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Advertisement-Financed Credit Ratings

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Abstract

Traditional business models of credit rating agencies (CRAs) are criticized for creating incentives for misreporting. This paper investigates a potential alternative in which CRAs receive revenue from advertisement only. We use a two-period Bayesian reputation model and show that CRAs will shirk when their reputation is either very high or very low. When reputation is at a medium level, the prospect of exploiting better reputation in the future might discipline CRAs to exert high effort in the present. However, when misreporting is possible, the CRA will always shirk and conduct either rating inflation or rating deflation when reputation is at a medium level.

JEL classification: G24, D82, L15.

Keywords: Credit rating agencies, advertisement, reputation.

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1 Introduction

Credit rating agencies (CRAs) are intermediaries on financial markets whose primary task is to provide independent assessments of debt securities' creditworthiness to potential investors by publishing ratings. Ideally, they provide an independent third opinion that reduces the information asymmetry on the debt market and improve its efficiency. However, in the recent years serious doubts were raised whether credit ratings are really independent or rather distorted by incentives to misreport. In the following we give a short summary over recent developments, which is adopted from White (2013) who provides a broad overview of the history of CRAs.

The activities of CRAs have not gained much public interest until the beginning of the new millennium. At the end of 2001 the American energy company Enron went bankrupt. Only a few days before this event, the three major CRAs had assessed Enron's creditworthiness with ratings that were considered to be investment-grade. Due to the Enron scandal the CRAs became subject of media attention across the world. For the first time, doubts were raised about the credibility of credit ratings. Only a few years after the Enron scandal, the CRAs were being accused of playing a crucial role in the emergence of the financial crisis of 2007-2008. The CRAs systematically failed to assess the high risk of structured assets like mortgage backed securities, which had been issued during the U.S. housing boom. More recently, the CRAs were also criticized for downgrading ratings of European government bonds, which can potentially threaten the stability of the European monetary union.

The issuer-pays model is currently the dominant business model of the big agencies.¹ It has been criticized for creating incentives to distort credit ratings and thus contributing to the crises mentioned above.² Since the issuer of a financial product pays for his own rating, he can put pressure on a CRA to publish a favorable rating by, for instance, threatening to cancel the business relationship and change to a competitor. Thus, CRAs might have an incentive to conduct rating inflation; namely to "whitewash" the issuer's

¹The three big CRAs are: Standard and Poor's Corporation (S&P), Moody's Investors Service and Fitch Ratings.

²See Mason and Rosner (2007), for instance.

creditworthiness by publishing ratings that are too high.³ The typical answer of the CRAs to such allegations is that they would not dare to conduct rating inflation because there is the threat of losing their good reputation. However, Mathis et al. (2009) argue that a high reputation can also have precisely the opposite effect. When the major part of CRA profit comes from issuer paid rating fees, reputation is no disciplining device but can be exploited by conducting rating inflation.

In the second traditional business model subscribers (potential investors) are charged for getting access to credit ratings. White (2013) points out that also the subscriber-pays model - which was used until the early 1970s - can provide incentives for publishing either inflated or deflated ratings. For instance, an investor might exert pressure on the CRA to publish a low rating, when he owns a short position.

Since the main incentive to publish incorrect ratings seems to come from direct payments between CRAs and either issuers or investors, economists should think about alternatives to finance credit ratings.⁴ A potential business model, which does not need direct payments between CRAs and other market participants, is the advertisement-based model.⁵ The main idea is that CRAs create ratings and publish them on their website free of charge. The cost of the audit process is covered by revenue from online advertisement.⁶ The advertisement-model is an example for a free-media model that is based on display advertisement.⁷ That means that the content is free of charge for the viewers and it is presented on a non-search website. Many industries that were originally based on a subscriber-based business model have switched to an advertisement-based business model or a mixture between these models. The most typical examples are TV-stations

³Even when there is no rating inflation but issuers can prevent the publication of unfavorable ratings, the issuer-pays model leads to the problem of “ratings shopping” and provide incentives for the creation of more complex financial products (see Skreta and Veldkamp (2009)).

⁴Mathis et al. (2009) pledge for eliminating direct commercial links between CRAs and issuers, and suggest to create a clearinghouse that pays rating fees independent of the rating outcome.

⁵This idea has been mentioned in White (2013).

⁶White (2013) also mentions a slightly variation of the advertisement-model in which the CRA provides fee based services that are not rating related and credit ratings only act as loss leaders. According to IAB (2014) 14% of total internet advertisement revenue in the first six month of 2014 came from the financial service industry.

⁷See Evans (2008) for an overview of the online advertisement industry.

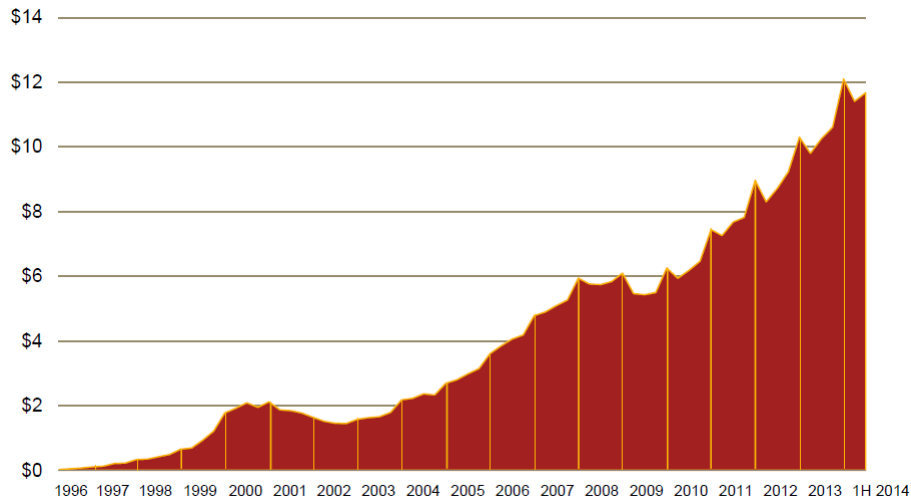


Figure 1: Quarterly online advertisement revenue growth trend 1996-2014 (\$ billions). Source: IAB/PwC Internet Ad Revenue Report, HY 2014.

and newspapers.⁸ Online advertisement emerges in 1994 for the first time and shows impressive growth rates in revenue since then (see Figure 1).

As long as there are no strong financial connections between advertisers and the players on the financial market, there are no incentives to publish ratings that are favorable for investors or issuers. Instead, one might conjecture that advertisement financed CRAs care more about their reputation and, therefore, put much effort into assessing the quality of financial products and report truthfully. This can be justified by the following argumentation. Online advertisement revenue is increasing in the online traffic that can be generated on the website. High CRA reputation increases the trustworthiness of the ratings. When ratings on the website are regarded as being truthful, there will be more traffic because the ratings are decision relevant for potential investors, which in turn increases the advertisement revenue of the CRA. Thus, there seem to be a strong incentive to maintain a good reputation.

Of course, there are also some drawbacks with the advertisement-model. First, it might be difficult to guarantee that there are no strong connections between advertisers and players on the financial market. When advertisers prefer either high or low ratings for specific assets and are able to put pressure on the CRA, we get the same conflicts of

⁸Whereby the newspaper industry typically uses a mix between advertisement and subscription fees and TV-stations often use a pure advertisement model.

interest that are already known from the other business models; however, they seem to be weaker, why we will not focus on them.

The second drawback is more important. When the rating is not solicited by an issuer, the ratings are based only on information that are publicly available. Whereas CRAs that are hired by issuers can get a deep inside into the issuing firm. The signals about the creditworthiness are noisier when the rating is unsolicited and more precise signals might be more expensive.⁹

This paper aims to explore: Whether the advertisement-based business model leads to the publication of credible credit ratings? Is a CRA incited to exert high effort into assessing financial products? What role does reputation play? And finally, is it desirable from a welfare perspective that CRAs switch from the issuer-pays model to the advertisement-based model?

In our analysis we will first abstract away from the possibility of misreporting, by which we mean reporting a signal dishonestly. We will rather concentrate on the CRA's incentive to invest in signal quality. We construct a two-period endogenous reputation model. In each period, a monopolistic CRA chooses a costly audit technology. After that the CRA receives a signal about the quality of a financial product and publishes a rating. The audit technology determines the signal accuracy: the signal is not informative and the audit costs are zero when the CRA shirks. In contrast, the signal can also be perfect when the investment in the audit process is high enough. The advertisement revenues are modeled as follows: the CRA only receives revenues from advertisement when the published ratings are regarded as informative by investors; namely, when different ratings induce different investment decisions.

We show that CRAs that use the advertisement-model may have a short run incentive for shirking; namely, reporting of ratings without making an elaborate assessment in advance. Shirking occurs in equilibrium when the CRA reputation is either very high or very low. Only when the reputation is at a medium level, the CRA may have an incentive to conduct a costly but thorough assessment. In an extended model, we allow the CRA

⁹The firms could also open their door for the CRAs and provide inside informations voluntarily without paying for the rating. The willingness to open the door alone could be seen as a positive signal about the creditworthiness.

to misreport. We show that the CRA will shirk and conduct either rating inflation or rating deflation.

The present paper is related to the literature on modeling reputation. We use the approach of Kreps and Wilson (1982) and Milgrom and Roberts (1982) to model reputation in a game of complete but imperfect information. Both papers introduce different types for one player (weak and strong, for instance) and interpret the belief of the other players about this type as reputation.¹⁰ We will introduce a CRA with two possible types: A committed type which has the strictly dominant strategy to conduct a costly rating process, and an opportunistic type which behaves strategically.

There is a growing economic literature on credit ratings. White (2013) provides a broad overview of the CRAs history, potential conflicts of interest and their regulation. In particular, the author describes the role of CRAs during the subprime and the European debt crisis. He also mentions the idea to finance credit rating with advertisement as a potential alternative to the issuer-pays or subscriber-pays models.

Some papers, like Mathis et al. (2009), Stolper (2009), or Opp et al. (2013) focus on the CRAs' incentive to conduct rating inflation. Mathis et al. (2009) investigate whether the threat of losing a good reputation disciplines CRAs to report truthfully. The authors use an infinitely repeated reputation model. Their main result is that a high reputation can discipline CRAs only when a large fraction of their revenue comes from other activities than rating. In contrast, when total revenue comes from rating activities, an opportunistic CRA will inflate ratings when the reputation is high enough. The crucial difference to our paper is that the CRA in Mathis et al. (2009) can perfectly observe the type (quality) of the financial assets at zero cost.¹¹ We consider a costly, imperfect screening technology and obtain different results. Even when the total CRA revenue comes from other sources than directly from rating activities, for instance from advertisement, an opportunistic CRA has an incentive to shirk and just report whatever

¹⁰With this approach it was possible to provide a formal justification for predatory pricing in the finite version of Selten's famous chain-store game (see Selten (1978)).

¹¹In the basic model of Mathis et al. (2009), it is even possible that the CRA is perfectly revealed as being opportunistic, because projects with a good type are successful for certain. In an extended version also good projects fail with positive probability. However, even in the extended version, the CRA gets a perfect signal about the project type.

the market believes. This might lead to rating inflation or deflation with probability one. Stolper (2009) argues that a regulator can discipline CRAs by designing an approval scheme. Rating inflation is detected when other agencies report truthfully at the same time. Thus, a misbehaving CRA can be identified and punished by denied approval. In our model, shirking or misreporting cannot be proved with certainty in equilibrium. Opp et al. (2013) investigate the connections between rating inflation and rating contingent regulation of capital requirements. In particular, the CRAs' audit technology and cost are similar to our approach.

Skreta and Veldkamp (2009) investigate a different source of inflated ratings. When issuers can deny the disclosure of so called shadow ratings,¹² they can solicit many ratings from different CRAs and disclose only the most favorable. Skreta and Veldkamp (2009) call this ratings shopping. Even when the CRAs report their signals honestly, the disclosed ratings are inflated as long as signals are noisy. For this reason, competition in the rating industry can lead to less informative ratings, in particular, for complex assets that are difficult to evaluate.¹³ Since our model considers unsolicited ratings of a monopolistic CRA, rating shopping does not play a role. Bolton et al. (2012) consider the coexistence of trusting investors, who take the CRA ratings at face value, and sophisticated investors, who anticipate opportunistic rating strategies. The authors compare a monopoly rating industry to a duopoly from a welfare point of view. Their main result is that although the monopolistic CRA conducts more rating inflation, the duopolistic rating industry is less efficient because ratings shopping is possible here. Thus, ratings shopping in duopoly is more crucial than efficiency loss due to monopolies. However, this result depends strongly on the existence of trusting investors.¹⁴ Fulghieri et al. (2013) point out that unsolicited ratings can be used by CRAs as a credible threat to issuers¹⁵ which makes it possible to charge higher fees for solicited ratings. In the advertisement-model, to solicit a rating might be seen as a self signaling device. When the firm open its door voluntary for CRA's

¹²Shadow ratings are preliminary ratings that are private information between the CRA and the issuer.

¹³This also leads to an incentive to create more complex financial products.

¹⁴Bolton et al. (2012) argue that pension fund manager can assumed to be trusting because their salary depend only marginally on asset returns. Since a thorough own assessment needs much effort it is easier to trust an external CRA rating.

¹⁵This does not mean that unsolicited ratings are necessarily deflated. In equilibrium, all creditworthy firms solicit a rating and every unsolicited rating just reflects low creditworthiness.

analysts it is most probably creditworthy. In the present paper, we focus on unsolicited ratings.

The remainder of this paper is structured as follows: the next section introduces the basic model. The analysis of the basic model is provided in Section 3. In Section 4 we present some numerical examples. Section 5 introduces and analyses our extension and Section 6 concludes. The proofs not included in the body of the paper are contained in the appendix.

2 Basic Model

Order of play and information structure

The model has two periods, which are denoted by $t \in \{1, 2\}$. Similar to Fulghieri et al. (2013), the players are a long living credit rating agency (CRA) and, in every period, a short living firm and many short living potential investors. The investors are assumed to be identical.¹⁶ Like other authors before,¹⁷ we use the standard approach of Kreps and Wilson (1982) and Milgrom and Roberts (1982) to model the CRA reputation. This is done by introducing a CRA type $\theta \in \{o, c\}$, which is randomly drawn by nature at the beginning of $t = 1$ and remains private information of the CRA. The type o stands for opportunistic and c for committed. Let the CRA be committed with probability φ_1 and opportunistic with probability $1 - \varphi_1$. At the beginning of the game φ_1 is the prior belief of the firm and the investors that the CRA is committed. It can be interpreted as reputation parameter.

As in Fulghieri et al. (2013), there is a firm in every period that seeks financing for its project via the credit market. The financial requirement is normalized to one. The firm promises to pay the return $R \in [0, X]$, where $X \in \mathbb{R}_+$, to the investor, as long as the project is financed and turns out to be successful. The probability of success depends on the project type $\tau \in \{g, b\}$, where g stands for good and b for bad. Let the project be good with probability β and bad with probability $1 - \beta$.¹⁸ Bad projects always fail. In

¹⁶That means, in particular, that all potential investors have the same prior beliefs and the same outside option.

¹⁷For instance Mathis et al. (2009), Bolton et al. (2012) or Fulghieri et al. (2013).

¹⁸A slightly different interpretation is that there is a population of projects with measure 1 and β

contrast, good projects are successful with probability α and fail with probability $1 - \alpha$. A project that fails generates zero profit, while successful projects generate X . Investors can either invest in the firm's risky project or invest in an outside option with certain return one. A project that has not been financed cannot be carried out and vanishes.

We make the following assumption:

Assumption 1.

$$\alpha\beta X < 1 < \alpha X.$$

Assumption 1 is standard in the literature on credit ratings. The first inequality implies that the financial market would collapse without further information about the project type.¹⁹ Without the rating, the certainty equivalent of the risky investment is lower than the riskless outcome of the outside option.²⁰ The second inequality ensures that good projects should be financed from a welfare perspective. Hence, when Assumption 1 is fulfilled truthful ratings can improve the efficiency of the market outcome.

Following Opp et al. (2013), the opportunistic CRA receives an imperfect signal $\sigma \in \{l, h\}$ about the project type:

$$\Pr(h|g) = \Pr(l|b) = \frac{1}{2} + s_t,$$

where the signal quality can be influenced by the CRA through choosing a costly audit technology $s_t \in [0, \frac{1}{2}]$ to the beginning of period t . The committed CRA is assumed to have the dominant strategy $s_t = \frac{1}{2}$ in every period. Accordingly, the signal of the committed type is always perfectly informative.

The cost of the audit process in period t is given by the function $c(s_t)$, which is assumed to have the properties: $c(0) = 0$, $c'(s_t) > 0$, $c''(s_t) \geq 0$. Hence, the cost and the marginal cost are increasing in rating accuracy s_t . Furthermore, the audit cost yields zero if the lowest technology $s_t = 0$ has been chosen. In contrast to Opp et al. (2013), we further assume that $c(\frac{1}{2}) < \pi$, where π is the per period advertisement revenue, which is

denotes the fraction of good types.

¹⁹This situation is similar to the adverse selection problem in the market for lemons in Akerlof (1970): The market breaks down when good and bad projects are pooled.

²⁰Since the investors are risk neutral, the certainty equivalent is equal to the expected return of financing the project.

explained in detail below. With the last property of the cost function we ensure that the CRA can have a positive per period payoff even when the audit process is conducted as thoroughly as possible.

Assumption 2.

$$\beta > \frac{1}{2}$$

Assumption 2 states that the investors rather expect to face good projects, which is the case of interest for our analysis.²¹ The CRA publishes a rating $r \in \{H, L\}$, where $H(L)$ is a high (low) rating that ideally signals investors a high (low) firms creditworthiness. It is obviously a simplification to allow only for two different rating scores. In the real world finer partitions of ratings can be observed.²² However, to use only two rating grades is a standard abstraction in the scientific literature on credit ratings. The $H(L)$ -rating might be interpreted as a level that is considered to be investment-grade (speculative-grade).²³

Assumption 3. $r = H$ iff $\sigma = h$. That is, the CRA publishes ratings according to the signal.

Assumption 3 is made in our basic model only. In our extension we will drop this assumption and also allow for misreporting by the CRA.

The order of play in period t can be summarized as follows:

0. If $t = 1$, nature determines the CRA type $\theta \in \{o, c\}$.
1. The opportunistic CRA chooses the rating technology $s_t \in [0, \frac{1}{2}]$, while the committed uses $s_t = \frac{1}{2}$.
2. Nature determines the project type $\tau \in \{g, b\}$. The opportunistic CRA receives an imperfect signal $\sigma \in \{l, h\}$ about τ , while the committed CRA obtains a perfect signal.

²¹Note that Assumption 2 does not imply that the investors will finance a project without further information (see Assumption 1).

²²Standard & Poor's, for instance, uses 24 rating grades, which reach from AAA to D. Hereby, the best ten rating grades, from AAA to BBB-, are considered to be investment grade. The remaining 14 grades, from BB- to D, are considered to be speculative-grade.

²³This interpretation can be justified as follows. Many institutional investors, for instance, pension funds, are not allowed to invest in financial assets that have speculative-grade ratings. Therefore, even a small decrease in the rating can have critical consequences, when the rating has fallen under the limit of investment-grade.

3. The CRA publishes the rating according to the signal, $r = H$ iff $\sigma = h$.
4. The firm chooses $R \in [0, X]$.
5. Investors make their investment decision.
6. The project outcome is realized.

The steps 1.-6. are repeated in $t = 2$. Firm and investors in $t = 2$ observe the rating, the investment decision and, provided that investment occurs, the project outcome of the first period. With this information structure we follow Mathis et al. (2009), who also argue that all relevant information for second period decisions can be summarized in the reputation parameter φ_2 , which in turn is the reputation prior of the second period. φ_2 is derived from exogenously given reputation prior φ_1 and all observable first period events.²⁴

Payoffs

By looking at the CRA payoff, the difference between the issuer-pay model and the advertisement-based model - which we investigate - becomes clear. When the issuer-pay model is used, the CRA revenue come from firms that pay for their own rating. This case has been analyzed often in the scientific literature. Examples are Bolton et al. (2012), where firms pay a fee for the publication of high ratings because they improve their lending conditions on the financial markets, or Fulghieri et al. (2013), where firms can be forced to pay for ratings by CRAs threat to publish a deflated unsolicited rating otherwise. In contrast to the issuer-pay model, in the advertisement-model neither issuer nor potential investors are charged for credit ratings. The CRA revenue comes from advertisement only, which is placed on the website where the rating is published. We use a rather simple revenue structure: the CRA revenue in period t is either zero or $\pi \in \mathbb{R}_+$. The revenue yields zero, when the observation of H -ratings and L -ratings both lead to the same investment decision. When ratings influence the investment decision, the CRA revenue yields π . The intuition behind this revenue structure is as follows. When different ratings lead to different market outcomes, the investors regard the rating as useful information. In this case, every investor visits the CRA website because useful

²⁴The details about the belief updates are explained in the next section.

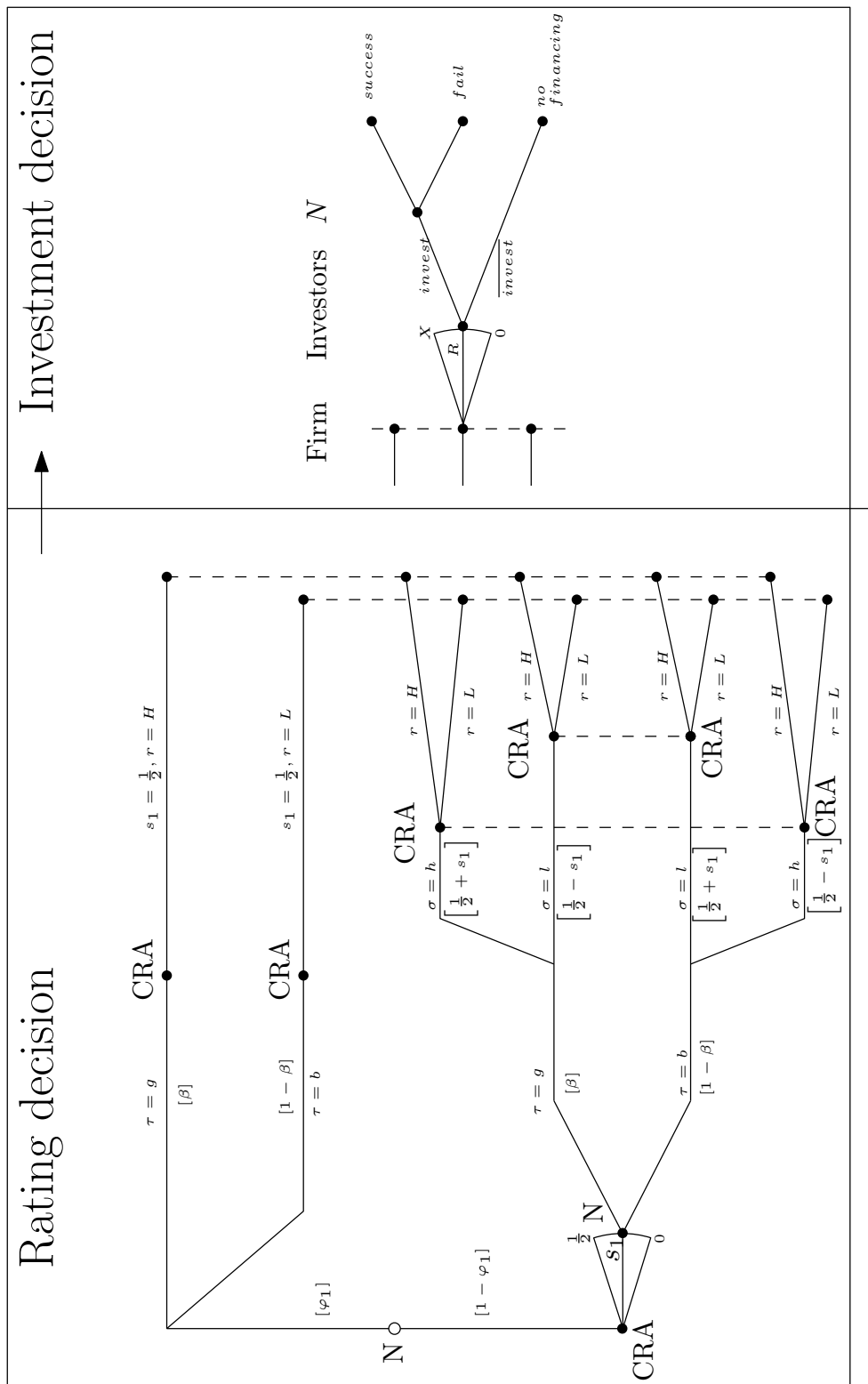


Figure 2: Game tree of the first period (probabilities in brackets, $N = \text{Nature}$).

information are provided free of charge there. Advertisers are willing to pay the amount π for placing their advertisement on the CRA website, since many investors will be exposed to it. The parameter π depends on the structure of the market for advertisement. In our model, π is exogenously given. In the other case, ratings have no influence on the market outcome. That means, the investors do not trust the rating and, therefore, do not visit the CRA website. Since advertisers will not pay for a website without traffic, the CRA revenue yields zero. We use the following definition:

Definition 1. *Let $\hat{\varphi}_t$ defined to be the lowest CRA reputation to the beginning of period t that induces the investors to make their decision in period t dependent on the published credit rating.*

When the reputation is below $\hat{\varphi}_t$, our analysis below shows that investors always choose the outside option. In contrast, when $\varphi_t > \hat{\varphi}_t$, the publication of H -ratings induce investment in the risky project. Thus, ratings are regarded as useful information for investment decisions and advertisement revenue is generated only when the threshold $\hat{\varphi}_t$ is exceeded. Note that Assumption 1 implies that $\hat{\varphi}_t$ is unique.

When per period profits are discounted with $\delta \in [0, 1]$, the opportunistic CRA's payoff is given by

$$\Pi = \sum_{t=1}^2 \delta^{t-1} [\mathbb{1}_{\{\varphi_t \geq \hat{\varphi}_t\}} \pi - c(s_t)],$$

where $\mathbb{1}_{\{\varphi_t \geq \hat{\varphi}_t\}}$ is the indicator function that yields 1 if $\varphi_t \geq \hat{\varphi}_t$ and 0 otherwise.

For the committed CRA we allow for all payoff functions that induce the CRA to choose always $s_t = \frac{1}{2}$. Accordingly, using the highest technology has to be a strictly dominant action in both periods.

Firm's and investors' payoffs depend on whether the project is financed and, when financing takes place, on the project outcome. Let Π_F denote the firm's and Π_I investors' payoff. When the project is not financed, the firm's and investors' payoffs are assumed to be $\Pi_F = 0$, and $\Pi_I = 1$, respectively. When investment occurs, the payoffs are risky and depend on the project outcome. When the project fails, payoffs are zero for both. In contrast, when the project is successful, the payoffs are given by $\Pi_F = X - R$ and $\Pi_I = R$.

In the next section we analyze the basic model. The equilibrium concept is perfect Bayesian equilibrium.

3 Analysis

A perfect Bayesian equilibrium (PBE) requires a consistency between a strategy profile and a system of beliefs. The strategy profile in our basic model prescribes investors' action, firm's action and the rating policy for the opportunistic CRA for every reachable information set. Beliefs to the beginning of period t are given by the CRA reputation prior φ_t and the technology that the opportunistic CRA is expected to choose, which we denote by \tilde{s}_t .

Prior beliefs

At the beginning of the first period, the reputation prior is given exogenously by $\Pr(\theta = c) = \varphi_1$. Without further information, the probability of having a good project and the probability of success conditional on investment are given by $\Pr(\tau = g) = \beta$ and $\Pr(\text{success}|\text{investment}) = \alpha\beta$, respectively.

Interim beliefs

After receiving the signal about the project type, the CRA updates the prior belief using Bayes rule. If $\sigma = h$, the probability of a good project is given by

$$\Pr(g|h) = \frac{\beta \left(\frac{1}{2} + s_t\right)}{\beta \left(\frac{1}{2} + s_t\right) + (1 - \beta) \left(\frac{1}{2} - s_t\right)} > \beta, \quad (1)$$

whereas it is given by

$$\Pr(g|l) = \frac{\beta \left(\frac{1}{2} - s_t\right)}{\beta \left(\frac{1}{2} - s_t\right) + (1 - \beta) \left(\frac{1}{2} + s_t\right)} < \beta, \quad (2)$$

if $\sigma = l$. Note that the CRA update is different from the firm's and investors' belief update because the CRA knows its own type and technology. After the rating has been published, the other market participants form interim beliefs about the CRA type, whereby the interim reputation is the reputation of the CRA after the rating has been observed but before the project outcome has been realized. Conditional on the rating, the interim

reputation yields

$$\Pr(c|H) = \frac{\beta\varphi_t}{\beta\varphi_t + (1 - \varphi_t)(\frac{1}{2} - \tilde{s}_t + 2\beta\tilde{s}_t)} \equiv \varphi_t^H \quad (3)$$

$$\text{and } \Pr(c|L) = \frac{(1 - \beta)\varphi_t}{(1 - \beta)\varphi_t + (1 - \varphi_t)(\frac{1}{2} + \tilde{s}_t - 2\tilde{s}_t\beta)} \equiv \varphi_t^L. \quad (4)$$

Given the interim beliefs, the probability that the project is good is updated as follows:

$$\Pr(g|H) = \varphi_t^H + \beta(1 - \varphi_t^H) \frac{1 + 2\tilde{s}_t}{1 + 4\beta\tilde{s}_t - 2\tilde{s}_t} \equiv \beta_t^H \quad \text{and} \quad (5)$$

$$\Pr(g|L) = \beta(1 - \varphi_t^L) \frac{1 - 2\tilde{s}_t}{1 - 4\beta\tilde{s}_t + 2\tilde{s}_t} \equiv \beta_t^L. \quad (6)$$

Accordingly, the rating contingent probability of success is given by $\Pr(\text{success}|r, \text{invest}) = \alpha\beta_t^r$. Since the interim beliefs are derived from all information available before the investment decision was made, they are determinative for the optimal actions of the firm and the investors.

Final beliefs

When no financing takes place in period t , the final reputation persists on the interim level. When the project has been financed and turns out to be successful, the final reputation becomes

$$\Pr(c|H, \text{ success}) = \frac{\varphi_t}{\varphi_t + (1 - \varphi_t)(\frac{1}{2} + \tilde{s}_t)} \equiv \varphi_t^{HS}. \quad (7)$$

By contrast, suppose investment occurs but the project is not successful. Then there are three possibilities: firstly, the CRA is committed and a good project has failed; secondly, the same can happen when the CRA is opportunistic;²⁵ and thirdly, the CRA is opportunistic and a bad project - which was rated high erroneously - failed. The CRA reputation is then given by

$$\Pr(c|H, \text{ fail}) = \frac{(1 - \alpha)\beta\varphi_t}{(1 - \alpha)\beta\varphi_t + (1 - \varphi_t)(\frac{1}{2} - \tilde{s}_t + 2\beta\tilde{s}_t - \alpha\beta(\frac{1}{2} + \tilde{s}_t))} \equiv \varphi_t^{HF}. \quad (8)$$

It is theoretically possible that also a project with a low rating is financed. However, we show below that this will not occur in equilibrium (see Lemma 2). We show the following lemma.

²⁵In contrast to the first case, the correctness of the rating could be mere coincidence, which depends on the technology that has been chosen.

Lemma 1.

$\varphi_t^L \leq \varphi_t \leq \varphi_t^H$, $\beta_t^L < \beta < \beta_t^H$ and $\varphi_t^{HF} \leq \varphi_t^H \leq \varphi_t^{HS}$. That is:

1. the interim reputation weakly increases after H-ratings and weakly decreases after L-ratings,
2. the interim probability of a good project strictly increases after H-ratings and strictly decreases after L-ratings,
3. the reputation weakly increases after successful projects and weakly decreases after failures.

For the proof: See Appendix A.

Investors' and firm's choice

Using backwards induction we first focus our analysis on the decision problems of the firm and the investors. Recall that both types are short-run players, which implies that they only care about actions of the actual period.

Lemma 2.

If $r = L$ is observed, investment never occurs in equilibrium.

For the proof: See Appendix A.

Lemma 2 is straightforward since Assumption 1 implies that investors never finance unrated firms and Lemma 1 implies that the expected payoff of the risky investment will decrease after the observation of low ratings.

Suppose the rating is high. The expected payoff of investment conditional on a high rating is larger than the riskless payoff when $\alpha\beta_t^H R \geq 1$ holds. From (3) and (5), this condition is equivalent to

$$R \geq \frac{\beta\varphi_t + (1 - \varphi_t) \left(\frac{1}{2} - \tilde{s}_t + 2\beta\tilde{s}_t\right)}{\alpha\beta \left(\varphi_t + (1 - \varphi_t) \left(\frac{1}{2} + \tilde{s}_t\right)\right)} \equiv \hat{R}. \quad (9)$$

Accordingly, investment occurs as long as the firm promises to pay at least \hat{R} in case of success. The firm anticipates this optimal investment decision rule by determining the return R . When the rating is low, it follows from Lemma 2 that the firm cannot induce investment. Hence, every $R \in [0, X]$ is optimal in this case because the firm's payoff is zero anyway. Without loss of generality, let $R = X$ when $r = L$. When the rating is high, the firm can induce investment with every $R \in [\hat{R}, X]$. If $R \geq \hat{R}$, the firm's expected payoff is $E(\Pi_F | R \geq \hat{R}) = \alpha\beta_t^H(X - R)$, which is maximized for $R = \hat{R}$. If $R < \hat{R}$, no

investment takes place and Π_F is zero for certain.

Obviously it is optimal to choose $R = \hat{R}$ as long as $\hat{R} \leq X$. If \hat{R} exceeds X , no investment can be induced even when the firm promises to pay the total profit.

Suppose, without loss of generality, the firm will choose $R = X$ if $\hat{R} > X$.

CRA's decision in $t = 2$

The opportunistic CRA cannot precommit in $t = 1$ to choose a specific technology s_2 in $t = 2$. In $t = 2$, the payoff of $t = 1$ is already realized and, therefore, not relevant for the optimal action in $t = 2$. Hence, the payoff function in $t = 2$ can be reduced to

$$\Pi^{(2)} = \mathbb{1}_{\{\varphi_2 \geq \hat{\varphi}_2\}} \pi - c(s_2).$$

The reputation threshold $\hat{\varphi}_t$ can be obtained by solving (9) for φ_t and setting $\hat{R} = X$, which leads to

$$\hat{\varphi}_t = \max \left\{ \frac{\tilde{s}_t - 2\tilde{s}_t\beta + \alpha\beta X (\tilde{s}_t + \frac{1}{2}) - \frac{1}{2}}{\tilde{s}_t + \beta - 2\tilde{s}_t\beta - \alpha\beta X (\frac{1}{2} - \tilde{s}_t) - \frac{1}{2}}, 0 \right\}. \quad (10)$$

Note that $\hat{\varphi}_t$ is unique and does not depend on s_t . Whether advertisement revenue is generated in $t = 2$ is determined by φ_2 and \tilde{s}_2 ; namely, the CRA reputation at the beginning of the second period and the belief of the other market participants about the CRA technology. Accordingly, the advertisement revenue of the actual period cannot be influenced by any action in that period with implies that the marginal advertisement revenue is zero in $t = 2$. In contrast, the marginal cost yields $c' > 0$. It follows directly that the unique optimal technology in $t = 2$ is $s_2 = 0$.

The following statement summarizes the discussion above and follows directly from $s_2 = \tilde{s}_2 = 0$ and the decision rules of firm and investors.

Proposition 1.

There is a unique equilibrium of the second period play. In this equilibrium, the opportunistic CRA chooses $s_2 = 0$ and reports $r = H$ if $\sigma = h$ and $r = L$ otherwise; the committed CRA type chooses $s_2 = \frac{1}{2}$; firm's and investors' belief is $\tilde{s}_2 = 0$. Furthermore,

1. *If the rating is H and*

$$\varphi_2 \geq \frac{1 - \alpha\beta X}{1 + \alpha\beta X - 2\beta} \equiv \varphi_2^*,$$

the firm sets

$$R = \frac{1 - \varphi_2 + 2\beta\varphi_2}{\alpha\beta(1 + \varphi_2)}$$

and investment takes place.

2. If the rating is $r = L$ or if $r = H$ and $\varphi_2 < \varphi_2^*$ holds, the firm sets $R = X$ and no investment takes place.

For the proof: See Appendix A.

The second period can be interpreted as a one-shot game. Here, the opportunistic CRA has no incentive to conduct a costly rating. Rational firms and investors will expect the opportunistic CRA to choose $s_2 = 0$. However, investment will occur whenever the rating is high and the CRA reputation is good enough.

CRA decision in $t = 1$

Analog to the second period, revenues in the first period are determined through expectations and the reputation to the beginning of the game. The crucial difference to the second period is that the CRA might care about its reputation. The final reputation of the first period is identical to the reputation prior of the second period, which in turn determines the second period revenues. Since this reputation can be influenced through the technology in $t = 1$, the CRA might have an incentive to choose $s_1 > 0$. The CRA optimization problem in $t = 1$ is given by

$$\max_{s_1 \in [0, \frac{1}{2}]} \Pi^{(1)} = (\mathbb{1}_{\varphi_1 \geq \hat{\varphi}_t} + \delta \mathbb{1}_{\varphi_2 \geq \varphi_2^*}) \pi - c(s_1). \quad (11)$$

Hereby, φ_2^* is already specified in Proposition 1. The updated prior reputation $\varphi_2 \in \{\varphi_1^H, \varphi_1^L, \varphi_1^{HS}, \varphi_1^{HF}\}$ is a random variable. The distribution of φ_2 - which depends on s_1 - is given by:

$$\begin{aligned} \Pr(\varphi_2 = \varphi_1^H | \varphi_1 \geq \hat{\varphi}_1(\tilde{s}_1)) &= 0, & \Pr(\varphi_2 = \varphi_1^H | \varphi_1 < \hat{\varphi}_1(\tilde{s}_1)) &= \frac{1}{2} + (2\beta - 1)s_1, \\ \Pr(\varphi_2 = \varphi_1^L | \varphi_1 \geq \hat{\varphi}_1(\tilde{s}_1)) &= \Pr(\varphi_2 = \varphi_1^L | \varphi_1 < \hat{\varphi}_1(\tilde{s}_1)) &= \frac{1}{2} - (2\beta - 1)s_1, \\ \Pr(\varphi_2 = \varphi_1^{HF} | \varphi_1 \geq \hat{\varphi}_1(\tilde{s}_1)) &= (1 - \alpha) \left(\frac{1}{2} + s_1 \right) + (1 - \beta) \left(\frac{1}{2} - s_1 \right), \\ \Pr(\varphi_2 = \varphi_1^{HF} | \varphi_1 < \hat{\varphi}_1(\tilde{s}_1)) &= 0, \\ \Pr(\varphi_2 = \varphi_1^{HS} | \varphi_1 \geq \hat{\varphi}_1(\tilde{s}_1)) &= \alpha\beta \left(\frac{1}{2} + s_1 \right), & \Pr(\varphi_2 = \varphi_1^{HS} | \varphi_1 < \hat{\varphi}_1(\tilde{s}_1)) &= 0. \end{aligned}$$

For the following analysis it is useful to define reputation thresholds:

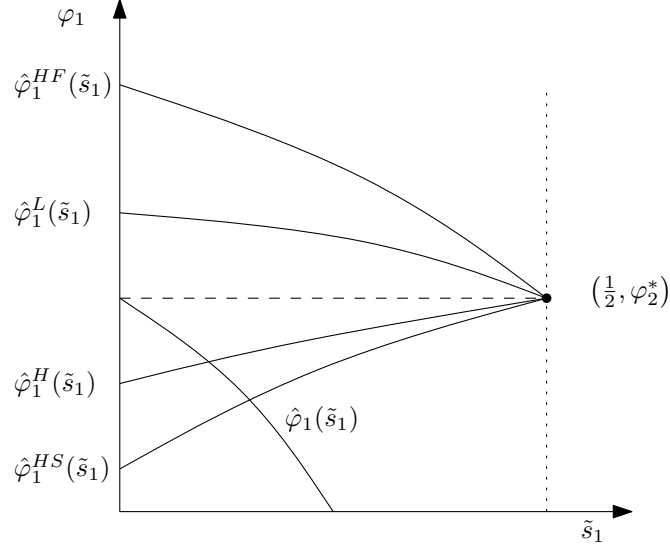


Figure 3: Reputation thresholds

Definition 2. Let $\hat{\varphi}_1^\omega = \min\{\varphi_1 : \varphi_1^\omega(\varphi) \geq \varphi_2^*\}$ for $\omega \in \{H, HF, L, HS\}$; that is, $\hat{\varphi}_1^\omega$ is the lowest reputation to the beginning of period 1 that leads to advertisement revenue in period 2 provided that the first period outcome is ω .

The threshold values $\hat{\varphi}_1^\omega$, $\omega \in \{H, HF, L, HS\}$, can be obtained by solving $\varphi_1^\omega(\varphi) = \varphi_2^*$, for φ_1 . The results can be found in Appendix B. The following lemma lists some useful properties of the reputation thresholds.

Lemma 3.

The reputation thresholds $\{\hat{\varphi}_1^H, \hat{\varphi}_1^L, \hat{\varphi}_1^{HS}, \hat{\varphi}_1^{HF}\}$ are functions of \tilde{s}_1 with properties:

1. $\hat{\varphi}_1^H(\frac{1}{2}) = \hat{\varphi}_1^L(\frac{1}{2}) = \hat{\varphi}_1^{HS}(\frac{1}{2}) = \hat{\varphi}_1^{HF}(\frac{1}{2}) = \varphi_2^*$.
2. $\hat{\varphi}_1^H$ and $\hat{\varphi}_1^{HS}$ are strictly increasing in \tilde{s}_1 .
3. $\hat{\varphi}_1^L$ is strictly decreasing in \tilde{s}_1 .
4. For every $\tilde{s}_1 \in (0, \frac{1}{2})$: $\hat{\varphi}_1^{HS} < \hat{\varphi}_1^H < \min\{\hat{\varphi}_1^L, \hat{\varphi}_1^{HF}\}$.

For the proof: see Appendix A.

The reputation thresholds functions can be illustrated by graphs in the $\varphi_1 \times \tilde{s}_1$ -space, see Figure 3 for an example.

For given beliefs (φ_1, \tilde{s}_1) , the opportunistic CRA's optimal audit technology s_1 can be determined by comparing the expected marginal revenues from using a slightly better technology with the marginal cost, which we denote by $mc =: c'(s_1)$.

Firstly, suppose $\varphi_1 < \min \{\hat{\varphi}_1(\tilde{s}_1), \hat{\varphi}_1^H(\tilde{s}_1)\}$. In this situation the credit ratings are not regarded as being informative in period two, even when the reputation rises after the publication of a high rating. Hence, independent from s_1 , there is neither investment in both periods nor advertisement revenue. Analog to the second period, since marginal costs are always positive, the optimal technology is $s_1 = 0$. The same argumentation can be applied in two other cases: firstly, when $\varphi_1 \geq \max \{\hat{\varphi}_1^L(\tilde{s}_1), \hat{\varphi}_1^{HF}(\tilde{s}_1)\}$ ²⁶; and secondly, when $\varphi_1 \in [\hat{\varphi}_1(\tilde{s}_1), \hat{\varphi}_1^{HS}(\tilde{s}_1)]$. In the first case, there will be advertisement revenue in the second period, whatever has happened in the first period. In the second case, there is revenue in the first period but not in the second.²⁷ Since in both cases the marginal revenue is zero again, the optimal technology is $s_1 = 0$.

Secondly, suppose $\varphi_1 \in [\hat{\varphi}_1^H(\tilde{s}_1), \hat{\varphi}_1(\tilde{s}_1)]$. The advertisement revenue π is then generated as long as $\varphi_2 = \varphi_1^H$, which occurs with probability $\frac{1}{2} + (2\beta - 1)s_1$. The marginal revenue is, therefore, $(2\beta - 1)\delta\pi > 0$ and the optimal technology is given implicitly by $s_1 = mc^{-1}((2\beta - 1)\delta\pi) > 0$.²⁸ The CRA has an incentive to increase the expected reputation - which raises the expected revenue in the future - by raising the audit technology above zero.

Thirdly, suppose $\varphi_1 \in [\max \{\hat{\varphi}_1^{HS}(\tilde{s}_1), \hat{\varphi}_1(\tilde{s}_1)\}, \min \{\hat{\varphi}_1^{HF}(\tilde{s}_1), \hat{\varphi}_1^L(\tilde{s}_1)\}]$. Accordingly, there is only revenue in $t = 2$ when $\varphi_2 = \varphi_1^{HS}$. The marginal revenue yields $\alpha\beta\delta\pi$ and the optimal technology is given by $s_1 = mc^{-1}(\alpha\beta\delta\pi)$, which is above zero again.

There are two cases left: first, $\varphi_1 \in [\max \{\hat{\varphi}_1^{HF}, \hat{\varphi}_1\}, \hat{\varphi}_1^L)$, which leads to $s_1 = mc^{-1}((2\beta - 1)\delta\pi)$; and second, $\varphi_1 \in [\hat{\varphi}_1^L, \hat{\varphi}_1^{HF})$, which leads to $s_1 = mc^{-1}((1 + \alpha\beta - 2\beta)\delta\pi)$.

The following proposition can be derived from the condition $s_1 = \tilde{s}_1$ and the analysis above.

Proposition 2.

The model admits three equilibrium types:

1. Everything Lost:

²⁶Note that it is not clear whether $\hat{\varphi}_1^{HF}$ is higher than $\hat{\varphi}_1^L$. This depends crucially on the parameters α and β . In particular, for high β and small α , $\hat{\varphi}_1^L < \hat{\varphi}_1^{HF}$ is possible.

²⁷Note that we also describe optimal CRA action outside from PBE. This second case cannot occur in PBE because $s_1^* = 0 < \tilde{s}_1$.

²⁸We focus on interior solutions in our discussion. Of course, it is also possible to obtain a corner solution where the marginal revenue exceeds the marginal cost at $s_1 = \frac{1}{2}$. It is optimal to choose $s_1 = \frac{1}{2}$ in this case.

For every $\varphi_1 < \hat{\varphi}_1^H(0)$, there exists a PBE in which the opportunistic CRA chooses $s_1^* = 0$ and gets the payoff $\Pi = 0$. This equilibrium is the unique equilibrium of the game for $\varphi_1 < \frac{(1-\beta)(1-\alpha\beta X)}{(1-2\beta+\alpha\beta X)^2} =: \underline{\varphi}$.

2. Resting on Laurels:

For every $\varphi_1 \geq \max\{\hat{\varphi}_1^{HF}(0), \hat{\varphi}_1^L(0)\}$, there exists a unique PBE in which the opportunistic CRA chooses $s_1^* = 0$ and gets the payoff $\Pi = (1 + \delta)\pi$.

3. Reputation Building:

For $\varphi_1 \in [\underline{\varphi}, \hat{\varphi}_1^H(0), \max\{\hat{\varphi}_1^{HF}(0), \hat{\varphi}_1^L(0)\})$, every PBE involves the opportunistic CRA choosing $s_1^* > 0$.

For $\varphi_1 \in [\underline{\varphi}, \hat{\varphi}_1^H(0))$ it is possible to have multiple PBE, where the opportunistic CRA sets $s_1^* = 0$ in one PBE and $s_1^* > 0$ in the other.

For the proof: See Appendix A.

When the CRA reputation is low, so that, independent of the first period outcome, the rating in period 2 is not regarded as being informative, the opportunistic CRA will always shirk. We call this situation an “everything-lost equilibrium”. The same technology will be chosen when the CRA reputation is very high. Even when the reputation breaks down in case of a failed project, in the first period, the reputation is still high enough so that the rating is regarded informative in the second period. We call this equilibrium type “resting on laurels” because the opportunistic CRA can exploit its high reputation and can allow itself to be lazy about conducting an elaborate assessment. Of course the first two equilibrium types are extreme cases, which are driven by an endgame-effect of our two-period model. Nevertheless, they provide some important insights into incentives that may occur in the real world. When the reputation is either very high or very low, there are, at least in the short run, strong incentives to shirk. Between these extreme cases it is possible to find PBE, where the CRA conducts a costly rating process. The only incentive not to shirk is to raise the probability that advertisement revenue is generated in the future. We call this reputation building.

4 Numerical Examples

Example 1

Let $\alpha = 0.7$, $\beta = 0.7$, $\delta = 0.5$, $\pi = 2$, $X = 1.7$ and $c(s) = s^2$. The PBE in pure strategies for this example are listed in Table 1.

TABLE 1: Audit technologies in perfect Bayesian equilibria for different prior reputations

Reputation area	$s_1^* = s_1$	Revenue if	$Pr(\text{Revenue in } t = 2)$	$E(\Pi)$	Equilibrium Class
[0.000, 0.310)	0.000	/	0.000	0.000	everything lost
[0.319, 0.411)	0.245	HS	0.365	2.305	reputation building
[0.415, 0.468)	0.200	HF, HS	0.580	2.540	reputation building
[0.511, 1.000]	0.000	L, HF, HS	1.000	3.000	resting on laurels

$$\alpha = 0.7, \beta = 0.7, \delta = 0.5, \pi = 2, X = 1.7, c(s) = s^2.$$

The highest decrease in reputation occurs when the rating is low in $t = 1$. As long as the prior reputation is higher than $\hat{\varphi}_1^L(0) = 0.51$, the reputation in the second period is still high enough to generate advertisement revenue, even when a low rating was published in the first period. Hence, the CRA can rest on its laurels and chooses $s_1^* = 0$. Point A in Figure 4 (1) and (2) illustrates an example with $\varphi_1 = 0.53$. When the prior reputation is below $\varphi_1^L(\tilde{s}_1)$ but at least $\varphi_1^{HF}(\tilde{s}_1)$, the probability of obtaining advertisement revenue in $t = 2$ can be increased by using technologies better than zero. That is, by decreasing the probability of a low rating. Marginal revenue is given by $(2\beta - 1)\delta\pi = 0.4$ and marginal cost by $c'(s_1) = 2s_1$. The interior solution to (11) is then $s_1^* = 0.2$ and prior reputation area where this occurs in equilibrium is given by $[\varphi_1^{HF}(0.2), \varphi_1^L(0.2)] = [0.415, 0.468)$. Point B in Figure 4 (1) and (2) refers to this case.

The second case of reputation building equilibria occurs when $\varphi_1 \in [0.319, 0.411)$. Revenue in $t = 2$ is generated only when the first period outcome is a successful project. Point C is one example.

For some prior beliefs, no pure PBE can be identified. For instance, let $\varphi \in [0.468, 0.511)$. When the investors belief is $\tilde{s}_1^{(1)} = 0$, CRA's profit is maximized for $s_1^{(2)} = 0.2$.²⁹ Conversely, when $\tilde{s}_1^{(2)} = 0.2$, CRA's profit is maximized for $s_1^{(1)} = 0$. The same happens when $\varphi_1 \in [0.310, 0.319) \cup [0.411, 0.415)$. In these cases it is always possible to identify PBE in mixed strategies. For our example we have to find a probability distribution over the beliefs $(\tilde{s}_1^{(1)}, \tilde{s}_1^{(2)})$ that makes the CRA indifferent between the technologies $(s_1^{(1)}, s_1^{(2)})$.

Let q denote the probability that the market belief is $s_1^{(1)}$. The CRA is indifferent

²⁹Which is the optimal action for all $\varphi_1 < [0.468, \varphi_1^L(\tilde{s}_1)]$.

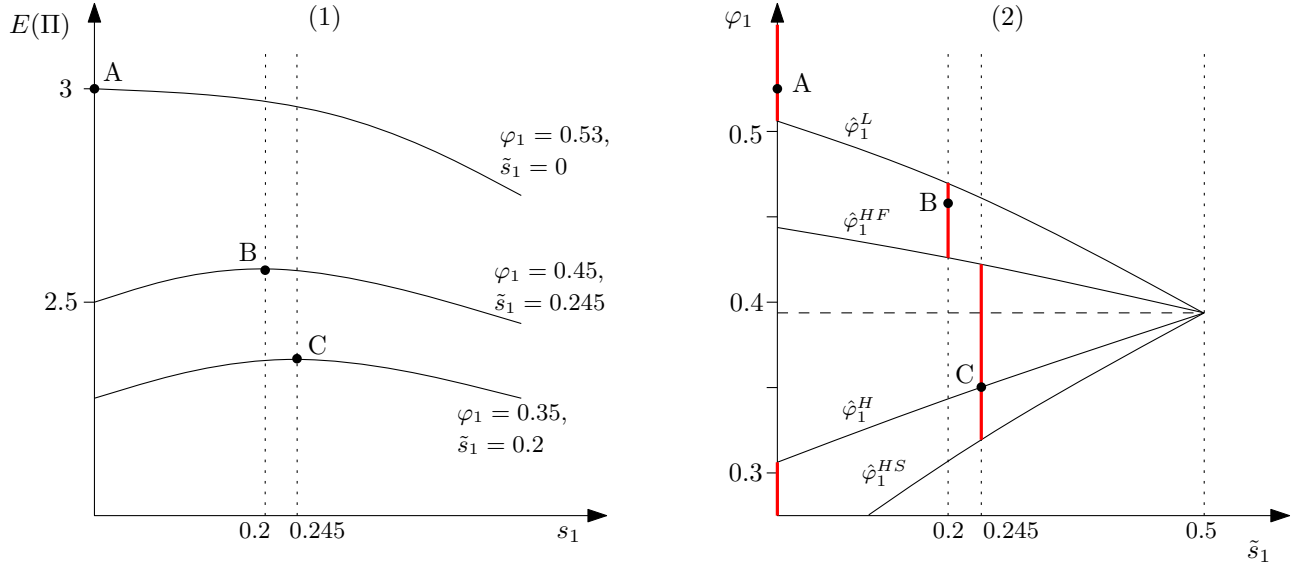


Figure 4: Expected CRA payoff, reputation thresholds and PBE (displayed in red). ($\alpha = 0.7, \beta = 0.7, \delta = 0.5, \pi = 2, X = 1.7$ and $c(s) = s^2$)

		Market Beliefs	
		$\tilde{s}_1^{(1)} = 0, [q]$	$\tilde{s}_1^{(2)} = 0.2, [1-q]$
CRA Action	$s_1^{(1)} = 0$	2.96	2.54
	$s_2^{(2)} = 0.2$	3	2.5

Figure 5: Expected CRA payoff for $\varphi \in [0.468, 0.511)$.

between $s_1^{(1)}$ and $s_1^{(2)}$ for $q = 0.5$. In a mixed PBE the CRA will randomize between these two technologies, so that both are chosen with probability 0.5, which is anticipated by the market. The expected CRA payoff yields 2.75.

Example 2

In the second numerical example, which is displayed in Figure 5 and Table 2, multiple equilibria emerge when the prior reputation is between 0.516 and 0.6.

TABLE 2: Example with multiple perfect Bayesian equilibria

Reputation area	$s_1^* = s_1$	Revenue if	$Pr(\text{Revenue in } t = 2)$	$E(\Pi)$	Equilibrium Class
[0.000, 0.600)	0.000	/	0.000	0.000	everything lost
[0.516, 0.641)	0.144	HS	0.248	1.655	reputation building
[0.642, 0.735)	0.107	L,HS	0.723	2.031	reputation building
[0.755, 1.000]	0.000	HF, L, HS	1.000	2.250	resting on laurels

$$\alpha = 0.7, \beta = 0.55, \delta = 0.5, \pi = 1.5, X = 1.7, c(s) = s^2$$

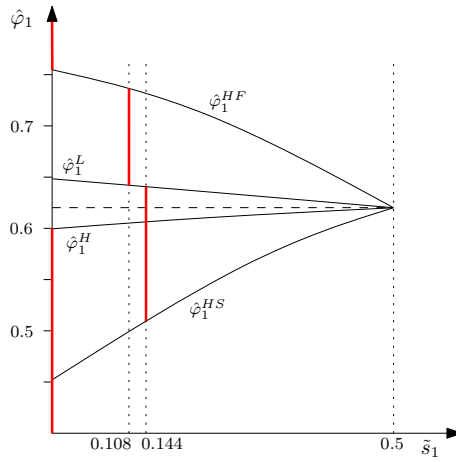


Figure 6: Reputation thresholds and multiple PBE (displayed in red). ($\alpha = 0.7, \beta = 0.55, \delta = 0.5, \pi = 1.5, X = 1.7, c(s) = s^2$)

The possibility of multiple equilibria was already mentioned in Proposition 2. The value $\underline{\varphi}$ for our example yields 0.506, which is the reputation at the intersection of $\hat{\varphi}_1^{HS}(\tilde{s}_1)$ and $\hat{\varphi}_1(\tilde{s}_1)$. Note that, by contrast to the first numerical example, $\hat{\varphi}_1^{HF}(\tilde{s}_1) > \hat{\varphi}_1^L(\tilde{s}_1)$ holds, which is caused by a much lower β . In consequence, when the first period outcome is a low rating (and no investment), the reputation decrease is smaller now.

5 Extension

In this section we analyze whether the opportunistic CRA has an incentive to misreport, whereby misreporting means that a rating is published which differs from the signal that the CRA has received. It is assumed that the committed CRA never misreports. Hence, compared to our basic model, we just have to drop Assumption 3 and extent the action

space of the opportunistic CRA and the belief system of firm and investors. Let $p_\sigma \in [0, 1]$ denote the probability that the CRA decides to publish a high rating when it reaches the information set $\sigma \in \{h, l\}$, and let \tilde{p}_σ denote the correspondent market belief. In our basic model it has been followed from Assumption 3 that $p_h = \tilde{p}_h = 1$ and $p_l = \tilde{p}_l = 0$. In what follows we first want to analyze whether the equilibria of the basic model survive. For that we analyze, in a first step, the case of naive investors who still believe that the CRA never misreports.³⁰ We search for profitable deviations by the CRA, which takes the naive beliefs as given. After that, we search for equilibria where misreporting can be anticipated by the market.

Naive Investors

Firstly, consider an everything-lost equilibrium. Due to the bad reputation, the CRA will never receive advertisement revenue in $t = 2$ anyway. The payoff yields $-c(s_1)$, which is independent from p_h and p_l . Taking the belief \tilde{p}_h and \tilde{p}_l as given, every $p_h, p_l \in [0, 1]$ is an optimal action. Of course, it also remains optimal to choose $s_1 = 0$. Accordingly, every equilibrium that belongs to the everything-lost-class survives in our extension.

Secondly, note that the analysis of the resting-on-laurel-case is very similar because the payoff always yields $(1 - \delta)\pi - c(s_1)$. It is straightforward that choosing $s_1 = 0, p_h = 1, p_l = 0$ remains optimal. The reputation-building equilibria are more interesting. Here it is possible to find profitable deviations. Consider, for instance, $\hat{\varphi}_1^{HF}(\tilde{s}_1^*) < \varphi_1 < \hat{\varphi}_1^L(\tilde{s}_1^*)$ and assume the CRA has chosen $s_1^* > 0$.³¹ Regardless of the received signal and the chosen technology, it is now optimal to publish a H-rating; i.e. the CRA deviates to $p_h = p_l = 1$. This deviation is strictly profitable because L -ratings never lead to revenue in the second period, while H -ratings lead to revenue with strictly positive probability.³²

Moreover, because the rating will be high anyway, it follows from backwards induction that $s_1 = 0$ is optimal now. Hence, the old equilibrium of the basic model will not survive, because, given the naive market beliefs, the CRA has an incentive to choose a low audit

³⁰However, even the naive investors consider the possibility of shirking.

³¹Point B in Figure 4 illustrates an example of this case.

³²Which is the signal contingent probability that the CRA assigns to the event that the project is successful. From (1) and (2) the probability yields $\frac{\alpha\beta(\frac{1}{2}+s_t)}{\beta(\frac{1}{2}+s_t)+(1-\beta)(\frac{1}{2}-s_t)}$ if $\sigma = h$ or $\frac{\alpha\beta(\frac{1}{2}-s_t)}{\beta(\frac{1}{2}-s_t)+(1-\beta)(\frac{1}{2}+s_t)}$ if $\sigma = l$.

technology and conduct rating inflation. It is straightforward that a profitably deviation can be identified for almost every reputation-building equilibrium. The CRA will always shirk and, depending on the market beliefs, either conduct rating inflation or deflation.

Proposition 3.

When investors and firms are naive ($\tilde{p}_h = 1, \tilde{p}_l = 0$):

1. *Every PBE of the basic model survives as long as it belongs either to the “everything-lost”- or the “resting on laurels”-class.*
2. *No PBE that belongs to the “reputation-building”-class survives because the CRA always has a strictly profitable deviation in conducting either rating inflation or rating deflation.*

For the proof: See Appendix A.

Thus, the advertisement-based model creates incentives to misreport when the investors are naive. In the following we investigate whether the incentives for misreporting maintain when they are considered by investors.

Sophisticated Investors

It is straightforward to see that truthful reporting can occur in equilibrium when CRA reputation is either very low or very high. Suppose sophisticated investors belief is $\tilde{s}_1 = 0, \tilde{p}_l = 0, \tilde{p}_h = 1$. Note that this is merely a special case of our basic model. It follows then by Proposition 2 and by Proposition 3 that truthful reporting is possible in every equilibrium that belongs to the resting-on-laurels or everything-lost class. Although the opportunistic CRA reports the signal honestly, the signal is not informative because both equilibrium-classes involve shirking.

Proposition 4.

When investors and firms are sophisticated:

1. *Truthful reporting occurs in equilibrium only when the CRA reputation is either very high or very low. The signal is not informative in this case.*
2. *The CRA will conduct either rating inflation or rating deflation with positive probability when reputation is at a medium level.*

Proof. The first part follows from Proposition 2, Proposition 3 and our discussion above. Assume, to the contrary of the second part, that reputation is at a medium level; namely that $\varphi_1 \in [\hat{\varphi}_1(0), \max \{ \hat{\varphi}_1^{HF}(0), \hat{\varphi}_1^L(0) \})$ holds, and further assume that the opportunistic CRA uses $p_h = 1, p_l = 0$ in equilibrium. Equilibrium requires then $\tilde{p}_h = 1, \tilde{p}_l = 0$.

But these are exactly the naive investors' beliefs and, therefore, Proposition 3 can be applied. It follows that the CRA always has a strictly positive deviation, which generates a contradiction to our assumption above. \square

Even when direct commercial connections between the CRA and the other market participants are eliminated, misreporting or shirking will occur in equilibrium. Compared to the traditionally used business models, the incentives for misreporting have a different source in the advertisement model. Rating inflation or deflation is not used to satisfy issuers or investors. Instead, the CRA will publish ratings that copy investors belief about the market situation; namely, conducting rating inflation (deflation) when β is sufficiently high (low). When β is sufficiently high, the decrease in reputation after a failed project is harmless compared to reputation loss after L -ratings. But also when β is low, it is optimal for the CRA to shirk and publish a rating that matched the market beliefs.

6 Conclusion

In the recent years, credit rating agencies (CRAs) have been criticized for publishing distorted credit ratings. It is argued that traditional business models of CRAs create incentives to publish either inflated or deflated ratings, since CRAs are paid by directly by either issuers of financial products or potential investors.

In this paper, we investigate an alternative business-model in which the CRAs receive revenue from online advertisement. The ratings are published free of charge on a website. As long as the ratings are regarded as useful information for investment decisions, many potential investors will visit the website and advertisement revenue is generated.

The obvious advantage of the advertisement-based model seems to be that the commercial connections between the CRA and the participants of the debt market are weaker. Hence, one may conjecture that incentives to misreport are mitigated. Moreover, CRAs might invest more effort into assessing the quality of financial products and care more about their reputation.

We use a two-period Bayesian reputation model and show that these conjectures are wrong. In our basic model we abstract away the possibility of misreporting. Instead we

focus on the possibility to shirk, which means that ratings are published without making an elaborate assessment in advance. We show that CRAs will shirk when their reputation is either sufficiently high or sufficiently low. In the short run, high reputation can be exploited for shirking since advertisement revenue is generated anyway. On the contrary, when reputation is low, investors do not regard ratings as valuable information and there is no short run incentive to conduct costly assessments. Only when the reputation is at a medium level, the prospect of exploiting better reputation in the future might discipline CRAs to work hard in the present.

In our extension we also allow for misreporting. We show that the CRA will conduct either rating deflation or rating inflation when the reputation is at a medium level. The reason is that the CRA can maintain its reputation by publishing ratings that match the market beliefs.

The main insight from this paper is that an advertisement-based model for CRAs will neither mitigate incentives for misreporting nor for shirking. Even when commercial connections between the CRA and issuers or investors are eliminated, truthful reporting cannot be induced.

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A Proofs

Proof of Lemma 1

1. For proving the first part, differentiate (3) and (4) with respect to \tilde{s}_t , which yields

$$\frac{d\varphi_t^H}{d\tilde{s}_t} = \frac{-\beta\varphi_t(1-\varphi_t)(2\beta-1)}{(\beta\varphi_t + (1-\varphi_t)(\frac{1}{2} - \tilde{s}_t + 2\beta\tilde{s}_t))^2} \quad \text{and} \quad (12)$$

$$\frac{d\varphi_t^L}{d\tilde{s}_t} = \frac{(1-\beta)\varphi_t(1-\varphi_t)(2\beta-1)}{((1-\beta)\varphi_t + (1-\varphi_t)(\frac{1}{2} + \tilde{s}_t - 2\tilde{s}_t\beta))^2}. \quad (13)$$

It follows then from Assumption 2 that $\frac{d\varphi_t^H}{d\tilde{s}_t} < 0 < \frac{d\varphi_t^L}{d\tilde{s}_t}$. Hence, if $\tilde{s}_t = \frac{1}{2}$, φ_t^H takes the maximum value and φ_t^L takes the minimum value. Inserting $\tilde{s}_t = \frac{1}{2}$ in (3) and (4) leads to

$$\varphi_t^H|_{\tilde{s}_t=\frac{1}{2}} = \varphi_t^L|_{\tilde{s}_t=\frac{1}{2}} = \varphi_t. \quad (14)$$

Accordingly, there is no interim reputation update if $\tilde{s}_t = \frac{1}{2}$. For every $\tilde{s}_t \in [0, \frac{1}{2})$, it follows from $\frac{d\varphi_t^H}{d\tilde{s}_t} < 0 < \frac{d\varphi_t^L}{d\tilde{s}_t}$ that $\varphi_t^L < \varphi_t < \varphi_t^H$ holds, which proves the first part of Lemma 1.

2. The second part can be proven similar to the first part. We obtain

$$\beta_t^H = \frac{\varphi_t\beta + (1-\varphi_t)\beta(\frac{1}{2} + \tilde{s}_t)}{\varphi_t\beta + (1-\varphi_t)(\beta(\frac{1}{2} + \tilde{s}_t) + (1-\beta)\beta(\frac{1}{2} - \tilde{s}_t))} \quad \text{and} \quad (15)$$

$$\beta_t^L = \frac{\varphi_t\beta + (1-\varphi_t)\beta(\frac{1}{2} + \tilde{s}_t)}{\varphi_t\beta + (1-\varphi_t)(\beta(\frac{1}{2} + \tilde{s}_t) + (1-\beta)\beta(\frac{1}{2} - \tilde{s}_t))}, \quad (16)$$

by inserting (3) in (5) and (4) in (6). The derivatives with respect to \tilde{s}_t are given by

$$\frac{d\beta_t^L}{d\tilde{s}_t} = \frac{-4\beta(1-\beta)(1-\varphi_t)}{(2\tilde{s}_t(2\beta-1)(\varphi_t-1) + \varphi_t + 1 - 2\beta\varphi_t)^2} < 0 < \frac{4\beta(1-\beta)(1-\varphi_t)}{(2\tilde{s}_t(2\beta-1)(\varphi_t-1) + \varphi_t - 1 - 2\beta\varphi_t)^2} = \frac{d\beta_t^H}{d\tilde{s}_t}. \quad (17)$$

Inserting $\tilde{s}_t = 0$ in (15) and (16) leads to

$$\beta_t^L|_{\tilde{s}_t=0} = \frac{(1+\varphi_t)\beta}{1+(1-2\beta)\varphi_t} < \beta < \frac{(1+\varphi_t)\beta}{1+(2\beta-1)\varphi_t} = \beta_t^H|_{\tilde{s}_t=0}. \quad (18)$$

(17) and (18) imply $\beta_t^L < \beta < \beta_t^H$, which proves the second part of Lemma 1.

3. What remains to be shown is that $\varphi_t^{HF} \leq \varphi_t \leq \varphi_t^{HS}$ holds. $\varphi_t^H \leq \varphi_t^{HS}$ can be obtained easily by comparing (3) with (7) and from $\beta < 1$. The inequality holds strictly for $\tilde{s}_t < \frac{1}{2}$. Furthermore, it is straightforward to show from (8) and (3) that $\varphi_t^{HF} \leq \varphi_t$ holds for $\alpha > 0$, which proof the lemma. \square

Proof of Lemma 2

Assume to the contrary that investors decide to invest a project that has obtained a L -rating. In equilibrium it is then required that

$$\alpha\beta_t^L R \geq 1; \quad (19)$$

namely, investors expected payoff from investment conditional upon a L -rating is higher than the certain payoff from the outside option. However, it can be shown that

$$\alpha\beta_t^L R < \alpha\beta R \leq \alpha\beta X < 1. \quad (20)$$

It is known from Lemma 1 that $\beta_t^L < \beta_t$ holds, which explains the first inequality of (20). The second inequality is obtained by $R \leq X$; i.e., the firm cannot promise to pay the investor more than the total profit in case of success. The third inequality comes directly from Assumption 1: without any rating published, the probability of success is too low to induce investment. Obviously, (19) and (20) build a contradiction, which proofs the lemma. \square

Proof of Proposition 1 It is already argued in Section 3 that the opportunistic CRA will choose $s_2 = 0$, which is the unique optimal action in $t = 2$. The committed CRA has the strictly dominant action $s_2 = \frac{1}{2}$. The CRA rating behavior follows then directly from Assumption 3. The remaining two statements can be proven as follows:

1. Equilibrium requires $\tilde{s}_2 = s_2$. Therefore, the critical reputation φ_2^* is obtained by inserting $\tilde{s}_2 = 0$ in (10). Analogously, R_2^* is obtained from (9) by inserting $\tilde{s}_2 = 0$ in \hat{R} .
2. The last part follows directly from the preceding analysis in Section 3. \square

Proof of Lemma 3

1. The first part is easily proven by inserting $\tilde{s}_1 = \frac{1}{2}$ in (21), (22), (23) and (24). When firm and investors believe that both CRA types pool in $t=1$, there is no belief update. In that case, the CRA revenues in $t=2$ are generated if and only if $\varphi_1 \geq \varphi_2^*$, which is independent on the first period outcome.

2. Deriving (21) and (22) with respect to \tilde{s}_1 yields

$$\frac{d\hat{\varphi}_1^{HS}}{d\tilde{s}_1} = \frac{8\beta(X\alpha - 1)(1 - X\alpha\beta)}{(2\tilde{s}_1 - 4\beta + 3X\alpha\beta - 2X\tilde{s}_1\alpha\beta + 1)^2}$$

and

$$\frac{d\hat{\varphi}_1^H}{d\tilde{s}_1} = \frac{8\beta^2(2\beta - 1)(X\alpha - 1)(1 - X\alpha\beta)}{(2\tilde{s}_1 - 4\tilde{s}_1\beta + 4\beta^2 + X\alpha\beta - 4X\alpha\beta^2 - 2X\tilde{s}_1\alpha\beta + 4X\tilde{s}_1\alpha\beta^2 - 1)^2}.$$

It follows from Assumptions 1 and 2 that both derivatives are strictly positive.

3. From (23) it follows that

$$\frac{d\hat{\varphi}_1^L}{d\tilde{s}_1} = \frac{8\beta(X\alpha - 1)(1 - X\alpha\beta)(3\beta - 2\beta^2 - 1)}{(2\tilde{s}_1 - 4\beta - 4\tilde{s}_1\beta + 4\beta^2 + 3X\alpha\beta - 4X\alpha\beta^2 - 2X\tilde{s}_1\alpha\beta + 4X\tilde{s}_1\alpha\beta^2 + 1)^2},$$

which is, from Assumptions 1 and 2, strictly smaller than 0.

4. Note that the following argumentation is true for $\tilde{s} < \frac{1}{2}$. For $\tilde{s} = \frac{1}{2}$, there is no belief update and all reputation thresholds are equal (see point 1 of this lemma).

The first inequality of Lemma 1.4 is obviously fulfilled, since reputation rises after the observation of successful projects. Reputation falls when a project fails, therefore, the threshold $\hat{\varphi}_1^{HF}$ has to be higher than $\hat{\varphi}_1^H$. Finally, it follows from Assumption 2 that $\varphi_1^H > \varphi_1^L$, which directly leads to $\hat{\varphi}_1^H < \hat{\varphi}_1^L$. \square

Proof of Proposition 2 A PBE for the basic model requires consistency between equilibrium strategies and beliefs in both periods. For every first period play, Proposition 1 specifies a unique equilibrium play of the continuation game in the second period. Hence, it is sufficient to focus on the first period. A PBE of the 2-period game is found when $s_1 = \tilde{s}_1$ and $s_1 \in \arg \max_{s_1 \in [0, \frac{1}{2}]} E(\Pi|\tilde{s}_1)$ holds.

1. Let $\tilde{s}_1 = 0$. Since $\hat{\varphi}_1^H(0) < \hat{\varphi}_1(0)$, it follows from $\varphi_1 < \hat{\varphi}_1^H(0)$ that $E(\Pi|\tilde{s}_1 = 0) = -c(s_1)$. From $c' > 0$, it is straightforward that $\tilde{s}_1^* = s_1^* = 0 = \arg \max_{s_1 \in [0, \frac{1}{2}]} E(\Pi|\tilde{s}_1)$. Accordingly, there is a PBE in which the opportunistic CRA chooses $s_1^* = 0$ and gets the payoff $\Pi = 0$. What remains to be shown is uniqueness for $\varphi_1 < \underline{\varphi}$. First note, that $\underline{\varphi}$ is the reputation at the intersection between $\hat{\varphi}_1(\tilde{s}_1)$ and $\hat{\varphi}_1^{HS}(\tilde{s}_1)$, which can be obtained from (10) and (21). From Lemma 3 it follows that $\hat{\varphi}_1(\tilde{s}_1)$ decreases, whereas $\hat{\varphi}_1^{HS}(\tilde{s}_1)$ increases; which ensures that $\underline{\varphi}$ is unique. Assume, to the contrary,

that $\varphi_1 < \underline{\varphi}$ holds and that the PBE from above is not unique. In that case there is another PBE in which the CRA chooses $s_1 > 0$. From $\varphi_1 < \underline{\varphi}$ it follows that $E(\Pi|\tilde{s}_1)$ is either $-c(s_1)$ or $\pi - c(s_1)$. It is straightforward to see that the CRA will choose $s_1 = 0$ because both expressions strictly decrease in s . This generates a contradiction to our above assumption and proves the first part of the proposition.

2. Let $\tilde{s}_1 = 0$ again. It follows from $\varphi_1 \geq \max\{\hat{\varphi}_1^{HF}(0), \hat{\varphi}_1^L(0)\}$ that $E(\Pi|\tilde{s}_1 = 0) = (1 - \delta)\pi - c(s_1)$. From $c' > 0$, it is straightforward that $\tilde{s}_1^* = s_1^* = 0 = \arg \max_{s_1 \in [0, \frac{1}{2}]} E(\Pi|\tilde{s}_1)$. Accordingly, there is a PBE in which the opportunistic CRA chooses $s_1^* = 0$ and gets the payoff $\Pi = (1 + \delta)\pi$. Uniqueness can be derived by an argumentation which is analog to the preceding point.
3. Suppose, to the contrary, there is a PBE in which the CRA chooses $s_1 = 0$ and the reputation prior is $\varphi_1 \in [\hat{\varphi}_1^H(0), \max\{\hat{\varphi}_1^{HF}(0), \hat{\varphi}_1^L(0)\})$. It follows from the analysis in Section 3 that the CRA has a strictly profitable deviation in increasing s_1 above zero, which generates a contradiction.

□

Proof of Proposition 3 The first part follows directly from the discussion in Section 5. For the second part we have to search for strictly profitable deviations, which we identify in what follows for every $\varphi_1 \in [\hat{\varphi}_1^H(\tilde{s}_1), \max\{\hat{\varphi}_1^{HF}(\tilde{s}_1), \hat{\varphi}_1^L(\tilde{s}_1)\})$.

There are two cases. Firstly, suppose $\varphi_1 < \hat{\varphi}_1^L(\tilde{s}_1)$. In this case it is profitable to deviate to $p_h = p_l = 0$, since L -ratings lead to revenue in $t = 2$ for certain and H -ratings only with probability $\alpha \Pr(g|\sigma) < 1$. Accordingly, the opportunistic CRA will conduct rating deflation. Secondly, suppose $\varphi_1 \geq \hat{\varphi}_1^L(\tilde{s}_1)$. In contrary to the preceding case, it is profitable to deviate to $p_h = p_l = 1$ because L -ratings never lead to revenue in $t = 2$ and H -ratings with probability $\alpha \Pr(g|\sigma) > 0$. Accordingly, the opportunistic CRA will conduct rating inflation.

□

B Reputation thresholds of the basic model

The reputation thresholds are obtained by equating (3), (4), (7) and (8) with φ_2^* , which is specified in Proposition 1, and solve for φ_1 , which yields:

$$\hat{\varphi}_1^{HS} = \frac{1 + 2\tilde{s}_1 - \alpha\beta X - 2\tilde{s}_1\alpha\beta X}{2\tilde{s}_1 - 4\beta + 3\alpha\beta X - 2\tilde{s}_1\alpha\beta X + 1}, \quad (21)$$

$$\hat{\varphi}_1^H = \frac{(1 - \alpha\beta X)(2\tilde{s}_1 - 4\tilde{s}_1\beta - 1)}{2\tilde{s}_1 - 4\tilde{s}_1\beta + 4\beta^2 + \alpha\beta X - 4\alpha\beta^2 X - 2\tilde{s}_1\alpha\beta X + 4\tilde{s}_1\alpha\beta^2 X - 1}, \quad (22)$$

$$\hat{\varphi}_1^L = \frac{(1 - \alpha\beta X)(2\tilde{s}_1 - 4\tilde{s}_1\beta + 1)}{2\tilde{s}_1 - 4\beta - 4\tilde{s}_1\beta + 4\beta^2 + 3\alpha\beta X - 4\alpha\beta^2 X - 2\tilde{s}_1\alpha\beta X + 4\tilde{s}_1\alpha\beta^2 X + 1} \quad (23)$$

and

$$\hat{\varphi}_1^{HF} = \frac{2(1 - \alpha\beta X)(\tilde{s}_1 - 2\tilde{s}_1\beta + \frac{1}{2}\alpha\beta + \tilde{s}_1\alpha\beta - \frac{1}{2})}{2\tilde{s}_1 - 4\tilde{s}_1\beta + 4\beta^2 + \alpha\beta - 4\alpha\beta^2 + 3\alpha^2\beta^2 X + \alpha\beta X + 2\tilde{s}_1\alpha\beta - 4\alpha\beta^2 X - 2\tilde{s}_1\alpha^2\beta^2 X - 2\tilde{s}_1\alpha\beta X + 4\tilde{s}_1\alpha\beta^2 X - 1}. \quad (24)$$