_c, \mathbf{s}_r, \lambda^0, \lambda_c, \lambda_r) \\ = \alpha U(\mathbf{s}_c, \mathbf{s}_r) + \lambda^0 \left( \sum_{i=1}^m (n_c^i s_c^i + n_r^i s_r^i) - T \right) + \sum_{i=1}^m \lambda_c^i (s_c^i - \gamma_c^i) + \sum_{i=1}^m \lambda_r^i (s_r^i - \gamma_r^i), \end{aligned}$$

where  $\lambda^0, \lambda_c = (\lambda_c^1, \dots, \lambda_c^m) \geq 0$  and  $\lambda_r = (\lambda_r^1, \dots, \lambda_r^m) \geq 0$  are parameters. At first, we calculate the partial differentiates of  $L$  with respects to  $s_c^i$  and  $s_r^i$  for all  $i = 1, \dots, m$ .

$$\frac{\partial L}{\partial s_c^i} = \alpha N_c \frac{\partial U_c}{\partial s_c^i} + \beta N_r \frac{\partial U_r}{\partial s_c^i} + \lambda^0 \left( n_c^i + \sum_{j=1}^m (s_c^j \frac{\partial n_c^j}{\partial s_c^i} + s_r^j \frac{\partial n_r^j}{\partial s_c^i}) \right) + \lambda_c^i, \quad (6.2)$$

$$\frac{\partial L}{\partial s_r^i} = \alpha N_c \frac{\partial U_c}{\partial s_r^i} + \beta N_r \frac{\partial U_r}{\partial s_r^i} + \lambda^0 \left( n_r^i + \sum_{j=1}^m (s_c^j \frac{\partial n_c^j}{\partial s_r^i} + s_r^j \frac{\partial n_r^j}{\partial s_r^i}) \right) + \lambda_r^i. \quad (6.3)$$

Since the total numbers of capitalists and residents are fixed, we have

$$\begin{aligned} \frac{\partial n_c^i}{\partial s_c^i} &= - \sum_{\substack{j=1, \dots, m \\ j \neq i}} \frac{\partial n_c^j}{\partial s_c^i}, & \frac{\partial n_r^i}{\partial s_c^i} &= - \sum_{\substack{j=1, \dots, m \\ j \neq i}} \frac{\partial n_r^j}{\partial s_c^i}, \\ \frac{\partial n_c^i}{\partial s_r^i} &= - \sum_{\substack{j=1, \dots, m \\ j \neq i}} \frac{\partial n_c^j}{\partial s_r^i}, & \frac{\partial n_r^i}{\partial s_r^i} &= - \sum_{\substack{j=1, \dots, m \\ j \neq i}} \frac{\partial n_r^j}{\partial s_r^i}. \end{aligned}$$

Therefore (6.2) and (6.3) can be reformed as follows.

$$\begin{aligned} \frac{\partial L}{\partial s_c^i} &= \alpha N_c \frac{\partial U_c}{\partial s_c^i} + \beta N_r \frac{\partial U_r}{\partial s_c^i} + \lambda^0 \left( n_c^i + \sum_{j=1}^m ((s_c^j - s_c^i) \frac{\partial n_c^j}{\partial s_c^i} + (s_r^j - s_r^i) \frac{\partial n_r^j}{\partial s_c^i}) \right) + \lambda_c^i, \\ \frac{\partial L}{\partial s_r^i} &= \alpha N_c \frac{\partial U_c}{\partial s_r^i} + \beta N_r \frac{\partial U_r}{\partial s_r^i} + \lambda^0 \left( n_r^i + \sum_{j=1}^m ((s_c^j - s_c^i) \frac{\partial n_c^j}{\partial s_r^i} + (s_r^j - s_r^i) \frac{\partial n_r^j}{\partial s_r^i}) \right) + \lambda_r^i, \\ & i = 1, \dots, m. \end{aligned}$$

By Kuhn-Tucker condition of optimization theory, we know that for an optimal solution  $(\mathbf{s}_c^*, \mathbf{s}_r^*)$  of (6.1), there exist suitable parameter  $\lambda^{0*}, \lambda_c^*$  and  $\lambda_r^*$  such that

$$\left\{ \begin{array}{l} \frac{\partial L(\mathbf{s}_c^*, \mathbf{s}_r^*, \lambda^{0*}, \lambda_c^*, \lambda_r^*)}{\partial s_c^i} = 0 \\ \frac{\partial L(\mathbf{s}_c^*, \mathbf{s}_r^*, \lambda^{0*}, \lambda_c^*, \lambda_r^*)}{\partial s_r^i} = 0 \\ s_c^{i*} < \gamma_c^i \Rightarrow \lambda_c^{i*} = 0 \\ s_r^{i*} < \gamma_r^i \Rightarrow \lambda_r^{i*} = 0 \\ s_c^{i*} = \gamma_c^i \Rightarrow \lambda_c^{i*} \geq 0 \\ s_r^{i*} = \gamma_r^i \Rightarrow \lambda_r^{i*} \geq 0 \end{array} \right. \quad (6.4)$$

Since  $\partial U_c / \partial s_c^i, \partial U_r / \partial s_c^i, \partial U_c / \partial s_r^i, \partial U_r / \partial s_r^i, \partial n_c^j / \partial s_c^i, \partial n_r^j / \partial s_c^i, \partial n_c^j / \partial s_r^i$  and  $\partial n_r^j / \partial s_r^i$  can be calculated from the results in Section 5, the optimal tax policy can be obtained from equations (6.4) and constraint (6.1).

## 7 Concluding remarks

This paper considers a new migration model, which includes residents and capitalists. After a model description in Section 2, we prove that at least one equilibrium exists in Section 3, in which there is no more migration of capitalists and residents. In Section 4, we impose some assumptions to assure the Hicks' stability of an equilibrium. Section 5 gives some comparative statics analyses. In Section 6, we provide a model for optimal tax policy.

In our model, the capitalists are supposed to be homogeneous and every firm is invested by one capitalist only. If a large firm can be treated as a composition of several small firms, then our model works more generally. For the convenience of model analysis, this paper recognizes a capitalist as a people or stockholder. However, it may be better to think of it as a representative of unit capital, or stock itself, which explains why it does not affect congestion directly.

The model in this paper does not involve any local public goods in explicit form. However, the URSF  $Q^i$  of each region  $i$  actually represents the level of local public goods there. Suppose that from the viewpoint of the local government in region  $i$ , it is rational to levy tax  $t_c^i$  on each capitalist there and tax  $t_r^i$  on each resident there. Then our model still works by replacing tax amount  $s_c^i$  by  $s_c^i + t_c^i$ , and amount  $s_r^i$  by  $s_r^i + t_r^i$ .

We give two future research topics. First, this research suppose that residents are free to migrate among regions. In China, to avoid congestion, the registered permanent residence system is enforced. In this way, migration of residents is restricted. However, as a fact, the congestion in Beijing and Shanghai remains severe. It is important to extend our model and results to include the migration cost. Second, the stability conditions in Section 4 are sufficient but not necessary. It is important to examine the stability when either  $w_i$  or  $z_i$  is, or both are positive at the considered equilibrium.

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## Appendix

We prove Theorem 4.1 here by several lemmas. First, we show that we can give the following assumption without loss of generality:

$$w_i x_m - x_i w_m \leq 0, \quad i = 1, \dots, m, \quad (\text{A.1})$$

where all the  $w_i$  and  $x_i$  are valued at the considered equilibrium. If  $x_i = 0$  for all  $i$ , then (A.1) holds evidently. If  $x_i \neq 0$  for some  $i$ , we can rename the regions so that

$$\frac{w_m}{x_m} = \max_{\substack{i=1, \dots, m \\ x_i \neq 0}} \frac{w_i}{x_i},$$

which implies (A.1).

**Lemma A. 1** *For integers  $k = 2, \dots, m - 1$ , the following results for  $2k \times 2k$  matrices hold.*

$$A_1(\epsilon, k) + B_1(\epsilon, k) = \quad (\text{A.2})$$

$$\begin{vmatrix} \epsilon w_m & 0 & w_m & x_m & \cdots & w_m & x_m \\ \epsilon y_m & 0 & y_m & z_m & \cdots & y_m & z_m \\ 0 & -x_1 & w_2 & x_2 & \cdots & 0 & 0 \\ 0 & -z_1 & y_2 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -x_1 & 0 & 0 & \cdots & w_k & x_k \\ 0 & -z_1 & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} + \begin{vmatrix} 0 & \epsilon x_m & w_m & x_m & \cdots & w_m & x_m \\ 0 & \epsilon z_m & y_m & z_m & \cdots & y_m & z_m \\ -w_1 & 0 & w_2 & x_2 & \cdots & 0 & 0 \\ -y_1 & 0 & y_2 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_1 & 0 & 0 & 0 & \cdots & w_k & x_k \\ -y_1 & 0 & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} \geq 0,$$

$$A_2(\epsilon_1, \epsilon_2, k) + B_2(\epsilon_1, \epsilon_2, k) = \quad (\text{A.3})$$

$$\begin{vmatrix} \epsilon_1 w_1 & 0 & w_m & x_m & \cdots & w_m & x_m \\ \epsilon_2 y_1 & 0 & y_m & z_m & \cdots & y_m & z_m \\ 0 & -x_1 & w_2 & x_2 & \cdots & 0 & 0 \\ 0 & -z_1 & y_2 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -x_1 & 0 & 0 & \cdots & w_k & x_k \\ 0 & -z_1 & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} + \begin{vmatrix} 0 & \epsilon_1 x_1 & w_m & x_m & \cdots & w_m & x_m \\ 0 & \epsilon_2 z_1 & y_m & z_m & \cdots & y_m & z_m \\ -w_1 & 0 & w_2 & x_2 & \cdots & 0 & 0 \\ -y_1 & 0 & y_2 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_1 & 0 & 0 & 0 & \cdots & w_k & x_k \\ -y_1 & 0 & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} \geq 0,$$

$$A_3(k) + B_3(k) = \quad (\text{A.4})$$

$$\begin{vmatrix} \frac{w_m}{m} + \frac{w_1 w_m}{m w_{k+1}} & 0 & w_m & x_m & \cdots & w_m & x_m \\ \frac{y_m}{m} + \frac{w_1 (y_m - y_{k+1})}{m w_{k+1}} & 0 & y_m & z_m & \cdots & y_m & z_m \\ 0 & -x_1 & w_2 & x_2 & \cdots & 0 & 0 \\ 0 & -z_1 & y_2 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -x_1 & 0 & 0 & \cdots & w_k & x_k \\ 0 & -z_1 & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} + \begin{vmatrix} 0 & \frac{x_m}{m} + \frac{x_1 w_m}{m w_{k+1}} & w_m & x_m & \cdots & w_m & x_m \\ 0 & \frac{z_m}{m} + \frac{x_1 (y_m - y_{k+1})}{m w_{k+1}} & y_m & z_m & \cdots & y_m & z_m \\ -w_1 & 0 & w_2 & x_2 & \cdots & 0 & 0 \\ -y_1 & 0 & y_2 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_1 & 0 & 0 & 0 & \cdots & w_k & x_k \\ -y_1 & 0 & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} \geq 0,$$

where  $\epsilon$ ,  $\epsilon_1$  and  $\epsilon_2$  are positive numbers.

**Proof:** We introduce notation  $A(\epsilon, k, j)$  by letting all the elements, except  $2j - 1$  and  $2j$  rows, in the second column of  $A_1(\epsilon, k)$  be zero, and notation  $B(\epsilon, k, j)$  by letting all the elements, except  $2j - 1$  and  $2j$  rows, in the first column of  $B_1(\epsilon, k)$  be 0, where  $j = 2, \dots, k$ .

$$\begin{aligned}
A(\epsilon, k, j) &= \begin{vmatrix} \epsilon w_m & 0 & \cdots & w_m & x_m & \cdots & w_m & x_m \\ \epsilon y_m & 0 & \cdots & y_m & z_m & \cdots & y_m & z_m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -x_1 & \cdots & w_j & x_j & \cdots & 0 & 0 \\ 0 & -z_1 & \cdots & y_j & z_j & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & w_k & x_k \\ 0 & 0 & \cdots & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} \\
&= \begin{vmatrix} \epsilon w_m & x_m \frac{w_j z_1 - y_j x_1}{w_j z_j - x_j y_j} + w_m \frac{z_j x_1 - x_j z_1}{w_j z_j - x_j y_j} \\ \epsilon y_m & z_m \frac{w_j z_1 - x_1 y_j}{w_j z_j - x_j y_j} + y_m \frac{z_j x_1 - z_1 x_j}{w_j z_j - x_j y_j} \end{vmatrix} \prod_{l=2}^k (w_l z_l - x_l y_l), \\
B(\epsilon, k, j) &= \begin{vmatrix} 0 & \epsilon x_m & \cdots & w_m & x_m & \cdots & w_m & x_m \\ 0 & \epsilon z_m & \cdots & y_m & z_m & \cdots & y_m & z_m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_1 & 0 & \cdots & w_j & x_j & \cdots & 0 & 0 \\ -y_1 & 0 & \cdots & y_j & z_j & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & w_k & x_k \\ 0 & 0 & \cdots & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} \\
&= \begin{vmatrix} w_m \frac{w_1 z_j - x_j y_1}{w_j z_j - x_j y_j} + x_m \frac{y_1 w_j - y_j w_1}{w_j z_j - x_j y_j} \\ y_m \frac{w_1 z_j - x_j y_1}{w_j z_j - x_j y_j} + z_m \frac{y_1 w_j - w_1 y_j}{w_j z_j - x_j y_j} \end{vmatrix} \begin{vmatrix} \epsilon x_m \\ \epsilon z_m \end{vmatrix} \prod_{l=2}^k (w_l z_l - x_l y_l).
\end{aligned}$$

For these two determinants, it holds that

$$\begin{aligned}
&A(\epsilon, k, j) + B(\epsilon, k, j) \\
&= \epsilon \frac{w_j z_1 - y_j x_1 + w_1 z_j - x_j y_1}{w_j z_j - x_j y_j} \begin{vmatrix} w_m & x_m \\ y_m & z_m \end{vmatrix} \prod_{l=2}^k (w_l z_l - x_l y_l) \\
&= \epsilon \frac{(w_j z_1 - x_j y_1) + (w_1 z_j - x_1 y_j)}{w_j z_j - x_j y_j} (w_m z_m - x_m y_m) \prod_{l=2}^k (w_l z_l - x_l y_l) \\
&\geq 0,
\end{aligned}$$

where the last inequality is from (4.4).

By a basic property of determinants, it holds that

$$A_1(\epsilon, k) = \sum_{j=2}^k A(\epsilon, k, j), \quad B_1(\epsilon, k) = \sum_{j=2}^k B(\epsilon, k, j).$$

Therefore we have proven (A.2). Similarly we can prove that

$$A_2(\epsilon_1, \epsilon_2, k) + B_2(\epsilon_1, \epsilon_2, k)$$

$$\begin{aligned}
&= \left( \sum_{j=2}^k \frac{\epsilon_1(w_j z_m - x_j y_m) + \epsilon_2(w_m z_j - x_m y_j)}{w_j z_j - x_j y_j} \right) \prod_{l=1}^k (w_l z_l - x_l y_l) \\
&\geq 0,
\end{aligned}$$

$$\begin{aligned}
&A_3(k) + B_3(k) \\
&= \left( \sum_{j=2}^k \frac{y_{k+1}(w_j x_m - x_j w_m) + w_j(w_m z_m - x_m y_m)}{w_j z_j - x_j y_j} \right) \frac{\prod_{l=1}^k (w_l z_l - x_l y_l)}{m w_{k+1}} \\
&\quad + \sum_{j=2}^k \frac{(w_1 z_j - x_1 y_j) + (w_j z_1 - x_j y_1)}{m(w_j z_j - x_j y_j)} (w_m z_m - x_m y_m) \prod_{l=2}^k (w_l z_l - x_l y_l) \\
&\geq 0,
\end{aligned}$$

where the last inequality uses (A.1).  $\square$

**Lemma A. 2** For integers  $k = 2, \dots, m-1$ , the following  $2k \times 2k$  determinant is positive.

$$C(k) = \begin{vmatrix} 0 & 0 & w_m & x_m & \cdots & w_m & x_m \\ 0 & 0 & y_m & z_m & \cdots & y_m & z_m \\ -w_1 & -x_1 & w_2 & x_2 & \cdots & 0 & 0 \\ -y_1 & -z_1 & y_2 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_1 & -x_1 & 0 & 0 & \cdots & w_k & x_k \\ -y_1 & -z_1 & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} > 0. \quad (\text{A.5})$$

**Proof:** By calculation, we have

$$C(k) = \begin{vmatrix} \delta_1 & \delta_2 & w_m & x_m & \cdots & w_m & x_m \\ \delta_3 & \delta_4 & y_m & z_m & \cdots & y_m & z_m \\ 0 & 0 & w_2 & x_2 & \cdots & 0 & 0 \\ 0 & 0 & y_2 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & w_k & x_k \\ 0 & 0 & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} = (\delta_1 \delta_4 - \delta_2 \delta_3) \prod_{l=2}^k (w_l z_l - x_l y_l),$$

where

$$\begin{aligned}
\delta_1 &= w_m \sum_{j=2}^k \frac{w_1 z_j - x_j y_1}{w_j z_j - x_j y_j} + x_m \sum_{j=2}^k \frac{y_1 w_j - y_j w_1}{w_j z_j - x_j y_j} \\
\delta_2 &= w_m \sum_{j=2}^k \frac{x_1 z_j - x_j z_1}{w_j z_j - x_j y_j} + x_m \sum_{j=2}^k \frac{w_j z_1 - x_1 y_j}{w_j z_j - x_j y_j} \\
\delta_3 &= y_m \sum_{j=2}^k \frac{w_1 z_j - x_j y_1}{w_j z_j - x_j y_j} + z_m \sum_{j=2}^k \frac{y_1 w_j - y_j w_1}{w_j z_j - x_j y_j} \\
\delta_4 &= y_m \sum_{j=2}^k \frac{x_1 z_j - x_j z_1}{w_j z_j - x_j y_j} + z_m \sum_{j=2}^k \frac{w_j z_1 - x_1 y_j}{w_j z_j - x_j y_j}.
\end{aligned}$$



Note that

$$\begin{aligned}
& \delta_1 \delta_4 - \delta_2 \delta_3 \\
&= (w_m z_m - x_m y_m) \\
& \quad \left( \sum_{j=2}^k \frac{w_j z_1 - x_1 y_j}{w_j z_j - x_j y_j} \sum_{j=2}^k \frac{w_1 z_j - x_j y_1}{w_j z_j - x_j y_j} - \sum_{j=2}^k \frac{y_1 w_j - y_j w_1}{w_j z_j - x_j y_j} \sum_{j=2}^k \frac{x_1 z_j - x_j z_1}{w_j z_j - x_j y_j} \right) \\
&= (w_m z_m - x_m y_m) (w_1 z_1 - x_1 y_1) \sum_{j_1=2}^k \sum_{j_2=2}^k \frac{w_{j_1} z_{j_2} - x_{j_2} y_{j_1}}{(w_{j_1} z_{j_1} - x_{j_1} y_{j_1})(w_{j_2} z_{j_2} - x_{j_2} y_{j_2})} \\
&= (w_m z_m - x_m y_m) (w_1 z_1 - x_1 y_1) \sum_{j_1=2}^k \sum_{j_2=2}^k \frac{w_{j_1} z_{j_2} - x_{j_1} y_{j_2}}{(w_{j_1} z_{j_1} - x_{j_1} y_{j_1})(w_{j_2} z_{j_2} - x_{j_2} y_{j_2})} \\
&> 0,
\end{aligned}$$

where the last inequality is from (4.4). By (4.4) again we obtain the conclusion.  $\square$

**Lemma A. 3** For integers  $k = 2, \dots, m-1$ , the following two  $2k \times 2k$  determinants are positive.

$$D(k) = \begin{vmatrix} (m-k)w_1 & (m-k)x_1 & w_m & x_m & \cdots & w_m & x_m \\ (m-k)y_1 & (m-k)z_1 & y_m & z_m & \cdots & y_m & z_m \\ -w_1 & -x_1 & w_2 & x_2 & \cdots & 0 & 0 \\ -y_1 & -z_1 & y_2 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_1 & -x_1 & 0 & 0 & \cdots & w_k & x_k \\ -y_1 & -z_1 & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} > 0, \quad (\text{A.6})$$

$$D_1(k) = \begin{vmatrix} (m-k-1)w_1 & (m-k-1)x_1 & w_m & x_m & \cdots & w_m & x_m \\ (m-k)y_1 & (m-k)z_1 & y_m & z_m & \cdots & y_m & z_m \\ -w_1 & -x_1 & w_2 & x_2 & \cdots & 0 & 0 \\ -y_1 & -z_1 & y_2 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_1 & -x_1 & 0 & 0 & \cdots & w_k & x_k \\ -y_1 & -z_1 & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} > 0. \quad (\text{A.7})$$

**Proof:** By decomposition of the two determinants, we have

$$\begin{aligned}
D(k) &= A_2(m-k, m-k, k) + B_2(m-k, m-k, k) + C(k) \\
& \quad + (m-k)^2 \prod_{l=1}^k (w_l z_l - x_l y_l) > 0, \\
D_1(k) &= A_2(m-k-1, m-k, k) + B_2(m-k-1, m-k, k) + C(k) \\
& \quad + (m-k-1)(m-k) \prod_{l=1}^k (w_l z_l - x_l y_l) > 0.
\end{aligned}$$

Here  $A_2$ ,  $B_2$  and  $C$  are defined by (A.3) and (A.5), and the inequality is from Lemmas A.1 and A.2 and (4.4).  $\square$

**Lemma A. 4** For integers  $k = 1, 2, \dots, m - 1$ , the following determinant is positive.

$$E(k) = \begin{vmatrix} a_{11} & b_{11} & a_{12} & b_{12} & \cdots & a_{1k} & b_{1k} \\ c_{11} & d_{11} & c_{12} & d_{12} & \cdots & c_{1k} & d_{1k} \\ a_{21} & b_{21} & a_{22} & b_{22} & \cdots & a_{2k} & b_{2k} \\ c_{21} & d_{21} & c_{22} & d_{22} & \cdots & c_{2k} & d_{2k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k1} & b_{k1} & a_{k2} & b_{k2} & \cdots & a_{kk} & b_{kk} \\ c_{k1} & d_{k1} & c_{k2} & d_{k2} & \cdots & c_{kk} & d_{kk} \end{vmatrix} > 0,$$

where  $a_{ij}$ ,  $b_{ij}$ ,  $c_{ij}$  and  $d_{ij}$  are defined by (4.8).

**Proof:** At first, when  $k = 1$ ,

$$E(1) = \begin{vmatrix} \frac{(m-1)w_1+w_m}{m} & \frac{(m-1)x_1+x_m}{m} \\ \frac{(m-1)y_1+y_m}{m} & \frac{(m-1)z_1+z_m}{m} \end{vmatrix} > 0$$

holds evidently by use of (4.4). Then consider the case of  $k = 2, \dots, m - 1$ . According to some basic properties of determinants, we have

$$\begin{aligned} E(k) &= \begin{vmatrix} \frac{(m-k)w_1+w_m}{m} & \frac{(m-k)x_1+x_m}{m} & \frac{w_m}{m} & \frac{x_m}{m} & \cdots & \frac{w_m}{m} & \frac{x_m}{m} \\ \frac{(m-k)y_1+y_m}{m} & \frac{(m-k)z_1+z_m}{m} & \frac{y_m}{m} & \frac{z_m}{m} & \cdots & \frac{y_m}{m} & \frac{z_m}{m} \\ -w_1 & -x_1 & w_2 & x_2 & \cdots & 0 & 0 \\ -y_1 & -z_1 & y_2 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_1 & -x_1 & 0 & 0 & \cdots & w_k & x_k \\ -y_1 & -z_1 & 0 & 0 & \cdots & y_k & z_k \end{vmatrix} \\ &= \frac{1}{m^2} D(k) + \frac{1}{m^2} (A_1(1, k) + B_1(1, k)) \\ &\quad + \left( \frac{m-k}{m^2} ((w_m z_1 - x_m y_1) + (w_1 z_m - x_1 y_m)) \right) \\ &\quad + \frac{1}{m^2} (w_m z_m - x_m y_m) \prod_{l=2}^k (w_l z_l - x_l y_l) \\ &> 0, \end{aligned}$$

where  $A_1$ ,  $B_1$  and  $D$  are defined by (A.2) and (A.6), the last inequality is from Lemmas A.1–A.3 and (4.4).  $\square$

Next, we consider the principal minors of odd size.

**Lemma A. 5** For integer  $k = 0, 1, \dots, m - 2$ , the following determinant is negative.

$$F(k) = \begin{vmatrix} a_{11} & b_{11} & a_{12} & b_{12} & \cdots & a_{1k} & b_{1k} & a_{1,k+1} \\ c_{11} & d_{11} & c_{12} & d_{12} & \cdots & c_{1k} & d_{1k} & c_{1,k+1} \\ a_{21} & b_{21} & a_{22} & b_{22} & \cdots & a_{2k} & b_{2k} & a_{2,k+1} \\ c_{21} & d_{21} & c_{22} & d_{22} & \cdots & c_{2k} & d_{2k} & c_{2,k+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{k1} & b_{k1} & a_{k2} & b_{k2} & \cdots & a_{kk} & b_{kk} & a_{k,k+1} \\ c_{k1} & d_{k1} & c_{k2} & d_{k2} & \cdots & c_{kk} & d_{kk} & c_{k,k+1} \\ a_{k+1,1} & b_{k+1,1} & a_{k+1,2} & b_{k+1,2} & \cdots & a_{k+1,k} & b_{k+1,k} & a_{k+1,k+1} \end{vmatrix} < 0,$$

where  $a_{ij}$ ,  $b_{ij}$ ,  $c_{ij}$  and  $d_{ij}$  are defined by (4.8).

**Proof:** When  $k = 0$ , the result holds from (4.3). The case of  $k = 1$  can be shown in a way similar to the following case of  $k \geq 2$ . When  $k \geq 2$ , we have the following result by basic properties of determinants:

$$F(k) = \begin{vmatrix} \frac{(m-k-1)w_1+w_m}{m} & \frac{(m-k-1)x_1+x_m}{m} & \frac{w_m}{m} & \frac{x_m}{m} & \dots & \frac{w_m}{m} & \frac{x_m}{m} & \frac{w_m}{m} \\ \frac{(m-k)y_1+y_m}{m} & \frac{(m-k)z_1+z_m}{m} & \frac{y_m}{m} & \frac{z_m}{m} & \dots & \frac{y_m}{m} & \frac{z_m}{m} & \frac{y_m-y_{k+1}}{m} \\ -w_1 & -x_1 & w_2 & x_2 & \dots & 0 & 0 & 0 \\ -y_1 & -z_1 & y_2 & z_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -w_1 & -x_1 & 0 & 0 & \dots & w_k & x_k & 0 \\ -y_1 & -z_1 & 0 & 0 & \dots & y_k & z_k & 0 \\ -w_1 & -x_1 & 0 & 0 & \dots & 0 & 0 & w_{k+1} \end{vmatrix} = w_{k+1} F_1(k),$$

where  $F_1(k)$  is the following determinant.

$$\begin{vmatrix} \frac{(m-k-1)w_1}{m} + \frac{w_m}{m} + \frac{w_1 w_m}{m w_{k+1}} & \frac{(m-k-1)x_1}{m} + \frac{x_m}{m} + \frac{x_1 w_m}{m w_{k+1}} & \frac{w_m}{m} & \frac{x_m}{m} & \dots & \frac{w_m}{m} & \frac{x_m}{m} \\ \frac{(m-k)y_1}{m} + \frac{y_m}{m} + \frac{w_1(y_m-y_{k+1})}{m w_{k+1}} & \frac{(m-k)z_1}{m} + \frac{z_m}{m} + \frac{x_1(y_m-y_{k+1})}{m w_{k+1}} & \frac{y_m}{m} & \frac{z_m}{m} & \dots & \frac{y_m}{m} & \frac{z_m}{m} \\ -w_1 & -x_1 & w_2 & x_2 & \dots & 0 & 0 \\ -y_1 & -z_1 & y_2 & z_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_1 & -x_1 & 0 & 0 & \dots & w_k & x_k \\ -y_1 & -z_1 & 0 & 0 & \dots & y_k & z_k \end{vmatrix}$$

By further decomposition of determinant  $F_1(k)$ , we have

$$\begin{aligned} F_1(k) &= \frac{D_1(k)}{m^2} + A_3(k) + B_3(k) \\ &+ \left\{ \frac{(w_1 + w_{k+1})(w_m z_m - x_m y_m) + (m-k)w_m(w_1 z_1 - x_1 y_1) - y_{k+1}(x_1 w_m - x_m w_1)}{m^2 w_{k+1}} \right. \\ &+ \left. \frac{(m-k)(w_m z_1 - x_m y_1) + (m-k-1)(w_1 z_m - x_1 y_m)}{m^2} \right\} \prod_{l=2}^k (w_l z_l - x_l y_l) \\ &> 0, \end{aligned}$$

where  $A_3$ ,  $B_3$ ,  $D_1$  are defined by (A.4) and (A.7), and the inequality is from conditions (4.3), (4.4), (A.1) and Lemmas A.1, A.3. By (4.3) again, we obtain  $F(k) < 0$ .  $\square$