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Optimal Structure and Dissolution of Partnerships*

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Abstract

We study a partnership model with non-identical type distributions and interdependent values. For any convex combination of revenue and social surplus in the objective function, we derive the optimal dissolution mechanism for arbitrary initial ownership and use this mechanism to determine the optimal initial ownership structures. These ownership structures are nontrivial because private information is a transaction cost that makes the model non-Coasian. Equal ownership is always optimal with identical distributions but not with non-identical distributions. When distributions are ranked by stochastic dominance, stronger agents receive higher initial ownership shares when the weight on revenue is small but not necessarily when it is large.

Keywords: partnership dissolution, optimal dissolution mechanisms, efficient frontier, optimal property rights, beyond the Coase irrelevance, interdependent values, non-identical distributions.

JEL-Classification: C72, D44, D61, D82

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1 Introduction

The Coase Theorem provides the important insight that the connection between the efficiency of the final allocation and the initial ownership structure depends on the ease or difficulty with which property rights can be reallocated. Accordingly, the final allocation will be efficient irrespective of the initial ownership structure if transaction costs are negligible and property rights are well-defined. There is ample evidence by now that initial misallocations are not always easily and quickly mended through subsequent transactions, indicating in the light of the Coase Theorem that transaction costs can be substantive.¹

The theoretical literature on the impossibility of ex post efficient bilateral trade in the tradition of Myerson and Satterthwaite (1983) has identified private information as an important and often insurmountable cost of transaction. Relaxing the assumption of extreme ownership structure that underlies the bilateral trade setup, the partnership literature, initiated by Cramton, Gibbons, and Klemperer (1987), has highlighted that with appropriately chosen ownership structures ex post efficient dissolution – that is, efficient reallocation of property rights – may be possible. While obviously important, ex post efficiency is only one of many possible and plausible objectives. In particular, one may wonder what is the second-best dissolution mechanism when ex post efficiency is not possible, and more generally and more fundamentally, what ownership structures are optimal when the mechanism has to generate positive revenue, for example, to cover legal expenses or taxes.

In this paper, we answer these questions. We analyze a general partnership model that permits an arbitrary number of agents, non-identical type distributions, interdependent values, and any convex combination of revenue and social surplus in the designer’s objective. For example, our model lends itself naturally to the problem of optimally allocating shares to cash-constrained partners in a start-up company.² For any initial ownership structure, we first derive the optimal dissolution mechanism, subject to incentive compatibility and individual rationality constraints. Then we choose the ownership structure to maximize the designer’s objective function and thereby determine the optimal structure of initial ownership.

Partnerships with shared initial ownership create countervailing incentives (Lewis and Sappington, 1989) at the dissolution stage insofar as an agent may end up buying additional shares or selling his share. An agent’s expected utility is typically minimized for types for whom the

¹For example, Bleakley and Ferrie (2014) show that initial land parcel size after the opening of the frontier in Georgia predicts farm size essentially one-for-one for 50-80 years after land opening, with the effect of initial conditions attenuating gradually and disappearing only after 150 years. Milgrom (2004) makes a similar point in the context of the allocation of radio spectrum licenses, and Che and Cho (2011) describe vividly the inefficiencies associated with the Oklahoma land rush at the turn to the 20th century. Interestingly, Coase’s own argument (Coase, 1959) favoring the use of auctions to allocate spectrum licenses is consistent with the notion that subsequent market transactions will not easily fix initial misallocations, which is the central premise of the insightful Theorem that bears his name (Coase, 1960) and that continues to be influential in public policy debates.

²The requirement that the dissolution mechanism generates positive revenue can be justified in a number of mutually non-exclusive ways as an additional transaction cost in the form of lawyers or government-levied taxes that need to be paid.

expected after-dissolution share equals the initial ownership share. These worst-off types do not get any information rent because they are indifferent between over- and underreporting. As they depend on the allocation rule, the worst-off types – which are the types for which the individual rationality constraint binds – are endogenous to the design problem. Overcoming the problem of simultaneously determining the optimal allocation rule and the endogenous worst-off types represents the main technical challenge when studying optimal dissolution mechanism.

Given a critical type for each agent, we define the virtual surplus as the value of the allocation in terms of virtual types. An agent's virtual type equals his virtual cost for types below the critical type and his virtual valuation for types above it, reflecting binding upward and downward incentive constraints. We then show that there is an essentially unique combination of critical types and an allocation rule such that, firstly, the allocation rule maximizes the virtual surplus given the critical types, and secondly, the critical types are worst-off types under the allocation rule. This is the allocation rule of all optimal dissolution mechanisms. Because virtual costs always exceed virtual valuations, the optimal dissolution mechanisms allocate based on ironed virtual type functions that are flat for types around the critical type. For some ownership structures, critical types are such that ties in terms of ironed virtual types happen with strictly positive probability. In this case, a suitably specified tie-breaking rule is an essential ingredient to the optimal allocation rule (ensuring that the critical type of each agent is worst off).

Because the initial ownership structure defines the agents' outside option at the dissolution stage, it affects the outcome. We show that any weighted sum of surplus and revenue generated by the optimal dissolution mechanisms is concave in the initial shares, allowing us to characterize optimal ownership structures by first-order conditions.

If types are identically distributed, we find that the value function is Schur-concave in property rights, i.e., more equal ownership structures are (weakly) better. Moreover, the set of optimal initial shares is larger the greater is the weight on revenue in the objective. This implies that with identical distributions equal shares are optimal for any weight on revenue. Thus, with ex ante identical agents a symmetric ownership structure is robust in a way that is analogous to the robustness obtained by Neeman (1999) in the context of a public good problem. However, Schur-concavity and robustness do not generalize beyond the setup with identical distributions.

If types are drawn from different distributions, we find that under private values the optimal ownership structures are such that each agent with a nonzero initial share has the same critical worst-off type, whereas the agents with zero shares have a higher critical worst-off type. As critical worst-off types vary with the distributions and the weight on revenue, asymmetric initial shares are typically optimal and, most importantly, these shares vary with the revenue weight. An agent that is optimally chosen to be the majority owner when revenue is relatively unimportant, may be optimally assigned a minority share when the weight on revenue is high. Under interdependent values, a similar characterization as under private values holds: to each critical worst-off type there is simply a constant added that reflects both the interdependence and the agent's type distribution.

To summarize, a symmetric ownership structure is always optimal for a partnership model in

which agents draw their types from the same distribution. This symmetric ownership structure is detail-free (Wilson, 1987) insofar as it does not depend on the specifics of the distribution, provided the distribution is the same for every agent.³ Further, for identical distributions an extreme ownership structure is never optimal, irrespective of the weight on revenue and of the severity of interdependencies. When partners draw their types from different distributions, for example because one is an expert while others are newcomers to the industry, the optimal ownership structures depend in subtle ways on the finer details of the environment, such as the distributions, the weight on revenue, and the importance of interdependencies. Symmetric ownership is typically not optimal, and even extreme ownership structures may be optimal.

With notable exceptions, which we discuss below, the literature on partnership dissolution has mainly focused on ex post efficient allocation rules and on the question under what conditions on distributions, valuations, and property rights ex post efficient reallocation is possible subject to incentive compatibility and individual rationality without running a deficit. For the case in which all agents draw their types from the same distribution, Cramton, Gibbons, and Klemperer (1987) and Fieseler, Kittsteiner, and Moldovanu (2003) analyzed, respectively, models with private values and with interdependent values. Cramton, Gibbons, and Klemperer showed that with equal ownership, ex post efficiency is always possible. In contrast, Fieseler, Kittsteiner, and Moldovanu established that if interdependence is positive and strong enough, ex post efficient reallocation may be impossible for any initial ownership structure. Their analysis gives thus additional salience to the question of what are optimal dissolution mechanisms, which is part of our study. Subsequent contributions with interdependent values were made by Kittsteiner (2003), Jehiel and Paudyal (2006), and Chien (2007). Kittsteiner (2003) performed a first attack to the problem of having to avoid deficits by providing a new mechanism – a double-auction with veto rights – that, albeit not ex post efficient, is individually rational, incentive compatible and balances the budget.

Focusing on private values, Che (2006) and Figueroa and Skreta (2012), with the latter building on the results of Schweizer (2006), extended the analysis to settings where each agent's type is drawn from a different distribution. When distributions can be ranked by stochastic dominance, Che and Figueroa and Skreta show that the ownership structure that maximizes revenue, given an ex post efficient allocation rule, assigns larger shares to stronger agents. Segal and Whinston (2011) provide, amongst other things, a generalization of the results of Schweizer (2006) to interdependent values.

To the best of our knowledge, the following are the only papers that analyze objectives other than ex post efficiency for partnership models with multilateral private information. Segal and Whinston (2014) study a second-best bargaining problem under a liability rule with two agents and private values. Our work complements theirs. While Segal and Whinston study a richer

³While the ownership structure is detail-free, the required dissolution mechanism need not be. For ex post efficiency and symmetric ownership, the $k+1$ -price auction of Cramton, Gibbons, and Klemperer (1987) provides a detail-free mechanism. Whether this can be extended to revenue extraction and interdependent values is an open question. For detail-free dissolution mechanisms for asymmetric bilateral partnerships with private values, see Wasser (2013).

class of property rights, called liability rules, their analysis in this part of the paper is confined to two agents, private values, and the second-best mechanism, taking as given the initial allocation of property rights. In contrast, we first characterize the efficient frontier for an arbitrary number of agents, allowing for interdependent values and asymmetric distributions, and then derive the optimal ownership structure for any such partnership. Mylovanov and Tröger (2014) solve the informed principal problem one obtains when maximizing one agent's payoff in a bilateral partnership with private values. Our analysis differs from theirs insofar as our designer is not a member of the partnership and his objective attaches the same welfare weight to all agents. Other precursors to our paper are Lu and Robert (2001) and the unpublished paper by Chien (2007). Lu and Robert study the same objective function as we do in the derivation of optimal dissolution mechanisms but they confine attention to private values and identical type distributions, and they do not address which allocation of initial shares is optimal. Chien solves for the second-best mechanism under given initial ownership, with the main results being confined to the special case of two agents. Our approach is both simpler and more general than Chien's because it characterizes the whole efficient frontier for an arbitrary number of partners. Moreover, unless types are identically distributed, the second best mechanism differs from what Chien's analysis suggests.

The remainder of this paper is organized as follows. Section 2 introduces the setup as well as basic mechanism design results. Section 3 derives and characterizes the optimal dissolution mechanisms. Section 4 determines the optimal initial ownership structures. Section 5 illustrates the main characterization results for the bilateral case. In Section 6, we illustrate the efficient tradeoff between revenue and social surplus when the ownership structure is fixed and when it can be chosen optimally. Section 7 concludes. The proof of the main theorem for the characterization of the optimal dissolution mechanisms is in the Appendix.

2 Model

2.1 Setup

There is a set of n risk-neutral agents $\mathcal{N} := \{1, 2, \dots, n\}$ who jointly own one indivisible object. Each agent $i \in \mathcal{N}$ owns share $r_i \in [0, 1]$ in the object, where $\sum_{i \in \mathcal{N}} r_i = 1$. Accordingly, the initial property rights are represented by a point $\mathbf{r} := (r_1, \dots, r_n)$ in the $(n - 1)$ -dimensional standard simplex $\Delta^{n-1} := \{\mathbf{r} \in [0, 1]^n : \sum_{i=1}^n r_i = 1\}$.

Each agent i privately learns his type x_i which is a realization of the continuous random variable X_i . Each X_i is independently distributed according to a twice continuously differentiable cumulative distribution function F_i with support $[0, 1]$ and density f_i . Agent i 's ex post valuation for the object is

$$v_i(\mathbf{x}) := x_i + \sum_{j \neq i} \eta(x_j)$$

where $\mathbf{x} := (x_1, \dots, x_n)$ and where η is a differentiable function with $\eta'(x_j) < 1$ for all x_j . Agent

i 's status-quo utility from owning share r_i is $r_i v_i(\mathbf{x})$.

For each i , let

$$\psi_{\alpha,i}^S(x_i) := x_i - \eta(x_i) + \alpha \frac{F_i(x_i)}{f_i(x_i)} \quad \text{and} \quad \psi_{\alpha,i}^B(x_i) := x_i - \eta(x_i) - \alpha \frac{1 - F_i(x_i)}{f_i(x_i)}$$

denote a family of virtual cost and virtual valuation functions, parametrized by $\alpha \in [0, 1]$. We make the regularity assumptions

$$\frac{d}{dx_i} \psi_{\alpha,i}^S(x_i) > 0 \quad \text{and} \quad \frac{d}{dx_i} \psi_{\alpha,i}^B(x_i) > 0 \quad \text{for all } i,$$

implying that $\psi_{\alpha,i}^S, \psi_{\alpha,i}^B$ are strictly increasing for all α . Moreover, for all i and $K \in \{S, B\}$ we define the cumulative distribution function of the random variable $Y = \psi_{\alpha,i}^K(X_i)$ as

$$G_{\alpha,i}^K(y) := \begin{cases} 0 & \text{if } y < \psi_{\alpha,i}^K(0), \\ F_i((\psi_{\alpha,i}^K)^{-1}(y)) & \text{if } y \in [\psi_{\alpha,i}^K(0), \psi_{\alpha,i}^K(1)], \\ 1 & \text{if } y > \psi_{\alpha,i}^K(1). \end{cases}$$

Observe that for every i and y , $G_{\alpha,i}^B(y) \geq G_{\alpha,i}^S(y)$.

In Section 3, we will take the initial property rights \mathbf{r} as given, and assume that the partnership is about to be dissolved, resulting in a reallocation of initial property rights \mathbf{r} and monetary transfers. By the Revelation Principle, it is without loss to focus on incentive compatible direct dissolution mechanisms. A direct dissolution mechanism (\mathbf{s}, \mathbf{t}) consists of an allocation rule $\mathbf{s}: [0, 1]^n \rightarrow \Delta^{n-1}$ and a payment rule $\mathbf{t}: [0, 1]^n \rightarrow \mathbb{R}^n$, where $\mathbf{s}(\mathbf{x}) = (s_1(\mathbf{x}), \dots, s_n(\mathbf{x}))$ and $\mathbf{t}(\mathbf{x}) = (t_1(\mathbf{x}), \dots, t_n(\mathbf{x}))$. The agents report their types \mathbf{x} whereupon agent i receives share $s_i(\mathbf{x})$ and pays the amount $t_i(\mathbf{x})$.

Define $S_i(x_i) := E[s_i(x_i, \mathbf{X}_{-i})]$ and $T_i(x_i) := E[t_i(x_i, \mathbf{X}_{-i})]$ to be the interim expected share and payment of agent i . Moreover, let

$$U_i(x_i) := E[v_i(x_i, \mathbf{X}_{-i}) (s_i(x_i, \mathbf{X}_{-i}) - r_i)] - T_i(x_i)$$

denote i 's interim expected net payoff from taking part in the dissolution. A direct dissolution mechanism is Bayesian incentive compatible if

$$U_i(x_i) \geq E[v_i(x_i, \mathbf{X}_{-i}) (s_i(\tilde{x}_i, \mathbf{X}_{-i}) - r_i)] - T_i(\tilde{x}_i) \quad \forall x_i, \tilde{x}_i \in [0, 1], i \in \mathcal{N} \quad (\text{IC})$$

and interim individually rational if

$$U_i(x_i) \geq 0 \quad \forall x_i \in [0, 1], i \in \mathcal{N}. \quad (\text{IR})$$

The designer's objective is to maximize a weighted sum of the ex ante expected social surplus $E[\sum_i v_i(\mathbf{X}) s_i(\mathbf{X})]$, which is the value of the final allocation, and the ex ante expected revenue

$E[\sum_i t_i(\mathbf{X})]$ subject to the incentive compatibility and individual rationality constraints. Suppose the designer puts weight $\alpha \in [0, 1]$ on revenue and let

$$W_\alpha(\mathbf{s}, \mathbf{t}) := (1 - \alpha) \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X}) s_i(\mathbf{X})] + \alpha \sum_{i \in \mathcal{N}} E[t_i(\mathbf{X})].$$

In Section 3, where we take the initial property rights \mathbf{r} as given, we will study *optimal dissolution mechanisms* that solve

$$\max_{\mathbf{s}, \mathbf{t}} W_\alpha(\mathbf{s}, \mathbf{t}) \quad \text{s.t. (IC) and (IR).} \quad (1)$$

Note that the initial shares \mathbf{r} enter this problem solely through the constraint (IR). Optimal dissolution mechanisms will be denoted by $(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$.

In Section 4, we will then turn to analyzing *optimal ownership structures* that solve

$$\max_{\mathbf{r}} W_\alpha(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}}) = \max_{\mathbf{r}, \mathbf{s}, \mathbf{t}} W_\alpha(\mathbf{s}, \mathbf{t}) \quad \text{s.t. (IC) and (IR).} \quad (2)$$

2.2 Incentive Compatibility and Worst-off Types

The standard characterization of Bayesian incentive compatibility applies to our environment (see, e.g., Myerson, 1981): (IC) holds if and only if

$$S_i \text{ is nondecreasing,} \quad (\text{IC1})$$

$$U_i(x_i) = U_i(\hat{x}_i) + \int_{\hat{x}_i}^{x_i} (S_i(z) - r_i) dz \quad \forall x_i, \hat{x}_i \in [0, 1]. \quad (\text{IC2})$$

For a given monotone allocation rule, payoff equivalence (IC2) pins down interim expected payoffs U_i and payments T_i up to a constant.

Consider a dissolution mechanism (\mathbf{s}, \mathbf{t}) that satisfies (IC1) and (IC2). Let the set of worst-off types of agent i be denoted by $\Omega_i(\mathbf{s}) := \arg \min_{x_i} U_i(x_i)$. By (IC2), U_i is differentiable almost everywhere and $U'_i(x_i) = S_i(x_i) - r_i$ wherever U_i is differentiable. The monotonicity of S_i implies the following characterization of the set of worst-off types (see also Cramton, Gibbons, and Klemperer, 1987, Lemma 2). If there is an x_i such that $S_i(x_i) = r_i$, then $\Omega_i(\mathbf{s})$ is a (possibly degenerate) interval and

$$\Omega_i(\mathbf{s}) = \{x_i : S_i(x_i) = r_i\}.$$

If $S_i(x_i) \neq r_i$ for all $x_i \in [0, 1]$, then $\Omega_i(\mathbf{s})$ is a singleton and

$$\Omega_i(\mathbf{s}) = \{x_i : S_i(z) < r_i \forall z < x_i \text{ and } S_i(z) > r_i \forall z > x_i\}.$$

Let $\Omega(\mathbf{s}) := \Omega_1(\mathbf{s}) \times \cdots \times \Omega_n(\mathbf{s})$.

In addition to identifying the set of worst-off types, the characterization of incentive com-

patibility also allows us to write the designer's objective in terms of virtual types. For a given *critical type* $\hat{x}_i \in [0, 1]$, define agent i 's virtual type function as

$$\psi_{\alpha,i}(x_i, \hat{x}_i) := \begin{cases} \psi_{\alpha,i}^S(x_i) & \text{if } x_i < \hat{x}_i, \\ \psi_{\alpha,i}^B(x_i) & \text{if } x_i > \hat{x}_i. \end{cases}$$

Define the virtual surplus given critical types $\hat{\mathbf{x}} = (\hat{x}_1, \dots, \hat{x}_n)$ as

$$\widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) := E \left[\sum_{i \in \mathcal{N}} (s_i(\mathbf{X}) - r_i) \psi_{\alpha,i}(X_i, \hat{x}_i) \right].$$

Lemma 1. *Suppose the dissolution mechanism (\mathbf{s}, \mathbf{t}) satisfies (IC1) and (IC2). Then,*

$$W_\alpha(\mathbf{s}, \mathbf{t}) = \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) - \alpha \sum_{i \in \mathcal{N}} U_i(\hat{x}_i) + (1 - \alpha) \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X})r_i] \quad \forall \hat{\mathbf{x}} \in [0, 1]^n. \quad (3)$$

Moreover,

$$\Omega(\mathbf{s}) = \arg \min_{\hat{\mathbf{x}}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}). \quad (4)$$

Proof. The definition of U_i implies

$$W_\alpha(\mathbf{s}, \mathbf{t}) = \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X})(s_i(\mathbf{X}) - r_i)] - \alpha \sum_{i \in \mathcal{N}} E[U_i(X_i)] + (1 - \alpha) \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X})r_i]. \quad (5)$$

Using the fact that $\sum_{i \in \mathcal{N}} (s_i(\mathbf{X}) - r_i) = 0$, we get

$$\begin{aligned} \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X})(s_i(\mathbf{X}) - r_i)] &= \sum_{i \in \mathcal{N}} E \left[\left(X_i - \eta(X_i) + \sum_{j \in \mathcal{N}} \eta(X_j) \right) (s_i(\mathbf{X}) - r_i) \right] \\ &= \sum_{i \in \mathcal{N}} E[(X_i - \eta(X_i))(S_i(X_i) - r_i)]. \end{aligned} \quad (6)$$

Integrating (IC2) by parts, we obtain for all $\hat{x}_i \in [0, 1]$

$$\begin{aligned} E[U_i(X_i)] &= U_i(\hat{x}_i) + \int_0^1 \int_{\hat{x}_i}^{x_i} (S_i(z) - r_i) dz f_i(x_i) dx_i \\ &= U_i(\hat{x}_i) - \int_0^{\hat{x}_i} F_i(z)(S_i(z) - r_i) dz + \int_{\hat{x}_i}^1 (1 - F_i(z))(S_i(z) - r_i) dz. \end{aligned} \quad (7)$$

Substituting (6) and (7) into (5) yields

$$\begin{aligned} W_\alpha(\mathbf{s}, \mathbf{t}) &= \sum_{i \in \mathcal{N}} \left(\int_0^{\hat{x}_i} \psi_{\alpha,i}^S(z)(S_i(z) - r_i) f_i(z) dz + \int_{\hat{x}_i}^1 \psi_{\alpha,i}^B(z)(S_i(z) - r_i) f_i(z) dz \right) \\ &\quad - \alpha \sum_{i \in \mathcal{N}} U_i(\hat{x}_i) + (1 - \alpha) \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X})r_i] \end{aligned}$$

which, by the definitions of $\psi_{\alpha,i}(x_i, \hat{x}_i)$ and $\widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}})$, is equivalent to (3).

Consider $\hat{\mathbf{x}}, \boldsymbol{\omega} \in [0, 1]^n$. By (3), we obtain

$$\widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) - \widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}) = \alpha \sum_{i \in \mathcal{N}} (U_i(\hat{x}_i) - U_i(\omega_i)).$$

Hence, for all $\boldsymbol{\omega} \in \Omega(\mathbf{s})$ and $\hat{\mathbf{x}} \notin \Omega(\mathbf{s})$, we have $U_i(\hat{x}_i) \geq U_i(\omega_i)$ for all i , where the inequality is strict for at least one i , and therefore $\widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) > \widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega})$. Consequently, $\Omega(\mathbf{s}) = \arg \min_{\hat{\mathbf{x}}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}})$. \square

3 Optimal Dissolution Mechanisms

In the following, we will determine the solution to the designer's problem stated in (1). The preceding section implies that we can replace the constraints (IC) and (IR) by (IC1), (IC2), and $U_i(\omega_i) \geq 0$ for all i and $\omega_i \in \Omega_i(\mathbf{s})$. Define $\mathfrak{S} := \{\mathbf{s} : S_i \text{ is nondecreasing for each } i \in \mathcal{N}\}$ such that (IC1) is equivalent to $\mathbf{s} \in \mathfrak{S}$.

Consider an allocation rule $\mathbf{s} \in \mathfrak{S}$ and some worst-off types $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n) \in \Omega(\mathbf{s})$. Under (IC2), (3) in Lemma 1 implies that we can write the objective as

$$\widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}) - \alpha \sum_{i \in \mathcal{N}} U_i(\omega_i) + (1 - \alpha) \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X})r_i].$$

Note that the individual rationality constraint $U_i(\omega_i) \geq 0$ is binding when choosing payments \mathbf{t} that maximize the above expression for a given \mathbf{s} . $U_i(\omega_i) = 0$ and (IC2) imply that any optimal \mathbf{t} has to be such that interim expected payments satisfy, for all i ,

$$T_i(x_i) = E[v_i(x_i, \mathbf{X}_{-i}) (s_i(x_i, \mathbf{X}_{-i}) - r_i)] - \int_{\omega_i}^{x_i} (S_i(z) - r_i) dz.$$

It remains to determine the optimal allocation rule. Since the second term in the objective above is zero under optimal payments and the third term is independent of the dissolution mechanism, we can restrict attention to maximizing $\widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}) = \min_{\hat{\mathbf{x}}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}})$, where the equality follows from (4) in Lemma 1. Consequently, an optimal allocation rule $\mathbf{s}^{\mathbf{r}}$ has to satisfy

$$\mathbf{s}^{\mathbf{r}} \in \arg \max_{\mathbf{s} \in \mathfrak{S}} \min_{\hat{\mathbf{x}} \in [0, 1]^n} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}). \quad (8)$$

Instead of directly solving (8), we will look for a saddle point $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ that satisfies

$$\mathbf{s}^* \in \arg \max_{\mathbf{s} \in \mathfrak{S}} \widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}^*), \quad (9)$$

$$\boldsymbol{\omega}^* \in \arg \min_{\hat{\mathbf{x}} \in [0, 1]^n} \widetilde{W}_\alpha(\mathbf{s}^*, \hat{\mathbf{x}}). \quad (10)$$

Note that if a saddle point $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ exists, then $\mathbf{s}^{\mathbf{r}}$ solves the problem in (8) if and only if $(\mathbf{s}^{\mathbf{r}}, \boldsymbol{\omega}^*)$

is a saddle point.⁴ For a saddle point, (9) requires that \mathbf{s}^* maximizes \widetilde{W}_α given critical types $\boldsymbol{\omega}^*$ whereas (10) requires that the critical types $\boldsymbol{\omega}^*$ are worst-off types under allocation rule \mathbf{s}^* , i.e., $\boldsymbol{\omega}^* \in \Omega(\mathbf{s}^*)$. In the following, we will show that a saddle point $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ exists and that \mathbf{s}^* is essentially unique.⁵ We will characterize \mathbf{s}^* and thereby identify the optimal dissolution mechanisms.

Consider the optimization problem in (9). Pointwise maximization of

$$\widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}^*) = E \left[\sum_{i \in \mathcal{N}} (s_i(\mathbf{X}) - r_i) \psi_{\alpha,i}(X_i, \omega_i^*) \right]$$

would require allocating the object to the agent i with the highest virtual type $\psi_{\alpha,i}(x_i, \omega_i^*)$. Yet, since $\psi_{\alpha,i}^S(x_i) > \psi_{\alpha,i}^B(x_i)$ for all x_i , $\psi_{\alpha,i}(x_i, \omega_i^*)$ is not monotone at ω_i^* , resulting in the monotonicity constraint $\mathbf{s} \in \mathfrak{S}$ to be violated. The solution to (9) hence involves ironing (Myerson, 1981): the object is allocated to an agent i with the highest *ironed virtual type*

$$\bar{\psi}_{\alpha,i}(x_i, z_i) := \begin{cases} \psi_{\alpha,i}^S(x_i) & \text{if } \psi_{\alpha,i}^S(x_i) < z_i, \\ z_i & \text{if } \psi_{\alpha,i}^B(x_i) \leq z_i \leq \psi_{\alpha,i}^S(x_i), \\ \psi_{\alpha,i}^B(x_i) & \text{if } z_i < \psi_{\alpha,i}^B(x_i) \end{cases}$$

where the ironing parameter $z_i \in [\psi_{\alpha,i}^B(\omega_i^*), \psi_{\alpha,i}^S(\omega_i^*)]$ is the unique solution to

$$E[\psi_{\alpha,i}(X_i, \omega_i^*)] = E[\bar{\psi}_{\alpha,i}(X_i, z_i)]. \quad (11)$$

It is straightforward to verify that $\frac{d}{d\omega_i^*} E[\psi_{\alpha,i}(X_i, \omega_i^*)] = \alpha$ and that $\frac{d}{dz_i} E[\bar{\psi}_{\alpha,i}(X_i, z_i)] = G_{\alpha,i}^B(z_i) - G_{\alpha,i}^S(z_i) > 0$. Moreover, note that for $\omega_i^* = 0$ and $\omega_i^* = 1$, (11) yields $z_i = \psi_{\alpha,i}^B(0)$ and $z_i = \psi_{\alpha,i}^S(1)$, respectively. Using implicit differentiation, we can solve (11) for ω_i^* , resulting in

$$\omega_i^* = \omega_{\alpha,i}(z_i) := \frac{1}{\alpha} \int_{\psi_{\alpha,i}^B(0)}^{z_i} (G_{\alpha,i}^B(y) - G_{\alpha,i}^S(y)) dy. \quad (12)$$

Note that $\omega_{\alpha,i}(\cdot)$ is a continuous and strictly increasing function. Agent i 's ironed virtual type $\bar{\psi}_{\alpha,i}(x_i, z_i)$ is constant and equal to z_i for an interval of types that contains the critical type $\omega_{\alpha,i}(z_i)$ whereas it is strictly increasing otherwise. Since several agents may tie for the highest $\bar{\psi}_{\alpha,i}(x_i, z_i)$, we have to specify a tie-breaking rule.

Let H denote the set of all $n!$ permutations $(h(1), h(2), \dots, h(n))$ of $(1, 2, \dots, n)$. We will call each $h \in H$ a hierarchy among the agents in \mathcal{N} . A *hierarchical tie-breaking rule* breaks ties in favor of the agent who is the highest in the hierarchy: If the set of agents $\mathcal{I} \subseteq \mathcal{N}$ tie for the highest ironed virtual type and there is hierarchical tie-breaking according to hierarchy h , the object is assigned to agent $\arg \max_{i \in \mathcal{I}} h(i)$. A *randomized hierarchical tie-breaking rule* ran-

⁴Suppose $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ satisfies (9) and (10). Then, $\min_{\mathbf{s}} \widetilde{W}_\alpha(\mathbf{s}^*, \hat{\mathbf{x}}) = \widetilde{W}_\alpha(\mathbf{s}^*, \boldsymbol{\omega}^*) \geq \widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}^*) \geq \min_{\mathbf{s}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}})$ for all $\mathbf{s} \in \mathfrak{S}$ and hence \mathbf{s}^* solves the problem in (8). Conversely, for all \mathbf{s}^r that satisfy (8), the above has to hold with equality, implying that $(\mathbf{s}^r, \boldsymbol{\omega}^*)$ is a saddle point.

⁵ \mathbf{s}^* is unique up to the exact specification of a tie-breaking rule.

domly selects a hierarchy $h \in H$ according to an exogenously specified probability distribution $\mathbf{a} := (a_1, \dots, a_n!) \in \Delta^{n!-1}$ over $H = \{h_1, \dots, h_n!\}$ and then breaks ties hierarchically according to h .⁶ The outcome in terms of interim expected shares S_1, \dots, S_n of any tie-breaking rule can equivalently be obtained by a randomized hierarchical tie-breaking rule \mathbf{a} .

Define the *ironed virtual type allocation rule* $\mathbf{s}^{\mathbf{z}, \mathbf{a}}$ with ironing parameters $\mathbf{z} = (z_1, \dots, z_n)$ and randomized hierarchical tie-breaking rule \mathbf{a} as, for all $i \in \mathcal{N}$,

$$s_i^{\mathbf{z}, \mathbf{a}}(\mathbf{x}) := \begin{cases} 1 & \text{if } \bar{\psi}_{\alpha, i}(x_i, z_i) > \max_{j \neq i} \bar{\psi}_{\alpha, j}(x_j, z_j), \\ \sum_{h \in \hat{H}_i} a_h & \text{if } \bar{\psi}_{\alpha, i}(x_i, z_i) = \max_{j \neq i} \bar{\psi}_{\alpha, j}(x_j, z_j), \\ 0 & \text{if } \bar{\psi}_{\alpha, i}(x_i, z_i) < \max_{j \neq i} \bar{\psi}_{\alpha, j}(x_j, z_j), \end{cases}$$

where $\hat{H}_i := \{h \in H : h(i) > h(k) \ \forall k \in \arg \max_{j \neq i} \bar{\psi}_{\alpha, j}(x_j, z_j)\}$. For a given $\boldsymbol{\omega}^*$, $\mathbf{s}^* = \mathbf{s}^{\mathbf{z}, \mathbf{a}}$ solves the problem in (9) for $\mathbf{z} = (\omega_{\alpha, 1}^{-1}(\omega_1^*), \dots, \omega_{\alpha, n}^{-1}(\omega_n^*))$ and any tie-breaking rule $\mathbf{a} \in \Delta^{n!-1}$.

Now consider (10), which is equivalent to $\boldsymbol{\omega}^* \in \Omega(\mathbf{s}^*)$, i.e., requiring $\boldsymbol{\omega}^*$ to be worst-off types under allocation rule \mathbf{s}^* . A simultaneous solution to (9) and (10) hence corresponds to \mathbf{z}, \mathbf{a} such that

$$(\omega_{\alpha, 1}(z_1), \dots, \omega_{\alpha, n}(z_n)) \in \Omega(\mathbf{s}^{\mathbf{z}, \mathbf{a}}).$$

Note that the interim expected share $S_i^{\mathbf{z}, \mathbf{a}}(x_i)$ under an ironed virtual type allocation rule is constant for an interval of types x_i that contains the critical type $\omega_{\alpha, i}(z_i)$. The characterization of the set of worst-off types in Section 2 then implies that the above requirement is equivalent to $S_i^{\mathbf{z}, \mathbf{a}}(\omega_{\alpha, i}(z_i)) = r_i$ for all $i \in \mathcal{N}$.

Let $\underline{z} := -\eta(0)$, $\bar{z} := 1 - \eta(1)$ and define the correspondence $\Gamma_n : [\underline{z}, \bar{z}]^n \rightarrow [0, 1]^n$ such that

$$\Gamma_n(\mathbf{z}) := \left\{ (S_1^{\mathbf{z}, \mathbf{a}}(\omega_{\alpha, 1}(z_1)), \dots, S_n^{\mathbf{z}, \mathbf{a}}(\omega_{\alpha, n}(z_n))) : \mathbf{a} \in \Delta^{n!-1} \right\}.$$

$\Gamma_n(\mathbf{z})$ yields the set of all vectors of expected shares for critical types $\omega_{\alpha, 1}(z_1), \dots, \omega_{\alpha, n}(z_n)$ that can be obtained with ironing parameters \mathbf{z} and some tie-breaking rule \mathbf{a} . If $z_i = z_j$ for two agents i, j , there is a strictly positive probability for a tie and the expected shares depend on tie-breaking. $\Gamma_n(\mathbf{z})$ is singleton-valued if and only if $z_i \neq z_j$ for all i and $j \neq i$.

The following theorem represents our main technical result. The proof is contained in Appendix A. There, we uncover a recursive structure to Γ_n by partitioning its domain in a suitable way. This then allows us to prove the theorem by induction, using the tractable two-agent case as the base case.

Theorem 1. *For each $\mathbf{r} \in \Delta^{n-1}$, there exists a unique $\mathbf{z} \in [\underline{z}, \bar{z}]^n$ such that $\mathbf{r} \in \Gamma_n(\mathbf{z})$.*

According to Theorem 1, there is a unique $\mathbf{z} \in [\underline{z}, \bar{z}]^n$ such that $\mathbf{s}^* = \mathbf{s}^{\mathbf{z}, \mathbf{a}}$ and $\boldsymbol{\omega}^* = (\omega_{\alpha, 1}(z_1), \dots, \omega_{\alpha, n}(z_n))$ constitute a saddle point satisfying (9) and (10) for some tie-breaking rule $\mathbf{a} \in \Delta^{n!-1}$. Any other optimal allocation rule \mathbf{s}^* may differ from $\mathbf{s}^{\mathbf{z}, \mathbf{a}}$ only with respect

⁶An alternative interpretation is that ownership of the object is split up into shares \mathbf{a} and that each share a_h is allocated according to hierarchy h .

to the tie-breaking rule. Theorem 1 also implies that the inverse correspondence $\Gamma_n^{-1}(\mathbf{r})$ is singleton valued for all initial shares \mathbf{r} .

Note that through restricting the definition of Γ_n and the statement of Theorem 1 to $z_i \in [\underline{z}, \bar{z}] = [\psi_{\alpha,i}^S(0), \psi_{\alpha,i}^B(1)] \subset [\psi_{\alpha,i}^B(0), \psi_{\alpha,i}^S(1)]$, we have confined attention to critical types $\omega_i^* \in [\omega_{\alpha,i}(\underline{z}), \omega_{\alpha,i}(\bar{z})] \subset [0, 1]$. This restriction is without loss when looking for optimal allocation rules. As is apparent from the proof of Theorem 1, for $\mathbf{z} = \Gamma_n^{-1}(\mathbf{r})$ we have $z_i = \underline{z}$ if and only if $r_i = 0$. Hence for all \mathbf{r} , $z_j > \underline{z}$ for at least one j . Accordingly, $S_i^{\mathbf{z}, \mathbf{a}}(\omega_{\alpha,i}(z_i)) = 0$ for all $z_i \leq \underline{z}$. If there is a saddle point involving critical type $\omega_i^* = \omega_{\alpha,i}(\underline{z})$ then there is also a saddle point for each $\omega_i^* \in [0, \omega_{\alpha,i}(\underline{z})]$. However, all these saddle points are equivalent in terms of the implied allocation rule \mathbf{s}^* and i 's worst-off types $\Omega_i(\mathbf{s}^*) = \{x_i : S_i^*(x_i) = 0\} = [0, (\psi_{\alpha,i}^B)^{-1}(\underline{z})]$. A similar line of arguments holds for $z_i \geq \bar{z}$, which only occurs if $r_i = 1$.

Summarizing the findings of this section, the following theorem presents our main result on optimal dissolution mechanisms.

Theorem 2. *Let $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r})$. All optimal dissolution mechanisms $(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$ that solve (1) consist of an allocation rule $\mathbf{s}^{\mathbf{r}}$ that allocates ownership of the object to an agent i with the greatest ironed virtual type $\bar{\psi}_{\alpha,i}(x_i, z_i^*)$, where ties are broken such that $S_i^{\mathbf{r}}(\omega_{\alpha,i}(z_i^*)) = r_i$ for all $i \in \mathcal{N}$, and a payment rule $\mathbf{t}^{\mathbf{r}}$ such that interim expected payments satisfy*

$$T_i^{\mathbf{r}}(x_i) = E[v_i(x_i, \mathbf{X}_{-i}) (s_i^{\mathbf{r}}(x_i, \mathbf{X}_{-i}) - r_i)] - \int_{\omega_{\alpha,i}(z_i^*)}^{x_i} (S_i^{\mathbf{r}}(y) - r_i) dy \quad \text{for all } i \in \mathcal{N}.$$

There is a randomized hierarchical tie-breaking rule \mathbf{a}^ such that $\mathbf{s}^{\mathbf{z}^*, \mathbf{a}^*}$ is an optimal allocation rule.*

4 Optimal Ownership Structures

Having identified the optimal dissolution mechanisms $(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$ for given initial property rights \mathbf{r} in the preceding section, we are now in a position to study optimal initial ownership structures. In the following we will consider the problem stated in (2), i.e., maximizing $W_\alpha(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$ over $\mathbf{r} \in \Delta^{n-1}$.

According to Section 3, we have

$$W_\alpha(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}}) = (1 - \alpha) \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X}) r_i] + \max_{\mathbf{s} \in \mathfrak{S}} \min_{\hat{\mathbf{x}} \in [0,1]^n} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}).$$

Since any solution to the max-min problem corresponds to a saddle point $(\mathbf{s}^{\mathbf{r}}, \boldsymbol{\omega}^*)$ of \widetilde{W}_α ,

$$\begin{aligned} \max_{\mathbf{s} \in \mathfrak{S}} \min_{\hat{\mathbf{x}} \in [0,1]^n} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) &= \min_{\hat{\mathbf{x}} \in [0,1]^n} \max_{\mathbf{s} \in \mathfrak{S}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) \\ &= \min_{\hat{\mathbf{x}} \in [0,1]^n} \left\{ - \sum_{i \in \mathcal{N}} r_i E[\psi_{\alpha,i}(X_i, \hat{x}_i)] + \max_{\mathbf{s} \in \mathfrak{S}} E \left[\sum_{i \in \mathcal{N}} s_i(\mathbf{X}) \psi_{\alpha,i}(X_i, \hat{x}_i) \right] \right\} \end{aligned}$$

After some rearrangements using $\sum_i E[v_i(\mathbf{X})r_i] = \sum_i E[r_i(X_i - \eta(X_i))] + \sum_i E[\eta(X_i)]$ we get

$$W_\alpha(\mathbf{s}^r, \mathbf{t}^r) = \min_{\hat{\mathbf{x}} \in [0,1]^n} \left\{ -\alpha \sum_{i \in \mathcal{N}} r_i E[\psi_{1,i}(X_i, \hat{x}_i)] \right. \\ \left. + (1 - \alpha) \sum_{i \in \mathcal{N}} E[\eta(X_i)] + \max_{\mathbf{s} \in \mathcal{S}} E \left[\sum_{i \in \mathcal{N}} s_i(\mathbf{X}) \psi_{\alpha,i}(X_i, \hat{x}_i) \right] \right\}.$$

In the following, it will be more convenient to represent the standard simplex by $\hat{\Delta}^{n-1} := \{\mathbf{r} \in [0,1]^{n-1} : \sum_{i=1}^n r_i \leq 1\}$. Note that using this definition, $(r_1, \dots, r_{n-1}) \in \hat{\Delta}^{n-1}$ is equivalent to $(r_1, \dots, r_{n-1}, 1 - \sum_{i=1}^{n-1} r_i) \in \Delta^{n-1}$. Define the value function $V_\alpha: \hat{\Delta}^{n-1} \rightarrow \mathbb{R}$ such that $V_\alpha(\hat{r}_1, \dots, \hat{r}_{n-1}) = W_\alpha(\mathbf{s}^r, \mathbf{t}^r)$ for each $\mathbf{r} = (\hat{r}_1, \dots, \hat{r}_{n-1}, 1 - \sum_{i=1}^{n-1} \hat{r}_i)$. Hence, for each $\mathbf{r} \in \hat{\Delta}^{n-1}$,

$$V_\alpha(\mathbf{r}) = \min_{\hat{\mathbf{x}} \in [0,1]^n} \left\{ \alpha \sum_{i=1}^{n-1} r_i \left(E[\psi_{1,n}(X_n, \hat{x}_n)] - E[\psi_{1,i}(X_i, \hat{x}_i)] \right) - \alpha E[\psi_{1,n}(X_n, \hat{x}_n)] \right. \\ \left. + (1 - \alpha) \sum_{i \in \mathcal{N}} E[\eta(X_i)] + \max_{\mathbf{s} \in \mathcal{S}} E \left[\sum_{i \in \mathcal{N}} s_i(\mathbf{X}) \psi_{\alpha,i}(X_i, \hat{x}_i) \right] \right\}.$$

Observe that $V_\alpha(\mathbf{r})$ is the minimum of a family of linear functions of \mathbf{r} (indexed by $\hat{\mathbf{x}}$). Consequently, $V_\alpha(\mathbf{r})$ is concave and differentiable almost everywhere. By the envelope theorem

$$\frac{\partial V_\alpha(\mathbf{r})}{\partial r_i} = \alpha \left(E[\psi_{1,n}(X_n, \omega_n^*)] - E[\psi_{1,i}(X_i, \omega_i^*)] \right)$$

where $\omega_i^* = \omega_{\alpha,i}(z_i^*)$ for $i \in \mathcal{N}$ and $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r}, 1 - \sum_{i=1}^{n-1} r_i)$. Because $E[\psi_{1,i}(X_i, \omega_i^*)] = \omega_i^* - E[\eta(X_i)]$, this is equivalent to

$$\frac{\partial V_\alpha(\mathbf{r})}{\partial r_i} = \alpha \left(\omega_{\alpha,n}(z_n^*) - E[\eta(X_n)] - \omega_{\alpha,i}(z_i^*) + E[\eta(X_i)] \right). \quad (13)$$

Note that since each $\omega_{\alpha,i}$ and Γ_n^{-1} are continuous functions, these partial derivatives are continuous. Therefore, V_α is differentiable on $\hat{\Delta}^{n-1}$.

Theorem 3. $W_\alpha(\mathbf{s}^r, \mathbf{t}^r)$ is concave in \mathbf{r} . The optimal ownership structures are all $\mathbf{r}^* \in \Delta^{n-1}$ such that $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r}^*)$ satisfies, for all $i \in \mathcal{N}$ and some Y ,

$$\omega_{\alpha,i}(z_i^*) - E[\eta(X_i)] = Y \quad \text{if } r_i^* > 0, \\ \omega_{\alpha,i}(z_i^*) - E[\eta(X_i)] \geq Y \quad \text{if } r_i^* = 0.$$

Proof. That $W_\alpha(\mathbf{s}^r, \mathbf{t}^r)$ is concave on Δ^{n-1} follows because we have shown that V_α is concave on $\hat{\Delta}^{n-1}$. Consider the problem of maximizing $V_\alpha(r_1, \dots, r_{n-1})$ subject to $(r_1, \dots, r_{n-1}) \in \hat{\Delta}^{n-1}$. As we maximize a concave and differentiable function over a convex set, a solution exists and can be identified using Kuhn-Tucker conditions. We represent the requirement $(r_1, \dots, r_{n-1}) \in$

$\widehat{\Delta}^{n-1}$ by the following n inequality constraints: For all $i \in \{1, \dots, n-1\}$, let λ_i denote the Lagrange multiplier on the constraint $r_i \geq 0$ and let λ_n denote the Lagrange multiplier on the constraint $1 - r_n = \sum_{i=1}^{n-1} r_i \leq 1$. Any solution corresponds to shares and non-negative multipliers satisfying

$$\frac{\partial V_\alpha(\mathbf{r})}{\partial r_i} + \lambda_i - \lambda_n = 0 \quad \text{and} \quad \lambda_i r_i = 0 \quad \text{for all } i \in \{1, \dots, n-1\}$$

as well as $(\sum_{i=1}^{n-1} r_i - 1)\lambda_n = 0$. Using (13) this implies that optimal shares \mathbf{r}^* satisfy

$$\begin{aligned} \omega_{\alpha,i}(z_i^*) - E[\eta(X_i)] &= Y \quad \text{for all } i \in \mathcal{N} \text{ where } r_i > 0, \\ \omega_{\alpha,i}(z_j^*) - E[\eta(X_j)] &\geq Y \quad \text{for all } j \in \mathcal{N} \text{ where } r_j = 0 \end{aligned}$$

where $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r}^*)$. Finally, note that for all \mathbf{r} and $\mathbf{z} = \Gamma_n(\mathbf{r})$, we have $r_i > (=) 0$ if and only if $z_i > (=) \underline{z}$. \square

For private values (i.e., $\eta'(x) = 0$ and therefore $E[\eta(X_i)] = E[\eta(X_j)]$ for all i, j), Theorem 3 shows that optimal ownership structures are such that the optimal dissolution mechanism induces the same critical worst-off type for all agents who own a nonzero share whereas each agent with a zero share has a higher critical worst-off type. For interdependent values a similar characterization holds after subtracting $E[\eta(X_i)]$ from the critical worst-off type $\omega_{\alpha,i}(z_i^*)$ for each agent i . We will show next that optimal ownership structures are in the interior of the simplex if types are identically distributed. With non-identical distributions, however, this need not hold, as we demonstrate below in Section 5.

Identical Distributions Environments with identically distributed types have received considerable attention in the literature, and so it is of interest to study optimal ownership structures in this special case. If $F_i = F$ for all i , the only potential source of ex ante asymmetry among agents are the initial property rights \mathbf{r} . In this case, the effect of the initial ownership structure on the combination of surplus and revenue that can be achieved through optimal dissolution can be conveniently studied using the theory of majorization.⁷ Given two vectors \mathbf{r} and \mathbf{q} with n components we say \mathbf{r} is *majorized* by \mathbf{q} , denoted by $\mathbf{r} \prec \mathbf{q}$, if

$$\sum_{i=1}^k r_{[i]} \leq \sum_{i=1}^k q_{[i]} \quad \text{for } k \in \{1, \dots, n-1\} \quad \text{and} \quad \sum_{i=1}^n r_{[i]} = \sum_{i=1}^n q_{[i]}$$

where $r_{[1]} \geq \dots \geq r_{[n]}$ denotes the components of $\mathbf{r} = (r_1, \dots, r_n)$ in decreasing order. Intuitively, $\mathbf{r} \prec \mathbf{q}$ is a notion of the components of \mathbf{r} being more equal (less diverse) than the components of \mathbf{q} . A real-valued function ϕ is *Schur-concave* if $\mathbf{r} \prec \mathbf{q}$ implies $\phi(\mathbf{r}) \geq \phi(\mathbf{q})$.

⁷For a comprehensive reference, see Marshall, Olkin, and Arnold (2011).

Corollary 1. *Suppose $F_i = F$ for all $i \in \mathcal{N}$. Then $W_\alpha(\mathbf{s}^\mathbf{r}, \mathbf{t}^\mathbf{r})$ is Schur-concave in \mathbf{r} . The optimal initial shares are all*

$$\mathbf{r}^* \in \Gamma_n(z^*, \dots, z^*) = \{\mathbf{r} \in \Delta^{n-1} : \mathbf{r} \prec \mathbf{r}^\alpha\}$$

where z^* is the unique solution to

$$\sum_{i \in \mathcal{N}} (G_\alpha^S(z^*))^{n-i} (G_\alpha^B(z^*))^{i-1} = 1$$

and where $\mathbf{r}^\alpha := (r_1^\alpha, \dots, r_n^\alpha)$ with $r_i^\alpha := (G_\alpha^S(z^*))^{n-i} (G_\alpha^B(z^*))^{i-1}$ for all $i \in \mathcal{N}$.

Proof. With identically distributed types, $W_\alpha(\mathbf{s}^\mathbf{r}, \mathbf{t}^\mathbf{r})$ is symmetric in \mathbf{r} , i.e., $W_\alpha(\mathbf{s}^\mathbf{r}, \mathbf{t}^\mathbf{r}) = W_\alpha(\mathbf{s}^{\mathbf{r}'}, \mathbf{t}^{\mathbf{r}'})$ if \mathbf{r}' is a permutation of \mathbf{r} . According to Marshall, Olkin, and Arnold (2011, p. 97) a function is Schur-concave if it is symmetric and concave.

If $F_i = F$, then $\omega_{\alpha,i} = \omega_\alpha$ and $E[\eta(X_i)]$ is symmetric across agents. Theorem 3 implies that all optimal initial shares \mathbf{r}^* are such that $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r}^*)$ satisfies $z_i^* = z^*$ for all $i \in \mathcal{N}$. As we will show next, there is a unique such z^* .

Consider an ironed virtual type allocation rule $\mathbf{s}^{\mathbf{z},h}$ with $z_i = z$ for all i and hierarchical tie-breaking according to $h \in H$. Under such an allocation rule, agent i 's critical type obtains the object if all agents j with $h(j) < h(i)$ have virtual valuations below z and all agents k with $h(k) > h(i)$ have virtual costs below z . Consequently, $S_i^{\mathbf{z},h}(\omega_\alpha(z)) = (G_\alpha^S(z))^{n-h(i)} (G_\alpha^B(z))^{h(i)-1}$. Each hierarchy $h' \neq h$ corresponds to a permutation of the components of $(S_1^{\mathbf{z},h}(\omega_\alpha(z)), \dots, S_n^{\mathbf{z},h}(\omega_\alpha(z)))$. $\Gamma_n(z, \dots, z)$ is the convex hull of the set of all permutations of $(S_1^{\mathbf{z},h}(\omega_\alpha(z)), \dots, S_n^{\mathbf{z},h}(\omega_\alpha(z)))$. Since G_α^S, G_α^B are strictly increasing, there is a unique z^* such that $\sum_{i \in \mathcal{N}} (G_\alpha^S(z^*))^{n-i} (G_\alpha^B(z^*))^{i-1} = 1$. It follows that $\Gamma_n(z^*, \dots, z^*) \subset \Delta^{n-1}$ whereas $\Gamma_n(z, \dots, z) \cap \Delta^{n-1} = \emptyset$ for all $z \neq z^*$.

Rado's Theorem (Marshall, Olkin, and Arnold, 2011, p. 34) implies that $\mathbf{r} \in \Gamma_n(z^*, \dots, z^*)$ is equivalent to $\mathbf{r} \prec (S_1^{\mathbf{z},h}(\omega_\alpha(z^*)), \dots, S_n^{\mathbf{z},h}(\omega_\alpha(z^*)))$. For $h(i) = i$, the RHS is equal to \mathbf{r}^α . \square

A direct implication of Schur-concavity is that $W_\alpha(\mathbf{s}^\mathbf{r}, \mathbf{t}^\mathbf{r})$ is minimized when property rights are concentrated at one agent ($r_i = 1$ for one i) whereas it is maximized for equal initial property rights ($r_1 = \dots = r_n = \frac{1}{n}$). Moreover, for all $\mathbf{r} \prec \mathbf{r}^\alpha$ the optimal allocation rule differs from that for initial shares $(\frac{1}{n}, \dots, \frac{1}{n})$ only with respect to the tie-breaking rule. As the tie-breaking rule does not affect the objective, $W_\alpha(\mathbf{s}^\mathbf{r}, \mathbf{t}^\mathbf{r})$ is maximized not only by equal initial ownership, but by all $\mathbf{r} \prec \mathbf{r}^\alpha$, i.e., by all initial shares in a convex subset of Δ^{n-1} . Increasing α increases the difference between G_α^S and G_α^B . In turn, the components of \mathbf{r}^α become more spread out, which makes the set of optimal initial shares larger.

Ex post efficiency as $\alpha \rightarrow 0$ Another important special case that has received attention in the literature is ex post efficiency. As $\alpha \rightarrow 0$, every optimal dissolution mechanism approaches a mechanism with ex post efficient allocation rule and transfers that maximize revenue under this

allocation rule. The optimal ownership structure for $\alpha \rightarrow 0$ hence yields the initial shares that allow for the highest revenue under ex post efficient allocation. Note that $\omega_{\alpha,i}(z_i^*) \in \Omega_i(s_i^{\mathbf{z}^*, \mathbf{a}})$ for $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r})$ for all \mathbf{r} . As $\alpha \rightarrow 0$, $s_i^{\mathbf{z}^*, \mathbf{a}}$ approaches the ex post efficient allocation rule and $\Omega_i(s_i^{\mathbf{z}^*, \mathbf{a}})$ shrinks to the singleton $\omega_{0,i}(r_i)$ that solves $\prod_{j \neq i} F_j(\omega_{0,i}) = r_i$, so that $\omega_{0,i}(r_i)$ is agent i 's unique worst-off type under the ex post efficient allocation rule. Theorem 3 then yields for the optimal ownership structure under $\alpha = 0$ the unique initial shares \mathbf{r}^* such that, for all $i \in \mathcal{N}$ and some Y ,

$$\begin{aligned} \omega_{0,i}(r_i) + E[\eta(X_i)] &= Y & \text{if } r_i^* > 0, \\ \omega_{0,i}(r_i) + E[\eta(X_i)] &\geq Y & \text{if } r_i^* = 0. \end{aligned}$$

For private values (where $\eta'(x) = 0$ for all x), this corresponds exactly to the revenue maximizing shares under ex post efficiency obtained by Che (2006) and Figueroa and Skreta (2012). For $\eta'(x) \neq 0$, this generalizes the results of those authors to interdependent values. In contrast to the private values case where, as observed by Figueroa and Skreta (2012), all agents have strictly positive shares, the asymmetry in $E[\eta(X_i)]$ under interdependent values may result in an extreme ownership structure where some agents get zero shares.

5 Bilateral Partnerships

To illustrate the working of the optimal dissolution mechanisms and the variety of optimal ownership structures that arise, we now specialize the setup to one with two agents.

5.1 Optimal dissolution mechanisms

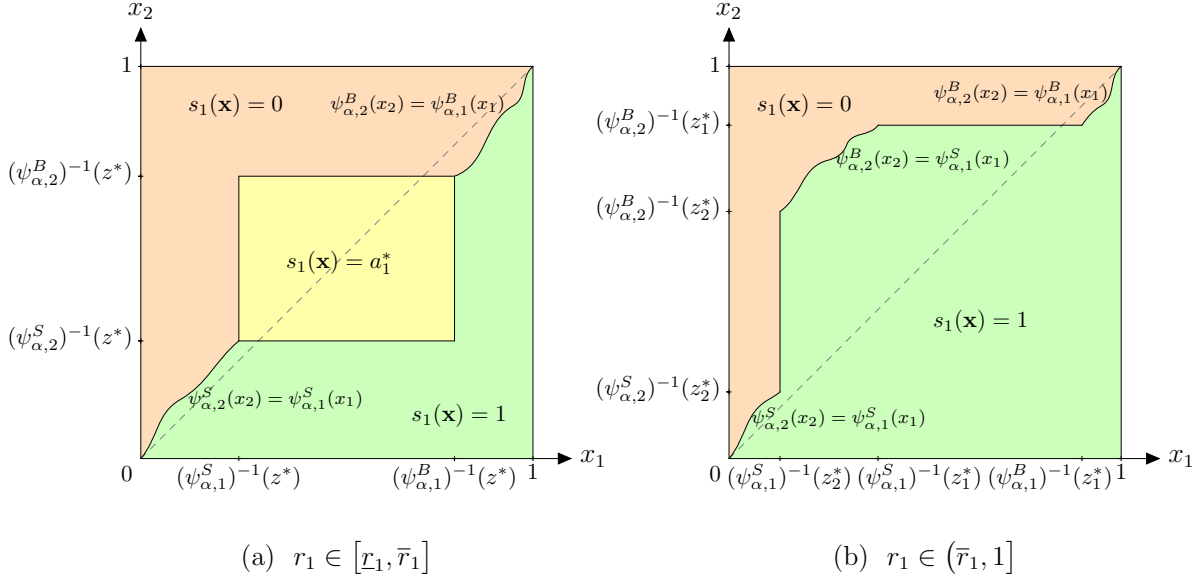
According to Theorem 2, an optimal dissolution mechanism allocates the object to the agent i with the higher ironed virtual type $\bar{\psi}_{\alpha,i}(x_i, z_i^*)$, where $(z_1^*, z_2^*) = \Gamma_2^{-1}(r_1, r_2)$. For bilateral partnerships, characterizing (z_1^*, z_2^*) further is possible at little additional cost.

Suppose $z_1^* > z_2^*$. Then, the critical type of agent 1 expects to obtain the object with probability $S_1(\omega_{\alpha,1}(z_1^*)) = G_{\alpha,2}^B(z_1^*)$ whereas the critical type of agent 2 expects to obtain the object with probability $S_2(\omega_{\alpha,2}(z_2^*)) = G_{\alpha,1}^S(z_2^*)$.⁸ Moreover, these probabilities are equal to the initial shares r_1 and $r_2 = 1 - r_1$, making the critical types worst-off types. Consequently, all initial shares that are consistent with $z_1^* > z_2^*$ satisfy $(G_{\alpha,2}^B)^{-1}(r_1) > (G_{\alpha,1}^S)^{-1}(r_2)$. This is true for all $r_1 \in (\bar{r}_1, 1]$, where \bar{r}_1 uniquely solves $(G_{\alpha,2}^B)^{-1}(\bar{r}_1) = (G_{\alpha,1}^S)^{-1}(1 - \bar{r}_1)$.

Similarly, we find that $z_1^* < z_2^*$ if and only if $r_1 \in [0, \underline{r}_1)$, where \underline{r}_1 is the unique solution to $(G_{\alpha,2}^S)^{-1}(\underline{r}_1) = (G_{\alpha,1}^B)^{-1}(1 - \underline{r}_1)$. Observe that $0 < \underline{r}_1 < \bar{r}_1 < 1$ for all $\alpha > 0$ and that \underline{r}_1 is decreasing and \bar{r}_1 is increasing in α .

It follows that for $r_1 \in [\underline{r}_1, \bar{r}_1]$ we must have $z_1^* = z_2^*$. In this case agents tie for the highest ironed virtual type with positive probability. If agent i wins ties with probability a_i , then i 's

⁸To see this, note that the cumulative distribution function of agent i 's ironed virtual type $Y_i = \bar{\psi}_{\alpha,i}(X_i, z_i^*)$ corresponds to $G_{\alpha,i}^S(y_i)$ for $y_i \leq z_i^*$ and $G_{\alpha,i}^B(y_i)$ for $y_i > z_i^*$.

Figure 1: Optimal allocation rule for $n = 2$.

critical type expects to obtain the object with probability $S_i(\omega_{\alpha,i}(z_i^*)) = a_i G_{\alpha,j}^B(z_i^*) + (1 - a_i) G_{\alpha,j}^S(z_i^*)$. The optimal allocation rule makes sure that this probability is equal to r_i . We thus obtain the following corollary to Theorem 2.

Corollary 2. *Suppose $n = 2$. The optimal allocation rule \mathbf{s}^* allocates full ownership to the agent i who has the higher ironed virtual type $\bar{\psi}_{\alpha,i}(x_i, z_i^*)$, where ties are broken in favor of agent 1 with probability a_1^* .*

(i) *If $r_1 \in [0, \underline{r}_1]$, then $z_1^* = (G_{\alpha,2}^S)^{-1}(r_1) < (G_{\alpha,1}^B)^{-1}(r_2) = z_2^*$ and $a_1^* \in [0, 1]$.*

(ii) *If $r_1 \in [\underline{r}_1, \bar{r}_1]$, then $z_1^* = z_2^* = z^*$, where z^* and a_1^* are the unique solution to*

$$a_1^* G_{\alpha,2}^B(z^*) + (1 - a_1^*) G_{\alpha,2}^S(z^*) = r_1, \quad a_1^* G_{\alpha,1}^S(z^*) + (1 - a_1^*) G_{\alpha,1}^B(z^*) = r_2.$$

(iii) *If $r_1 \in (\bar{r}_1, 1]$, then $z_1^* = (G_{\alpha,2}^B)^{-1}(r_1) > (G_{\alpha,1}^S)^{-1}(r_2) = z_2^*$ and $a_1^* \in [0, 1]$.*

In cases (i) and (iii) of Corollary 2, ties occur with probability 0, which explains why ties can be broken arbitrarily, i.e., why any $a_1^* \in [0, 1]$ is optimal. In contrast, for case (ii) the tie-breaking rule a_1^* of the optimal allocation rule is unique.

The optimal allocation rule described in Corollary 2 is illustrated in Figure 1. Panel (a) depicts case (ii) of Corollary 2 and Panel (b) case (iii), which after interchanging the agents' names also applies to case (i). The figures are drawn for a situation where $F_1 \neq F_2$, i.e., where agents draw their types from different distributions. From the figures we can infer how the optimal allocation rule for $\alpha > 0$ differs from the ex post efficient allocation rule that assigns the object to agent 1 (2) if (x_1, x_2) is below (above) the dashed 45-degree line.

Suppose the ownership structure is sufficiently symmetric such that $r_1 \in (\underline{r}_1, \bar{r}_1)$, which corresponds to Panel (a) of Figure 1. Types $x_1 \in [(\psi_{\alpha,1}^S)^{-1}(z^*), (\psi_{\alpha,1}^B)^{-1}(z^*)]$ of agent 1 and types $x_2 \in [(\psi_{\alpha,2}^S)^{-1}(z^*), (\psi_{\alpha,2}^B)^{-1}(z^*)]$ of agent 2 all have the same ironed virtual type z^* . If both type realizations are within these intervals, the object is assigned to agent 1 with probability $a_1^* \in (0, 1)$, as represented by the yellow rectangle. This inefficiency of the allocation is reminiscent of the traditional under-supply by a monopolist and of auctions with revenue-maximizing reserve prices. If both agents draw a sufficiently high type, the object is allocated to the agent with the highest virtual valuation, whereas for sufficiently low types the allocation is based on comparing virtual costs. Thus the object may end up in the hands of the agent who values it less, resulting in a second kind of inefficiency, like in the optimal of auction of Myerson (1981) with asymmetric bidders. Whereas the first kind of inefficiency is always present for $\alpha > 0$, the second kind vanishes if the agents' types are identically distributed.

As we increase r_1 within $[\underline{r}_1, \bar{r}_1]$, the probability a_1^* increases and z^* may change (it stays constant if $F_1 = F_2$), until we reach \bar{r}_1 where $a_1^* = 1$. At this point, we leave the case underlying Panel (a) of Figure 1 and switch to the situation depicted in Panel (b). As we increase r_1 further, z_1^* increases and z_2^* decreases, eventually reaching \bar{z} and \underline{z} , respectively, when $r_1 = 1$.

Now, consider $r_1 \in (\bar{r}_1, 1]$ as in Panel (b) of Figure 1. If types $(x_1, x_2) \in [(\psi_{\alpha,1}^S)^{-1}(z_2^*), 1] \times [0, (\psi_{\alpha,2}^B)^{-1}(z_1^*)]$ realize, the optimal allocation rule assigns the object to agent 1 if his virtual cost $\psi_{\alpha,1}^S(x_1)$ is higher than the virtual valuation $\psi_{\alpha,2}^B(x_2)$ of agent 2. Otherwise, the object is assigned to agent 2. For type realizations within this region, the optimal allocation thus corresponds exactly to the allocation rules derived by Myerson and Satterthwaite (1983), giving rise to the same inefficiency.⁹ If $x_1 < (\psi_{\alpha,1}^S)^{-1}(z_2^*)$, the object is allocated on the basis of virtual costs whereas if $x_2 > (\psi_{\alpha,2}^B)^{-1}(z_1^*)$, the object is assigned to the agent with the higher virtual valuation. In those cases, we obtain again the second kind of inefficiency that disappears if types are drawn from the same distribution. Note that for $r_1 = 1$, where $(\psi_{\alpha,1}^S)^{-1}(z_2^*) = 0$ and $(\psi_{\alpha,2}^B)^{-1}(z_1^*) = 1$, the optimal allocation rule coincides with the solution of Myerson and Satterthwaite (1983) on the entire type space $[0, 1]^2$. This is, of course, consistent with the partnership model approaching a bilateral trade setting where agent 1 is the seller and agent 2 the buyer as r_1 approaches 1.

As α increases while r_1 is kept fixed, the inefficiency of the optimal allocation increases: In Panel (a) the yellow rectangle with tie-breaking becomes larger and in Panel (b) the demarcation line where 1's virtual cost coincides with 2's virtual valuation moves upward and to the left. This is because a higher α makes the difference between virtual types and actual net types $x_i - \eta(x_i)$ larger. The comparative static effects of increasing the (positive) interdependence of valuations on the optimal allocation are similar to the effects of increasing α under private values. This is easiest to see for the case with linear interdependence $\eta(x) = ex$ with $e < 1$. In this case, i 's virtual type $\psi_{\alpha,i}^K(x_i)$ is larger than j 's virtual type $\psi_{\alpha,j}^L(x_j)$ with $K, L \in \{B, S\}$

⁹While their paper is best known for showing that revenue is negative when $\alpha = 0$ and $r_1 = 1$, Myerson and Satterthwaite (1983) also derive the optimal direct mechanism for $\alpha = 1$ and the value of α such that expected revenue is 0.

if and only if for private values (i.e., $\eta'(x) = 0$) $\psi_{\alpha/(1-e),i}^K(x_i) \geq \psi_{\alpha/(1-e),j}^L(x_j)$. The effect of increasing e in the model with linear interdependence will thus be qualitatively the same as increasing α in the private values model.

5.2 Optimal ownership structure

We now turn to studying the optimal ownership structures for bilateral partnerships. The tractability of the bilateral case allows for a more detailed characterization of the optimal initial shares identified in Theorem 3. In particular, we show as part of the following proposition that all optimal ownership structures in a given environment correspond to a unique vector of ironing parameters \mathbf{z} for the associated dissolution mechanisms.

Proposition 1. *For $n = 2$, exactly one of the following statements is true:*

(i) *The extreme ownership structure $(r_1^*, r_2^*) = (0, 1)$ is optimal and*

$$\omega_{\alpha,1}(\underline{z}) - E[\eta(X_1)] \geq \omega_{\alpha,2}(\bar{z}) - E[\eta(X_2)].$$

(ii) *There is a unique $\mathbf{z}^* \in (\underline{z}, \bar{z})^2$ such that*

$$\omega_{\alpha,1}(z_1^*) - E[\eta(X_1)] = \omega_{\alpha,2}(z_2^*) - E[\eta(X_2)] \quad \text{and} \quad \Gamma_2(\mathbf{z}^*) \cap \Delta^1 \neq \emptyset.$$

All ownership structures $(r_1^, r_2^*) \in \Gamma_2(\mathbf{z}^*) \cap \Delta^1$ are optimal and non-extreme.*

(iii) *The extreme ownership structure $(r_1^*, r_2^*) = (1, 0)$ is optimal and*

$$\omega_{\alpha,1}(\bar{z}) - E[\eta(X_1)] \leq \omega_{\alpha,2}(\underline{z}) - E[\eta(X_2)].$$

Proof. Define the function $L(r_1) := \omega_{\alpha,1}(z_1^*) - \omega_{\alpha,2}(z_2^*) - E[\eta(X_1) - \eta(X_2)]$ where $(z_1^*, z_2^*) = \Gamma_2^{-1}(r_1, 1 - r_1)$. From (13), we have $\alpha L(r_1) = -\frac{\partial V_\alpha(r_1)}{\partial r_1}$, so that the concavity of V_α implies that $L(r_1)$ is nondecreasing. According to Theorem 3, an optimal ownership structure $r_1^* = 1 - r_2^*$ satisfies either $r_1^* \in (0, 1)$ and $L(r_1^*) = 0$, or $r_1^* = 0$ and $L(0) \geq 0$, or $r_1^* = 1$ and $L(1) \leq 0$.

Recall that $\omega_{\alpha,i}$ is strictly increasing for $i = 1, 2$. Moreover, observe that the characterization of the optimal bilateral dissolution mechanism in Corollary 2 implies that z_1^* is strictly increasing and z_2^* is strictly decreasing in r_1 for all $r_1 \in [0, \underline{r}_1)$ and $r_1 \in (\bar{r}_1, 1]$. Consequently $L(r_1)$ is strictly increasing on $[0, \underline{r}_1)$ and $(\bar{r}_1, 1]$. If the optimal ownership structure r_1^* is such that $L'(r_1^*) > 0$, then r_1^* is unique and so is the corresponding \mathbf{z}^* .

We will now show that $L'(r_1) = 0$ for $r_1 \in [\underline{r}_1, \bar{r}_1]$ if and only if also $z_1^* = z_2^* = z$ does not change with r_1 in that range, implying uniqueness of \mathbf{z}^* . From Corollary 2 follows that z solves

$$a(G_2^B(z) + G_1^S(z)) + (1 - a)(G_2^S(z) + G_1^B(z)) = 1$$

for some a that is continuous and strictly increasing in r_1 , with $a = 0$ for $r_1 = \underline{r}_1$ and $a = 1$ for

$r_1 = \bar{r}_1$. Let the solution to the above equation for a given a be denoted by z^a . If $z^0 = z^1$, then z^a is the same for all a and therefore z does not change with r_1 and $L'(r_1) = 0$ for $r_1 \in [\underline{r}_1, \bar{r}_1]$. If $z^0 < z^1$ ($z^0 > z^1$), then z^a is strictly increasing (decreasing) in a (and r_1) as well as

$$G_2^B(z^a) + G_1^S(z^a) < (>) 1 \quad \text{and} \quad G_2^S(z^a) + G_1^B(z^a) > (<) 1 \quad \text{for all } a \in (0, 1).$$

for all $a \in (0, 1)$. Using (12), it follows that

$$L'(r_1) = (\omega'_{\alpha,1}(z) - \omega'_{\alpha,2}(z)) \frac{dz}{dr_1} = \frac{1}{\alpha} (G_1^B(z) - G_1^S(z) - G_2^B(z) + G_2^S(z)) \frac{dz}{dr_1} > 0. \quad \square$$

The first and the third case in Proposition 1 describe corner solutions in which agent 1 and agent 2, respectively, has an optimal share of 0. Accordingly, the second case captures the situations in which the optimal ownership structure has strictly interior shares. In what situations can we expect an extreme ownership structure to be optimal? We will next derive a result that excludes some types of corner solutions in environments where we can rank the agents' type distributions. To prove the result, we will make use of the following lemma.

Lemma 2. *For all $\alpha \in [0, 1]$ and $i \in \mathcal{N}$, $\omega_{\alpha,i}(z) < E[X_i] < \omega_{\alpha,i}(\bar{z})$.*

Proof. First observe that

$$\begin{aligned} \alpha \omega_{\alpha,i}(z) &= \int_{\psi_{\alpha,i}^B(0)}^z (G_{\alpha,i}^B(y) - G_{\alpha,i}^S(y)) dy \\ &= (G_{\alpha,i}^B(z) - G_{\alpha,i}^S(z))z - G_{\alpha,i}^B(z)E[\psi_{\alpha,i}^B(X_i) | \psi_{\alpha,i}^B(X_i) \leq z] + G_{\alpha,i}^S(z)E[\psi_{\alpha,i}^S(X_i) | \psi_{\alpha,i}^S(X_i) \leq z]. \end{aligned}$$

Using the fact that $G_{\alpha,i}^B(z)E[\psi_{\alpha,i}^B(X_i) | \psi_{\alpha,i}^B(X_i) \leq z] + (1 - G_{\alpha,i}^B(z))E[\psi_{\alpha,i}^B(X_i) | \psi_{\alpha,i}^B(X_i) \geq z] = E[\psi_{\alpha,i}^B(X_i)] = (1 - \alpha)E[X_i] - E[\eta(X_i)]$, we obtain

$$\begin{aligned} \alpha \omega_{\alpha,i}(z) &= (G_{\alpha,i}^B(z) - G_{\alpha,i}^S(z))z + (1 - G_{\alpha,i}^B(z))E[\psi_{\alpha,i}^B(X_i) | \psi_{\alpha,i}^B(X_i) \geq z] \\ &\quad + G_{\alpha,i}^S(z)E[\psi_{\alpha,i}^S(X_i) | \psi_{\alpha,i}^S(X_i) \leq z] - (1 - \alpha)E[X_i] + E[\eta(X_i)]. \end{aligned}$$

Consequently, after some rearranging, we find

$$\begin{aligned} \alpha(\omega_{\alpha,i}(z) - E[X_i]) &= E[\max\{z, \psi_{\alpha,i}^B(X_i)\}] - E[X_i - \eta(X_i)] < 0 \\ \text{and } \alpha(\omega_{\alpha,i}(\bar{z}) - E[X_i]) &= E[\min\{\psi_{\alpha,i}^S(X_i), \bar{z}\}] - E[X_i - \eta(X_i)] > 0 \end{aligned}$$

where the inequalities follow from $\max\{z, \psi_{\alpha,i}^B(x)\} < x - \eta(x) < \min\{\psi_{\alpha,i}^S(x), \bar{z}\} \forall x \in (0, 1)$. \square

We obtain the following result for environments where agent 1 is stronger than agent 2.

Proposition 2. *Suppose $n = 2$ and $E[X_1 - \eta(X_1)] \geq E[X_2 - \eta(X_2)]$. Then $r_1^* < 1$ for all $\alpha \in [0, 1]$. Moreover, if $F_1(x) \leq F_2(x)$ and $\eta'(x) \in (-1, 1)$ for all $x \in [0, 1]$, then there is an $\tilde{\alpha} > 0$ such that $r_1^* > 0$ for all $\alpha \in [0, \tilde{\alpha})$.*

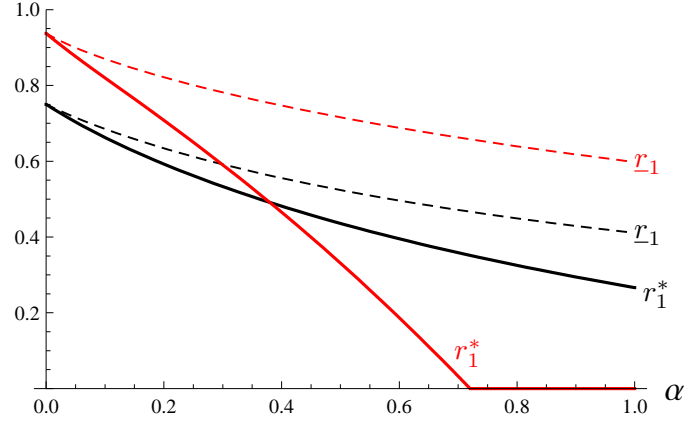


Figure 2: Optimal bilateral ownership structure under private values for $F_1(x) = x^2$ and $F_2(x) = 1 - (1 - x)^2$ in black as well as $F_1(x) = x^4$ and $F_2(x) = 1 - (1 - x)^4$ in red.

Proof. For a corner solution with $r_1^* = 1$, we would need $-\omega_{\alpha,1}(\bar{z}) + E[\eta(X_1)] \geq -\omega_{\alpha,2}(\underline{z}) + E[\eta(X_2)]$. However, this is impossible, because Lemma 2 implies

$$-\omega_{\alpha,1}(\bar{z}) + E[\eta(X_1)] < -E[X_1 - \eta(X_1)] \leq -E[X_2 - \eta(X_2)] < -\omega_{\alpha,2}(\underline{z}) + E[\eta(X_2)].$$

Now, consider a corner solution with $r_1^* = 0$ and suppose $F_1(x) \leq F_2(x)$ and $\eta'(x) \in (-1, 1)$ for all $x \in [0, 1]$. In this case, we need $-\omega_{\alpha,1}(\underline{z}) + E[\eta(X_1)] \leq -\omega_{\alpha,2}(\bar{z}) + E[\eta(X_2)]$, i.e.,

$$\omega_{\alpha,2}(\bar{z}) - \omega_{\alpha,1}(\underline{z}) \leq E[\eta(X_2) - \eta(X_1)] = \int_0^1 \eta'(x)(F_1(x) - F_2(x))dx < 1.$$

Recall that for $\alpha = 0$, $\omega_{0,i}(r_i)$ is the unique worst-off type under the ex post efficient allocation rule. Because $\lim_{\alpha \rightarrow 0} \omega_{\alpha,2}(\bar{z}) = \omega_{0,2}(1) = 1$ and $\lim_{\alpha \rightarrow 0} \omega_{\alpha,1}(\underline{z}) = \omega_{0,1}(0) = 0$, the above condition is violated for α small enough. \square

According to Proposition 2, if agent 1's expected net type is higher than agent 2's, then giving full ownership to agent 1 will never be optimal, independent of the weight on revenue. Moreover, if agent 1's type first-order stochastically dominates agent 2's and the interdependence of valuations is not strongly negative, then also giving full ownership to agent 2 is not optimal for revenue weights close enough to zero.

In the remainder of this section we will study specific examples where agent 1's type distribution first-order stochastically dominates that of agent 2, i.e., $F_1(x) \leq F_2(x)$ for all $x \in [0, 1]$. Among other things, these examples illustrate that extreme ownership can be obtained in the cases that are not excluded by Proposition 2.

Figure 2 shows the optimal share r_1^* of agent 1 as a function of α with private values (i.e., $\eta(x) = 0$ for all x) for the case with $F_1(x) = x^2$ and $F_2(x) = 1 - (1 - x)^2$ depicted in black and for the case with $F_1(x) = x^4$ and $F_2(x) = 1 - (1 - x)^4$ depicted in red. In both cases, agent

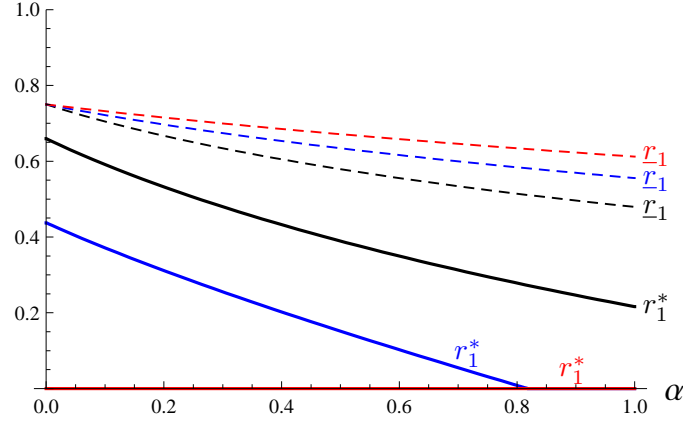


Figure 3: Optimal bilateral ownership structure with negative interdependence and $F_1(x) = x^2$, $F_2(x) = 1 - (1 - x)^2$ for $\eta(x) = -0.5x$ in black, $\eta(x) = -1.5x$ in blue, and $\eta(x) = -3x$ in red.

1 is the strong agent in the sense that F_1 stochastically dominates F_2 , but in latter case the dominance is much more pronounced. For both cases, Figure 2 also depicts how \underline{r}_1 changes with α . Note that r_1^* is decreasing, unique, and always below \underline{r}_1 , which means that under optimal ownership, the optimal dissolution mechanism features ironing parameters $z_1^* < z_2^*$.

When α is small, the optimal ownership structure favors the strong agent. However, as the weight on revenue increases, the strong agent is eventually discriminated against and ultimately obtains a smaller share than the weak agent for large α . In the second case, this goes so far that the optimal ownership structure gives the strong agent an ownership share of 0 for α in excess of three quarters. Hence, extreme ownership structures, which are at the heart of the bilateral trade model of Myerson and Satterthwaite (1983), can be optimal even with private values, provided the weight on revenue is large enough and agents are ex ante sufficiently different. Intuitively, giving more or even full initial ownership to the agent who is expected to have the lower valuation, increases the potential gains from trade at the dissolution stage. Initially favoring the weak agent is optimal if generating revenue, which is extracted from the gains from trade, is important.

Interestingly, negative interdependence in valuation may make an agent who draws his type from the stronger distribution effectively weak. Figure 3 illustrates this possibility for the distributions $F_1(x) = x^2$ and $F_2(x) = 1 - (1 - x)^2$ and three levels of negative interdependence: $\eta(x) = -0.5x$ in black, $\eta(x) = -1.5x$ in blue, and $\eta(x) = -3x$ in red. For a given α , the stronger is the negative interdependence, the smaller is the strong agent's (i.e., agent 1's) optimal share. If the negative interdependence is strong enough, the difference between the agents in terms of $E[\eta(X_i)]$ fully outweighs any difference in critical worst-off types $\omega_{\alpha,i}$, resulting in $r_1^* = 0$ for all values of α , including $\alpha = 0$, as shown for the case $\eta(x) = -3x$. As in the private values case, r_1^* is decreasing in α and always below \underline{r}_1 .

As our last example, we will consider positively interdependent values. In Figure 4 the optimal ownership structures r_1^* are displayed, together with \underline{r}_1 and \bar{r}_1 , for $\eta(x) = 0.3x$ and

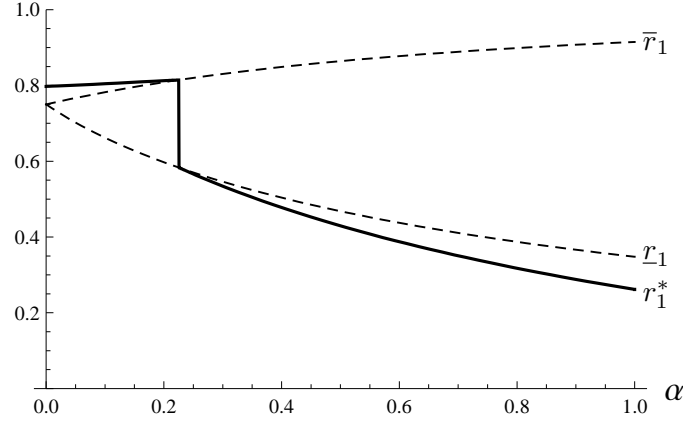


Figure 4: Optimal bilateral ownership structure under positive interdependence, assuming $\eta(x) = 0.3x$, $F_1(x) = x^2$ and $F_2(x) = 1 - (1 - x)^2$

again for the distributions $F_1(x) = x^2$ and $F_2(x) = 1 - (1 - x)^2$. In contrast to the examples above, r_1^* is now increasing in α , unique, and above \bar{r}_1 (implying $z_1^* > z_2^*$) for small α . Moreover, there is a unique α for which all $r_1^* \in [\underline{r}_1, \bar{r}_1]$ are optimal ownership structures.¹⁰ For higher α , r_1^* has similar properties as in the other figures: it is decreasing, unique, and below \underline{r}_1 . Again, it turns out to be optimal to let the strong agent be the majority owner for small α and the minority owner for large α .

6 Efficient Frontiers

We now briefly use the results obtained above to illustrate the tradeoff between social surplus and revenue. We define the efficient frontier to be the collection of all maximally achievable combinations of revenue and social surplus. Depending on whether the mechanism designer has to take the initial property rights as given, or whether he can also choose the ownership structure optimally, we distinguish between the efficient *fixed-ownership frontier* for a given \mathbf{r} and the efficient *optimal-ownership frontier*.

Any point along the fixed-ownership frontier given \mathbf{r} corresponds to an optimal dissolution mechanism that maximizes a weighted average of the expected revenue and expected surplus.¹¹

¹⁰The examples studied here all belong to the class of bilateral environments where $F_2(x) = 1 - F_1(1 - x)$ for all x and where $\eta(x) = ex$ for some $e < 1$. For such environments, the virtual type distributions satisfy $G_{\alpha,2}^B(z) = 1 - G_{\alpha,1}^S(1 - e - z)$ and $G_{\alpha,2}^S(z) = 1 - G_{\alpha,1}^B(1 - e - z)$ for all z . For the optimal dissolution mechanism of Corollary 2, this symmetry property implies that we have the same ironing parameters $z_1^* = z_2^* = \frac{1-e}{2}$ for all $r_1 \in [\underline{r}_1, \bar{r}_1]$. Hence, if the optimal ownership structures are such that $z_1^* = z_2^*$, then the entire interval $[\underline{r}_1, \bar{r}_1]$ is optimal. In contrast, for bilateral environments where the value of $z_1^* = z_2^*$ varies with r_1 , r_1^* is unique.

¹¹Let $\lambda^*(R) \geq 0$ be the value of the Lagrange-multiplier associated with the constraint of achieving an expected revenue of at least R . Then the optimal mechanism that is derived by solving this constrained optimization problem corresponds to the mechanism we have derived with $\alpha = \lambda^*(R)/(1 + \lambda^*(R))$. Much of the analysis in Myerson and Satterthwaite (1983) and Gresik and Satterthwaite (1989) rests on this insight, as does part of Tatur (2005)'s. In the context of public goods, it has been used by, among others, Mailath and Postlewaite (1990), Neeman (1999), Hellwig (2003) and Norman (2004). The literature has mainly focused on “second-best

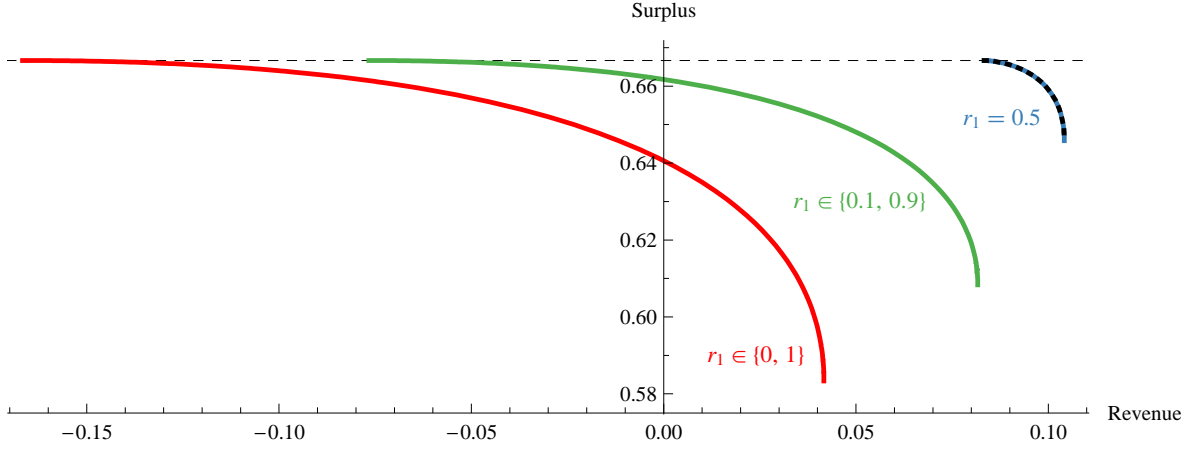


Figure 5: Efficient frontiers for $n = 2$, private values, and uniformly distributed types. Fixed-ownership frontiers are represented by solid colored lines, the optimal-ownership frontier by the dashed black line.

This frontier is constructed as follows. Let $(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}})$ denote the optimal dissolution mechanism that maximizes W_α under ownership structure \mathbf{r} , thereby generating expected social surplus $W_0(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}})$ and expected revenue $W_1(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}})$. The fixed-ownership frontier then corresponds to the set $\{(W_1(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}}), W_0(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}})) : \alpha \in [0, 1]\}$.

Figure 5 depicts fixed-ownership frontiers for bilateral partnerships with identically and uniformly distributed private values. Revenue is on the horizontal and social surplus on the vertical axis. The red curve represents the fixed-ownership frontier for the classical bilateral trade setting of Myerson and Satterthwaite (1983) with $r_i = 1$ for one agent i . The green curve corresponds to the fixed-ownership frontier for $r_i = 0.9$ for one i , whereas the blue curve is the fixed-ownership frontier for symmetric ownership $r_1 = r_2 = 1/2$. As is apparent from Figure 5, the frontier moves outward as initial ownership becomes less asymmetric. According to Corollary 1, this holds for arbitrary n and F as long as types are identically distributed. Moreover, Corollary 1 also implies that in this case the optimal-ownership frontier coincides with the fixed-ownership frontier for $r_i = \frac{1}{n}$. In Figure 5, the optimal-ownership frontier is represented by the dashed black curve.

The second-best mechanism that maximizes social surplus subject to budget balance corresponds to the point on the efficient frontier that intersects with the vertical axis. The slope at every point of an efficient frontier is $-\alpha/(1 - \alpha)$ for the α that generates this point. This is because the efficient frontier is tangent to the indifference curve associated with the designer's utility function $(1 - \alpha)\text{Surplus} + \alpha\text{Revenue}$.¹² Whereas the lower endpoint ($\alpha = 1$) of a fixed-ownership frontier is always to the right of the vertical axis (revenue zero is always feasible by

mechanisms" with a revenue constraint of $R = 0$. However, there is no need to confine attention to this particular revenue constraint.

¹²The red curve in Figure 5 is consistent with the analysis of bilateral trade under uniform valuations by Myerson and Satterthwaite (1983): the second best mechanism generates a surplus of $23/36$ (instead of $2/3$ under ex post efficiency) and is associated with $\alpha = 1/3$, yielding a slope of the frontier at this point of $-1/2$.

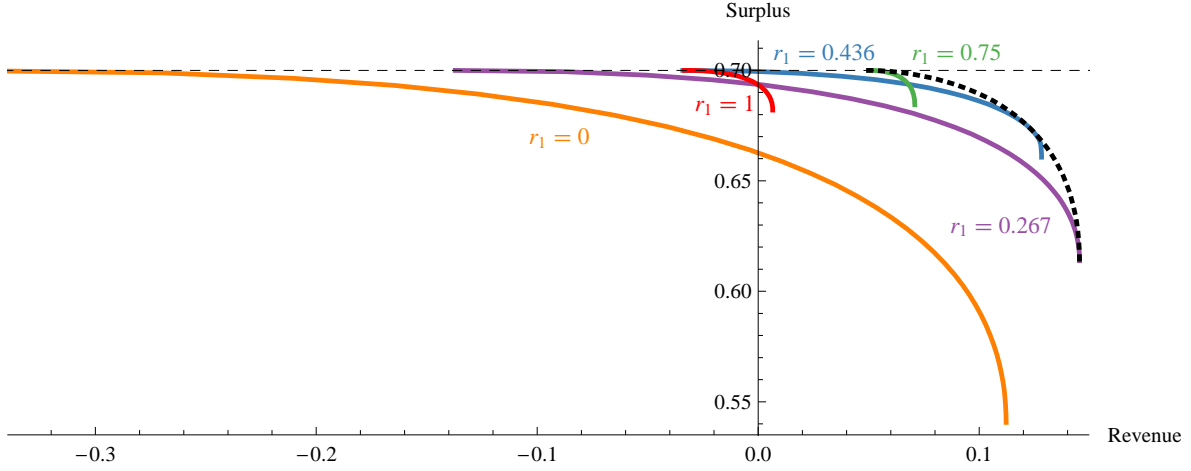


Figure 6: Efficient frontiers for $n = 2$, private values, and $F_1(x) = x^2$, $F_2(x) = 1 - (1 - x)^2$. Fixed-ownership frontiers are represented by solid colored lines, the optimal-ownership frontier by the dashed black line.

sticking to the initial allocation), the relative position of the upper endpoint ($\alpha = 0$) depends on the ownership structure. In Figure 5, as is consistent with Cramton, Gibbons, and Klemperer (1987), ownership structures with $r_i = 1$ or $r_i = 0.9$ for one i are too asymmetric for ex post efficient dissolution under budget balance, while the upper endpoint of the symmetric ownership frontier (the blue curve) is to the right of the vertical axis. In the last case, the frontier does not intersect with the vertical axis and the second-best mechanism coincides with the first-best.

With identically distributed types and nonpositive interdependence ($\eta'(x) \leq 0$), the second-best mechanism never induces $z_i^* = z^*$ for all i . In particular, for $n = 2$ second-best allocations look like the one displayed in Panel (b) of Figure 1. The reason for this is as follows. Fieseler, Kittsteiner, and Moldovanu (2003) have shown that with $r_i = \frac{1}{n}$ for all i and nonpositive interdependence there is an ex post efficient mechanism that generates nonnegative revenue. Consequently, our optimal mechanism for equal shares generates strictly positive revenue for all $\alpha > 0$. By Corollary 1, the optimal mechanism for a given α hence generates strictly positive revenue for all $\mathbf{r} \prec \mathbf{r}^\alpha$. Because the second-best mechanism generates zero revenue by definition, the corresponding α must be such that $\mathbf{r} \not\prec \mathbf{r}^\alpha$, implying an allocation rule where not all z_i^* coincide. In contrast, if under positive interdependence ex post efficiency is impossible without running a deficit under symmetric ownership, then the second-best mechanism corresponds to $z_i^* = z^*$ for all i (as for symmetric agents the optimal allocation rule is symmetric).

Now, suppose agents draw their types from different distributions. In Figure 6, efficient frontiers are drawn for bilateral partnerships with private values under the distributional assumptions $F_1(x) = x^2$ and $F_2(x) = 1 - (1 - x)^2$. The fixed-ownership frontiers are plotted for the extreme ownership $r_1 = 0$ and $r_1 = 1$ in orange and red, respectively, for $r_1 = 0.267$ (the optimal ownership structure if $\alpha = 1$) in purple, for $r_1 = 0.436$ (the optimal ownership structure if $\alpha = 0.5$) in blue, and $r_1 = 0.75$ (the revenue-maximizing ownership under ex post efficiency) in green. Observe that with non-identical type distributions, fixed-ownership fron-

tiers may intersect. The optimal-ownership frontier is plotted in dashed black. It corresponds to the upper envelope of the fixed-ownership frontiers for all possible \mathbf{r} .

As Figure 6 is drawn for private values, the optimal-ownership frontier illustrates the results of Schweizer (2006), Che (2006), and Figueroa and Skreta (2012) that ex post efficiency can always be achieved without a deficit if property rights are chosen optimally. With interdependent values, this need not hold and the optimal-ownership frontier may intersect with the vertical axis. Moreover, Figure 6 shows that the resulting shortfalls in revenue and surplus can be substantive if property rights are chosen sub-optimally, as is for example illustrated by the ownership structure $r_1 = 0$. Put differently, initial ownership structures matter.

7 Conclusions

We analyze a general partnership model that permits an arbitrary number of agents, non-identical type distributions, interdependent values, and any convex combination of revenue and social surplus in the designer's objective. Agents' private information about their types is the source of a transaction cost that, to paraphrase Che (2006), places our model in a world that is "beyond the Coasian irrelevance". For any initial ownership structure, we first solve for the optimal dissolution mechanism, subject to incentive compatibility and individual rationality constraints. Then we choose the ownership structure to maximize the designer's objective function and thereby determine the optimal initial ownership structures. We show that symmetric ownership is always optimal when all agents draw their types from the same distribution but typically not otherwise.

Embedding the static problem studied here in a dynamic setup seems a particularly valuable avenue for future research because agents who use a common resource interact repeatedly in many settings. Another important issue that relates to the allocation of property rights concerns their effects on the incentives to invest (Schmitz, 2002; Segal and Whinston, 2011, 2013), which we have abstracted away from in the present paper. Incorporating investment aspects into a model that permits constrained efficient partnership dissolution seems another natural and important avenue for future research. Finally, given the prominent role transaction costs and information play in the theory of the firm, a natural step for further research is to use our methods and results to develop models that permit a unified approach of bargaining within and across firms.

Appendix A: Proof of Theorem 1

For any $A \subseteq [\underline{z}, \bar{z}]^n$, let $\Gamma_n(A) = \{\mathbf{y} \in [0, 1]^n : \mathbf{y} \in \Gamma_n(\mathbf{z}) \text{ for some } \mathbf{z} \in A\}$ denote the image of A under Γ_n . We will prove Theorem 1 by showing that Γ_n has the following two properties:

Property 1: For every $\mathbf{y} \in \Gamma_n([\underline{z}, \bar{z}]^n)$, there is a unique \mathbf{z} such that $\mathbf{y} \in \Gamma_n(\mathbf{z})$.

Property 2: $\Delta^{n-1} \subset \Gamma_n([\underline{z}, \bar{z}]^n)$.

Property 1 implies the uniqueness part of Theorem 1. It says that every point in the image of Γ_n corresponds to exactly one \mathbf{z} . Put differently, the inverse correspondence $\Gamma_n^{-1}(\mathbf{y}) := \{\mathbf{z} \in [\underline{z}, \bar{z}]^n : \mathbf{y} \in \Gamma_n(\mathbf{z})\}$ is singleton valued for all $\mathbf{y} \in \Gamma_n([\underline{z}, \bar{z}]^n)$. *Property 2* implies the existence part of Theorem 1. It says that the image of Γ_n contains the standard simplex Δ^{n-1} .

The proof proceeds as follows. After some definitions and preliminary results in Subsection A.1, we show in Subsection A.2 that *Property 1* and *Property 2* hold for $n = 2$. In Subsection A.3, we first uncover the recursive structure of Γ_n . This then allows us to prove by induction that the two properties hold for all n , using $n = 2$ as the base case.

A.1 Preliminaries

Recall the virtual cost distributions $G_{\alpha,i}^S$ and virtual valuation distributions $G_{\alpha,i}^B$ defined in Section 2. In the following, we will drop the subscript α and write G_i^S, G_i^B instead. Suppose $z_i > z_j$. Then agent i 's critical type $\omega_{\alpha,i}(z_i)$ interim expects that his ironed virtual type $\bar{\psi}_{\alpha,i}(\omega_{\alpha,i}(z_i), z_i) = z_i$ is greater than the ironed virtual type $\bar{\psi}_{\alpha,j}(x_j, z_j)$ of agent j with probability $G_j^B(z_i)$. Similarly, the critical type $\omega_{\alpha,j}(z_j)$ of agent j interim expects to have a higher ironed virtual type than agent i with probability $G_i^S(z_j)$. Note that G_i^S and G_i^B are strictly increasing, $G_i^S(z_i) < G_i^B(z_i)$ for all $z_i \in [\underline{z}, \bar{z}]$, $G_i^S(\underline{z}) = 0$, and $G_i^B(\bar{z}) = 1$.

Consider agent i and a vector of ironing parameters \mathbf{z} . Let the set of agents other than i that have an ironing parameter less than z_i be denoted by $\mathcal{L}_i(\mathbf{z}) := \{j : j \neq i \text{ and } z_j < z_i\}$. Similarly, let the sets of agents with ironing parameter equal to and greater than z_i be denoted by $\mathcal{E}_i(\mathbf{z}) := \{j : j \neq i \text{ and } z_j = z_i\}$ and $\mathcal{G}_i(\mathbf{z}) := \{j : j \neq i \text{ and } z_j > z_i\}$, respectively. If $\mathcal{E}_i(\mathbf{z}) \neq \emptyset$ for some i , ties in terms of ironed virtual type have strictly positive probability.

Suppose ties are broken hierarchically according to h . For each agent i , let $\underline{\mathcal{E}}_i(\mathbf{z}, h) := \{j \in \mathcal{E}_i(\mathbf{z}) : h(j) < h(i)\}$ and $\bar{\mathcal{E}}_i(\mathbf{z}, h) := \{j \in \mathcal{E}_i(\mathbf{z}) : h(j) > h(i)\}$ denote the set of other agents with the same ironing parameter against whom agent i wins and loses ties, respectively. Hence, under hierarchy h , the expected share of critical type $\omega_{\alpha,i}(z_i)$ of agent i is

$$S_i(\omega_{\alpha,i}(z_i)) = p_i(\mathbf{z}, h) := \prod_{j \in \mathcal{L}_i(\mathbf{z}) \cup \underline{\mathcal{E}}_i(\mathbf{z}, h)} G_j^B(z_i) \prod_{k \in \mathcal{G}_i(\mathbf{z}) \cup \bar{\mathcal{E}}_i(\mathbf{z}, h)} G_k^S(z_i).$$

Let $\mathbf{p}(\mathbf{z}, h) := (p_1(\mathbf{z}, h), \dots, p_n(\mathbf{z}, h))$. The outcome $(S_1^{\mathbf{z}, \mathbf{a}}(\omega_{\alpha,1}(z_1)), \dots, S_n^{\mathbf{z}, \mathbf{a}}(\omega_{\alpha,n}(z_n)))$ of every randomized hierarchical tie-breaking rule \mathbf{a} is equal to a convex combination of $\mathbf{p}(\mathbf{z}, h)$ for different hierarchies $h \in H$. Consequently, the set of all possible expected shares given \mathbf{z} is equal to the convex hull of the expected shares under fixed hierarchies, i.e.,

$$\Gamma_n(\mathbf{z}) = \text{Conv}(\{\mathbf{p}(\mathbf{z}, h) : h \in H\}).$$

Note that depending on \mathbf{z} , we may have $\mathbf{p}(\mathbf{z}, h_1) = \mathbf{p}(\mathbf{z}, h_2)$ for some $h_1 \neq h_2$. In particular, if all n elements of \mathbf{z} are distinct, i.e., $\mathcal{E}_i(\mathbf{z}) = \emptyset$ for all i , then tie-breaking has no bite and all $\mathbf{p}(\mathbf{z}, h)$ coincide. In this case, $\Gamma_n(\mathbf{z})$ is a singleton. On the other hand, if \mathbf{z} is such that $z_i = z$

for all i , i.e., $\mathcal{L}_i(\mathbf{z}) = \mathcal{G}_i(\mathbf{z}) = \emptyset$, then all $n!$ points $\mathbf{p}(\mathbf{z}, h)$ are distinct extreme points of the convex hull $\Gamma_n(\mathbf{z})$. In general, if \mathbf{z} is such that its elements take $k \leq n$ distinct values z^1, \dots, z^k , then $\Gamma_n(\mathbf{z})$ is equal to the convex hull of $\prod_{l=1}^k m_l!$ distinct extreme points, where m_l denotes the number of agents i with $z_i = z^l$.

Lemma 3. *The correspondence Γ_n has the following properties:*

(i) *For all $\mathbf{z} \in [\underline{z}, \bar{z}]^n$, $\Gamma_n(\mathbf{z})$ is nonempty and convex.*

(ii) *Γ_n is upper hemicontinuous.*

Proof. (i) immediately follows from the discussion above. For (ii), we have to show that for any two sequences $\mathbf{z}^q \rightarrow \mathbf{z}$ and $\mathbf{y}^q \rightarrow \mathbf{y}$ such that $\mathbf{y}^q \in \Gamma_n(\mathbf{z}^q)$, we have $\mathbf{y} \in \Gamma_n(\mathbf{z})$. Note that if \mathbf{z} is such that all its components are distinct, $\Gamma_n(\mathbf{z})$ is a singleton that is continuous at \mathbf{z} . Moreover, if the sequence $\mathbf{z}^q \rightarrow \mathbf{z}$ is such that the sets of agents for which ironing parameters coincide stay the same over the whole sequence, $\Gamma_n(\mathbf{z}^q)$ and $\Gamma_n(\mathbf{z})$ are all equal to the convex hull of the same number of extreme points. Since these extreme points are continuous in \mathbf{z}^q , $\mathbf{y}^q \in \Gamma_n(\mathbf{z}^q)$ and $\mathbf{y}^q \rightarrow \mathbf{y}$ imply $\mathbf{y} \in \Gamma_n(\mathbf{z})$ in this case. Finally, suppose there are some i, j for which $z_i^q > z_j^q$ but $z_i = z_j$. Then, if $\mathbf{y}^q \rightarrow \mathbf{y}$ such that $\mathbf{y}^q \in \Gamma_n(\mathbf{z}^q)$, there exists a hierarchical tie-breaking rule for \mathbf{z} where $h(i) > h(j)$ for all i, j with $z_i^q > z_j^q$ and $z_i = z_j$ that induces \mathbf{y} . Hence, $\mathbf{y} \in \Gamma_n(\mathbf{z})$. \square

Partitioning the domain of Γ_n In order to study properties of the image of Γ_n , it will prove useful to consider the following partition of the domain $[\underline{z}, \bar{z}]^n$. Define

$$\xi_n(z) := \{\mathbf{z} \in [\underline{z}, \bar{z}]^n : z_i = z \text{ for at least one } i \in \mathcal{N}\}.$$

Note that $\xi_n(z) \cap \xi_n(z') = \emptyset$ for all $z \neq z'$. Moreover, $\bigcup_{z \in [\underline{z}, \bar{z}]} \xi_n(z) = [\underline{z}, \bar{z}]^n$. Consequently, ξ_n represents a partition of the domain of Γ_n . In addition, define

$$O_n(z) := \Gamma_n(\xi_n(z)).$$

Hence, the image of Γ_n can be written as $\Gamma_n([\underline{z}, \bar{z}]^n) = \bigcup_{z \in [\underline{z}, \bar{z}]} O_n(z)$. Below, we will determine properties of $O_n(z)$ and their implications for $\Gamma_n([\underline{z}, \bar{z}]^n)$.

A.2 Proof of Theorem 1 for $n = 2$

Suppose $n = 2$. There are only two possible hierarchies between two agents, i.e., $H = \{h_1, h_2\}$. Let h_1 (h_2) be the hierarchy where agent 1 (2) wins ties. Define $\zeta_1(z) := (G_2^B(z), G_1^S(z))$ and $\zeta_2(z) := (G_2^S(z), G_1^B(z))$. Hence, $\mathbf{p}(z, z, h_k) = \zeta_k(z)$ for $k = 1, 2$. The general description of Γ_n

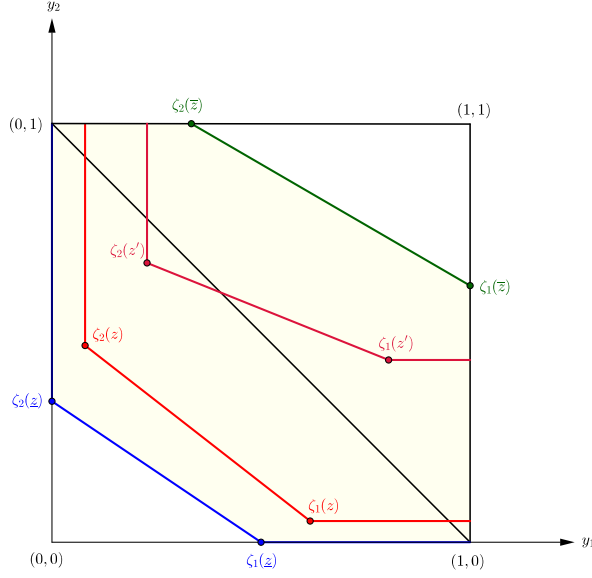


Figure 7: The image of Γ_2 and its components for $\underline{z} < z < z' < \bar{z}$, with $O_2(\underline{z})$ in blue, $O_2(z)$ in light red, $O_2(z')$ in dark red, and $O_2(\bar{z})$ in green.

in the preceding subsection implies

$$\Gamma_2(z_1, z_2) = \begin{cases} (G_2^B(z_1), G_1^S(z_2)) & \text{if } z_1 > z_2, \\ \text{Conv}(\{\zeta_1(z), \zeta_2(z)\}) & \text{if } z_1 = z_2 = z, \\ (G_2^S(z_1), G_1^B(z_2)) & \text{if } z_1 < z_2. \end{cases}$$

Suppose $z_1 = z_2 = z$. Geometrically, $\Gamma_2(z, z)$ is equal to all the points on the line segment from $\zeta_1(z)$ to $\zeta_2(z)$, i.e., all points in $\{a\zeta_1(z) + (1-a)\zeta_2(z) : a \in [0, 1]\}$, where a is the probability with which agent 1 wins ties.

Now consider $O_2(z) = \Gamma_2(\xi_2(z))$ for some $z \in [\underline{z}, \bar{z}]$. In Figure 7, $O_2(z)$ is represented by the polygonal chain in light red. Geometrically, $O_2(z)$ consists of the line segment $\Gamma_2(z, z)$ with two line segments attached to its endpoints: a vertical line segment from $\zeta_1(z)$ to $(1, G_1^S(z))$ that represents $\Gamma_2(z_1, z)$ for all $z_1 \in (z, \bar{z}]$ and a horizontal line segment from $\zeta_2(z)$ to $(G_2^S(z), 1)$ that represents $\Gamma_2(z, z_2)$ for all $z_2 \in (z, \bar{z}]$.

Observe that both coordinates of the vertices $\zeta_1(z)$ and $\zeta_2(z)$ are continuous and strictly increasing in z . Hence, for $z' > z$, $O_2(z') \cap O_2(z) = \emptyset$ and $O_2(z')$ is further away from the origin than $O_2(z)$ (cf. the dark red line in Figure 7). Put differently, O_2 has the following monotonicity property: If $z > z'$, then for all $\mathbf{y} \in O_2(z)$ and $\mathbf{y}' \in O_2(z')$ we have $y_i > y'_i$ for at least one i .

Hence, for every $\mathbf{y} \in \Gamma_2([\underline{z}, \bar{z}]^2)$, there is a unique z such that $\mathbf{y} \in O_2(z)$. Moreover, note that for each $\mathbf{y} \in O_2(z)$ there is a unique point $(z_1, z_2) \in \xi_2(z)$ such that $\mathbf{y} \in \Gamma_2(z_1, z_2)$. Consequently, for every $\mathbf{y} \in \Gamma_2([\underline{z}, \bar{z}]^2)$ there is a unique $\mathbf{z} \in [\underline{z}, \bar{z}]^2$ such that $\mathbf{y} \in \Gamma_2(\mathbf{z})$, i.e., *Property 1* holds for $n = 2$.

Consider $O_2(\underline{z})$ and note that $\zeta_1(\underline{z}) = (G_2^B(\underline{z}), 0)$ and $\zeta_2(\underline{z}) = (0, G_1^B(\underline{z}))$. Hence, the points $\mathbf{y} \in \Gamma_2(\underline{z}, \underline{z})$ all lie below the simplex Δ^1 , which is represented by the black line segment from $(1, 0)$ to $(0, 1)$ in Figure 7. Moreover, the vertical and horizontal parts of $O_2(\underline{z})$ intersect with the simplex exactly at its boundary since $(1, G_1^S(\underline{z})) = (1, 0)$ and $(G_2^S(\underline{z}), 1) = (0, 1)$, respectively.

Let us increase z . For z small enough, the line segment $\Gamma_2(z, z)$ still lies below the simplex such that the vertical and horizontal part of $O_2(z)$ intersect with the simplex since the endpoints $(1, G_1^S(z))$ and $(G_2^S(z), 1)$ of $O_2(z)$ are above and to the left of the simplex for all $z > \underline{z}$. As z increases, the two intersection points move inwards on the simplex. As z becomes large enough, one of the two vertices ζ_1 and ζ_2 crosses the simplex, such that one intersection point lies in $\Gamma_2(z, z)$. The two intersection points approach each other until they coincide when the second vertex also crosses the simplex. Finally, for z sufficiently close to \bar{z} , both $\zeta_1(z)$ and $\zeta_2(z)$ and therefore the entire polygonal chain $O_2(z)$ lie above the simplex. To see this, note that $\zeta_1(\bar{z}) = (1, G_1^S(\bar{z}))$ and $\zeta_2(\bar{z}) = (G_2^S(\bar{z}), 1)$.

We have just shown that for every $\mathbf{y} \in \Delta^1$, there is a z such that $\mathbf{y} \in O_2(z)$. Consequently, $\Delta^1 \subset \Gamma_2([\underline{z}, \bar{z}]^2) = \bigcup_{z \in [\underline{z}, \bar{z}]} O_2(z)$, i.e., *Property 2* holds for $n = 2$. In Figure 7, $\Gamma_2([\underline{z}, \bar{z}]^2)$ is the yellow area between $O_2(\underline{z})$ and $O_2(\bar{z})$, representing a hexagon.

A.3 Proof of Theorem 1 for $n > 2$

In the following, we will extend the approach of the previous subsection to $n > 2$. Characterizing O_n and Γ_n turns out to be significantly more complex in this case. To handle this complexity, we will first uncover the underlying recursive structure of Γ_n : one can construct Γ_n using modified versions of Γ_m for $m < n$. Exploiting this recursive structure will then allow us to prove Theorem 1 by induction, using $n = 2$ as the base case. We will show that *Property 1* and *Property 2* hold for n if they hold for all $m < n$.

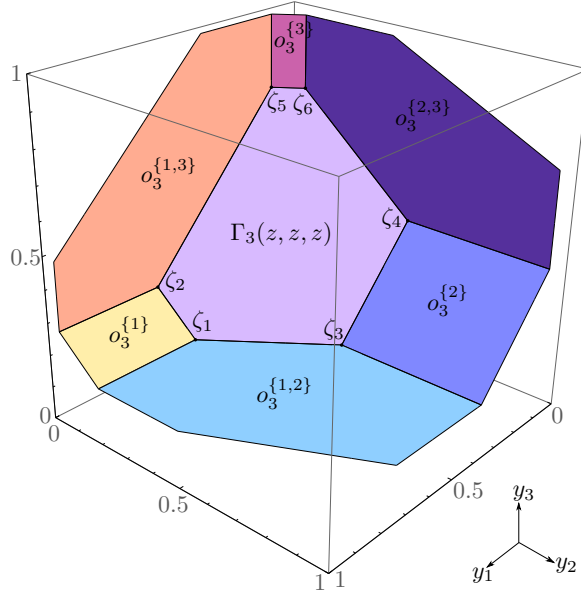
Suppose $z_1 = z_2 = \dots = z_n = z$ and consider $\Gamma_n(z, \dots, z) = \text{Conv}(\{\mathbf{p}(z, \dots, z, h) : h \in H\})$. For each of the $n!$ different hierarchies $h \in H$,

$$\mathbf{p}(z, \dots, z, h) = \left(\overbrace{\prod_{j \in \mathcal{E}_1(h)} G_j^B(z) \prod_{k \in \bar{\mathcal{E}}_1(h)} G_k^S(z)}^{=p_1(z, \dots, z, h)}, \dots, \overbrace{\prod_{j \in \mathcal{E}_n(h)} G_j^B(z) \prod_{k \in \bar{\mathcal{E}}_n(h)} G_k^S(z)}^{=p_n(z, \dots, z, h)} \right)$$

where we have simplified the notation by writing $\bar{\mathcal{E}}_i(h)$ instead of $\bar{\mathcal{E}}_i(z, \dots, z, h)$. Note that if $z > \underline{z}$, each $h \in H$ yields a distinct $\mathbf{p}(z, \dots, z, h)$. It can be shown that all points $\mathbf{p}(z, \dots, z, h)$ lie in the same $(n - 1)$ -dimensional hyperplane: For all $h \in H$,

$$\mathbf{p}(z, \dots, z, h) \in \left\{ \mathbf{y} \in \mathbb{R}^n : \sum_{i \in \mathcal{N}} (G_i^B(z) - G_i^S(z)) y_i = \prod_{j \in \mathcal{N}} G_j^B(z) - \prod_{j \in \mathcal{N}} G_j^S(z) \right\}$$

Consequently, $\Gamma_n(z, \dots, z)$ is a $(n - 1)$ -dimensional convex polytope (in the hyperplane defined above) with vertices $\{\mathbf{p}(z, \dots, z, h) : h \in H\}$. Each vertex is connected to $n - 1$ other vertices through an edge.

Figure 8: $O_3(z)$ and its components.

Now consider a nonempty subset of agents $\mathcal{K} \subset \mathcal{N}$ and denote its complement by $\mathcal{K}' := \mathcal{N} \setminus \mathcal{K}$. Define the set of hierarchies $H_{\mathcal{K}} \subset H$ such that for all $h \in H_{\mathcal{K}}$, we have $h(i) > h(j)$ for all $i \in \mathcal{K}$ and $j \in \mathcal{K}'$. If ties are broken by randomly choosing a hierarchy in $H_{\mathcal{K}}$, agents in \mathcal{K} always win ties against agents in \mathcal{K}' . The $(n-2)$ -dimensional polytope $\text{Conv}(\{\mathbf{p}(z, \dots, z, h) : h \in H_{\mathcal{K}}\})$ is a facet (i.e. an $(n-2)$ -face) of the $(n-1)$ -dimensional polytope $\Gamma_n(z, \dots, z)$. The boundary of $\Gamma_n(z, \dots, z)$ consists of $2^n - 2$ such facets, one for each possible nonempty $\mathcal{K} \subset \mathcal{N}$.¹³

Example with three agents In the preceding subsection we have seen that $\Gamma_2(z, z)$ is a line segment. Assuming $n = 3$, there are 6 possible hierarchies, i.e., $H = \{h_1, \dots, h_6\}$. Hence, $\Gamma_3(z, z, z)$ is a hexagon (with opposite sides parallel). Let $\zeta_l := \mathbf{p}(z, z, z, h_l)$ and suppose the hierarchies are enumerated in such a way that

$$\begin{aligned} \zeta_1 &= (G_2^B(z)G_3^B(z), G_1^S(z)G_3^B(z), G_1^S(z)G_2^S(z)), \zeta_2 = (G_2^B(z)G_3^B(z), G_1^S(z)G_3^S(z), G_1^S(z)G_2^B(z)), \\ \zeta_3 &= (G_2^S(z)G_3^B(z), G_1^B(z)G_3^B(z), G_1^S(z)G_2^S(z)), \zeta_4 = (G_2^S(z)G_3^S(z), G_1^B(z)G_3^B(z), G_1^B(z)G_2^S(z)), \\ \zeta_5 &= (G_2^B(z)G_3^S(z), G_1^S(z)G_3^S(z), G_1^B(z)G_2^B(z)), \zeta_6 = (G_2^S(z)G_3^S(z), G_1^B(z)G_3^S(z), G_1^B(z)G_2^B(z)). \end{aligned}$$

For example, $h_1(1) > h_1(2) > h_1(3)$ and $h_2(1) > h_2(3) > h_2(2)$. As shown in Figure 8, ζ_1, \dots, ζ_6 are the vertices of the hexagon $\Gamma_3(z, z, z)$. The six edges $\overline{\zeta_1\zeta_3}$, $\overline{\zeta_3\zeta_4}$, $\overline{\zeta_4\zeta_6}$, $\overline{\zeta_6\zeta_5}$, $\overline{\zeta_5\zeta_2}$, and $\overline{\zeta_2\zeta_1}$ correspond to tie-breaking using $H_{\{1,2\}}$, $H_{\{2\}}$, $H_{\{2,3\}}$, $H_{\{3\}}$, $H_{\{1,3\}}$, and $H_{\{1\}}$, respectively.¹⁴

¹³There are $\binom{n}{k}$ facets where $|\mathcal{K}| = k$, each having $k!(n-k)!$ vertices.

¹⁴For $n = 4$, $\Gamma_4(z, z, z, z)$ is a truncated octahedron. In general, $\Gamma_n(z, \dots, z)$ is reminiscent of a *permutahedron* (see, e.g., Ziegler, 1995), but its facets exhibit less symmetry (unless $F_i = F$ for all i).

Modified Γ_n correspondences and auxiliary definitions Below, we will use the following two modified versions of Γ_n . Let $\mathcal{M} = \{j_1, j_2, \dots, j_m\} \subseteq \mathcal{N}$ be a subset of $m \geq 2$ agents. First, we denote by $\hat{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$ the correspondence Γ_m for a partnership among the m agents in \mathcal{M} with modified virtual type distributions

$$\hat{G}_i^J(z) := G_i^J(z) \left(\prod_{k \in \mathcal{N} \setminus \mathcal{M}} G_k^B(z) \right)^{\frac{1}{m-1}} \quad \text{for } i \in \mathcal{M} \text{ and } J = S, B.$$

Note that all the properties of virtual type distributions G_i^J carry over to modified virtual type distributions \hat{G}_i^J . In particular, $\hat{G}_i^B(z) > \hat{G}_i^S(z)$ for all $z \in [\underline{z}, \bar{z}]$, $\hat{G}_i^B(\bar{z}) = 1$, and $\hat{G}_i^S(\underline{z}) = 0$. Hence all results for Γ_m extend to $\hat{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$.

Second, we denote by $\check{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$ the correspondence Γ_m for a partnership among the m agents in \mathcal{M} with modified virtual type distributions

$$\check{G}_i^J(z) := G_i^J(z) \left(\prod_{k \in \mathcal{N} \setminus \mathcal{M}} G_k^S(z) \right)^{\frac{1}{m-1}} \quad \text{for } i \in \mathcal{M} \text{ and } J = S, B.$$

Most properties of G_i^J carry over to their modified versions \check{G}_i^J , including $\check{G}_i^B(z) > \check{G}_i^S(z)$ for all $z \in (\underline{z}, \bar{z}]$ and $\check{G}_i^S(\underline{z}) = 0$. The only differences are $\check{G}_i^B(\bar{z}) < 1$, and $\check{G}_i^B(\underline{z}) = 0$. Again, all results for Γ_m extend to $\check{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$, except for those relying on $\check{G}_i^B(\bar{z}) = 1$ or $\check{G}_i^B(\underline{z}) > 0$. In particular, note that $\check{\Gamma}_m^{\mathcal{M}:\mathcal{N}}(z, \dots, z)$ is equivalent to $\Gamma_m(z, \dots, z)$ multiplied by the scalar $\prod_{k \in \mathcal{N} \setminus \mathcal{M}} G_k^S(z)$ (except for the m agents potentially being labeled differently).

We will also make use of the following auxiliary definitions for one-agent partnerships where \mathcal{M} is a singleton: $\hat{\Gamma}_1^{\{j\}:\mathcal{N}}(z) := \prod_{i \in \mathcal{N} \setminus j} G_i^B(z)$ and $\check{\Gamma}_1^{\{j\}:\mathcal{N}}(z) := \prod_{i \in \mathcal{N} \setminus j} G_i^S(z)$ for all $z \in [\underline{z}, \bar{z}]$.

Recursive structure of O_n Let us now study $O_n(z) = \Gamma_n(\xi_n(z))$. Define

$$\xi_n^K(z) := \{\mathbf{z} \in [z, \bar{z}]^n : z_i > z \text{ for } i \in K \text{ and } z_j = z \text{ for } j \in K'\}$$

for all $K \subset \mathcal{N}$, yielding a partition of $\xi_n(z)$ into $2^n - 1$ sets. Hence, $O_n(z) = \bigcup_{K \subset \mathcal{N}} \Gamma_n(\xi_n^K(z))$.

Consider a specific $K \subset \mathcal{N}$ and suppose $z_i > z$ for $i \in K$ and $z_j = z$ for $j \in K'$. Then, we can treat agents in K separately from agents in K' . For the former, their critical type's expected share is as in a partnership among $k := |K|$ agents with modified virtual type distributions \hat{G}_i^J as defined above. For the latter, expected shares are as in $\Gamma_{n-k}(z, \dots, z)$ but multiplied by the scalar $\prod_{i \in K} G_i^S(z)$, i.e., as in a partnership with $n - k$ agents and modified virtual type distributions \check{G}_i^J . Given $\mathbf{y} \in [0, 1]^n$, define $\mathbf{y}_K := (y_{i_1}, y_{i_2}, \dots, y_{i_k})$ for $K = \{i_1, i_2, \dots, i_k\}$ and $\mathbf{y}_{K'} := (y_{j_1}, y_{j_2}, \dots, y_{j_{n-k}})$ for $K' = \{j_1, j_2, \dots, j_{n-k}\}$. Hence, the closure of $\Gamma_n(\xi_n^K(z))$ is

$$o_n^K(z) := \left\{ \mathbf{y} \in [0, 1]^n : \mathbf{y}_K \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}([z, \bar{z}]^k) \text{ and } \mathbf{y}_{K'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(z, \dots, z) \right\}.$$

Note that $\Gamma_{n-k}(z, \dots, z)$ is an $(n - k - 1)$ -dimensional convex polytope. If, in addition, $\Gamma_m([z, \bar{z}]^m)$ is an m -dimensional convex polytope for all $m < n$ (as we have already shown for $m = 2$ above), then $o_n^{\mathcal{K}}(z)$ is an $(n - 1)$ -dimensional convex polytope for all \mathcal{K} .

With the definition above, $O_n(z) = \bigcup_{\mathcal{K} \subset \mathcal{N}} o_n^{\mathcal{K}}(z)$. Note that $o_n^{\emptyset}(z) = \Gamma_n(z, \dots, z)$. Consequently, $O_n(z)$ is a polytopal complex that consists of $2^n - 1$ polytopes of dimension $(n - 1)$: $\Gamma_n(z, \dots, z)$ with a polytope $o_n^{\mathcal{K}}(z)$ with nonempty \mathcal{K} attached to each of its $2^n - 2$ facets.

Example with three agents (continued) $O_3(z)$ consists of the hexagon $\Gamma_3(z, z, z)$ with one polygon attached to each of its six edges, as shown in Figure 8. Those polygons can be divided into two groups: $o_3^{\{1\}}(z)$, $o_3^{\{2\}}(z)$, and $o_3^{\{3\}}(z)$ each represent a convex quadrilateral whereas $o_3^{\{1,2\}}(z)$, $o_3^{\{1,3\}}(z)$, and $o_3^{\{2,3\}}(z)$ are hexagons. For example,

$$o_3^{\{1\}}(z) = \left\{ \mathbf{y} \in [0, 1]^3 : y_1 \in \hat{\Gamma}_1^{\{1\}:\mathcal{N}}([z, \bar{z}]) \text{ and } (y_2, y_3) \in \check{\Gamma}_2^{\{2,3\}:\mathcal{N}}(z, z) \right\}.$$

Since both $\hat{\Gamma}_1^{\{1\}:\mathcal{N}}([z, \bar{z}])$ and $\check{\Gamma}_2^{\{2,3\}:\mathcal{N}}(z, z)$ are line segments, $o_3^{\{1\}}(z)$ is a convex quadrilateral, sharing the edge $\overline{\zeta_2 \zeta_1}$ with the hexagon $\Gamma_3(z, z, z)$. Moreover,

$$o_3^{\{1,2\}}(z) = \left\{ \mathbf{y} \in [0, 1]^3 : (y_1, y_2) \in \hat{\Gamma}_2^{\{1,2\}:\mathcal{N}}([z, \bar{z}]^2) \text{ and } y_3 = \check{\Gamma}_1^{\{3\}:\mathcal{N}}(z) \right\}.$$

Note that y_3 is constant whereas $\hat{\Gamma}_2^{\{1,2\}:\mathcal{N}}([z, \bar{z}]^2)$ is a hexagon, which follows from the preceding subsection (cf. Figure 7). Hence, $o_3^{\{1,2\}}(z)$ is also a hexagon, sharing the edge $\overline{\zeta_1 \zeta_3}$ with the hexagon $\Gamma_3(z, z, z)$.

Monotonicity of O_n Observe that all coordinates of each $\mathbf{p}(z, \dots, z, h)$ are continuous and strictly increasing in z . Hence, if $\hat{z} > z$, then for all $\hat{\mathbf{y}} \in \Gamma_n(\hat{z}, \dots, \hat{z})$ and $\mathbf{y} \in \Gamma_n(z, \dots, z)$ we have $\hat{y}_i > y_i$ for at least one i . The following lemma shows that the monotonicity property of $\Gamma_n(z, \dots, z)$ extends to $O_n(z)$.

Lemma 4. *If $\hat{z} > z$, then for all $\hat{\mathbf{y}} \in O_n(\hat{z})$ and $\mathbf{y} \in O_n(z)$, $\hat{y}_i > y_i$ for at least one i .*

Proof. We will show that $\hat{y}_i > y_i$ for at least one i for all $\mathcal{K}, \mathcal{M} \subset \mathcal{N}$ and $\hat{\mathbf{y}} \in o_n^{\mathcal{M}}(\hat{z})$, $\mathbf{y} \in o_n^{\mathcal{K}}(z)$.

Note that each $\hat{\mathbf{y}} \in o_n^{\mathcal{M}}(\hat{z})$ corresponds to a $\hat{\mathbf{z}} \in [\hat{z}, \bar{z}]^n$ and a tie-breaking rule. Now, consider the $\tilde{\mathbf{y}}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(\hat{z}, \dots, \hat{z})$ that is obtained when breaking ties among agents in \mathcal{K}' in such a way that the same rule as for $\hat{\mathbf{y}}$ is applied for all $j, l \in \mathcal{K}'$ where $\hat{z}_j = \hat{z}_l$, whereas j wins against l for all $j, l \in \mathcal{K}'$ where $\hat{z}_j > \hat{z}_l$. This tie-breaking implies $\hat{y}_j > \tilde{y}_j$ for all $j \in \mathcal{K}' \cap \mathcal{M}$ since $p_j(\hat{\mathbf{z}}, h) > p_j(\hat{z}, \dots, \hat{z}, h)$ for all relevant hierarchies h . Moreover $\hat{y}_l \geq \tilde{y}_l$ for all $l \in \mathcal{K}' \cap \mathcal{M}'$. Hence, we conclude that for all $\hat{\mathbf{y}} \in o_n^{\mathcal{M}}(\hat{z})$ there is a $\tilde{\mathbf{y}}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(\hat{z}, \dots, \hat{z})$ such that $\hat{y}_i \geq \tilde{y}_i$ for all $i \in \mathcal{K}'$.

Since $\hat{z} > z$, there is for all $\tilde{\mathbf{y}}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(\hat{z}, \dots, \hat{z})$ and $\mathbf{y}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(z, \dots, z)$ at least one $i \in \mathcal{K}'$ such that $\tilde{y}_i > y_i$. Combining this with the conclusion of the preceding paragraph implies that for all $\hat{\mathbf{y}} \in o_n^{\mathcal{M}}(\hat{z})$ and $\mathbf{y} \in o_n^{\mathcal{K}}(z)$ there is at least one $i \in \mathcal{K}'$ such that $\hat{y}_i \geq \tilde{y}_i > y_i$. \square

For the three-agent example displayed in Figure 8, Lemma 4 implies that $O_3(z)$ moves towards the observer as we increase z . See also Figure 9 below that depicts $O_3(z)$ for four different values for z .

Induction step for Property 1 Monotonicity of O_n implies that for each $\mathbf{y} \in \Gamma_n([z, \bar{z}]^n) = \bigcup_{z \in [z, \bar{z}]} O_n(z)$ there is a unique z such that $\mathbf{y} \in O_n(z)$.

Lemma 5. *If Property 1 holds for all Γ_m with $m < n$, then Property 1 holds for Γ_n .*

Proof. Lemma 4 implies that for every $\mathbf{y} \in \Gamma_n([z, \bar{z}]^n)$ there is a unique z such that $\mathbf{y} \in O_n(z)$.

We will next show that for every $\mathbf{y} \in O_n(z)$, there is a unique $\mathcal{K} \subset \mathcal{N}$ such that $\mathbf{y} \in \Gamma_n(\xi_n^{\mathcal{K}}(z))$. Consider $\mathcal{K}, \mathcal{M} \subset \mathcal{N}$ such that $\mathcal{K} \neq \mathcal{M}$. Without loss of generality, suppose $\mathcal{K} \cap \mathcal{M}' \neq \emptyset$. Then, for all $\mathbf{y} \in \Gamma_n(\xi_n^{\mathcal{K}}(z))$ and $\tilde{\mathbf{y}} \in \Gamma_n(\xi_n^{\mathcal{M}}(z))$, $y_i > \tilde{y}_i$ for at least one $i \in \mathcal{K} \cap \mathcal{M}'$. To see this, consider the corresponding $\mathbf{z} \in \xi_n^{\mathcal{K}}(z)$ and $\tilde{\mathbf{z}} \in \xi_n^{\mathcal{M}}(z)$. For $i \in \mathcal{K} \cap \mathcal{M}'$ and $j \in \mathcal{K}'$, we have $z_i > z_j = z$ but $\tilde{z}_i = z \leq \tilde{z}_j$. Hence, in the first case the critical type of agent i has a strictly higher winning probability against agents in \mathcal{K}' than in the second case. The same is true for $j \in \mathcal{K} \cap \mathcal{M}$, since $z_i > z$ whereas $\tilde{z}_i = z < \tilde{z}_j$. Finally, the winning probability of agent i 's critical type against other agents in $\mathcal{K} \cap \mathcal{M}'$ cannot be lower for all $i \in \mathcal{K} \cap \mathcal{M}'$ when considering $\mathbf{z} \in \xi_n^{\mathcal{K}}(z)$ than when considering $\tilde{\mathbf{z}} \in \xi_n^{\mathcal{M}}(z)$. Consequently, $y_i > \tilde{y}_i$ for at least one i .

So far we have shown that for every $\mathbf{y} \in \Gamma_n([z, \bar{z}]^n)$, there are unique z, \mathcal{K} such that $\mathbf{y} \in \Gamma_n(\xi_n^{\mathcal{K}}(z))$. This already partially pins down \mathbf{z} : for all $i \in \mathcal{K}'$, we have $z_i = z$. Moreover, $\mathbf{y} \in \Gamma_n(\xi_n^{\mathcal{K}}(z))$ implies $\mathbf{y} \in o_n^{\mathcal{K}}(z)$ and therefore $\mathbf{y}_{\mathcal{K}} \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}([z, \bar{z}]^k)$. If Property 1 holds for $k < n$, there is a unique $\mathbf{z}_{\mathcal{K}}$ such that $\mathbf{y}_{\mathcal{K}} \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}(\mathbf{z}_{\mathcal{K}})$. This pins down z_i also for $i \in \mathcal{K}$. \square

Convexity of $\Gamma_n([z, \bar{z}]^n)$ Suppose $\Gamma_m([z, \bar{z}]^m)$ is a convex polytope for all $m < n$. As observed above, this implies that $O_n(z)$ is a polytopal complex consisting of $2^n - 1$ convex polytopes $o_n^{\mathcal{K}}$ of dimension $n - 1$, one for each $\mathcal{K} \subset \mathcal{N}$. If $\mathcal{K} \cap \mathcal{M} \neq \emptyset$, then the two polytopes $o_n^{\mathcal{K}}$ and $o_n^{\mathcal{M}}$ are adjacent, i.e., they share a facet (of dimension $n - 2$). Let the boundary of the polytopal complex $O_n(z)$ be defined as all the facets of each polytope $o_n^{\mathcal{K}}$ that are not shared with some other polytope $o_n^{\mathcal{M}}$, where $\mathcal{K} \neq \mathcal{M}$. Each point $\mathbf{y} \in \Gamma_n(\mathbf{z})$ on the boundary of $O_n(z)$ corresponds to a \mathbf{z} where, for some $\mathcal{K} \subset \mathcal{N}$, $z_i = \bar{z}$ for $i \in \mathcal{K}$ whereas $z_j = z$ for $j \in \mathcal{K}'$.

In a similar manner as we constructed O_n above, define

$$Q_n(z) := \Gamma_n(\{\mathbf{z} \in [z, \bar{z}]^n : z_i = \bar{z} \text{ for at least one } i \in \mathcal{N}\}) = \bigcup_{\mathcal{K} \subset \mathcal{N}} q_n^{\mathcal{K}}(z)$$

where

$$q_n^{\mathcal{K}}(z) := \left\{ \mathbf{y} \in [0, 1]^n : \mathbf{y}_{\mathcal{K}} \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}(\bar{z}, \dots, \bar{z}) \text{ and } \mathbf{y}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}([z, \bar{z}]^{n-k}) \right\}.$$

$Q_n(z)$ represents the image under Γ_n of the set of all \mathbf{z} where $z_i \geq z$ for all i and $z_i = \bar{z}$ for at least one i . Observe that $Q_n(z)$ contains all the boundary points of $O_n(\tilde{z})$ for each $\tilde{z} \in [z, \bar{z}]$. Moreover, $Q_n(\bar{z}) = O_n(\bar{z})$.

Lemma 6. $\Gamma_n([z, \bar{z}]^n)$ is an n -dimensional convex polytope for all $z < \bar{z}$. The boundary of this polytope is $O_n(z) \cup Q_n(z)$.

Proof. From the preceding subsection we know that $\Gamma_2([z, \bar{z}]^2)$ is a hexagon. We will now show that if $\Gamma_m([z, \bar{z}]^m)$ is a convex polytope for all $m < n$, then $\Gamma_n([z, \bar{z}]^n)$ is a convex polytope. Consequently, the first statement in the lemma follows by induction.

Suppose $\Gamma_m([z, \bar{z}]^m)$ is a convex polytope for all $m < n$ and recall that $\Gamma_n([z, \bar{z}]^n) = \bigcup_{\tilde{z} \in [z, \bar{z}]} O_n(\tilde{z})$. As derived above, $O_n(z)$ is a polytopal complex. As all coordinates of the extreme points of $\Gamma_n(z, \dots, z)$ are continuous and strictly increasing in z , Lemma 4 implies that $O_n(z)$ continuously moves further away from the origin as z increases. Hence, $O_n(z)$ is part of the boundary of $\Gamma_n([z, \bar{z}]^n)$.

In addition to $O_n(z)$, all boundary points of $O_n(\tilde{z})$ for each $\tilde{z} \in (z, \bar{z})$ are also part of the boundary of $\Gamma_n([z, \bar{z}]^n)$ whereas all interior points of $O_n(\tilde{z})$ are in the interior of $\Gamma_n([z, \bar{z}]^n)$. Lastly, note that $O_n(\bar{z})$ consists of only one convex polytope (namely $\Gamma_n(\bar{z}, \dots, \bar{z})$) and that all its points are part of the boundary of $\Gamma_n([z, \bar{z}]^n)$.

$Q_n(z)$ represents all points on the boundary of $\Gamma_n([z, \bar{z}]^n)$ described in the preceding paragraph, i.e., boundary points that are not in $O_n(z)$. Consequently, $O_n(z) \cup Q_n(z)$ represents the entire boundary of $\Gamma_n([z, \bar{z}]^n)$. Like $O_n(z)$, $Q_n(z)$ is also a polytopal complex that consists of $2^n - 1$ convex polytopes of dimension $n - 1$. The boundary of $\Gamma_n([z, \bar{z}]^n)$ therefore consists of $2^{n+1} - 2$ convex polytopes ($o_n^K(z)$ and $q_n^K(z)$ for all $K \subset \mathcal{N}$), making $\Gamma_n([z, \bar{z}]^n)$ an n -dimensional polytope with $2^{n+1} - 2$ facets.

Recall that for all $z < \bar{z}$, $O_n(z)$ consists of $\Gamma_n(z, \dots, z)$ with a $o_n^K(z)$ attached to each facet. The points in each $o_n^K(z)$ are further away from the origin than the points on the corresponding facet of $\Gamma_n(z, \dots, z)$. Because of the monotonicity and continuity properties of $O_n(z)$, for all $\mathbf{y} \in \text{Conv}(O_n(z))$ such that $\mathbf{y} \notin O_n(z)$, there is a $\tilde{z} \in (z, \bar{z}]$ such that $\mathbf{y} \in O_n(\tilde{z})$. Hence, the polytope $\Gamma_n([z, \bar{z}]^n) = \bigcup_{\tilde{z} \in [z, \bar{z}]} O_n(\tilde{z})$ is convex. \square

Induction step for Property 2 Consider $O_n(\underline{z})$. This represents a special case since $\Gamma_n(\underline{z}, \dots, \underline{z})$ is a general $(n - 1)$ -simplex with only n vertices rather than a polytope with $n!$ vertices. In particular, note that for each vertex $\mathbf{p}(\underline{z}, \dots, \underline{z}, h) = (p_1, \dots, p_n)$, $p_i \in (0, 1)$ for one i whereas $p_j = 0$ for all $j \neq i$, resulting in only n distinct vertices. Since $\sum_{i=1}^n p_i < 1$, the general simplex $\Gamma_n(\underline{z}, \dots, \underline{z})$ does not intersect with standard simplex Δ^{n-1} : the former lies closer to the origin than the latter.¹⁵

It follows that $O_n(\underline{z})$ consists of only $n + 1$ polytopes of dimension $(n - 1)$: The general simplex $\Gamma_n(\underline{z}, \dots, \underline{z})$ with a polytope o_n^i attached to each of its n facets (each corresponding to a general $n - 2$ -simplex), where, for each $i \in \mathcal{N}$,

$$o_n^i := \left\{ \mathbf{y} \in [0, 1]^n : \mathbf{y}_{\mathcal{N} \setminus i} \in \hat{\Gamma}_{n-1}^{\mathcal{N} \setminus i: \mathcal{N}}([\underline{z}, \bar{z}]^{n-1}) \text{ and } y_i = 0 \right\}.$$

¹⁵In the three-agent example above, we obtain, for $z = \underline{z}$, $\zeta_1 = \zeta_2 = (G_2^B(\underline{z})G_3^B(\underline{z}), 0, 0)$, $\zeta_3 = \zeta_4 = (0, G_1^B(\underline{z})G_3^B(\underline{z}), 0)$, $\zeta_5 = \zeta_6 = (0, 0, G_1^B(\underline{z})G_2^B(\underline{z}))$.

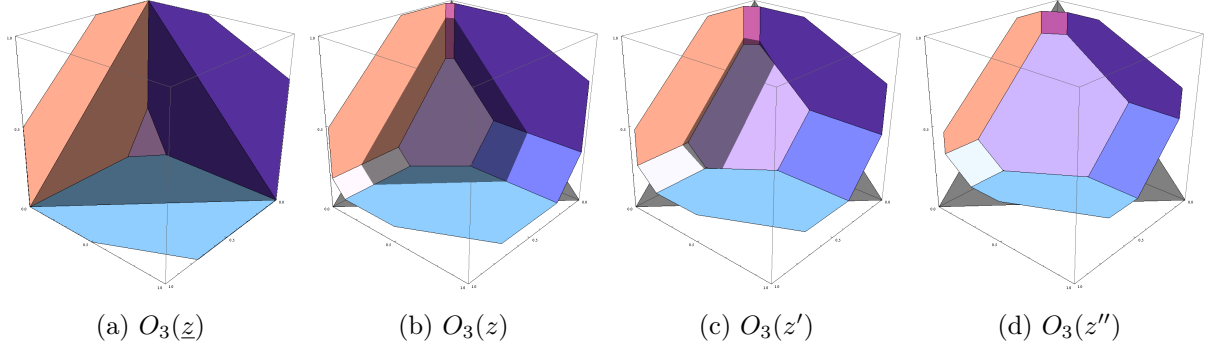


Figure 9: Increasing z in the three-player example. O_3 for some $\underline{z} < z < z' < z'' < \bar{z}$ and the simplex Δ^2 (semitransparent black triangle).

Lemma 7. *If Property 2 holds for all Γ_m with $m < n$, then $O_n(\underline{z})$ contains the entire boundary (all n facets) of Δ^{n-1} .*

Proof. $O_n(\underline{z})$ is the union of $\Gamma_n(\underline{z}, \dots, \underline{z})$ and n polytopes o_n^i as defined above. Property 2 for $m < n$ implies in particular $\Delta^{n-2} \subset \Gamma_{n-1}([\underline{z}, \bar{z}]^{n-1})$ and therefore $\Delta^{n-2} \subset \hat{\Gamma}_{n-1}^{\mathcal{N} \setminus i: \mathcal{N}}([\underline{z}, \bar{z}]^{n-1})$. Moreover, the n facets of Δ^{n-1} all correspond to one coordinate being set to zero, i.e., $\mathbf{y}_{\mathcal{N} \setminus i} \in \Delta^{n-2}$ and $y_i = 0$. \square

Panel (a) of Figure 9 illustrates Lemma 7 in the three-agent example. It shows how $O_3(\underline{z})$ intersects with the boundary of the semitransparent black triangle that represents the simplex Δ^2 . Figure 9 also conveys that as we increase z , the intersection of $O_2(z)$ with Δ^2 moves inward (Panels (b) and (c)) until the entire simplex has been covered and for all higher z , $O_2(z)$ does not intersect with Δ^2 (Panel (d)).¹⁶ Hence, Property 2 holds for Γ_3 .

Using the convexity of $\Gamma_n([\underline{z}, \bar{z}]^n)$, it is now straightforward to obtain the following lemma.

Lemma 8. *If Property 2 holds for all Γ_m with $m < n$, then Property 2 holds for Γ_n .*

Proof. If Property 2 holds for all Γ_m with $m < n$, then, according to Lemma 7, $O_n(\underline{z})$ contains the entire boundary of Δ^{n-1} . By Lemma 6, $\Gamma_n([\underline{z}, \bar{z}]^n)$ is convex and $O_n(\underline{z})$ is part of the boundary of $\Gamma_n([\underline{z}, \bar{z}]^n)$. Consequently, the boundary of Δ^{n-1} being contained in the boundary of $\Gamma_n([\underline{z}, \bar{z}]^n)$ implies Property 2 for Γ_n . \square

Final step As shown in the preceding subsection, Property 1 and Property 2 hold for $n = 2$. By induction, using Lemmata 5 and 8, Property 1 and Property 2 hold for all n .

¹⁶Let \hat{z} be the smallest z such that $\sum_{i \in \mathcal{N}} p_i(z, \dots, z, h) \geq 1$ for all $h \in H$. Similarly, let \tilde{z} be the greatest z such that $\sum_{i \in \mathcal{N}} p_i(z, \dots, z, h) \leq 1$ for all $h \in H$. Observe that $\underline{z} < \tilde{z} \leq \hat{z} < \bar{z}$ (with $\tilde{z} = \hat{z}$ if $F_i = F$ for all i). $O_n(z)$ intersects with Δ^{n-1} if and only if $z \leq \hat{z}$ whereas $\Gamma_n(z, \dots, z)$ intersects with Δ^{n-1} if and only if $z \in [\tilde{z}, \hat{z}]$. Panels (b), (c), and (d) of Figure 9 correspond to $z < \tilde{z} < z' < \hat{z} < z''$.

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