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## Conference Paper

# Socio-spatial formalism and nonlinear interaction in the diffusion of epidemics in regional space

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**Socio Spatial Formalism and Nonlinear Interaction in Regional Diffusion of Epidemics.  
The case of Whooping cough in Denmark**

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**Abstract**

Latest advances in the field of socio-spatial methodology is used to study the possible existence of nonlinear interaction in the spread of Whooping cough epidemics in Denmark and provide a qualitative description of the process. It is important to underline that the paper, at its present form, should be mainly regarded as a contribution to the field of Relative Socio Spatial Dynamics. The focus is on elaboration on two aspects namely the socio-spatial characteristics of the spread of the epidemics and thereby the relative allocation of infected subjects in the space of social interaction. By relying on the socio-spatial formalism, the log linear versions of the universal map of discrete relative dynamics suggested by Dendrinos & Sonis is used to replicate the dynamics of relative allocation of notified cases among urban and rural districts. Statistical measurements of the parameters of the suggested model are presented. It is argued that, in contrast to the dynamics of slow moving dynamics, the fast moving dynamics of the relative allocation of the infected subjects, justify the expansion of the deterministic set up by random interventions. Linear bifurcation analysis is applied on the modified model. The results of linear bifurcation analysis illustrate a wide range of possible

scenarios such as competitive exclusion, stability, flutter invariant curves and chaos.

## **I. Introduction:**

The existence of nonlinear interaction in the spread of various infectious diseases is extensively discussed in mathematical biology literature. Particularly the case of measles, mumps, rubella, chicken pox, scarlet fever and whooping cough epidemics have been subject to various investigations. Comparative studies, such as Schaffer & Kot (1985), Olsen & Schaffer (1990), based on real world data and epidemiological models, suggest that the dynamics of measles epidemics in certain urban areas are virtually the same and inherently chaotic. Rand & Wilson (1991) and Bolker & Grenfell (1993) provide new insights that challenge the traditional view that underlying dynamics of recurrent epidemics outbreaks are predominantly random and stochastic. Olsen et.al.(1988) focuses on the complexities that are observed in the occurrence of a group of childhood diseases in Copenhagen. It is particularly concluded that in the case of measles, mumps and rubella, the dynamics suggest low dimensional chaos. It is argued that the spread of chicken pox follows an annual cycle with noise superimposed, while no satisfying explanation is given for the case of scarlet fever and whooping cough.

Various other studies, based on application of SEIR models provide similar conclusions. SEIR models are among the best known epidemiological models, with an inherent ability to replicate the irregularities that characterize the outbreak of different epidemics. This approach divides the population in into four categories namely Susceptible, Exposed, Infective and Recovered. The dynamics of the outbreak of epidemics is replicated through a set of first order nonlinear differential equations. The irregular or periodic patterns in the dynamic behaviour of the process are then linked to the parametric variations in the parameters of the model. Parameters and variables of the models are purely biologically founded and defined. Among these parameters, one can mention birth rate, contact rate, latent and infectious periods etc. Some of these parameters are seasonally dependent and some are asserting constant values after some periods, but non of these variables indicate the socio-spatial impact of the process. In these studies the focus is on the absolute figures and actual abundance of infected subjects, while relative figures and the degree of relative abundance (deduced by social or spatial desaggregation criteria) are often neglected. SEIR models are predominantly based on the

studies of biological objects with almost no reference to the inherent complexity of social structures. Therefor the negligence of socio-spatial indicators or parameters in these models is essentially coherent with the analytical framework of these models.

The present paper does not intend to elaborate on the characteristics of various epidemiological models or focus on the epidemiological aspects of the process but rather intends to contribute to the understanding of epidemical diffusion process from a socio-spatial perspective. The emphasize is on providing a qualitative description of the diffusion of epidemics in a regional set up in order to incorporate the spatial aspects and relative nonlinear dynamics in to the temporal evolution of the diffusion process of epidemics.

At the theoretical level, the incorporation of the spatial aspects involves a regional perspective which is predominantly based on the socio-spatial formalism. This part of the analysis consists of the application of universal map of discrete relative dynamics suggested by Dendrinos & Sonis (1990) . Certain theoretical modifications of the initially deterministic structure of the universal map of discrete relative dynamics are proposed. The linear bifurcation analysis suggested by Sonis (1997) is then applied to the modified model. The bifurcation analysis results illustrate a wide range of possible scenarios such as competitive exclusion, stability, flutter invariant curves and chaos.

At the empirical level, the paper confine its effort to the study of monthly notification of whooping cough cases in Danish regions in the period 1912-1962.

## **II. Methodology:**

### **II.i. Socio-Spatial formalism:**

Socio-spatial methodology intends to describe the temporal evolution and spatial distribution of stocks in space. This formalism is predominantly based on a comprehensive interaction between relative discrete locational dynamics, ecologically inspired competition theory and nonlinear system theory. The theoretical set up is essentially based on the conception of

temporal and locational advantages attached to every stock and location in the space combined with the calculus of relative discrete dynamics. The mathematical model attributed to this approach can replicate and simulate a wide range of dynamic deterministic stock allocation operations. The stability analysis of the model provides all different scenarios which are observed in nonlinear systems such as stable and unstable competitive equilibria, periodic and nonperiodic fluctuation, stable fix points, strange attractors, bifurcation and chaotic phase transition. If the model is defined in a log linear form, it can be transformed to a standard linear least square regression model which consequently makes it possible to conduct statistical test of specific hypotheses. Another important characteristic of the model is the ability of the mapping to be dis aggregated among various location and stocks such as the case when  $m$ - stocks are allocated in  $n$ -locations.

The space of social interaction is assumed to be embedded in a spatial boundary. The absolute temporal allocation of  $m$ -stocks in  $n$ - regions is denoted by  $X_{ij}^t$ . The relative level of present of stock  $i$  in region  $j$  in time  $t$  is defined as  $x_{ij}^t = X_{ij}^t / \sum_k X_{kj}^t$  which can be also interpreted as the probability of the occurrence of a certain event in a given location in time  $t$ .

## II. ii. The Universal Map of Discrete Relative Dynamics

Dendrinis & Sonis (1990) suggests a model for replicating the dynamics of  $x_{ij}^t$  denoted as The Universal Map of Discrete Relative Dynamics which is defined by:

$$x_{ij}^{t+1} = \frac{F_{ij}(x_{ij}^t)}{\sum_{k=1}^n F_{jk}} , F_{ij} > 0 , \sum_{j=1}^n x_{ij}^{t+1} = 1 , 0 < x_{ij}^t < 1 , \sum_{j=1}^n x_{ij}^0 = 1 , 0 < x_{ij}^0 < 1 \quad (1)$$

Function  $F$  is the comparative advantages function attached to the allocation of stock  $i$  in region  $j$  in time  $t$ . A version of the model, that address the temporal relative allocation of a given stock in a heterogeneous discrete space (characterized by  $m$ -locations), is defined by:

$$x_i^{t+1} = \frac{F_i(\mathbf{x}^t)}{\sum_j F_j(\mathbf{x}^t)} \quad (2)$$

The model can be also defined by a log linear comparative advantages function  $F$  given by :

$$F_i = A_i f_i, f_i = \prod_j (x_j^t)^{a_{ij}}, A_i \in R^+, a_{ij} \in R \quad (3)$$

Following relation (3), function  $f_i$  is interpreted as region  $i$ 's regional advantage function. Parameter  $A_i$  is interpreted as bifurcation parameters related to the occurrence of bifurcation in the state variable  $x_i$  and parameter  $a_{ij}$  is interpreted as the spatial elasticity of regional advantages produced in region  $i$  in reference to the relative stock abundance at region  $j$ .

Another version of the model is the log-log linear version, characterized by  $F_i$  defined as:

$$F_i = e^{A_i f_i}, f_i = \prod_j (x_j^t)^{a_{ij}}, a_{ij} \in R \quad (4)$$

Mathematical transformations of the log linear case (3), leads to a standard linear least square regression model. This process provide options for conducting statistical test of specific hypotheses. Consider region I as a reference region and define following new variables,  $C_i = \ln(A_i/A_I)$ ,  $X_i^{t+1} = \ln(x_i^{t+1}/x_i^t)$ ,  $b_{ij} = a_{ij} - a_{Ij}$  and  $y_j^t = \ln x_j^t$ .

Since region I is the reference region,  $A_I=1$  and  $a_{Ij}=0$ . These manipulations transforms the model (3) to  $X_i^{t+1} = C_i + \sum_{j=I}^n b_{ij} y_j^t$  and consequently make it possible to estimate the parameters of the model as a simple regression model. Linear least square estimates of this model can be used to provide estimates for the parameters of the original model.

### II. iii. Stability Analysis of the Universal Map of Discrete Relative Dynamics

Building on the stability conditions first proposed by von Neuman, Dendrinos & Sonis (1990) present a detailed analysis of the stability conditions of the one stock 3-location version of the Universal Map of Discrete Relative Dynamics. The analysis is based on the consideration of Jacobi Matrix  $J_{t,t+1}$ , its values on a given equilibrium  $x^*$ , denoted by  $J^*$  and the characteristic polynomial of the  $J^*$  denoted by  $P(\mu)$ . The Jacobi Matrix  $J_{t,t+1}$  is defined as:

$$J_{t,t+1} = \|s_{ij}^{t,t+1}\| = \left\| \frac{\partial x_i^{t+1}}{\partial x_j^t} \right\| \quad i,j=1,2,3,\dots,n \quad (5)$$

The characteristic polynomial is given by  $P(\mu) = \mu^3 + \text{Tr } J^* \mu^2 + \Delta^* \mu + \det J^*$  in which,  $\Delta^* = \Delta_1 + \Delta_2 + \Delta_3$  is the sum of the main second order determinants of  $J^*$  and  $\text{Tr } J^*$  is the trace of  $J^*$ .

Von Neuman stability condition states that a given equilibrium  $x^*$  is asymptotic stable if for all its eigenvalues  $\mu$ , the absolute value of  $\mu$  is less than one.

Dendrinos & Sonis (1990) deduce the inequality  $-1 \pm \text{Tr } J^* < \Delta^* < 1$  for the description of the domain of stability of equilibria for the one stock 3-location case. The geometrical domain of asymptotic stability are within boundaries given by equations:

- Divergence boundary  $\text{Tr } J^* = \Delta^* + 1$ ,
- Flip boundary  $\text{Tr } J^* = -(\Delta^* + 1)$  and
- Flutter boundary  $\Delta^* = 1$ .

Variations in the space of parameters of the comparative advantages function  $F$  shifts the boundaries of the domain of stability and impose the emergence of new dynamic behaviour in the system. Dendrinos & Sonis (1990) illustrate how different dynamic behaviour can either emerge as a result of consequent variations in elasticities (structural parameters) or in the bifurcation parameters of model. By doing so, the dynamic features of the model is defined in terms of deterministic variations in the space of the parameters of the model.

#### **II. iv. Linear Bifurcation Analysis and Universal Map of Discrete Relative Dynamics**

Sonis (1997) shifts the focus of the analysis from the space of the parameters of the model to the space of the orbits. The approach is based on the notion of the travel of equilibrium, in which the movement of equilibrium from one point to another along a line segment is used to illustrate the change in the qualitative properties of a given model of the universal map.

Consider the one stock 3-location log linear case of the universal map and let the first region be the region of reference. Assume that  $\mathbf{x}^*$  denote an equilibrium point. Following relations (6) and (7) holds in all given time periods denoted by  $t$ .

$$\frac{x_3^t}{x_1^t} = A_3 f_3 = A_3 (x_1^t)^{a_{31}} (x_2^t)^{a_{32}} (x_3^t)^{a_{33}} \Rightarrow A_3 = (x_1^t)^{-1-a_{31}} (x_2^t)^{-a_{32}} (x_3^t)^{1-a_{33}} \quad (6)$$

$$\frac{x_2^t}{x_1^t} = A_2 f_2 = A_2 (x_1^t)^{a_{21}} (x_2^t)^{a_{22}} (x_3^t)^{a_{23}} \Rightarrow A_2 = (x_1^t)^{-1-a_{21}} (x_2^t)^{1-a_{22}} (x_3^t)^{-a_{23}} \quad (7)$$

Since  $\mathbf{x}^*$  is an equilibrium point then  $\mathbf{x}_i^* = \mathbf{x}_i^t$  for all  $t$ . It is concluded that :

$$A_1 = 1 \quad , \quad A_2 = (x_1^*)^{-1-a_{21}} (x_2^*)^{1-a_{22}} (x_3^*)^{-a_{23}} \quad , \quad A_3 = (x_1^*)^{-1-a_{31}} (x_2^*)^{-a_{32}} (x_3^*)^{1-a_{33}} \quad (8)$$

In other words given the matrix of elasticities and the position of an equilibrium point, these relations can be used to calculate the numerical values of the bifurcation parameters.

The linear bifurcation analysis is formalized by considering the movement of the equilibrium along a given line segment between equilibrium points  $\mathbf{z}$  and  $\mathbf{y}$ . The movement is performed in  $T$  discrete steps. Let  $z_{i,b}$  denotes the  $i$ 'th coordinate of the moving equilibrium after  $b$  consecutive steps, while  $z_i$  and  $y_i$  denotes the  $i$ 'th coordinate of equilibrium points  $\mathbf{z}$  and  $\mathbf{y}$ . The movement from  $\mathbf{z}$  to  $\mathbf{y}$  is formalized by the iterative procedure  $z_{i,b} = z_i(1 - b/T) + y_i(b/T)$ , for any given  $b$  between 0 and  $T$ .

Assume that the matrix of structural parameters  $(a_{ij})$  is known. Given that each  $z_{i,b}$  denote the  $i$ 'th coordinate of an equilibrium point, in any step of the procedure, the equations given in (8) are used to calculate the value of bifurcation parameters  $A_{i,b}$ . Knowing all the parameters of the log linear model, a point  $\mathbf{x}_0$  is chosen as the initial point of an orbit, which is supposed to be generated in every step of the procedure. Consequently  $T$  different orbits, each with starting point in  $\mathbf{x}_0$  and by the length  $l$  are generated.

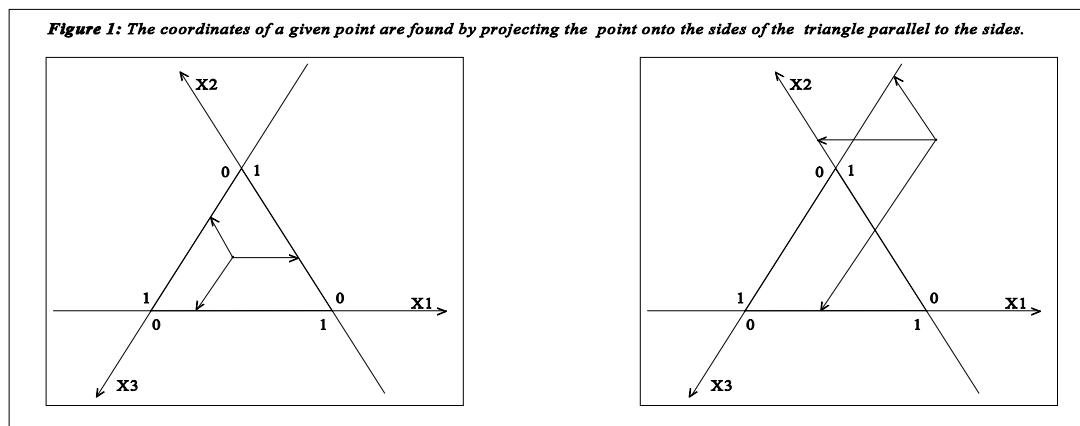


The focus is then on the tails of these orbits. It is initially assumed that the matrix of structural parameters ( $a_{ij}$ ) is known and each  $z_b$  denotes an equilibrium point, consequently the tail of the orbit  $O_b$  (initiated in point  $x_0$  in step  $b$ ) replicate the dynamic behaviour of the equilibrium point  $z_b$ .

A bifurcation diagram can be organized for all dimensions  $i$  by illustrating the coordinates of the points of the tail as a function of  $(b/T)$ . The total number of the steps are in fact the total number of the control bifurcation steps.

By performing this approach for different line segments between points  $z$  and  $y$  the focus of the analysis is shifted from the space of the parameters of the model to the space of the orbits.

A simple visualization of this approach, for the one stock 3-location case, can be illustrated in

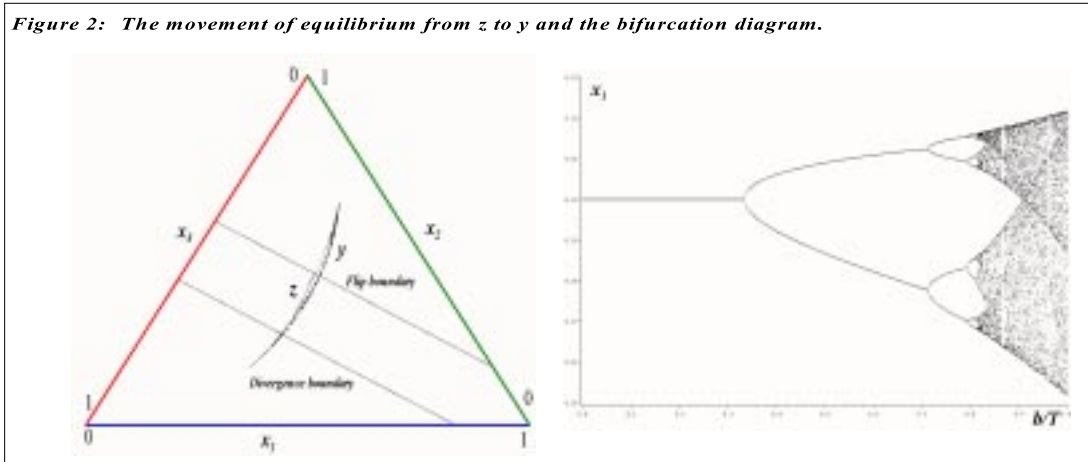


the space of Mubios barycentric coordinate system in Mubios equilateral triangle with the unit scale situated on its sides. As it is illustrated in Figure1 the coordinates of a given point in the space of Mubios barycentric coordinate system are found by projecting the point onto the sides of the Mubios equilateral triangle parallel to the sides.

In the following part assume that the matrix of structural parameters ( $a_{ij}$ ) is characterized by the conditions  $a_{1j}=0$ ,  $a_{2j}=2$  and  $a_{3j}=-a_{2j}$ . The focus is on the visualization of the movement of equilibrium from  $z=(0.3,0.4,0.3)$  to  $y=(0.3,0.5,0.2)$ . The movement is supposed to be conducted by 1000 equal jumps along the line segment from  $z$  to  $y$ .

Orbits are initially generated from the point  $x_0=(0.33,0.33,0.34)$  conducting numerous consecutive iterations of the log linear version of the universal map for the given parameters.

Figure 2: The movement of equilibrium from  $z$  to  $y$  and the bifurcation diagram.



The last 5% of the points on any orbit is regarded as the tail.

Figure 2 visualize the movement and illustrate the consequent bifurcation process that is occurring during the movement of equilibrium.

### III. Empirical application:

#### III. i. Background and data material

To the best knowledge of the authors, Dendrions & Sonis (1988) and Sonis (1996) are so far the only two studies that have addressed the empirical applications of the universal map. Dendrions & Sonis (1988) apply the loglinear version of the universal map into a historical study of the U.S. populations dynamics and by fitting the loglinear model to the logarithm of the relative populations of the U.S. regions, provide statistical evidence for the relevancy of the applied model. Sonis (1996) apply the same frame work into the study of empirical data of series of regional share of Gross National Product and successfully fit the loglinear model to the logarithm of the U.S. Gross National Product 1963-89. Both these studies confine their efforts to the statistical measurements of the parameters of the proposed model.

These studies share a common attribute, characterized by an inherent (relatively) slow moving dynamics in the both processes under consideration. To the best knowledge of the authors, no

study have so far applied the universal map to the cases in which the process under the study is influenced by a relatively fast moving dynamics.

The case of the spread of epidemics in geographical space is an obvious example of a process in which the dynamics of the relative diffusion is influenced by a relatively fast moving dynamics. These process can be viewed through an spatial desaggregation of the space. The desaggregation of the human geographical space into urban and rural areas and studying the relative allocation of the infected subjects allocated within the boundaries of these areas, is an example of such a process. By following this approach the focus is shifted towards the complex nonlinear interaction mechanism among various interacting subpopulations that are allocated in different regions of the space .

In the present paper, a data set consisting of monthly notification of observed whooping cough cases among Danish regions in the period 1913-1962 is used to demonstrate that the dynamics of the diffusion of whooping cough epidemics in Danish regions can be fairly replicated by the universal discrete relative dynamic map.

Two levels of spatial desaggregation criteria are used to study the process. In the first level, the total socio geographic space of Danish regions is divided into two components namely urban and rural regions. Consequently the total population is divided into two interacting sub populations. The one stock two location version of the universal map is used to replicate the dynamics of diffusion process among these two interacting populations.

Census statistics retrieved from Johansen (1985) shows that during the study period, the population is approximately distributed by a (54%, 46%) weight vector among rural and urban regions.

A second level of desaggregation of urban areas refine and improve the performance of the first model. Consequently the urban areas are divided into urban areas around Copenhagen and all other remaining urban areas . In other words, the second model, intends to replicate the behaviour of three interacting populations. Census statistics shows that during the study period, the total urban population is approximately distributed by a (48%,52%) weight vector among urban areas around Copenhagen and all other urban areas.

In both cases, parameter estimations are conducted by using the first 75% of the available data. The last 25% is then used to confirm the qualitative consistency of the approach.

### III. ii. Urban-Rural diffusion dynamics:

Consider a disaggregation of the total population into two interacting subpopulations in urban and rural regions. Let  $x_u$  and  $x_r$  represent the relative occurrence of whooping cough cases in urban and rural regions. Consider the one stock, two location case of model (3), by rural regions regarded as reference region. The consequent transformations and OLS regressions of the model provides following estimates and regression results.

**Table 1: Regression results for urban-rural process**

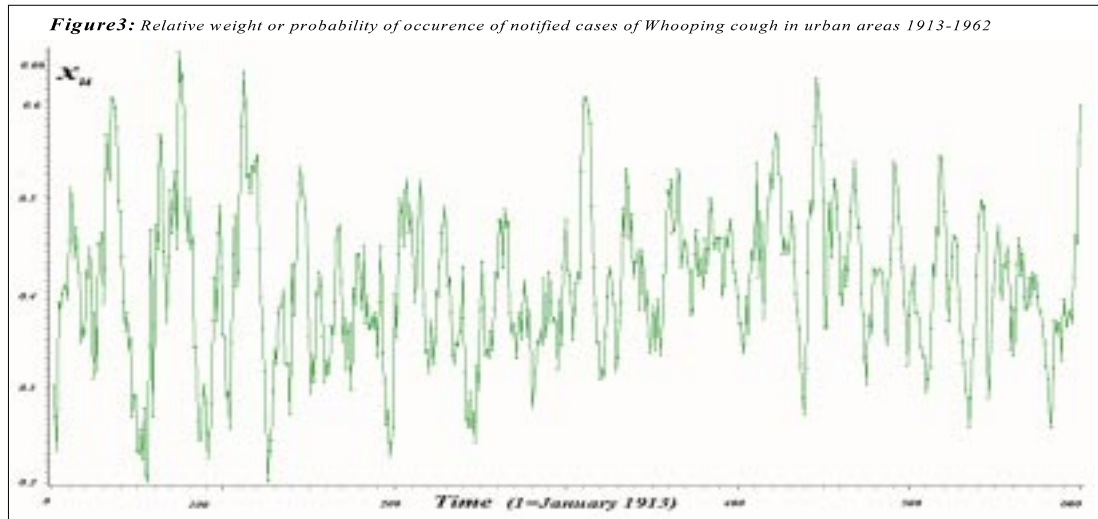
F Value 488.550, Prob>F 0.0001, R-square 0.6871

Parameter Estimates

Parameter	Estimate	Error	Std. T for H <sub>0</sub> : Parameter=0	rob >  T
$A_u$	0.673573	0.30238558	-1.306	0.1923
$a_{ur}$	0.627880	0.18270133	3.437	0.0006
$a_{uu}$	-1.113300	0.25569521	-4.354	0.0001

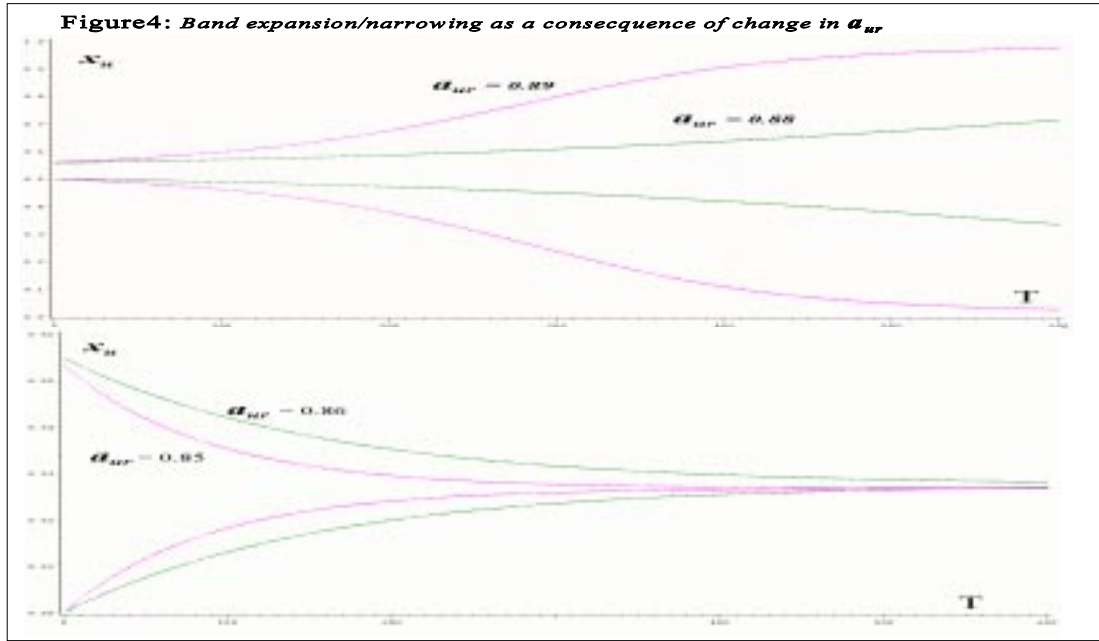
Since the parameters estimates are generated by using the first 75% of the data, it is expected that, a simulation of the estimated model provide a relatively good level of fit to the actual process. A simulation study of the model, generate a Voltera-Lutka type of dynamics similar to the May type logistic map with damping oscillation, which is not consistent with the actual dynamics of the process. Figure 3 present the dynamics of the relative occurrence of the whooping cough cases in urban areas in the study period.

The process is characterized by a very fast moving dynamics which is not consistent with the simulation results. The simulation study raise the question that wether the application of a *deterministic*, one stock two region loglinear case of the universal map is appropriate. In order to answer this question, it is important to underline that the fast moving dynamics of the process is not only the consequence of a series of iterations of the loglinear case, but most probably a manifestation of a deterministic system influenced by a random element.



For instance, the occurrence of the first case of whooping cough in an urban settlement or in rural regions, combined with many other factors, seemingly follow a stochastic dynamics which is obviously not predetermined by the parameters of the model. In order to illustrate this point, consider the regression results of the urban-rural process. The parameter  $a_{ur}$  is statistically supposed to belong to the interval  $[0.2697, 0.9859]$ . Varying  $a_{ur}$  in this interval provide a wide range of scenarios from damping oscillations to periodic motions.

The stability of equilibrium due to the parametric variations in the model is extensively discussed in Dendrinos&Sonis (1990). The aspect which is not raised before, involve the case in which one (or more) parameter(s) is (are) subject to a random perturbation in a given interval  $[a, b]$ . A example is the case in which  $a_{ur}$  is randomly shifted from 0.88 to 0.89. This random shift impose a change in the behavior of the system that leads to a rapid expansion of the band in which period two is occurring. Similar random shift from 0.86 to 0.85 leads to a rapidly damping oscillation. Figure 4 visualize the consequences of infinitesimal shifts of  $a_{ur}$ . Consequent random shifts in the parameters, as it is illustrated in figure 4, combined with the deterministic structure of the map can provide an explanation for the seemingly unstable dynamics of the process visualised in figure 3. Further analytical implications of these sort of interventions leads to the focus on the movements of the boundaries of the stability region as a consequence of random shifts in the space of parameters. Random shifts in the space of parameters impose changes in the composition of the elements of the Jacobi matrix and consequently the equations of divergence, flutter and flip boundaries.



In the next part of the paper, following a second level of spatial desaggregation, the one stock three location case of model (3) is used to argue and demonstrate that random perturbations in the space of parameters change the trace and second order determinants of the Jacobi matrix and consequently impose movements in the boundaries of divergence, flip and flutter.

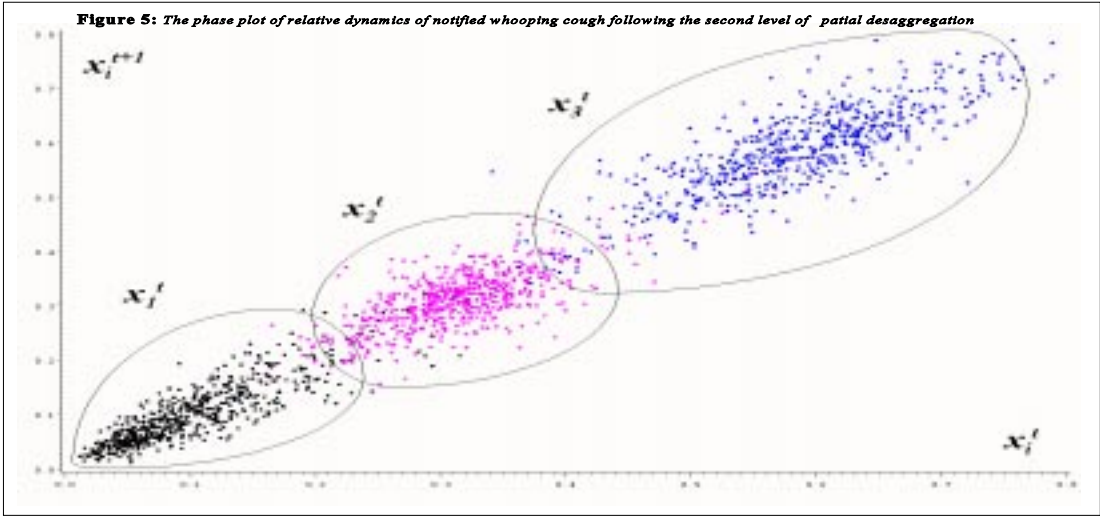
### III. iii. Desaggregation of urban population and consequent diffusion dynamics:

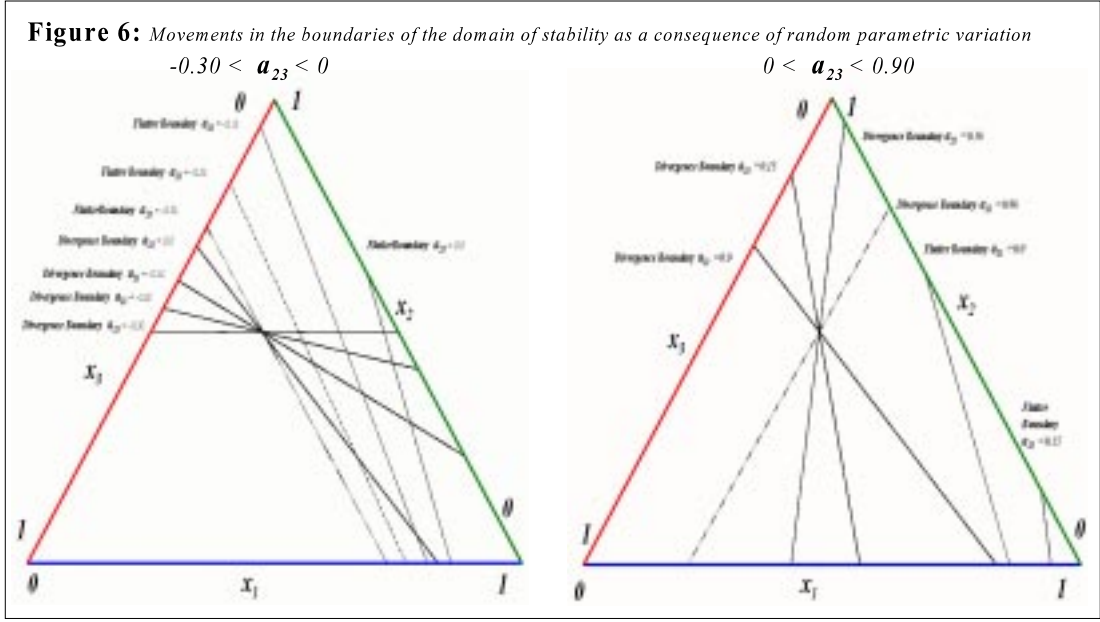
Following a second level of desaggregation, the total population is divided into three interacting subpopulations allocated in (1) urban areas around Copenhagen, (2) the remaining urban settlements and (3) rural areas. Utilizing the already known notation of the previous sections and based on this desaggregation criteria, let  $x_1$  denote the relative occurrence of whooping cough cases in urban areas around Copenhagen, while  $x_2$  and  $x_3$  represent the same variable in the remaining urban settlements and rural regions. Figure 5 visualizes the relative occurrence of whooping cough cases inside the boundaries of these regions.

Census statistics suggests an approximated distribution of population by a (22%, 24% 54%) weight vector for the above mentioned regions during the study period. It is interesting to notify that despite the approximately equal distribution of population among urban and rural areas (46% vs. 54%), the probability of getting infected in urban settlements (region 1 and 2) is consistently less than the probability of getting infected in rural areas (region 3). Similar phenomena is observed in region 1 (Copenhagen) and region 2 (other urban areas). Since Copenhagen has ( probably always had ) a larger population density than the other two regions, one can intuitively expect a larger rate of contact among individuals in region 1 and consequently a larger probability for getting infected in this region. Again, it is interesting to observe that, although the relative distribution of population in region 1 (Copenhagen) and region 2 (other urban areas) is approximately the same (22% vs. 24%), the probability of getting infected in region 1 (or the relative occurrence of notified cases in Copenhagen) is consistently lower than the probability of getting infected in region 2 (or relative occurrence of notified cases in other urban areas).

It is our understanding that the epidemiological arguments alone, can not provide satisfactory explanations for the above phenomena. Incorporating socio-spatial implications of the process into the considerations will probably provide additional insights which are seemingly neglected in epidemiological model design considerations.

Application of one stock, three region, loglinear case of the universal map, to the first 75% of the available data, regarding region 1 as reference region yields following estimation results:





Comparing the results presented in Table 1 and Table 2, reveals an improvement in the performance and statistical measurements of the second model. But using the results of Table 2 in a simulation study indicate that consequent iteration of log linear case of universal map leads to a sequence of points which are rapidly converging towards a fixed point. This is similar to the previous case, mentioned in the application of one stock two location case, and seemingly not consistent with the historical data. As it was the case in the first model, a random shift in the space of parameters can probably explain this inconsistency.

**Table 2: Regression results for urban-rural process following the second criteria of spatial disaggregation**

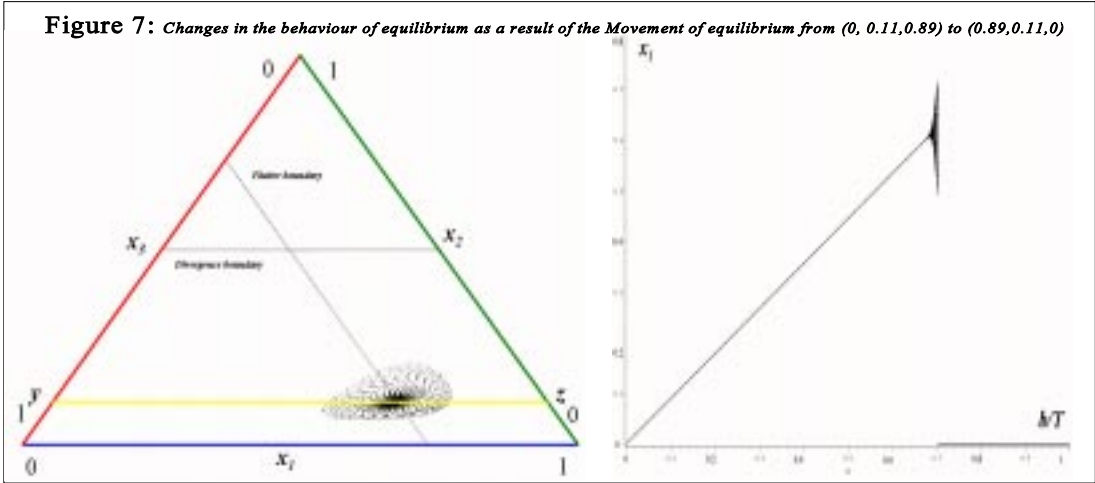
Model 1, Dependent variable $\ln(x_2^{t+1}/x_1^{t+1})$					Model 2, Dependent variable $\ln(x_3^{t+1}/x_1^{t+1})$				
F Value, 338.016, Prob>F 0.0001, R-square 0.6955					F Value, 486.763, Prob>F 0.0001, R-square 0.7668				
Parameter Estimates					Parameter Estimates				
Parameter	Estimate	Error	Std. T for $H_0$ : Parameter=0	Prob >  T	Parameter	Estimate	Error	Std. T for $H_0$ : Parameter=0	Prob >  T
$a_{21}$	-0.790056	0.05533691	-14.277	0.0001	$a_{31}$	-0.838801	0.05792521	-14.481	0.0001
$a_{22}$	0.906811	0.17831390	5.085	0.0001	$a_{32}$	0.362074	0.18665424	1.940	0.0530
$a_{23}$	0.292492	0.30967831	0.945	0.3454	$a_{33}$	1.200073	0.32416300	3.702	0.0002



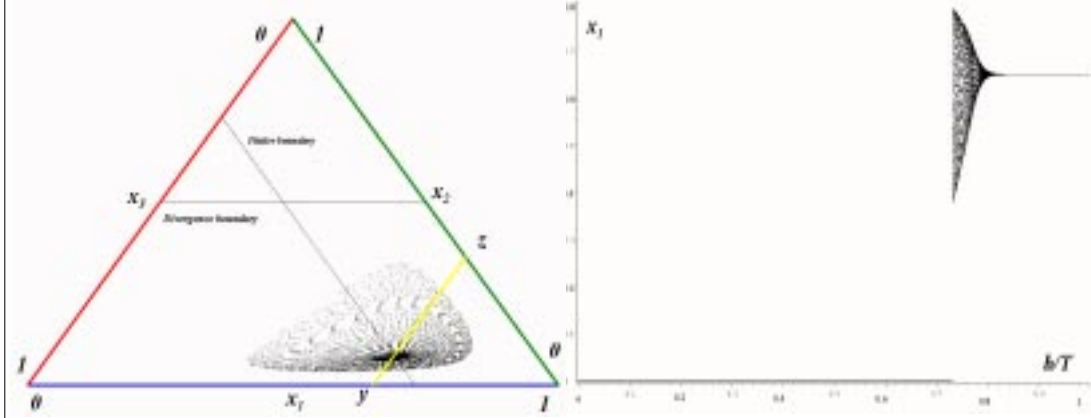
For instance, based on the results presented in Table 2, the estimate for parameter  $a_{23}$  should statistically belong to interval  $[-0.30272, 0.88770]$ . A random shift of  $a_{23}$  in this interval, impose changes in the composition of the Jacobi matrix, and consequently move the boundaries of the stability region. Figure 6 illustrate this process for varying  $a_{23}$  and visualize portions of divergence, flutter and flip boundary that lies inside the Mubios triangle. Following this path, the focus is then shifted towards the visualization of the changes in the qualitative behaviour of the equilibrium, as a consequence of random shifts in parametric space. Initially the linear bifurcation analysis suggested by Sonis (1997) is applied and the related changes in the qualitative properties of the model is studied by following the movement of equilibrium along a strait line segment.

Later on, a modified version of Sonis linear bifurcation approach is used to visualize the changes in the qualitative properties of the model by following the movement of equilibrium along a circle (or any other conic curve). Following this approach, a fixed point is regarded as the centre of the circle and for varying radius, the behaviour of equilibrium is studied along a given circle with radius  $r$  and centre given by coordinates  $(o_1, o_2, o_3)$  inside the Mubios triangle.

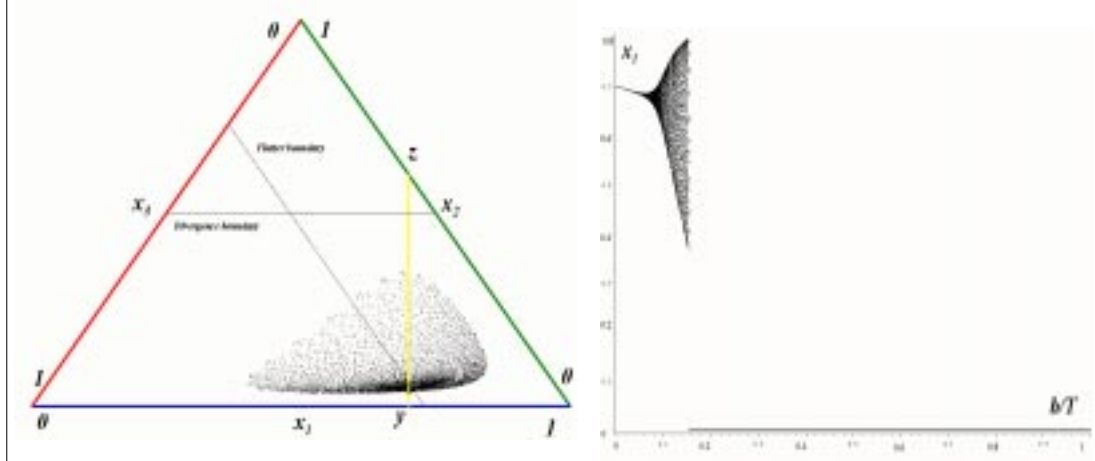
Initially, consider the results given in Table 2, and let  $a_{23} = -0.292$  while other elements of  $(a_{ij})$  remains as given in Table 2. In order to illustrate the consequence of this random shift, various points  $y$  and  $z$  are selected .Figures 7-12 visualize various movements of equilibrium along line segment  $yz$  and the consequent bifurcation diagrams.



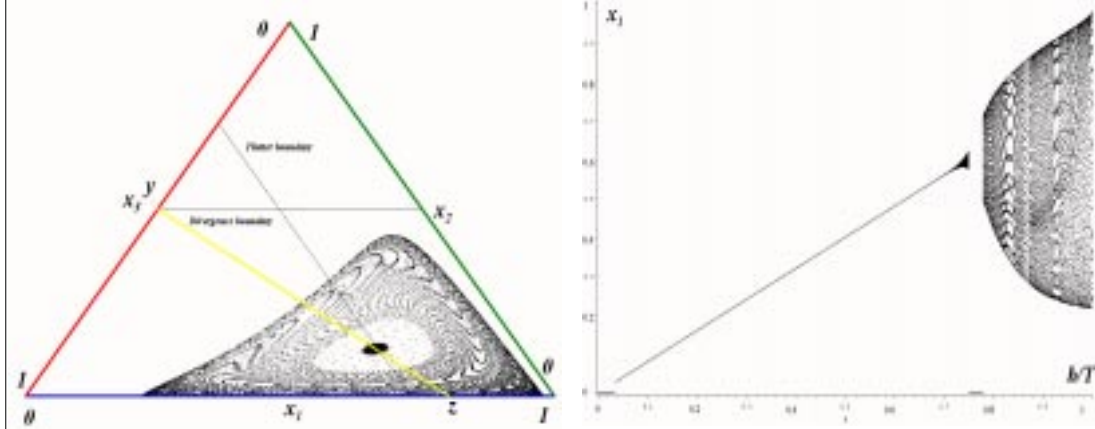
**Figure 8:** Changes in the behaviour of equilibrium as a result of the Movement of equilibrium from  $(0.65, 0, 0.35)$  to  $(0.65, 0.35, 0)$



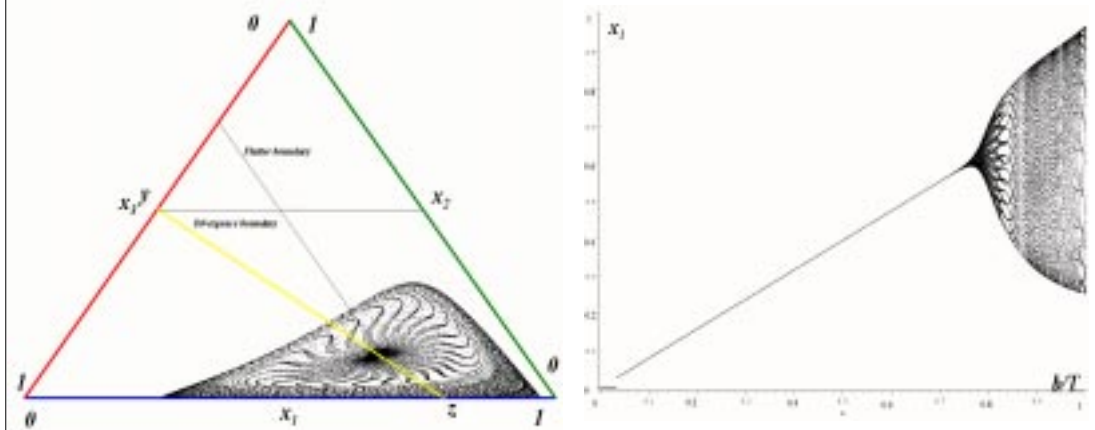
**Figure 9:** Changes in the behaviour of equilibrium as a result of the Movement of equilibrium from  $(0.70, 0, 0.30)$  to  $(0.40, 0.60, 0)$

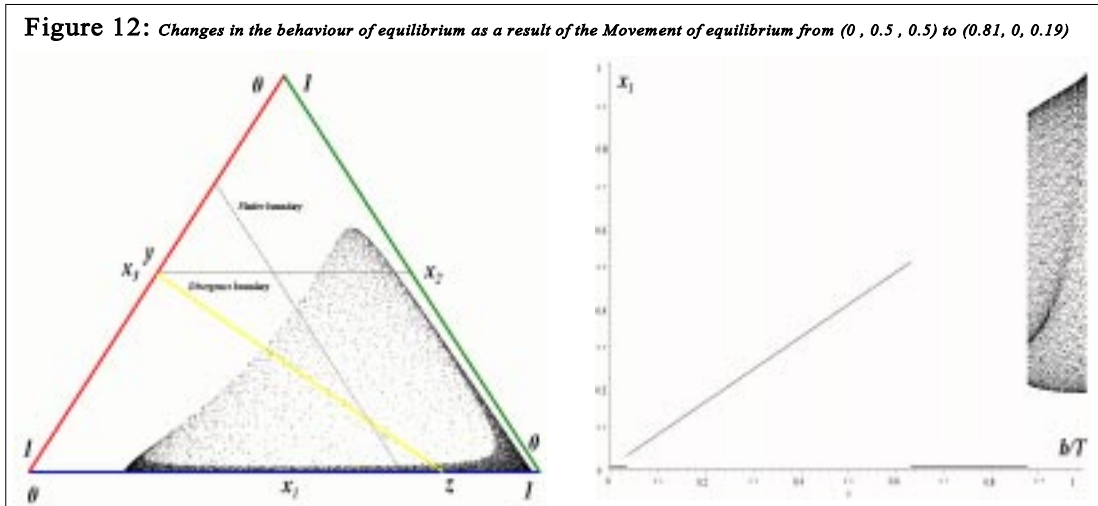


**Figure 10:** Changes in the behaviour of equilibrium as a result of the Movement of equilibrium from  $(0, 0.5, 0.5)$  to  $(0.8, 0, 0.2)$



**Figure 11:** Changes in the behaviour of equilibrium as a result of the Movement of equilibrium from  $(0, 0.5, 0.5)$  to  $(0.79, 0, 0.21)$





The cases visualized in Figure 7-12 illustrate the changes that are occurring in the behaviour of the system when the direction of the movement of equilibrium is given. Figures 10-12 specially address the changes due to infinitesimal alteration in the direction of the movement.

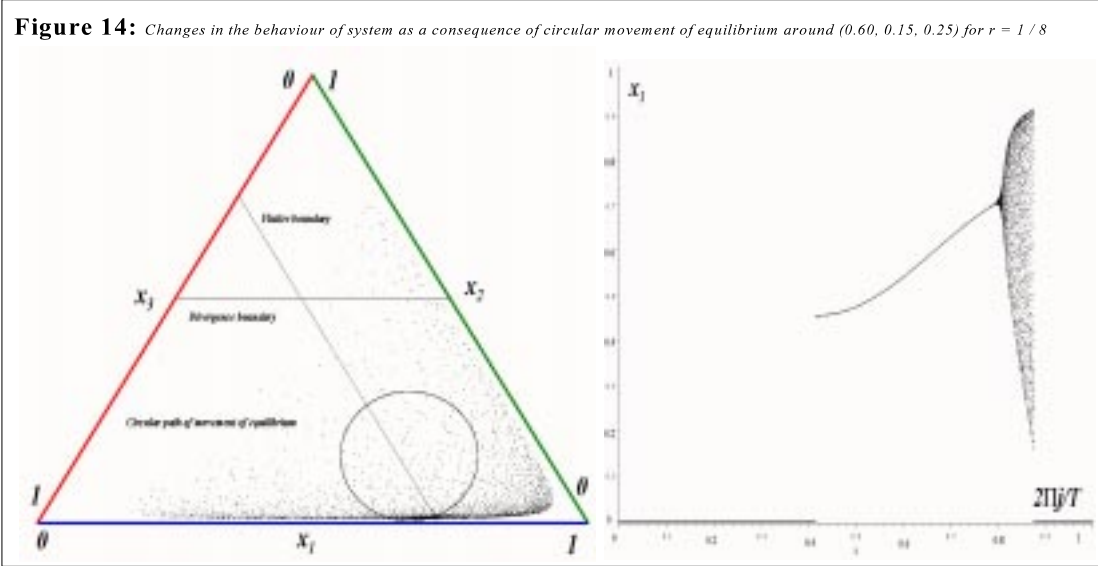
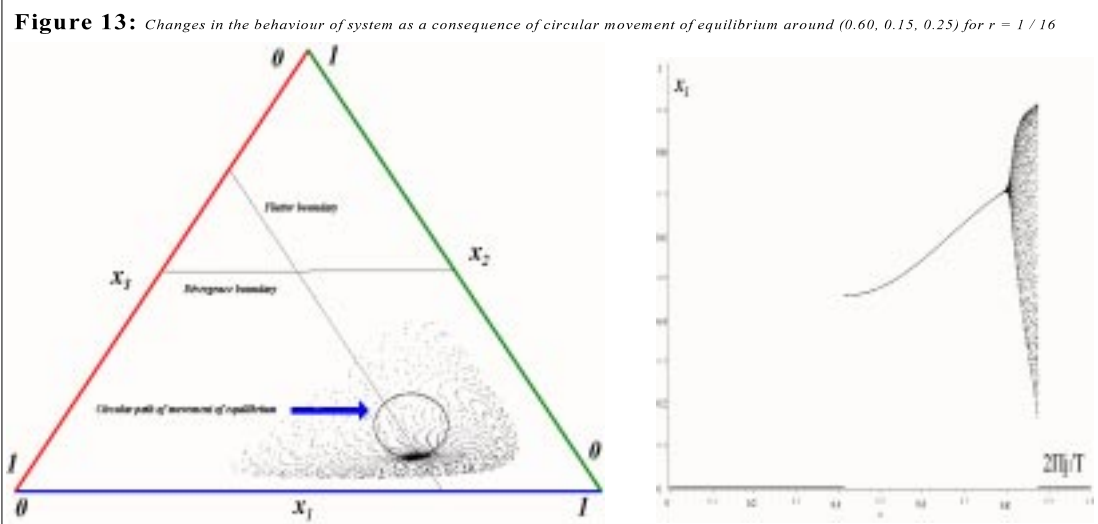
These figures raise the issue of the changes and behavioural impacts due to the alteration in the direction of the movement of equilibrium. Two scenarios are interesting. The first case suggested by Sonis (1997) consider the scanning of the space by fixing a point  $x$  as the centre, and following the movement from  $x$  to different points  $y$ . The position of point  $y$  can follow an appropriate criteria suitable for the model at hand. For instance, in the space of Mubios barrycentric coordinates, one can choose to put  $y$  on the sides of the triangle and then follow the movement of equilibria for all possible  $xy$  rays.

Present paper suggest a second scenario, in which there is no precondition for the straitness of the line segment that is connecting equilibrium points  $x$  and  $y$ . For instance one can consider a path following a conic function, with start in  $x$  and ending in  $y$ .

Following this idea, assume a point  $c$  as the centre of a circular path with radius  $r$  and let  $x=y$ . For a given  $r$ , the circular path is supposed to be traversed in  $T$  equally long steps connecting  $x$  to  $y$ . Scanning of the space involve a variation in  $r$ , in which incremental changes in  $r$  leads to expanding circular paths and consequently the scanning of the space. Using this method provide a tool for studying the behaviour of the system in the vicinity of a given point. In order to visualize the dynamics of change, for any given  $r$  a bifurcation diagram is generated. Since this method can easily be adopted to other paths of movement (such as conic and

trigonometric functions), the applicability of this approach can be regarded as its main attribute.

Figures 13-14 visualize the application of this approach to the same model as in Figures 7-12.



## IV. Conclusions

This study focus on the consequences of alteration in the space of parameters in the domain of socio-spatial formalism. It is illustrated that random alteration in the space of parameters is an inherent characteristics for socio-spatial systems. It is argued that since the socio-spatial systems are often prone to inherent parametric variations, their parameters do not remain the same for long time periods. It is suggested that, the incorporation of random shifts in the frame work of the model can be an appropriate analytical option for studying these parametric variations .Consequences of imposing random shifts in the space of parameters and orbits is discussed and visualized. Although appreciating the grate significance of the Sonis linear bifurcation approach, a modified version of this approach is discussed to address the behavioural change in the space of orbits when the movement of equilibrium does not follow a strait line segment.

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