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Production decisions under demand uncertainty: The high-low search approach

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No. 223

**PRODUCTION DECISIONS UNDER
DEMAND UNCERTAINTY:
THE HIGH-LOW SEARCH APPROACH**

Steve Alpern and Dennis J Snower



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ABSTRACT

Production Decisions under Demand Uncertainty: The High-Low Search Approach

The paper presents a model of 'high-low search' under uncertainty, in which a 'conservative' firm 'searches' for an unknown product demand by making a sequence of production decisions. After each production decision and the concomitant sales, the firm infers whether its supply is 'too high' or 'too low'. We show how the production decision reduces the firm's demand uncertainty interval and how this reduced uncertainty (in turn) affects its future production decisions.

JEL classification: 022, 026

Keywords: demand uncertainty, high-low search, inventories, production decisions

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NON-TECHNICAL SUMMARY

This paper is concerned with the way in which firms learn about the demand for their products, and introduces a new approach to the analysis of production decisions under demand uncertainty. In conventional analyses, firms are assumed to set their levels of production so as to maximize their profits, given the probability distribution of product demands which they face. In the Bayesian literature, firms begin with a probability distribution which reflects their 'prior' beliefs concerning the behaviour of product demand shocks. Firms base their production decisions on this distribution, after which they observe sales and update their prior distribution. In this framework the process of search for new information concerning demand is a sampling problem. By contrast, firms in our analysis do more than take random samples from the distribution of product demands. Instead, they use their production decisions to determine what sort of information about demands they will receive. Each production decision may be interpreted as an 'experiment' which is designed to generate new information concerning demand.

We assume that the firm does not produce to order and that the product demand which the firm faces is constant through time, so that a firm's current observation about product demand is relevant to its future production decisions. We analyse the firm's production decisions within a framework which captures the effects of past supply decisions on the information currently available about demand. We consider a very simple model in which prices do not play a role: demand is a given quantity which the firm attempts to determine. Initially the firm knows only that the demand for its product lies within a specified interval. We assume that the firm faces uncertainty rather than risk, in the sense that it does not know even the probability distribution of demand for its product. We picture the firm learning about the demand for its product by supplying output and observing how much of this output remains unsold. We call this behaviour 'high-low search' since the firm learns that its supply is 'too high' when the quantity sold falls short of the quantity put up for sale and inventories remain, and that its supply is 'too low' when its inventories are exhausted. When the supply is 'too high', the firm can infer the exact level of demand: demand is equal to the quantity sold. When the supply is 'too low', the firm is unable to infer the exact level of demand. In short, the information which the firm receives is asymmetric: positive inventories yield quantitative information about product demand whereas only qualitative information is available when inventories are exhausted.

We examine a minimax production strategy for the firm, one which minimizes the costs in the current and future time periods of overproduction and underproduction under the most adverse demand conditions. The firm chooses

a 'supply strategy', a sequence of quantities supplied in each time period. The quantities supplied follow this sequence until demand is known and thereafter supply is set equal to demand. Firms make their production decisions not only with a view to maximizing their current profits given their current information, but also with a view to improving their knowledge of demand so as to increase their profits in the future.

We derive a formula and numerical simulations which illustrate how the optimal sequence of supply decisions depends on the rate at which the firm discounts costs in future time periods and on the relative costs of overproduction and underproduction:

In the traditional microeconomic theory of production, the firm's information about demand is assumed to be independent of its supply decisions. By contrast, this paper argues that, in the presence of uncertainty, this independence cannot in general be upheld. When demand is uncertain, information about demand is revealed by the activities of supplying goods and observing how many of them are sold. If firms are aware of this, production decisions will be made with a view to revealing such information.

PRODUCTION DECISIONS UNDER DEMAND UNCERTAINTY:
THE HIGH-LOW SEARCH APPROACH

by

Steve Alpern* and Dennis J. Snower**

1. Introduction

This paper presents a new approach to the analysis of sequential production decisions under uncertain product demand. We analyze the behavior of a firm which learns about the unknown product demand it faces by observing the sales following its sequence of supply decisions. In other words, we picture the firm as "searching" for its product demand by supplying output and observing how much of this output remains after its sales have taken place. We call this behavior "high-low search", since the firm learns that its supply is "too high" when the quantity sold falls short of the quantity put up for sale and that its supply is "too low" when there is a stock-out. To fix ideas, we will assume that product demand is constant, but unknown to the firm. Thus, when supply is "too high", the firm can infer the exact level of demand (which is equal to the quantity sold). We will consider production strategies which not only use currently available "high-low" information about demand but moreover actively incorporate the anticipated acquisition of this feedback from sales.

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Our approach is to view supply levels as "guesses" about the level D of demand "hidden" by nature in a given interval. We employ techniques motivated by the mathematical theory of "search games", and in particular the theory of optimal high-low search, to solve a model of the firm's production problem under unknown demand.

In the traditional microeconomic theory of production, the firm's information about demand is assumed to be independent of its supply decisions. This is the case in both the standard deterministic models (where the firm knows the demand curve it faces and chooses the profit-maximizing point on this curve) and the standard stochastic models (where the firm's demand curve contains a random disturbance term, whose distribution is assumed to be known and independent of the quantities supplied). By contrast, this paper argues that in the presence of uncertainty (as distinguished from risk, where the densities of all relevant variables are known), the independence of demand information and supply decisions can generally not be upheld. The reason is that when demand is uncertain, information about this demand is revealed by the activities of supplying goods and observing how many of them are sold. If firms are aware of this, production decisions will be made with a view to such information revelation.

In a previous paper (Alpern and Snower (1986)), we considered the implications of this information revelation in the context of a Bayesian model of inventory holding, assuming a known distribution of demand. Here, however, we consider the problem of uncertainty rather than risk, as no prior demand distribution is assumed, we derive "conservative" production strategies for the firm.

The paper is organized as follows. In Section 2 we briefly describe how the firm's production problem is related to the mathematical theory of high-low search, which we survey. Section 3 formalizes our production model. In Section 4 we develop a recursive procedure, summarized in Theorems 1 and 2, for determining the firm's conservative (minimax cost) production levels. Section 5 examines the dependence of the optimal supply levels on the discount factor and the relative cost of overproduction to underproduction. In Section 6 we take a numerical economic example and investigate the comparative dynamics of production levels with respect to the product's production cost, price, holding cost and depreciation rate. Our conclusions are in Section 7.

2. High-low search

A search game is one in which a Searcher tries to find (geometrically coincide with) a Hider in minimum time or cost, subject to given motion and information constraints. (See Gal (1980)). In the problem of "high-low" search, the searcher (the firm, in this paper) wishes to locate a number, say D (for demand), which is hidden in a known set of real numbers H . The set H is an interval either of the discrete kind $\{1, 2, \dots, 43\}$ or a continuous one of type $[D, \bar{D}]$ (as in our model below). The searcher makes a sequence of guesses G_1, G_2, \dots in the interval H and after each guess is told whether $G_t < D$ (the guess is "too low") or $G_t > D$ (it is "too high"). In the case of a discrete interval $H = \{1, 2, \dots, n\}$, eventually some guess G_N equals D , and the time N needed to find D is taken to be the payoff. This game was solved in Johnson (1964) for small values of n , but the general problem is still open. An elegant general solution to this discrete problem was obtained

by Gal (1974) assuming a binary information feedback of $G_t \leq D$ or $G_t > D$ (stockout or surplus stock, in our model).

In the economic problem of this paper, the supplies G_1, G_2, \dots are interpreted as guesses about the demand D which is known only to be in a given interval $[D, \bar{D}]$. The aim of the firm, however, is not simply to find (or rather converge to) D as rapidly as possible. Rather, the firm wishes to minimize the opportunity costs arising from overproduction and underproduction relative to the "full information optimum" production level of D . The search game model closest to this problem is that of Baston and Bostock (1985). In their model the searcher picks a sequence of points G_1, G_2, \dots to find a point D hidden in the interval $[0, 1]$, subject to triadic high-low feedback ($G_t < D$, $G_t = D$, or $G_t > D$). Yet unlike our model, the cost function which the searcher tries to minimize is the "sum of errors" cost $C = \sum_{t=1}^{\infty} |G_t - D|$. The minimax pure search strategy determined by Alpern (1985) finds any point D with total cost of $C \approx .625$.

In this paper, the model of Baston and Bostock is modified, with respect to information and cost, so as to fit the supply problem faced by the firm described in the introduction. First, the information feedback must be made assymmetric: If G_t is less than or equal to D (stockout) then the searcher will be told this, as in the Baston-Bostock model (except he will no longer be told if he hits D exactly, since this too is a stockout). However if G_t exceeds D (excess supply) then the searcher will be told not only that $G_t > D$ but will be told the exact value of D (which the firm would know in this case by observing sales). The cost function must also be modified to

$C = \sum_{t=1}^{\infty} \alpha^{t-1} [\phi(D-G_t)^+ + \psi(G_t-D)^+]$, where $Z^+ = \max(Z, 0)$, α is the discount factor, ϕ is the unit opportunity cost of underproduction (i.e. the unit price

minus the unit cost), and ∇ is the unit opportunity cost of overproduction (which involves holding costs, depreciation costs and opportunity costs related to discounting, as shown in Section 5). If $\alpha = \phi = \psi = 1$ this cost function reduces to the "sum of errors" as in Baston and Bostock (1985).

3. The Production Model

In this section we describe a production model which captures the dependence of current demand information on past supply levels, as discussed in the introduction. Initially, all the firm knows about demand is that it lies within a specified demand uncertainty interval $[D, \bar{D}]$, where $\bar{D} > D$. At the beginning of the first time period - which we call the "morning" of period $t = 1$ - the firm produces an amount Q_1 . The quantity of output which the firm puts up for sale (the "supply", for short) is given by $G_1 = Q_1 + (1 - \delta) I_0$, where I_t is the inventory stock remaining at the end of period t , and δ is the rate of inventory depreciation. At the end of the first time period - which we call the "afternoon" of period $t = 1$ - sales occur. The amount sold is $Z_1 = \min(G_1, D)$. As a result, the inventory stock remaining at the end of the first period is $I_1 = G_1 - Z_1$. This temporal sequence of activities is repeated in all subsequent time periods.

The firm's information set J_t^M in the morning ("M") of period t , before the production level has been set, contains the initial demand uncertainty interval as well as all its previous supplies and inventory stocks:

$$J_t^M = [G_1, \dots, G_{t-1}; I_0, I_1, \dots, I_{t-1}; D, \bar{D}] .$$

Its information set J_t^A in the afternoon ("A") of period t , just before sales have taken place, contains G_t as well as J_t^M :

$$J_t^A = [C_1, \dots, C_t, I_0, \dots, I_{t-1}, \underline{D}, \bar{D}].$$

Now that we have described the dynamics and information transfers of the model, let us consider the firm's objectives. If the actual demand D were known to the firm, then of course profits would be maximized by setting every supply level C_t equal to D . In this case, there would never be any inventory stocks. Since demand is not known, there are opportunity costs of underproduction and overproduction, which we assume are linear. Normalizing costs, we let the unit cost of underproduction be unity and the unit cost of overproduction be b (where $b = \frac{\gamma}{\phi}$ is obtained as the ratio of overproduction to underproduction costs). Finally, let there be a discount factor α , $0 \leq \alpha \leq 1$, so that the cost function which the firm seeks to minimize is given by

$$(1) \quad C = \sum_{C_t < D} \alpha^{t-1} \cdot (D - C_t) + \sum_{C_t > D} \alpha^{t-1} \cdot b \cdot (C_t - D).$$

With the definition of the cost (1), the formal description of our model is complete. However, in order to identify a reasonable class of strategies available to the firm, we must mention the inferences about demand that the firm may make by observing (past) sales. If there is a stockout at the end of period 1 ($I_1 = 0$), then the firm infers that its supply is "too low", i.e. that $C_1 \leq D$. (Strictly speaking, the supply is "too low" or "just right".) Consequently the demand uncertainty interval is reduced from $[\underline{D}, \bar{D}]$ to $[C_1, \bar{D}]$, and the next supply level C_2 should be set somewhere in the latter interval. In fact, as shown below, C_2 should be set above C_1 . As long as stockouts occur, the supply levels increase and the demand uncertainty interval shrinks. If, in some first period N , a positive inventory stock is

left ($I_N > 0$), then the firm may infer the exact level of demand

$D = Z_N = G_N - I_N$. In all subsequent periods, the firm should meet this known demand exactly, $D = G_{N+1} = G_{N+2} = \dots$. (This is possible only if the N^{th}

period inventory stock can be exhausted by satisfying the demand in period $(N+1)$, in other words, $(1-\delta) \cdot I_N \leq D$. Otherwise, regardless of the production

level Q_{N+1} , the amount put up for sale in period $(N+1)$ would exceed demand:

$G_{N+1} = Q_{N+1} + (1-\delta) \cdot I_N \geq (1-\delta) \cdot I_N > D$. To ensure that any inventory stock

can be sold off in the next period, we assume that $\bar{D} \leq 2 \cdot \underline{D}$, which implies

that $(1-\delta) \cdot I_N \leq I_N = G_N - D \leq \bar{D} - \underline{D} \leq \underline{D} \leq D$.) Thus in all periods after N ,

no costs will be incurred as demand will be met exactly.

A "supply strategy" for the firm is a sequence $S = (S_1, S_2, \dots)$

with $\underline{D} \leq S_1 \leq S_2 \leq \dots \leq \bar{D}$. The set of all such strategies is denoted by \mathcal{S} .

The supply strategy S determines the supply sequence G_1, G_2, \dots according to the following rule:

$$(2a) \quad \text{Define } N = N(S, D) = \min \{t \mid S_t > D\}$$

$$(2b) \quad G_t = S_t \text{ for } t \leq N$$

$$(2c) \quad G_t = D \text{ for } t > N.$$

In other words the supply follows the levels prescribed by S until

demand is known (from surplus stock) and thereafter supply is set equal to

demand. Of course if $D \geq \sup S_t$ then $N = \infty$ and the exact level of demand is never determined.

The strategic form of the problem can now be stated. If the firm adopts the supply strategy S in \mathcal{S} and the exact level of demand is D , then the cost to the firm is given by

$$(3) \quad C(S, D) = \sum_{t=1}^{N-1} a^{t-1} (D - S_t) + a^N \cdot b(S_N - D),$$

where $N = N(S, D)$. (The sum is infinite and the overproduction term missing in the case $N = \infty$.) Of course there are strategies available to the firm other than those in \mathcal{J} , but it is clear that any strategy not in \mathcal{J} is dominated by a strategy in \mathcal{J} .

We call a strategy S in \mathcal{J} "terminating" if $S_t = \bar{D}$ for some finite t . For such a terminating strategy we define the termination time $T = T(S)$ to be the last period t such that $S_t < \bar{D}$ (so that $S_T < \bar{D} = \bar{S}_{T+1} = \bar{S}_{T+2} = \dots$). The termination time T is the maximum number of stockouts permitted by the supply strategy. (Note that $N-1$ is the actual number of stock-outs when demand is D and thus N is a function of S and D ; by contrast, T is the maximum number of stock-outs resulting from strategy S and thus T is a function of S only.)

4. A Conservative Strategy

In this section we derive the firm's optimal supply decisions (concerning the amounts it offers for sale through time) when it pursues a strategy of minimizing its opportunity cost under the most adverse demand conditions. Specifically, we derive a recursive formula for the conservative supply strategy $\bar{S} = (\bar{S}_1, \bar{S}_2, \dots)$ in \mathcal{J} satisfying

$$(4) \max_{0 \leq D \leq 1} c(\bar{S}, D) = \min_{S \in \mathcal{J}} \max_{0 \leq D \leq 1} c(S, D) = W,$$

where $c(S, D)$ is given by (3) and the demand interval is normalized to $[0, 1]$. (Of course, the actual level of demand must satisfy our condition $\bar{D} \leq 2 \cdot D$.) The minimax value W represents the smallest opportunity cost

which can be guaranteed by the firm. \bar{S} is the (unique) strategy which guarantees that the cost will not exceed W . We show that, for any values of the parameters a and b (which characterize the cost function $c(S,D)$), \bar{S} is unique and terminating.

Since the derivation of the formula for \bar{S} has a revealing economic interpretation, we will explain each step intuitively in terms of the firm's production and information-acquisition activities. For this purpose, it is convenient to assume that $a = 1$ (i.e., no discounting) so that (3) reduces to

$$(5) \quad c(S,D) = (D-S_1) + (D-S_2) + \dots + (D-S_{N-1}) + b \cdot (S_N - D) .$$

(The case concerning arbitrary a , $0 \leq a \leq 1$, will be covered later in Theorem 2.)

To derive the conservative strategy satisfying (4), we choose an arbitrary period of time $t=K+1$ and find the firm's optimal supply decision $G_{K+1} = \bar{S}_{K+1}$, assuming that all its previous supply decisions $G_1 = X_1, G_2 = X_2, \dots, G_K = X_K$ have resulted in stock-outs. (This is the only substantive problem to be solved since, as we have seen, once there is no stock-out, the firm proceeds to satisfy its demand exactly.) As noted above, the supply decision G_{K+1} is made in the morning of period $t=K+1$, when the relevant demand uncertainty interval is $[X_K, 1]$. To find the optimal quantity to be put up for sale ($G_{K+1} = \bar{S}_{K+1}$), the firm needs to consider the consequences of making any arbitrary supply decision $G_{K+1} = X$, where X lies within $[X_K, 1]$.

For this purpose, suppose that the firm now finds itself in the afternoon of period $t=K+1$, when its supply decision $G_{K+1} = X$ has already

been made, but sales have not yet taken place. The set of strategies available to the firm is

$$(6) \mathcal{S}_{K+1}(X_1, \dots, X_K, X) = \{S \in \mathcal{D}: S_1 = X_1, \dots, S_K = X_K, S_{K+1} = X\}.$$

The least cost which the firm can guarantee through optimal supply decisions in the future (viz, from the morning of period $K+2$ onwards) is

$$(7) C_{K+1}^A(X_1, \dots, X_K, X) = \min_{S \in \mathcal{S}_{K+1}(X_1, \dots, X_K, X)} \max_{X_K \leq D \leq 1} c(S, D).$$

Observe that if the firm sets $G_{K+1} = X$ at its optimal level ($X = \bar{S}_{K+1}$), then the optimal cost on the afternoon of period $K+1$ ($C_{K+1}^A(X_1, \dots, X_K, X)$) must be equal to the corresponding optimal cost on the morning of period $K+1$ (before G_{K+1} has been set). We define the latter cost as follows:

$$(8) C_{K+1}^M(X_1, \dots, X_K) = \min_{S \in \mathcal{S}_K(X_1, \dots, X_K)} \max_{X_K \leq D \leq 1} c(S, D)$$

which is the least cost which the firm can guarantee by optimal supply decisions from the morning of period $K+1$ onwards. In other words, this cost is

$$(9) C_{K+1}^M(X_1, \dots, X_K) = \min_{X_K \leq X \leq 1} C_{K+1}^A(X_1, \dots, X_K, X).$$

At this point it is useful to interpret definitions (7) and (8) and their relation (9) in terms of the following two person zero-sum game of perfect information. The firm (the minimizer) sets a supply level G_1 and then Nature (the maximizer) either says "stockout" or reveals a demand D in the interval $(G_1, 1)$. As soon as the demand is revealed, the game stops with payoff given by (1). Otherwise the firm sets a new supply level

$C_2, C_2 \geq C_1$, and again Nature either says "stockout" or reveals a demand. In the case of non-termination, demand D is taken to be 1 in the payoff formula

(1). All nonterminal "nodes" for the minimizing firm are of type

$[X_1, \dots, X_K]^M$ interpreted as the situation in the morning of the $K+1$ day after supply levels $G_1 = X_1, \dots, G_K = X_K$ have resulted in stockout.

Nonterminal nodes of Nature are of the form $[X_1, \dots, X_K, X]^A$, interpreted as the situation faced by Nature (or buyers) on the afternoon of the $K+1$ 'st day, after supply levels X_1, \dots, X_K resulted in stockouts, and after a current supply of X is offered for sale. The game-tree structure is pictured in

Figure 1. In a finite two-person zero sum game of perfect information, all

nodes may be recursively assigned values. In this analogy

$C_{K+1}^A(X_1, \dots, X_K, X)$ (of Equation (7)) is the value assigned to Nature's node $[X_1, \dots, X_K, X]^A$ and $C_{K+1}^M(X_1, \dots, X_K)$ (of equation (8)) is the value assigned to the Firm's node $[X_1, \dots, X_K]^M$. Similarly, the fact that a

minimizer's node is always valued at the minimum value of its successor nodes, is seen in Equation (9). The reason our solution is somewhat more complicated than in the finite graph analogy is that paths may be nonterminating

($N = \infty$) and the cardinality of successors to a node is the cardinality of the continuum.

Substituting (5) into (8) gives

$$(10a) \quad C_{K+1}^M(X_1, \dots, X_K)$$

$$= \min_{S \in \mathcal{D}_K(X_1, \dots, X_K)} \max_{X_K \leq D \leq 1} \left[\sum_{t=1}^K (D - X_t) + \sum_{t=K+1}^{N-1} (D - S_t) + b(S_N - D) \right]$$

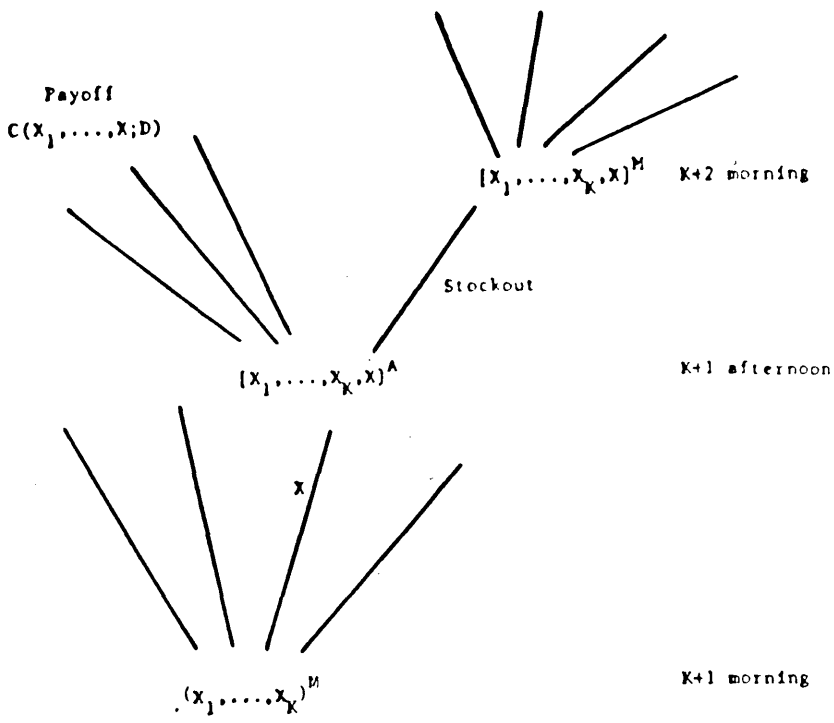


Figure 1: The Game-tree Structure

$$\begin{aligned}
 = \min_{S_t} \max_{X_K \leq D \leq 1} & \left\{ \sum_{t=1}^K (X_K - X_t) + \sum_{t=1}^K (D - X_K) \right. \\
 & \left. + \left[\sum_{t=K+1}^{N-1} (D - S_t) + b \cdot (S_N - D) \right] \right\}
 \end{aligned}$$

(where we recall that past decisions are denoted by X_t , $t=1, \dots, K$, and the future decisions are denoted by S_t , $t=K+1, \dots, N$).

This cost is illustrated as the sum of the lengths of the line segments in Figure 2a. The line segment for $t=1$ stands for $(D-X_1)$, the next for $(D-X_2)$, and so on. The segment at $t=N$ is doubled to describe the case where $b=2$.

The right-hand side of the last equation of (10a) consists of three terms, each describing a different cost resulting from the firm's temporal sequence of decisions. The first term, which we denote by

$\sigma_K = \sum_{t=1}^K (X_K - X_t)$, may be called the "past cost of past decisions". This is a sunk cost, illustrated by the sum of the dotted line segments in Figure 2b. It arises because, on the morning of period $t=K+1$, the firm knows that it has already made the supply decisions X_1, \dots, X_K and that demand (D) must lie in the interval $[X_K, 1]$. Thus, the opportunity cost which has already accrued as result of these past decisions is σ_K .

The second term, $\sum_{t=1}^K (D - X_K) = K \cdot (D - X_K)$, is the "future cost of past decisions". This arises because in the future (viz, after period $t=K+1$) the firm may discover that demand lies above the lower bound of the interval $[X_K, 1]$. In that event, the firm's past decisions X_1, \dots, X_K will turn out to be more costly than σ_K (the first term). The second term is illustrated by the sum of the dashed line segments in Figure 2b.

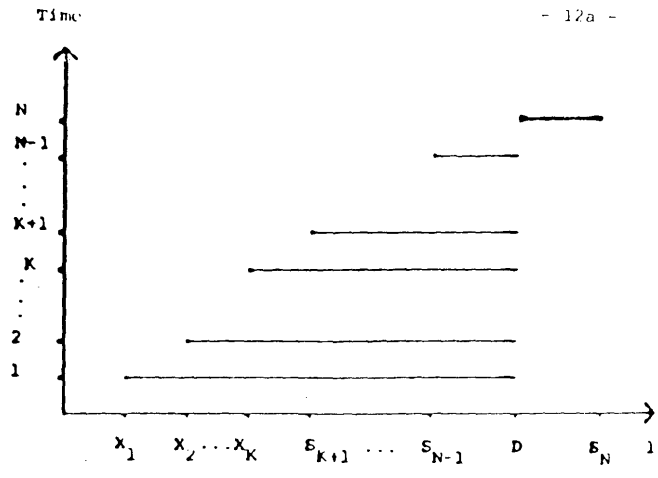


Figure 2a

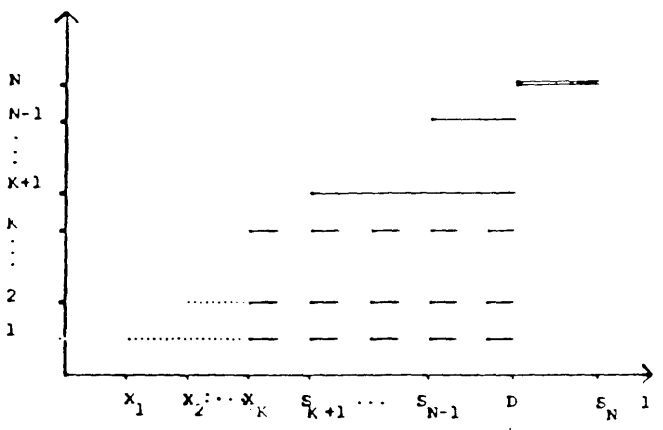


Figure 2b

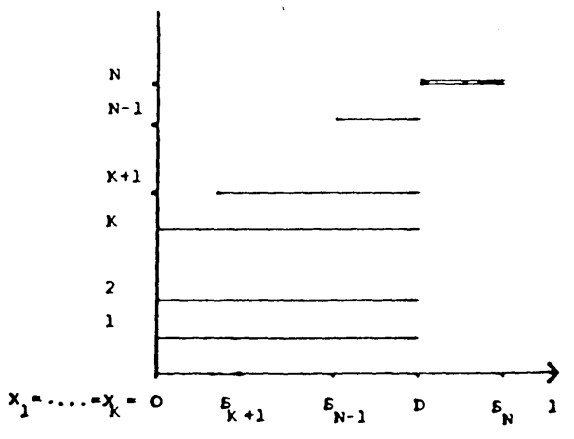


Figure 2c

Finally, the third term $\left[\sum_{t=K+1}^{N-1} (D-S_t) + b \cdot (S_N - D) \right]$, is the "future cost of future decisions". This is the sum of the discrepancies between each of the future decisions $S_{K+1}, S_{K+2}, \dots, S_N$ and the demand D . It is given by the sum of the solid line segments in Figure 2b.

In short, the optimal cost (10a) on the morning of period $K+1$ may be rewritten as follows:

$$(10b) \quad C_{K+1}^M(x_1, \dots, x_K) = o_K + \min_{S \in \mathcal{S}_K(x_1, \dots, x_K)} \max_{x_K \leq D \leq 1} \left\{ \sum_{t=1}^K (D - x_K) + \left[\sum_{t=K+1}^{N-1} (D - S_t) + b \cdot (S_N - D) \right] \right\}$$

We now rescale the demand uncertainty interval $[x_K, 1]$ linearly to the original demand interval $[0, 1]$, so that the second and third right-hand terms above (illustrated by the dashed and solid line segments in Figure 2b) are magnified to extend from 0 to 1 (as pictured by the line segments in Figure 2c). The rescaling function is $g(y) = (y - x_K)/(1 - x_K)$, where y is any point in the interval $[x_K, 1]$ and $g(y)$ is the corresponding point in $[0, 1]$. Defining $D' = g(D)$ and $S'_t = g(S_t)$, the optimal cost (10b) becomes

$$(10c) \quad C_{K+1}^M(x_1, \dots, x_K) = o_K + (1 - x_K) \cdot \min_{0 \leq S'_{K+1} \leq S'_{K+2} \leq \dots} \max_{0 \leq D' \leq 1} \left\{ \sum_{t=1}^K (D' - 0) + \sum_{t=K+1}^{N-1} (D' - S'_t) + b \cdot (S'_N - D') \right\} = o_K + (1 - x_K) \cdot \min_{0 = S'_1 = \dots = S'_K \leq S'_{K+1} \leq \dots} \max_{0 \leq D' \leq 1} \left\{ \sum_{t=1}^K (D' - S'_t) \right\}$$

$$\begin{aligned}
 & + \sum_{t=K+1}^{N-1} (D'_t - S'_t) + b \cdot (S'_N - D'_t) \} \\
 & = \sigma_K + (1-X_K) \cdot \min_{S'_t \in D_K(0, \dots, 0)} \max_{0 \leq D'_t \leq 1} c(S', D') \\
 & = \sigma_K + (1-X_K) \cdot C_{K+1}^M(0, \dots, 0) \quad \text{by (8)}
 \end{aligned}$$

In other words, the optimal cost on the morning of period $K+1$, conditional on the past decisions X_1, \dots, X_K , has two components: (i) the sunk cost σ_K and (ii) the optimal future cost (of past and future decisions). These latter costs are identical to what the optimal cost (rescaled by $(1-X_K)$) would have been if the firm had failed to supply anything in the first K periods (as shown in Figure 2c). Let $W_K = C_{K+1}^M(0, \dots, 0)$ be the optimal future cost, conditional on the first K supplies having been set at zero. Then (10c) becomes

$$(10d) \quad C_{K+1}^M(X_1, \dots, X_K) = \sigma_K + (1-X_K) \cdot W_K.$$

By (4) and (10d), we find that the optimal supply decision $C_{K+1} = X^*$ must satisfy $C_{K+1}^A(X_1, \dots, X_K, X^*) = \sigma_K + (1-X_K) \cdot W_K$.

We are now in a position to derive this optimal supply decision. On the afternoon of period $K+1$ (after an amount X has been supplied but before sales have taken place), the firm anticipates that two things may happen:

- (a) There may be a stock-out ($I_{K+1} = 0$), from which the firm infers that demand must be at least as large as the quantity put up for sale ($X \leq D \leq 1$). In this case, the least cost which the firm can

guarantee through optimal supply decisions from period $K+2$ onwards is

$$\begin{aligned}
 (11) \quad C_{K+1}^A(X_1, \dots, X_K, X; I_{K+1} = 0) &= \min_{S \in \mathcal{J}_{K+1}} \max_{X \leq D \leq 1} c(S, D) \\
 &= C_{K+2}^H(X_1, \dots, X_K, X) \\
 &= \sigma_{K+1} + (1 - X) \cdot w_{K+1} \quad \text{by (8) and (10d)} \\
 &= \sigma_K + K(X - X_K) + w_{K+1}(1 - X)
 \end{aligned}$$

since $\sigma_{K+1} = \sigma_K + K(X - X_K)$

(b) A positive inventory stock may be left over ($I_{K+1} > 0$), from which the firm infers the exact level of demand D in $[X_K, X]$. In this case $N = K+1$. Here, the least cost which the firm can guarantee through optimal future decisions (meeting demand exactly) is

$$C_{K+1}^A(X_1, \dots, X_K, X; I_{K+1} > 0) = \min_{S \in \mathcal{J}_{K+1}} \sup_{X_K \leq D \leq X} c(S, D)$$

where $c(S, D) = \sum_{t=1}^K (D - X_t) + b \cdot (X - D)$

$$\begin{aligned}
 &= \sigma_K + K(D - X_K) + b(X - D) \\
 &= \sigma_K + D \cdot (K - b) - K \cdot X_K + b \cdot X.
 \end{aligned}$$

Observe that the value of the worst-possible demand depends on whether $(K-b)$ is positive or negative. If $K < b$ (so that an increase in D raises the future cost of underproduction,

$\sum_{t=1}^K (D-X_t) = K \cdot D + \sum_{t=1}^K X_t$, by less than it reduces the future cost of overproduction, $b \cdot (X-D)$, then the demand which maximizes the cost function is the lowest demand $D = X_K$. Thus, the cost function becomes

$$(12i) \quad C_{K+1}^A (X_1, \dots, X_K; I_K > 0) = \sigma_K + b \cdot (X - X_K) \quad \text{if } K < b.$$

On the other hand, if $K \geq b$ (so that demand raises the future cost of overproduction by at least as much as it reduces the future cost of underproduction, the worst-possible demand is the lowest demand; $D=X$. In this case, the cost function is

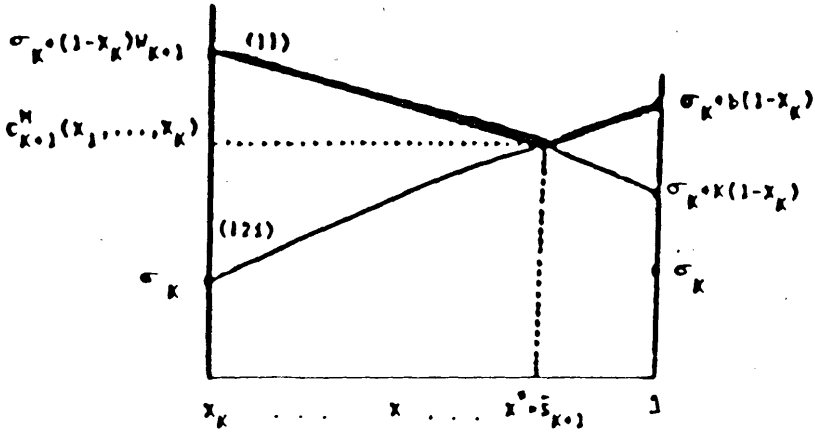
$$(12ii) \quad C_{K+1}^A (X_1, \dots, X_K; I_K > 0) = \sigma_K + K \cdot (X - X_K) \quad \text{if } K \geq b$$

The optimal supply decision $C_{K+1}^* = X^* = \bar{S}_{K+1}$ is the one which minimizes the maximum of the two costs (11) and (12 i or ii) above:

$$(13) \quad \min_{X_K \leq X \leq 1} C_{K+1}^A (X_1, \dots, X_K, X) \\ = \min_{X_K \leq X \leq 1} \max [C_{K+1}^A (X_1, \dots, X_K, X; I_K = 0), \\ C_{K+1}^A (X_1, \dots, X_K, X; I_K > 0)].$$

Figure 3i deals with the case in which $K < b$, where (13) is evaluated using (11) and (12i), as

Figure 3ii: $k < b$

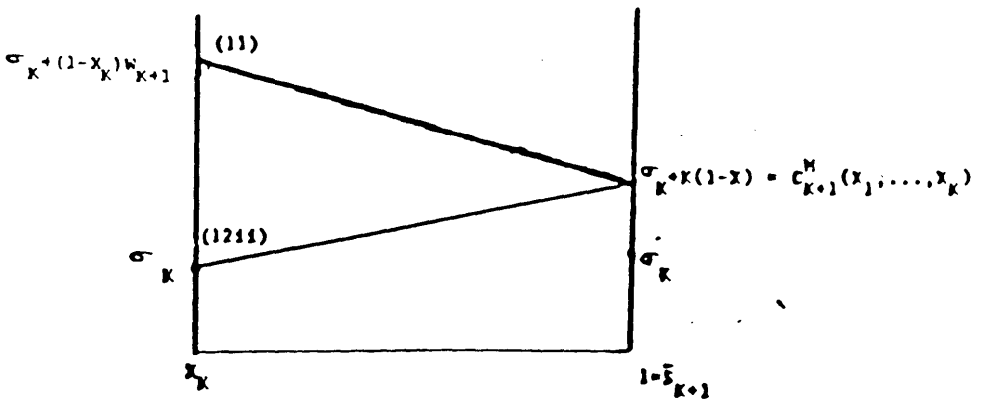


$$(11) \quad Y = \sigma_K + k(x - x_K) + w_{K+1}(1 - x)$$

$$(12i) \quad Y = \sigma_K + b(x - x_K)$$

$$(12ii) \quad Y = \sigma_K + k(x - x_K)$$

Figure 3iii: $k \geq b$



$$(14) \min_{X_K \leq X \leq 1} C_{K+1}^A(X_1, \dots, X_K, X)$$

$$= \min_{X_K \leq X \leq 1} \max \left\{ \left[\sigma_K + K \cdot (X - X_K) + (1 - X) \cdot W_{K+1} \right], \right.$$

$$\left. \left[\sigma_K + b \cdot (X - X_K) \right] \right\}.$$

The maximum of the two costs is illustrated by the thick line segments in the Figure 3. The supply decision $C_{K+1}^A = X$ which minimizes this maximal cost is illustrated by the value of $X = \bar{S}_{K+1}$ which lies at the intersection of the two cost functions (11) and (12i):

$$(15) X = \bar{S}_{K+1} = \frac{W_{K+1} + (b - K) \cdot X_K}{b - K + W_{K+1}}$$

The height of this intersection ($\min_{X_K \leq X \leq 1} C_{K+1}^A(X_1, \dots, X_K, X)$, which is the optimal afternoon cost in period $K+1$, given that the firm has made the optimal supply decision $X = \bar{S}_{K+1}$) is identical with the optimal morning cost in period $K+1$ (by 9) and its value (substituting (15) into (14)) is

$$(16) C_{K+1}^M(X_1, \dots, X_K) = \sigma_K + \frac{b \cdot W_{K+1} \cdot (1 - X_K)}{b - K + W_{K+1}}$$

Equating the formulae for $C_{K+1}^M(X_1, \dots, X_K)$ of (10d) and (16) and solving for W_K , we obtain

$$(17) W_K = \frac{b \cdot W_{K+1}}{b - K + W_{K+1}} \quad \text{for } K < b$$

Figure 3b deals with the case in which $K \geq b$.

Now (13) becomes (using 12ii instead of 12i)

$$(18) \quad C_{K+1}^M(X_1, \dots, X_K) = \min_{X_K \leq X \leq 1} \max \{ (\sigma_K + K \cdot (X - X_K) + (1-X) \cdot W_{K+1}), \\ (\sigma_K + K \cdot (X - X_K)) \}$$

The maximum cost is minimized when

$$(19) \quad X = \bar{S}_{K+1} = 1 \quad \text{for } K \geq b.$$

Thus,

$$(20) \quad C_{K+1}^M(X_1, \dots, X_K) = \sigma_K + K \cdot (1 - X_K) \quad \text{for } K \geq b.$$

Equating (10d) and (20), and solving for W_K yields

$$(21) \quad W_K = K \quad \text{for } K \geq b.$$

In sum, the firm's optimal supply decision in any time period, assuming that all previous decisions have resulted in stock-outs, may be derived as follows: As shown in Figure 2b, each past decision is associated with a future cost of underproduction (given by the dashed line segments) which depends positively on demand. The greater the number (K) of past decisions which the firm has made, the more a given increase in demand will raise the overall future cost of underproduction. When K is sufficiently large, the effect of demand on the overall future cost of underproduction is greater than the effect on the cost of overproduction. In that event, the

worst-possible demand is $D=1$ and thus the associated optimal supply decision is $\bar{S}_{K+1} = 1$. Thus, \bar{S} is a terminating strategy, with some finite termination time $T = T(\bar{S})$. By (21), we then can find the associated optimal cost W_K . By (17), we obtain the previous optimal cost W_{K-1} and, by (15) we thereby find the previous optimal supply decision \bar{S}_K , and so on.

These results are summarised in the following theorem.

Theorem 1: When there is no discounting ($\alpha = 1$) the unique solution to the problem

$$\max_{0 \leq D \leq 1} C(\bar{S}, D) = \min \max_{S \in \mathcal{S} \ 0 \leq D \leq 1} C(S, D) = W$$

is computed as follows:

First the optimal costs $W = W_0, W_1, W_2, \dots$, may be recursively computed (downwards) by the formulae

$$(21) \quad W_K = K \quad \text{for } K \geq b, \text{ and}$$

$$(17) \quad W_K = b \cdot W_{K+1} / (b - K + W_{K+1}) \quad \text{for } K < b.$$

Once the costs W_K are known, the conservative supply levels $\bar{S}_1, \bar{S}_2, \dots$ may be recursively computed (upwards) by letting $X_K = \bar{S}_K$ and using (15):

$$(15) \quad \bar{S}_1 = W_1 / (b + W_1)$$

$$\bar{S}_{K+1} = \frac{W_{K+1} + (b-K)\bar{S}_K}{b - K + W_{K+1}} \quad 1 < K < b,$$

$$(19) \quad \bar{S}_K = 1 \quad \text{for } K \geq b.$$

Thus, the termination time $T = T(\bar{S})$ is such that $T+1 = \min\{i: i \geq b\}$.

It should be noted in particular that the conservative strategy \bar{S} sets supply equal to the maximum possible level of demand if stockouts have occurred for more than b periods.

The analysis presented above for the case of no discounting ($\alpha = 1$) can be generalized to cover any discount rate α , $0 \leq \alpha \leq 1$. No new ideas are needed, although the algebra becomes more complicated. The solution in the general case is given below.

Theorem 2: For any discount rate α and over- to under-production cost ratio b , with $0 \leq \alpha \leq 1$ and $0 \leq b < \infty$, there is a unique solution to

$\max_{0 \leq D \leq 1} C(\bar{S}, D) = \min_{S \in \mathcal{S}} \max_{0 \leq D \leq 1} C(S, D) = W$. The conservative strategy \bar{S} may be

computed as follows: For $K = 0, 1, 2, \dots$, define γ_K by $\gamma_0 = 0$ and, for

$K \geq 1$, $\gamma_K = 1 + \alpha + \dots + \alpha^{K-1} = (1 - \alpha^K) / (1 - \alpha)$. (Note that for $\alpha = 1$,

$\gamma_K = K$, for comparison with Theorem 1.) The strategy \bar{S} will have a

termination time T such that $T+1 = \min \{i: \gamma_i \geq \alpha^i b\}$ which is always finite

(set always nonempty). The optimal costs $W = W_0, W_1, W_2, \dots$ can be

recursively computed (downwards) by

$$W_K = \gamma_K \quad \text{if } K > T$$

$$W_K = \frac{\alpha^K b W_{K+1}}{b \alpha^K - \gamma_K + W_{K+1}} \quad \text{if } K \leq T.$$

Then the optimal supply levels $\bar{S}_1, \bar{S}_2, \dots$ may be determined recursively upwards by

$$\bar{S}_1 = W_1 / (b + W_1)$$

$$\bar{S}_{K+1} = \frac{W_{K+1} + (a^K b - \gamma_K) \bar{S}_K}{b a^K - \gamma_K + W_{K+1}} \quad \text{for } K \leq T$$

$$\bar{S}_K = 1 \quad \text{for } K > T.$$

Proof of Theorem 2: Put briefly, the proof with discounting is obtained from the proof without discounting by attaching a factor a^{t-1} to every term with subscript t . For example, the sunk cost σ_K is now a function of

$$\sigma_K = \sum_{t=1}^K a^{t-1} (X_K - X_t). \quad \text{Similarly, the "future cost of past decisions" is}$$

$$\sum_{t=1}^K a^{t-1} (D - X_K) = (1 + a + \dots + a^{K-1}) (D - X_K) = \gamma_K (D - K)$$

rather than $K \cdot (D - K)$. The equations from which the recursion formulae are obtained have the form

$$(11)' \quad C_{K+1}^A(X_1, \dots, X_K, X; I_{K+1} = 0) = \sigma_K + \gamma_K (X - X_K) + W_{K+1} (1 - X)$$

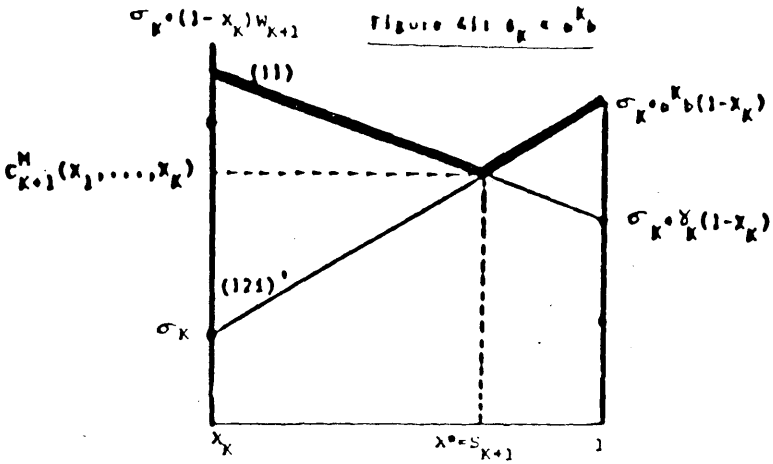
$$(12i)' \quad C_{K+1}^A(X_1, \dots, X_K, X, I_{K+1} > 0) = \sigma_K + a^K \cdot b \cdot (X - X_K) \quad \text{if } \gamma_K < a^K b$$

$$(12ii)' \quad C_{K+1}^A(X_1, \dots, X_K, X, I_{K+1} > 0) = \sigma_K + \gamma_K (X - X_K) \quad \text{if } \gamma_K \geq a^K b.$$

These lines are drawn in figures 4i and 4ii.

5. Analysis of the Conservative Strategy

The recursive formulae of Theorem 2 enable us to compute the conservative strategy $\bar{S} = (\bar{S}_1, \bar{S}_2, \dots)$ in terms of the two parameters

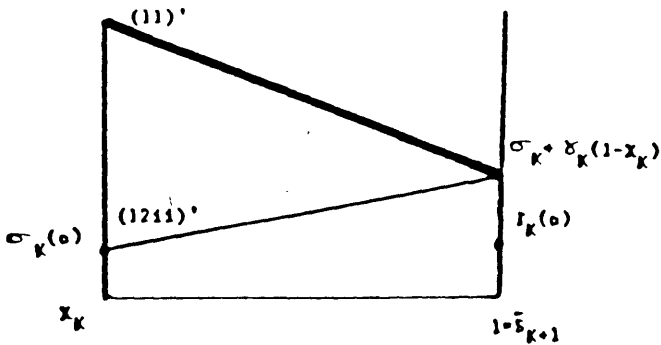


$$(11)' \quad Y = \sigma_K + a^K b (x - x_K) \cdot w_{K+1} (1 - x)$$

$$(12i)' \quad Y = \sigma_K + \delta_K (x - x_K)$$

$$(12ii)' \quad Y = \sigma_K + \delta_K (x - x_K)$$

Figure 4iii: $\delta_K \geq a^K b$



$\delta_K \cdot (1 + a + a^2 + \dots + a^{K-1})$, so that $\delta_K = K$ when $a=1$

$$\sigma_K(x_1, \dots, x_K) = \sum_{t=1}^K a^{t-1} (x_K - x_t)$$

α (discount factor) and b (overproduction-underproduction ratio). In this section we examine what the strategy \bar{S} looks like and analyze the dependence of T (termination time) and each \bar{S}_i on α and b .

The termination times $T = T(\alpha, b)$ partition the $\alpha - b$ plane as shown in Figure 5. The lines L_i are drawn according to the equation

$$b = \gamma_i / \alpha^i = (1 + \alpha + \alpha^2 + \dots + \alpha^{i-1}) / \alpha^i.$$

Below the line $L_1 = L_1(\alpha) = 1/\alpha$, the conservative strategy is to supply at maximum level, i.e., 1, from the start. That is, $\bar{S} = (1, 1, \dots)$, and hence $T=0$. For all points above L_1 but on or below L_2 , T equals 1. (In other words, if a stock-out occurs in the first period, then the second period supply is set at the maximum level.)

Similarly for all points above L_{i-1} but on or below L_i , $T = i-1$. The conservative strategy may be described algebraically for small values of T , using the formulae of Theorem 2. When $T=1$ we calculate that

$$W_1 = \gamma_1 = 1 \text{ and } W_0 = (\alpha^0 \cdot b \cdot W_1) / (W_1 + b \cdot \alpha^0 - \gamma_0) = b \cdot (1+b)^{-1}.$$

$$\bar{S}_1 = W_0 / \alpha^0 \cdot b = (1+b)^{-1}.$$

Since $T = 1$, it follows that $\bar{S}_t = 1$ for $t > 1$.

It is instructive to consider $c(\bar{S}, D)$ as a function of D in this case. For $D < (1+b)^{-1}$ there is overproduction of $(1+b)^{-1} - D$ in the first period, and no further costs. So $C(\bar{S}, D) = b \cdot ((1+b)^{-1} - D)$. For

$D \geq (1+b)^{-1}$ there is underproduction of $D - (1+b)^{-1}$ in the first period and overproduction of $1-D$ in the second period, and a total cost of

$$(D - (1+b)^{-1}) + \alpha b (1-D),$$

as shown in Figure 6. The assumption that $T = 1$ guarantees that the line above $[(1+b)^{-1}, 1]$ is upward sloping, since its slope is $1 - \alpha b$, which is at least one.

For large values of T it is not feasible to derive rational functions for \bar{S}_t in terms of α and b , but it is an easy matter to compute the optimal supply levels \bar{S}_t numerically. This is done in Figure 7 for

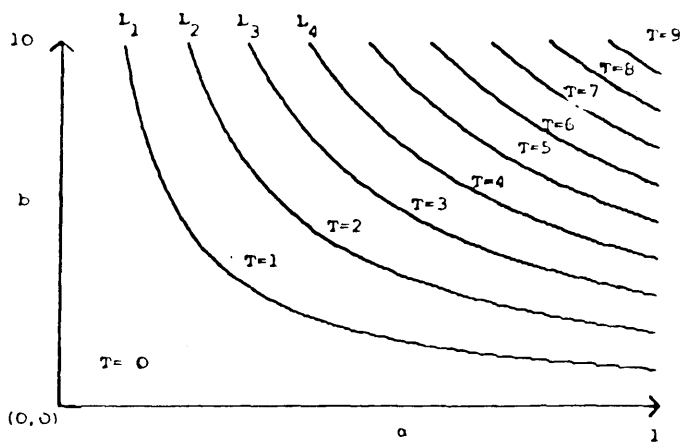


Figure 5

$$T(\alpha, b) = \min \left\{ i : \sum_i \geq \alpha^i \cdot b \right\}$$

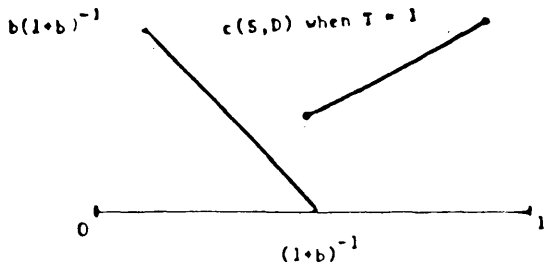
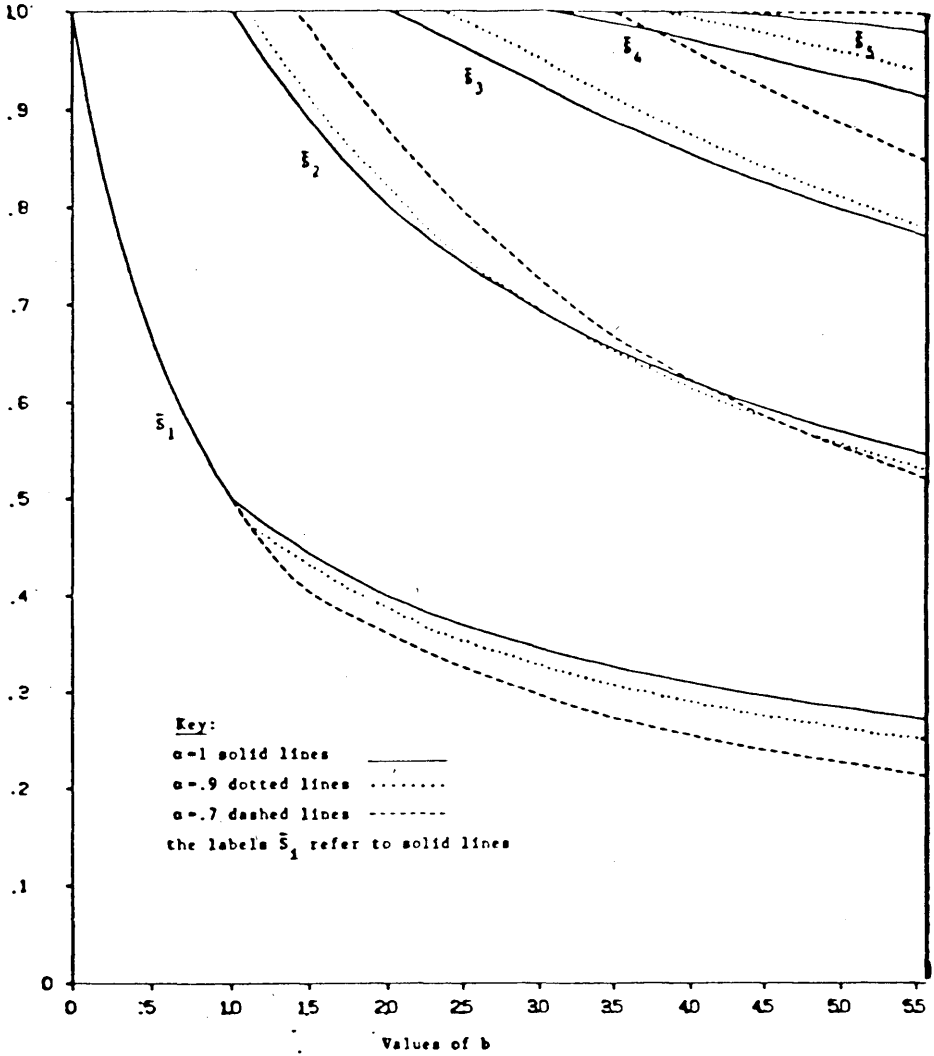


Figure 6

$\alpha = 1$, $\alpha = .9$ and $\alpha = .7$. Observe that for fixed α and t , the t 'th supply level \bar{S}_t is decreasing in b : the higher the relative cost of overproduction to underproduction, the lower the amount produced.

The dependence of \bar{S}_t on α is more subtle. When $t = 1$, the first supply level \bar{S}_1 is increasing in α . This is easy to explain. The higher α , the greater the relative importance of future costs, so the greater the incentive to gain information about demand. Higher supply levels give more information about demand since they are more likely to yield a surplus (thus revealing demand exactly) and in the event of a stockout give a higher lower bound on demand. Thus \bar{S}_1 is increasing in α . For $t \geq 2$ the dependence of \bar{S}_t on α has an additional "hedging" component. Consider the second supply level \bar{S}_2 , assuming the initially supplied level \bar{S}_1 resulted in a stockout. The demand interval is $[\bar{S}_1, 1]$ and there has already been one "bet" made (namely \bar{S}_1) that demand is "low" (\bar{S}_1 is the least point in the current demand interval). Since the strategy \bar{S} is conservative, i.e., minimax, the second guess \bar{S}_2 must "hedge" against the low guess \bar{S}_1 . The bigger the size of the first bet \bar{S}_1 , the higher the value of \bar{S}_2 must be. The "size" of the "bet" \bar{S}_1 decreases as α increases, since the relative importance of first period costs decreases with α . Thus the "hedging factor" causes all guesses \bar{S}_t except \bar{S}_1 to decrease with respect to α . The interplay of the information-gathering motive and the hedging motive means that \bar{S}_t , $t \geq 2$, may be increasing or decreasing in α , as shown in Figure 7. (The greater the number $t-1$ of previous low guesses, the more significant the hedging factor).

Figure 7 Graphs of \bar{E}_i as a function of b, for $\alpha = 1, .9, .7$



6. Numerical Example

Suppose the firm sells a perfectly durable good ($\delta = 0$) which costs 8 to produce and sells for 10. The cost of holding inventory from one period to the next is 2 per unit. The discount factor α is $1/2$. Initially, all the firm knows about demand is that it lies in the interval $[20, 31]$.

In the above example the opportunity cost of a unit of underproduction is simply the unit (foregone) profit $10 - 8 = 2$. A unit of overproduction has two associated costs. There is a holding cost of 2 and a discounting cost of $(1 - (1/2))8 = 4$ (which represents the savings that could have been achieved by producing the extra unit in the next period which is discounted). This gives a total overproduction cost of 6 and hence a ratio of over- to under-production costs of $b = 3$. Thus the parameters of this example are $\alpha = 1/2$ and $b = 3$.

To calculate the optimal supply sequence \bar{S} for this example we use the formulae of Theorem 2 with $\alpha = 1/2$ and $b = 3$. First observe that the optimal strategy has a termination time $T=2$ since $(\alpha=1/2, b=3)$ lies above L_1 and below L_2 in Figure 5. Thus we begin the recursion with

$W_2 = \gamma_2 = 1 + \alpha$, $W_1 = b(1 + \alpha)/(1 + b)$ and $W_0 = b(1 + \alpha)/(2 + \alpha + b)$. For $t > 2 = T$, we have $\bar{S}_t = 1$. The remaining two supply levels are computed from Theorem 2 as

$$(13a) \quad \bar{S}_1 = (1 + \alpha)/(2 + \alpha + b)$$

and then

$$(13b) \quad \bar{S}_2 = (1 + \alpha)^2 / \alpha(2 + \alpha + b).$$

Since $\alpha = 1/2$ and $b = 3$ this gives a first-period supply level of $\bar{S}_1 = 3/11$ and (in the event of a stock-out) a second-period supply level of $\bar{S}_2 = 9/11$ and a worst-case cost of $W = W_0 = 9/11$. If the normalized interval $[0, 1]$ is linearly rescaled to the actual demand interval $[20, 31]$ this gives a supply

strategy $\bar{S} = (23, 29, 31, 31, \dots)$ and a worst-case cost of 18.

The calculation of total costs for a firm employing this supply strategy \bar{S} at certain critical demands levels is shown in Table 1, where overproduction is indicated by square brackets and under-production by round ones. The resulting graph of $C(\bar{S}, D)$ for D in $[20, 31]$ is pictured in Figure 8. Observe that $C(\bar{S}, D)$ is piecewise linear with each piece having a maximum of 18. Any other supply strategy S will have a maximum cost strictly higher than 18.

The ratio (b) of over- to under-production costs may be specified in terms of more basic economic parameters. In particular, if the production cost is Z , the selling price is P , the holding cost is h , then the ratio b may be expressed as

$$(14) \quad b = [h + (1-a \cdot (1-\delta)) \cdot Z] / (P-C).$$

It is interesting to observe how the two nontrivial supply levels

\bar{S}_1 and \bar{S}_2 vary as the parameters p , h , δ , Z and a are changed individually in a neighbourhood of the numerical values in our example. These comparative statics are graphed in Figure 9, with the parameter value of the example underlined, and the two optimal supply levels of the example denoted by A and B. In each case, the domain of the parameter is taken to be the maximal interval such that the termination time T remains, as in the example, at 2. Hence, Equations (13a) and (13b) are relevant.

Figure 9i shows the firm's "supply curves" $\bar{S}_1(P)$ and $\bar{S}_2(P)$, giving the conservative supply levels in the first two periods (assuming stockout in period 1) in terms of an exogenously given price. These levels increase with price because opportunity costs of underproduction increase with price. An increase in holding cost or depreciation rate increases the cost of

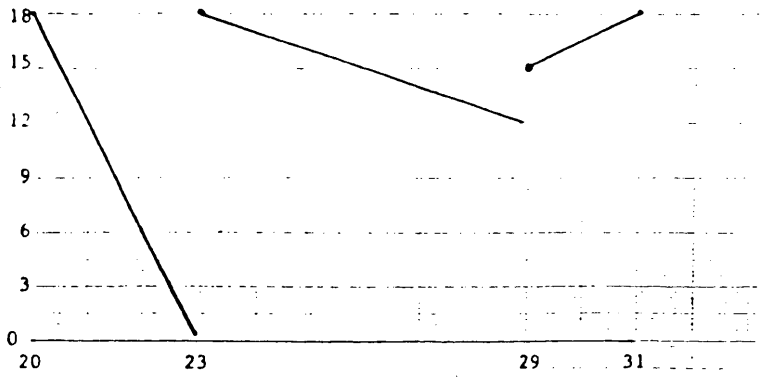


Figure 8: $c(S,D)$

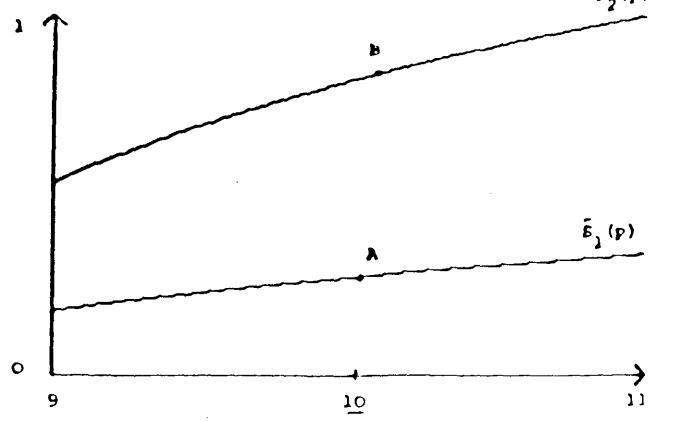


Figure 9i

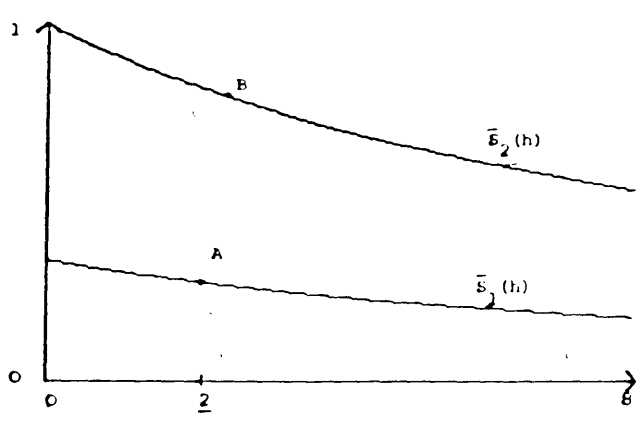


Figure 9ii

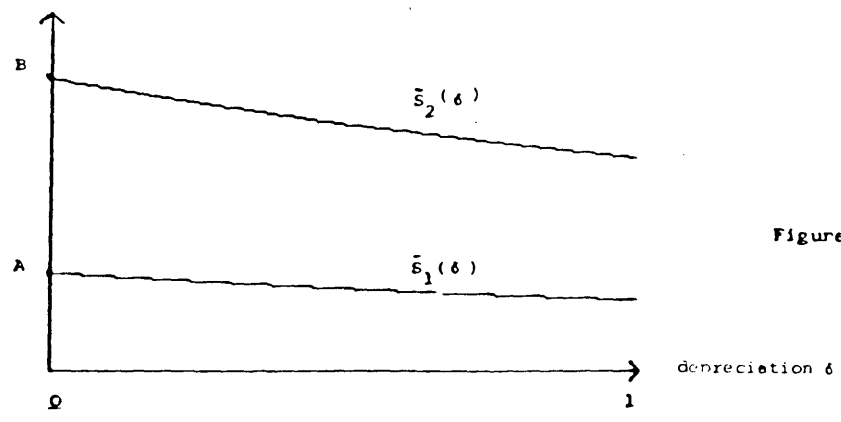


Figure 9iii

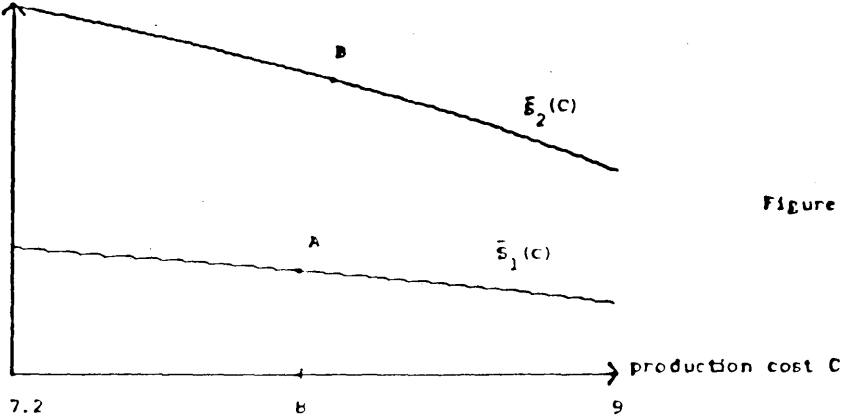


Figure 9 iv

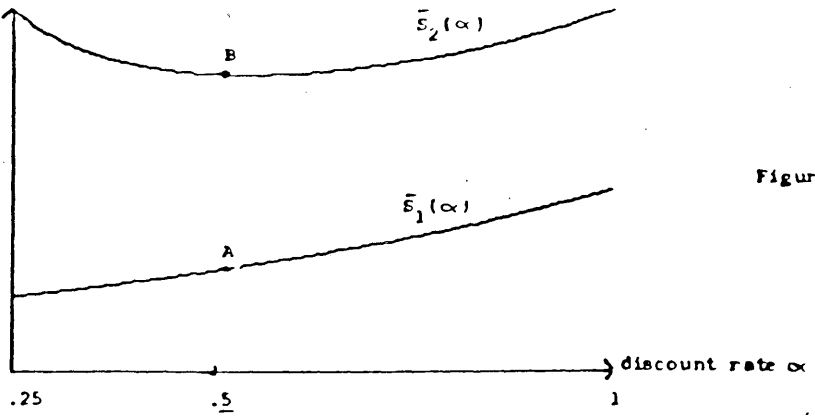


Figure 9 v

overproduction and hence decreases the conservative supply levels, as indicated in 9ii and 9iii. If production costs increase, the opportunity cost of underproduction decreases, and the opportunity cost of overproduction increases. Hence supply levels are also decreasing with respect to production costs, as shown in Figure 9iv.

The dependence of conservative supply levels on the discount factor α is the most complicated. This has already been discussed at the end of section 5, with reference to Figure 7. But in that context α and b were considered as independent. Figure 9v, on the other hand, takes into consideration the dependence (14) of the ratio of over- to under-production costs, b , on α . It can be seen from Figure (9v), however, that the qualitative description of the supply levels' dependence on α is the same as in the earlier discussion, at least for the parameter values of this example.

7. Conclusions

Our model of sequential production decisions highlights the interdependence of a firm's supply decisions and its information set. By supplying output and observing how much of it is sold, the firm may gain information about its product demand, and this information is relevant to its future supply decisions. We suggest that this interdependence may be expected to arise under conditions of demand uncertainty. Our model analyzes the optimal sequence of production decisions for a conservative firm in the face of this interdependence.

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