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**Conference Paper**

## A Two Sector Endogenous Growth Model with Two Accumulating Factors- Implementing Trade

Beiträge zur Jahrestagung des Vereins für Socialpolitik 2015: Ökonomische Entwicklung - Theorie und Politik - Session: Institutions, environment and economic development, No. D02-V1

**Provided in Cooperation with:**

Verein für Socialpolitik / German Economic Association

*Suggested Citation:* Kirschbaum, Birgit (2015) : A Two Sector Endogenous Growth Model with Two Accumulating Factors- Implementing Trade, Beiträge zur Jahrestagung des Vereins für Socialpolitik 2015: Ökonomische Entwicklung - Theorie und Politik - Session: Institutions, environment and economic development, No. D02-V1, ZBW - Deutsche Zentralbibliothek für Wirtschaftswissenschaften, Leibniz-Informationszentrum Wirtschaft

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# A two sector endogenous growth model with two accumulating factors-implementing trade

February 2015

## **Abstract**

This paper is about trade in an endogenous growth model with accumulation of physical and human capital. The convergence to the ray of steady states is proved for two different development stages of similar small countries. Free trade benefits the education sector of a relatively less developed country and thus stimulates the human capital accumulation. Whereas a technological more developed country declines in favour of the world market.

Only if the small country is relatively less developed then there is a convergence to the balanced growth path of the world market and the country improves its situation. Without any trade policy a less developed country will lose necessary factors of production. A more developed country deals with stagnation or even fluctuation of its development stage.

JEL-Classification: D51, D91, F10, O11

Keywords: endogenous growth, accumulating factors, trade

# 1 Introduction

Inequality is an useful incentive to economic activities. Not only the big differences in the world can be an engine for free trade but also a slight imbalance may be responsible for large consequences in similar regions. The goal of this paper is to analyze the effects of free trade between industrialized developed countries on their economic growth.

Motivated by the role of the EU there is the question if trade affects a relatively homogenous area like Europe. For many people Europe embodies a coalescent example which stands for freedom, wealth and culture. As similar as the countries may appear, actually they are quite different. For a successfully constant economic growth Europe's countries have to grow together strongly, despite all diversities. Does free trade, induced by these inequalities, support the globalisation? Is there an advantage of trade considering the economic growth rate of the Member States?

To answer these questions this work looks at a model of the neoclassical growth theory, which is expanded to a trade component. As expected trade could have a similar effect like an exogenous shock with a subsequently convergence to the world markets steady state. By having a closer look on the education sector of a country it is questionable if varieties in this sector cause trade and if so does free trade affect economic growth?

The latest events of the last years have shown that there is still a transition between success and setbacks. In 2012 Spain reduced the investments in the education sector, as part of a cost-cutting program. This implicates less money for teachers by working more hours a day and even the funds of research institutes and universities has been shortened. As known from is another engine of economic growth is the human capital accumulation as known from Lucas.1988.

The present work is based on the Lucas-Uzawa Model of endogenous growth, which shows the importance of human capital accumulation for economic growth. Romer (1990) and Barro (1991) contributed the empirical confirmation of the important and distinct role of human capital.

Galor and Mountford (2006, 2008) showed in theory and evidence, that trade causes the "Great Divergence" in income per capita for industrialized and non-industrialized economies. They argue, that the former comparative advantages shaped the trade pattern and thus the development divergences. A comparative advantage in the production of labour intensive good leads to a specialisation in that sector in particular. Therefore there is a higher incentive to invest in the size of population. Differently from the present industrialized countries they had a comparative advantage for capital intensive goods, resulting a specialization in capital intensive goods. Assuming, that labor intensive goods are produced with low skill labour and capital intensive goods with high skill labour, the industrialized economies concentrated on the investments in education to generate high skill labour.

The different investment approaches influence the income per capita in contradictory ways. More

developed industrialized countries invest in human capital accumulation and thus the impact of trade is a higher income per capita. Less developed countries focus on the size of population and in turn the income per capita drops.

But even this theory does not consider physical capital as a factor of production for human capital accumulation. The following models include physical capital as an input of the human capital accumulation like in Bond et al. (1996), Mino (1992), Mulligan and Sala-i-Martin (1993), Stokey and Rebelo (1995). They showed that expenditures in goods and physical capital account for the human capital technology. To open an economy to the rest of the world implies changes of supply and demand at the factors and goods markets. The above mentioned expenditures can be compensated or interpreted as the additional welfare caused by free trade.

By looking only on economics with a similar development stages it is interesting that they use for the education sector as well as for the manufacturing both factors of production. This growth model is an endogenous growth model with two accumulating factors, physical and human capital. But the trade literature has concentrated almost exclusively on physical capital as the only accumulation factor. Bond et. al. (2003) created one of the first models with trade in an endogenous growth model with two accumulating factors.

The model of Lucas (1988) neglected the role of physical capital in the human capital sector. The models of the paper "Equilibrium dynamics in two-sector models of endogenous growth" written by Landrón-de-Guevara, Ortigueira and Santos from (1997) close this gap in the literature. It is another essential example for this paper. On the one hand an equilibrium which includes physical capital as an input to the education sector is analyzed and on the other hand they show an the positive influence of leisure in the utility function on the equilibrium.

The present paper looks only at the first model and questions if there is an effect of trade on the balanced growth path. Is the state of development crucial for the impact of trade between similar regions?

But the following work shows, that trade induce a convergence of income per capita to the development stages. A less industrialized country exports physical capital and is able to invest all available human capital in education. Beside this trade generate an additional human capital increase by importing human capital intensive goods. Unlike a relatively more developed country, there is a negative effect of trade on human capital accumulation. The specialisation in the consumption sector reduces the investment in human capital and there is also a loss of human capital by exporting human capital indirectly and thus reducing the development gap. Here the decisive reason for trade differs. It is based on different endowment ratios of the countries, with the same technological standard.

To sum up: this model combines the both most influential factors for economic growth, human capital accumulation and trade.

The paper is organized as follows. It begins with a description of the autarky model in section 2 by taking a closer look to the production and consumption of the considered economy. Section 3 deals with the introduction of trade and differ between the two types of countries. First a less developed country is looked at and a more developed country is examined afterward. The stability of both frameworks is discussed in section 4 and section 5 concludes.

## 2 Intuition

The example of the textile industry should simplify the construction and mechanism of the model, more precisely the production process of a handbag. When we look at the small economy there is a consumption sector and an education sector. The consumption sector produces a consumption good which benefits the households by a given utility. Here the handbag should satisfy the demand of the consumers. In particular the special need of the consumers is to carry something from one place to another. This commodity good can be produced in different ways depending on the ratio of production factors. In general human capital and physical capital are needed for the production process. On the one hand it can be produced with relatively more human capital or on the other hand relatively physical capital intensive.

On the first case a bag is considered which is handmade and accurate in every detail, such as the Hermes Birkin Bag, which is a classical design item of France. The production process is very complex and needs high skilled bag makers. Thus relatively more human capital is needed for the manufacture.

Having a closer look at the relatively physical capital intensive production process, an imitation of the mentioned bag can be produced with relatively more machines than specific tailors knowledge. A country like Bangladesh has workers which are skilled in sewing and are also used to sewing machines. Even the cuttings of the pattern are mainly made by machines. In contrast to France the pieces are cut by tailors their selves.

The same example is used for the education sector. The produced good is the education of the workers, like skills and abilities. There are also two different ways how to accumulate human capital. The workers can learn how to sew and produce the bag only with the help of a teacher. Here human capital is the exclusively production factor in the education sector. The workers learn while listening to the teacher and get an imagination of how it would work.

If both factors are used in the education sector, then the teachers training is supported with a sewing machine, for example. Now then the workers are able to observe the manufacture and know more precisely the processing steps. It is also conceivable that technical equipment supports the learning process. A TV or tablet PC could be used to watch some tutorial videos. Even in

this case human capital and physical capital is used in the production process<sup>1</sup>.

This paper looks at a small country which trade with the world market at the time of different development stages. A country is less developed, as soon as the education sector is less developed compared to the rest of the world. For more developed countries the contrary is the case. The impact of the various development stages appears in the production processes in each sector.

A less developed country has relatively more physical capital than human capital. That is why the production of the consumption good is physical capital intensive. A closer look to the education sector which is relatively less developed shows, that human capital is the only production factor. Unlike the situation which dominates a relatively more developed country. Both factors are used for both sectors. But the countries' consumption sector uses relatively more human capital than physical capital compared to the world market. Bangladesh produces the handbag in a complete different way than France does. It starts with the education of the workers and end up with the manufacturing itself.

So far that is the situation without any trade. These differences lead to trade with the consumption good and the production factor physical capital. The idea is that a country could benefit from the import of the consumption good by analysing this product. Especially the human capital accumulation is stimulated by trade. Here the handbag could be measured out exactly, the nature and position of the seams could be determined and also the nature and quality of the material could perfect the imitation. The processed human capital is a new source for the education sector. The consumption good in an exchange for physical capital could improve the education sector and help to develop.

### 3 Autarky

This paper is focused on a general model economy of endogenous growth. Starting with an autarkic economy two kinds of agents are assumed in one observed country, representative identical firms and representative identical households. The two sector model consists of education and consumption. There are no transport costs or market barriers in the considered country.

#### 3.1 Production

Generally there are two factors of production, labor  $L$  and capital  $K$ . They can be used for the consumption and education sector and can move freely between these sectors thus they face to the same factor prices. The factor price of capital is the interest  $r$  and the rate of return for the

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<sup>1</sup>A teacher can be every skilled worker who already produced a few bags. Per definition human capital is not necessarily linked to workers. There exists a stock of knowledge which is applied by the workers.

supplied labor is the wage  $w$ . There are more details about the production factors.

$$L = N(t)u(t)\tilde{h}(t)$$

Labor depends on the average amount of human capital  $h(t)$ ,  $u(t)$  is the amount which is used for the production of consumption goods and the population size  $N(t)$ .

$$K = v(t)\tilde{k}(t)$$

Capital consists of the average amount of physical capital  $k(t)$  and the portion which is used in the consumption sector  $v(t)$ .

The concave production function for the consumption sector is  $F[K, L]$  which is increasing and linear homogenous in both factors of production. This is the same for the production function of the education sector:  $G[K, L]$ . Firms borrow two kinds of capital from the households, physical and human capital. The cost of producing the consumption good can be expressed as a function of  $w$  and  $r$ ,  $P = p(w, r)$ . The profit maximization problems for the consumption sector at the factor and at the goods market is:

$$\max \Pi = F[v(t)\tilde{k}(t), u(t)\tilde{h}(t)] - rv(t)\tilde{k}(t) - wu(t)\tilde{h}(t) \quad (1)$$

$$\frac{\partial F}{\partial \tilde{k}(t)} = rv(t) \quad (2)$$

$$\frac{\partial F}{\partial \tilde{h}(t)} = wu(t) \quad (3)$$

The factors of production are substitutable and the same good can be produced physical capital intensive of a less developed country or human capital intensive in a relatively more developed country.

### 3.2 Households

The households  $N(t) = N_0 e^{nt}$  grow with an exogenous rate  $n$ . These consumers are factor owner and supply both physical capital and human capital to the factor markets. They choose between the accumulation of the produced good or to sell them for consumption. Therefore the income is used to buy consumption goods and to invest in human capital accumulation. All consumers have the same preferences and discount their future utility with the rate  $\rho$ .

$$U[\tilde{c}(t)] = \frac{\tilde{c}(t)^{1-\sigma}}{1-\sigma}$$

The consumption affects the current utility. For simplicity the effect of education on the households' utility will be ignored. It is shown in Caballè and Santos (1993), that  $\tilde{c}(t)$ ,  $\tilde{k}(t)$  and  $\tilde{h}(t)$

grow at the same rate  $g$  in a steady-state equilibrium.

The small country is in a steady-state equilibrium with  $\tilde{C}^i(t)$ ,  $\tilde{k}^i(t)$  and  $\tilde{h}^i(t)$  which grow at a rate  $g^i$ .

$$k^i(t) = \tilde{k}^i(t)e^{-g^i t}, h^i(t) = \tilde{h}^i(t)e^{-g^i t}, c^i(t) = \tilde{c}^i(t)e^{-g^i t}$$

The variables of choice for the optimization problem is the amount of consumption  $c(t)$ , the amount of physical capital allocated to production  $v(t)$  and education  $(1 - v(t))$  as well as the amount of human capital allocated to production  $u(t)$  and education  $(1 - u(t))$  and the maximal stream of discounted utility.

$$\max_{c(t), u(t), v(t)} V[k_0, h_0] = \int_0^\infty e^{-(\rho - n - (1 - \sigma)g)t} \frac{c(t)^{1 - \sigma}}{1 - \sigma} dt \quad (4)$$

subject to

$$\dot{k} = rv(t)k(t) + wu(t)h(t) - (\pi + n + g)k(t) - c(t) \quad (5)$$

$$\dot{h} = G[(1 - v(t))k(t), (1 - u(t))h(t)] - (\theta + g)h(t) \quad (6)$$

$$c(t) \geq 0, \quad k(t) \geq 0, \quad h(t) \geq 0,$$

$$0 \leq u(t) \leq 1, \quad 0 \leq v(t) \leq 1,$$

$$\rho - n - (1 - \sigma)g > 0, \quad k_0, h_0 \text{ exogenous}$$

Here  $\pi$  with  $\pi \geq 0$  is the depreciation of physical capital and the depreciation of human capital is  $\theta$  with  $\theta \geq 0$ .

The Hamiltonian for the representative household is

$$\begin{aligned} \mathbb{H} = & \frac{c(t)^{1 - \sigma}}{1 - \sigma} \\ & + \gamma_1(rv(t)k(t) + wu(t)h(t) - (\pi + n + g)k(t) - c(t)) \\ & + \gamma_2(G[(1 - v(t))k(t), (1 - u(t))h(t)] - (\theta + g)h(t)) \end{aligned} \quad (7)$$

where  $\gamma_1$  and  $\gamma_2$  are the shadow prices of physical and human capital in utility units.

The first order conditions for the optimal solution are

$$c(t)^{-\sigma} = \gamma_1(t) \quad (8)$$

$$\gamma_1(t)r = \gamma_2(t)G_K[(1 - v(t))k(t), 1 - u(t))h(t)] \quad (9)$$

$$\gamma_1(t)w = \gamma_2(t)G_L[(1 - v(t))k(t), 1 - u(t))h(t)] \quad (10)$$

$$\hat{\gamma}_1(t) = \rho + \sigma g + \pi - rv(t) \quad (11)$$

$$\hat{\gamma}_2(t) = \rho - n + \sigma g + \theta - G_L[(1 - v(t))k(t), 1 - u(t))h(t)] \quad (12)$$

The condition for pareto efficiency for consumption demand  $c(t)$  is (8) and the inter temporal arbitrage conditions for physical and human capital are (11) and (12). The transversality conditions at infinity are

$$\lim_{t \rightarrow \infty} e^{-(\rho - n - (1 - \sigma)g)t} \gamma_1(t)k(t) = 0 \quad (13)$$



$$\lim_{t \rightarrow \infty} e^{-(\rho - n - (1 - \sigma)g)t} \gamma_2(t) k(t) = 0 \quad (14)$$

It arises a Keynes-Ramsey Rule for the autarky situation.

$$\hat{c} = \frac{1}{\sigma} (rv(t) - \pi - \sigma g - \rho) \quad (15)$$

## 4 Trade

The trade model consists of two different regions, the world market "WM" and a home country "\*". In the following part two different development stages are looked at. The considered country is economical small. This means that the decisions of the country does not affect the world market. The agents of both regions have the same preferences and work with the same technologies. There are still no transport costs and no market barriers. The wide variation considers the endowment with factors of production. More precisely the world market has an absolute advantage in both production factors, physical and human capital, but the physical capital- human capital ratio  $k/h$  differs in both regions.

$$(k(t)/h(t))^{WM} \neq (k(t)/h(t))^* \quad (16)$$

Two cases are distinguished: The world market trades with either a relatively less developed country or with a relatively more developed country. Physical capital is tradeable and the only mobile factor of production. The other commodity is the consumption good. In the open economy as well as in the world market are identical households  $N^i(t) = N_0^i e^{nt}$ . The equations of the two factors of production in each country  $i$  are:  $L^i = N^i(t) u^i(t) h^i(t)$  and  $K^i = v^i(t) k^i(t)$ . The modified utility function is as follows.

$$U^i[C^i(t)] = \frac{C^i(t)^{1-\sigma}}{1-\sigma}$$

$C^i$  is the total consumption in a small country  $i$ . The total consumption consist of the domestic produced goods and the imported goods with  $C^i = c^i + \bar{c}^i$ . The households do not know and do not care about the origin of the commodity.

### 4.1 Case 1: Trade in a less developed country

First a relatively less developed country trades with the world market. Less developed means there is a smaller technological development stage of a country compared to the world market and is unrelated to developing countries. The development stage has impact on the education sector and the used technology. The initial situation is the autarky situation without trade. The Home country has relatively more physical capital than human capital compared to the world market. That is why the education sector exists in a smaller way. According to that the physical capital-human capital ratio of the considered country is higher than the ratio at the world market.

$$(k(t)/h(t))^* > (k(t)/h(t))^{WM}$$

Both regions produce a consumption good which benefits the consumers in the same way. But in the production process the less developed country uses less human capital and substitute it with physical capital. Conditioned by the different  $(k/h)^i$  ratios, there are also different relative prices for the commodities. A higher supply of physical capital leads to a relatively lower factor price in the less developed country. In this country the interest rate is relatively smaller than at the world market for the autarky situation. But the commodity good is relatively more expensive in the small economy than at the world market. Based on the relatively lower equipment of consumption goods the price is relatively higher in the small economy.

$$(r/p(w, r))^* < (r/p(w, r))^{WM}$$

The following passage describes the short run reactions brought by free trade under the terms of the Heckscher-Ohlin Model. Remembering there is no trade of two consumption goods in this model. They exchange the consumption good for physical capital.

At free trade there is only one ruling price for the whole world. The small economy adjusts the price of the world market. For this state of development the country's lower price ratio has to go up to the level of the world market. A higher interest rate leads to higher production costs and so a higher commodity price, by neglecting the impact of the wage to the price, with  $\frac{\partial p}{\partial r} > 0$ . But the reactions of the labour market counteracts this effect and compensate this developments. The human capital-physical capital ratios allows inferences about the comparative endowments. According to that the wage of the small country is relatively more expensive in the autarky situation and with trade the wage level decreases. A lower wage causes lower costs and in the end it partially reduces the consumption good price, with  $\frac{\partial p}{\partial w} < 0$ . It arises the following condition:

$$\left| \frac{\partial p}{\partial r} \right| \leq \left| \frac{\partial p}{\partial w} \right|$$

The price relationship of the less developed economy must hold

$$(\hat{r} - p(\hat{w}, r)) > 0$$

If the price difference of the interest rate is higher than the difference of the commodity good, then the less developed country adopts the world market's price relationship. In this case there is a rising price ratio  $\frac{r}{p(w, r)}$ .

This happens by an increase of the interest rate, a decrease of the consumption goods price or as a result of both effects. A higher interest rate has two reactions, direct on the factor market as well as indirect on the goods market. On the one hand an increasing interest rate leads to a higher supply and a lower demand of physical capital.

But on the other hand the factor price reaction effects the good price. The commodity is produced with physical capital, which is now relatively more expensive than it was before trade. The accrued

higher costs lead to a higher price of the consumption good. But in total the price ratio increases because of trade. Therefore a decreasing wage level is mandatory by adopting of the world market price relationship.

The indirect impact on the factor markets are as follows. Labor is relatively cheap thus the supply decrease and the demand increases for human capital. The consumers do not spend their time in the production of the consumption good anymore, caused by the low wage. Instead they invest their workload to the education sector and increase the human capital accumulation. The stock of physical capital decreases because of a higher supply und lower demand of physical capital.

Considering the consumption good sector, a less price for the produced good implies a smaller supply caused by the domestic producers. It is more attractive for the producers to offer the factors of production to a higher price to the factor markets instead of producing the consumption goods. The unexploited physical and human capital raises the supply at the respective factor market. Even here the consumers spend more time in human capital accumulation, because the firms offer the possibility to work in a smaller way than before. Since the less developed country produces the commodity relatively physical capital intensive, there is a relatively higher available amount of physical capital than of human capital by the same waiver of consumption goods.

The other reaction to the reduced price is a higher consumers' demand in the home country for the commodity. Higher demand and less supply leads to an excess demand and can be satisfied by importing the consumption good.

To sum up the stock of physical capital is higher than before trade. Additionally there is less demand for physical capital, the oversupply will be exported to the world market. The additional labor is used for education. An other positive impact to human capital accumulation is the imported consumption good. It can be used for human capital accumulation by analyzing the imported good.

This paper neglect the price effect and so the reaction of the terms of trade, because a small country is looked at, which has no influence on the proceedings at the world market.

To conclude, the pattern of foreign trade is as follows for relatively less developed countries. The home country exports physical capital and imports the consumption good. Thus the world market imports the factor of production capital and exports the commodity.

After the reactions through trade the dynamics of growth gets priority in the exposition of the model. Relating to the education sector there is a new variation of the countries' dynamics. The less developed country does not spent physical capital on human capital accumulation, but the world market does. Responsible for this different allocation is the development stage of the world market's education sector, which is relatively more developed than the one of the small economy. The less developed country has the same technology for the consumption good as the world market indeed, but the production function of the education sector differs.

This results by following the approach of Galor and Mountford (2006). In this work less developed countries have no incentives to invest additionally physical capital in the education sector. But the trade pattern makes clear, that there are also no incentives to invest in the population size, because the relatively less developed country exports physical capital and no good which is produced labour intensive. This raises the question if trade leads to exploiting the resources.

Both regions have the same technology for the consumption sector:

$$F^i[K^i, L^i] = rk^i(t) + wu^i(t)h^i(t)$$

but different technologies in the education sector :

$$G^*[L^*] = B^*(1 - u^*(t))h^*(t)\bar{c}$$

$$G^{WM}[K^{WM}, L^{WM}] = B^{WM}((1 - v^{WM}(t))k^{WM}(t))^\alpha((1 - u^{WM}(t))h^{WM}(t))^{1-\alpha}$$

Where  $\bar{c}$  is the amount of imports which is produced in the rest of the world. To target a steady-state equilibrium  $\tilde{C}^i(t)$ ,  $\tilde{k}^i(t)$  and  $\tilde{h}^i(t)$  grow at the same rate  $g^i$  in a region. The growth rate of a relatively less developed country is smaller than the rate of the world market.

$$g^{WM} > g^h$$

The consumers pay the same price  $p$  for the imported good and the home manufactured one. For less developed countries the Hamiltonian is:

$$\begin{aligned} \mathbb{H}^* &= \frac{(c^*(t) + \bar{c}(t))^{1-\sigma}}{1-\sigma} \\ &+ \gamma_1^*(rk^*(t) + wu^*(t)h^*(t) - (\pi + n + g^*)k^*(t) - p(c^*(t) + \bar{c})) \\ &+ \gamma_2^*(B^*(1 - u^*(t))h^*(t)\bar{c} - (\theta + g^*)h^*(t)) \end{aligned} \quad (17)$$

with the laws of motion of physical and human capital

$$\dot{k}^* = rk^*(t) + wu^*(t)h^*(t) - (\pi + n + g^*)k^*(t) - p(c^*(t) + \bar{c}) \quad (18)$$

$$\dot{h}^* = B^*(1 - u^*(t))h^*(t)\bar{c} - (\theta + g^*)h^*(t) \quad (19)$$

$$\bar{c}(t) > 0, \quad c^*(t) \geq 0, \quad k^*(t) \geq 0, \quad h^*(t) \geq 0,$$

$$0 \leq u^*(t) \leq 1, \quad 0 \leq v^*(t) \leq 1,$$

$$\rho - n - (1 - \sigma)g^* > 0, \quad k_0^*, h_0^* \text{ exogenous}$$

The equilibrium allocations will be fully specify with the following first order conditions and both transversality conditions(24) and(25).

$$\frac{(c^*(t) + \bar{c}(t))^{-\sigma}}{p} = \gamma_1^*(t) \quad (20)$$

$$\gamma_1^*(t)w = \gamma_2^*(t)B^*\bar{c}(t) \quad (21)$$

$$\hat{\gamma}_1^*(t) = \rho + \sigma g^* + \pi - r \quad (22)$$

$$\hat{\gamma}_2^*(t) = \rho - n + \sigma g^* + \theta - B^*\bar{c}(t) \quad (23)$$

and

$$\lim_{t \rightarrow \infty} e^{-(\rho - n - (1 - \sigma)g^*)t} \gamma_1^*(t)k^*(t) = 0 \quad (24)$$

$$\lim_{t \rightarrow \infty} e^{-(\rho - n - (1 - \sigma)g^*)t} \gamma_2^*(t)k^*(t) = 0 \quad (25)$$

The allocation of the consumption good is efficiently if the marginal utility is equal to the shadow price of physical capital accumulation. Thus the optimum of output allocation is shown in (20). Since the consumers do not differ between domestic produced or imported goods, there is no difference to the situation in the autarky model.

The derivation of  $u^*$  (21) shows the condition for the best solution for splitting the available human capital into the production sector and the education sector. The marginal productivity of human capital depends on the amount of imports.

The last two equations (22) and (23)are the intertemporal arbitrage conditions and are defined by the growth rates of the shadow prices of the associated factors of production. Equation (23) is also interesting considering the changes induced by trade. The import of consumption goods influences the intertemporal arbitrage in a negative way, considering the human capital accumulation. The more human capital is available the less imports are necessary and the other way round. For a relatively less developed country with a small endowment of human capital, it is to recommend to import human capital intensive consumption goods to improve their situation.

In addition to these direct trade effects, there are also indirect level effects through trade. The level of human capital rised because of two reasons. First the education sector benefits by importing the relatively human capital intensive produced consumption good. Second there is more labour available, which comes from the commodity good sector and is no longer needed there, because of the reduced total domestic production. Beside this positive indirect effects, there is even a negativ effect on the physical capital accumulation cause by exporting this factor. It follows a similar Keynes Ramsey Rule like in a closed economy and without any direct trade effects. The only difference is the level of the balanced growth path, considered to the country's growth rate  $g^*$ . But there is no direct influence of the imports in the growth rate of consumption.

$$\hat{C}^* = \frac{1}{\sigma}(r - \pi - \sigma^* - \rho) \quad (26)$$

To sum up this section, trade leads to human capital intensive import goods. In this case the small economy despecialize in the consumption good sector. Less domestic produced goods are needed and thus more labour can be used for the education sector. The released capital can be exported and hence additionally finance the growing education sector.

## 4.2 Case 2: Trade in a more developed country

The second case deals with the small country, which is relatively more developed compared to the world market. It is assumed, that the technological development stage of the world market is the average technological progress of the remaining world. Therefore a more developed country is a very industrialized country which produces high tech manufacturing by using lots of human capital. As well the world market has an absolute advantage in both production factors. The rate of both production factors is the critical item of the development stage.

$$(k(t)/h(t))^* < (k(t)/h(t))^{WM}$$

Since the high proportion of human capital in the developed country, the physical capital-human capital ratio of the world market is higher than the one of the home country. Compared to the world market physical capital is relatively short, this means to the factor price, that the interest rate of the more developed country is relatively high. Because of a higher developed and hence better equipped education sector, the consumption good is relatively cheaper in the observed small economy than in the rest of the world. That justifies why the consumption good is produced with relatively more human capital than physical capital.

$$(r/p)^* > (r/p)WM$$

The different price ratios are again the reason for trade. But the trade pattern of a more developed country is different from the above considered less developed case. This results from the following short run reactions induces through trade. The more developed small country adopts the price ratio of the world market. For this the price of the commodity good is more expensive and/or the interest rate decreases.

On the one hand a higher price for the consumption good is more attractive for the domestic producers, because of a higher marginal product of labor. The marginal product is higher than the wage for labor and that is why the supply of the product increases. The reactions on the factor markets are a higher demand and a lower supply of both factors of production compared with the situation before trade.

On the other hand a higher price leads to less consumers' demand and an oversupply arises, which can be exported to the world market.

But these reactions are conditioned by  $(\hat{r} - p(\hat{w}, r)) < 0$ . If the difference of the counter is smaller

than the difference of the denominator then the ratio does not decrease and thus does not adapt the world's market's price ratio. This is valid for an inverse connection between the impact of the factor prices to the consumption good price:  $\frac{\partial p}{\partial r} < 0$  and  $\frac{\partial p}{\partial w} > 0$  so that following applies:

$$\left| \frac{\partial p}{\partial r} \right| \leq \left| \frac{\partial p}{\partial w} \right|$$

This describes the fact that the rising wage level needs to impact the consumption good price more than the lower interest rate does. A higher wage level leads to a higher demand of labour in the consumption good sector. A higher supply of labor results, thus the households want to spend their time in producing the commodity. Consequently this is a negative effect on the education sector, where is less labour available than before trade. There is a second negative effect on the human capital accumulation brought about the increased consumption price. If more households are needed for the production processes in the consumption sector, then there is also less available labor for the education sector.

Compared to the less developed country the more developed country uses both factors of production for human capital accumulation. That is why this second effect has a stronger impact. The higher production of the consumption good needs more of both factors, which are also production factors of the education sector. So the higher demand of labour is not the only fact which reduces the human capital accumulation. The higher demand of physical capital also reduces the share of the education sector.

Whereas the interest rate decreases, the demand of physical capital increases. The supply of physical capital decrease as it would be more profitable by using it in the production sector for the consumption good production. The demand surplus is eliminated by import the over demand of the factor production from the world market. The intention to import physical capital is supported by the risen amount of the consumption good, caused by the higher commodity price. This induces also a over demand of physical capital, which increase the imports.

Concluding after following the Heckscher-Ohlin Model again, a more developed country exports the consumption good and imports physical capital from the world market,  $\bar{c}(t) < 0$ . The other way round, the world market exports physical capital and imports the consumption good.

Because of trade even in this second case there are no price differences between the commodities. But by looking at the dynamics of the growth model it differs from the situation of a less developed country. In this case both regions, home country and world market, are more similar to each other. Both regions produce the education good with human capital and additional physical capital. The consequence of trade is a specialisation on the production of the consumption good. For the production process both factors are needed and the incentive to invest in the education sector to expand technological progress is now lower than before. The more developed country provides the opportunity for technology leap frogging. Furthermore the distance to the world technology

frontier could be closed and the leading position of the relatively more developed country gets lost. The consumption sector has priority over the education sector.

So this case is more similar to the general autarky situation, because of the same types of production functions in both sectors.

Education sector:

$$G^*[K^*, L^*] = B^*((1 - v^*(t))k^*(t))^{\alpha^*}((1 - u^*(t))h^*(t))^{1-\alpha^*}$$

$$G^{WM}[K^{WM}, L^{WM}] = B^{WM}((1 - v^{WM}(t))k^{WM}(t))^{\alpha^{WM}}((1 - u^{WM}(t))h^{WM}(t))^{1-\alpha^{WM}}\bar{c}$$

Although they use the same technology in the education sector the technology of the world market is a little bit different from the other. The elasticities of productivities are different,  $\alpha^* < \alpha^{WM}$ . The world market use relatively more physical capital than human capital for the production in the education sector. Whereas in the small economy the technology of the education sector needs relatively more human capital than physical capital. Implementing trade, the world market benefits from the imports of the human capital intensive produced consumption good  $\bar{c}$ . It acts like an indirect export of knowledge.

For the consumption good sector it is exactly the same in both regions.

$$F^i[K^i, L^i] = rv(t)k^i(t) + wu^i(t)h^i(t)$$

$$g^{WM} < g^h$$

The laws of motions for the two accumulating factor are:

$$\dot{k}^i = rv^i k^i(t) + wu^i(t)h^i(t) - (\pi + n + g^i)k^i(t) - pc^i(t)$$

$$\dot{h}^* = B^*[(1 - u^*(t))h^*(t)]^{\alpha^*}[(1 - v^*(t))k^*(t)]^{1-\alpha^*} - (\theta + g^* + \bar{c}(t))h^*(t)$$

$$\dot{h}^{WM} = B^{WM}[(1 - u^{WM}(t))h^{WM}(t)]^{\alpha^{WM}}[(1 - v^{WM}(t))k^{WM}(t)]^{1-\alpha^{WM}} - (\theta + g^{WM} + \bar{c}(t))h^{WM}(t)$$

The physical capital accumulation is not effected by trade directly. Free trade affects like a exogenous shock with some level effect, but no continuously impact. That is different from the human capital accumulation. There is a negative effect on the accumulation caused by exporting the consumption good. The more developed country specialises in the production of the consumption sector. For this both factors of production are necessary in a larger scale than before trade. These factors are missing in the education sector and it is less productive. Furthermore is there an indirect trade of human capital to the rest of the world. This does not mean, that they "lose" human capital, they just generate less new human capital in the economy.

The resulting Keynes Ramsey Rule is different affected by trade. It depends not only on the interest rate  $r$  but also on the amount of physical capital  $v^*(t)$  and also on the export good in the considered country.

$$\hat{C}^* = \frac{1}{\sigma}(rv^*(t) + r(1 - v^*(t))^{1-\alpha^*} v^{*\alpha^*}(t) - \pi - \sigma g^* - \bar{c}(t) - \rho) \quad (27)$$



Higher exports reduce the marginal product of physical capital and can lead to a smaller consumption growth rate. There is a direct impact of trade to the economic growth, if a country is relatively more developed than the world market.

The growth rate of consumption is derived from the Hamiltonian for relatively more developed countries:

$$\begin{aligned} \mathbb{H} = & \frac{(C^*(t))^{1-\sigma}}{1-\sigma} \\ & + \gamma_1^*(rv^*(t)k^*(t) + wu^*(t)h^*(t) - (\pi + n + g^* + \bar{c}(t))k^*(t) - p(c^*(t) + \bar{c}(t))) \\ & + \gamma_2^*(B^*[(1-u^*(t))h^*(t)]^{\alpha^*}[(1-v^*(t))k^*(t)]^{1-\alpha^*} - (\theta + g^* + \bar{c}(t))h^*(t)) \end{aligned} \quad (28)$$

on condition that:

$$\begin{aligned} \bar{c}(t) < 0, \quad c^*(t) \geq 0, \quad k^*(t) \geq 0, \quad h^*(t) \geq 0, \\ 0 \leq u^*(t) \leq 1, \quad 0 \leq v^*(t) \leq 1, \\ \rho - n - (1-\sigma)g^* > 0, \quad k_0^*, h_0^* \text{ exogenous} \end{aligned}$$

The first order conditions are:

$$\frac{(C^*(t))^{-\sigma}}{p} = \gamma_1^*(t) \quad (29)$$

$$\gamma_1^*(t)r = \gamma_2^*(t)B^*(1-\alpha^*)(1-u^*(t))^{\alpha^*}v^{-\alpha^*}(t)\left(\frac{h^*(t)}{k^*(t)}\right)^\alpha \quad (30)$$

$$\gamma_1^*(t)w = \gamma_2^*(t)B^*\alpha^*\left(\frac{1-v^*(t)}{u^*(t)}\right)^{1-\alpha^*}\left(\frac{k^*(t)}{h^*(t)}\right)^{1-\alpha^*} \quad (31)$$

$$\hat{\gamma}_1^*(t) = -rv^*(t) - r(1-v^*(t))^{1-\alpha^*}(v^*(t))^{\alpha^*} + \pi + \rho + \sigma g^* + \bar{c}(t) \quad (32)$$

$$\hat{\gamma}_2^*(t) = -B^i\alpha^*\left(\frac{k^*(t)}{h^*(t)}\right)^{1-\alpha^*}(1-v^*(t))^{1-\alpha^*} + \theta - n + \rho + \sigma g^* + \bar{c}(t) \quad (33)$$

For the sake of completeness, the transversality conditions are again:

$$\lim_{t \rightarrow \infty} e^{-(\rho-n-(1-\sigma)g^*)t}\gamma_1^*(t)k^*(t) = 0 \quad (34)$$

$$\lim_{t \rightarrow \infty} e^{-(\rho-n-(1-\sigma)g^*)t}\gamma_2^*(t)h^*(t) = 0 \quad (35)$$

## 5 Trade policy

The arising problems through trade in both cases urge to have a closer look at relevant or possible trade policy instruments.

Trade leads to the exploitation of physical capital in a less developed country. The government could avoid this by giving an export limit for physical capital and stipulates to invest physical capital in education like human capital. This change in strategy supports a stronger de-specialisation

in the production of the consumption good and expand the education sector. A smaller amount of exports reduces the amount of import as well caused by balanced trade. That is why there is also a smaller gain of human capital by the high skill produced goods. This is compensated by the physical capital investments. But this political intervention leads to a smaller growth rate than there is with free trade.

Considering the more developed country the rising problem is the fluctuation or stagnation of the technological development stage. The progress of the former constantly expanding education sector is lower or non existent compared to the situation without trade. Both production factors are especially used for the consumption good sector, because of the trade specialisation. Even in this case a restriction on exports could reduce the problem.

## 6 Stability

The equilibrium stability of the autharky model is shown in Landrón-de-Guevara, Ortigueira and Santos (1997). The following part studies only the trade situation for both different situations. When a country opens for trade there is an imbalance in the short run on the world market and the previous steady state is disturbed. This situation implies good condition to study the global stability with the methodology which is used by Caballé and Santos (1993). It is an combination of techniques from Dynamic Programming and the Maximum Principle.

As shown in Benveniste and Scheinkman (1982) the vector of co-state variables and the derivative of the value function  $DV(k, h)$  are equal to each other. This fact is used to show the stability for both situations.

$$\frac{\partial V}{\partial k} = \frac{\gamma_1^i}{\gamma_2^i} = \frac{\partial V}{\partial h}$$

As Landrón-de-Guevara, Ortigueira and Santos (1997) an invariant  $\gamma_1^i/\gamma_2^i$  is assumed and the autarky situation is an interior stationary solution. It is questionable if the open economies reach a new steady state and convergence to the equilibrium of the world market.

### 6.1 Case 1: Stability of a less developed country

Compared to the situation on the world market the less developed country has a higher  $k(t)^i/h(t)^i$  ratio and a smaller growth rate without trade. Does the country converge to the steady state of the world market thru trade?

Hence the  $k^*(t)/h^*(t)$  ratio must go down, with  $\dot{k}^*(t) < 0$  and  $\dot{h}^*(t) = 0$ . This happens because the less developed country exports physical capital. The human capital accumulation is unchanged, because the reduction of physical capital does not affect the education sector. The imbalance

which enables trade has to compensate fully. The allocation of human capital and physical capital should be the same in both regions after trade.

$$\left(\frac{v(t)k(t)}{u(t)h(t)}\right)^* = \left(\frac{v(t)k(t)}{u(t)h(t)}\right)_{WM} \text{ and } \left(\frac{(1-v(t))k(t)}{(1-u(t))h(t)}\right)^* = \left(\frac{(1-v(t))k(t)}{(1-u(t))h(t)}\right)_{WM}$$

A less developed country will converge to the steady state of the world market. As a result from trade the home economy grew with the same rate like the world market.

## 6.2 Case 2: Stability of a more developed country

A more developed country has relatively more human capital than physical capital. In this case the  $k^*(t)/h^*(t)$  ratio is smaller than at the world market. Caused by trade the small country is not any more in its previous balanced growth path but in an arbitrary point of the world market. Free trade induces the export of the consumption good and the import of physical capital. According to that  $\dot{k}^*(t) > 0$  and  $\dot{h}^*(t) > 0$ . The human capital accumulation is supported by additional physical capital. Thus more physical capital means more human capital as well. Depending on the intensities of the two effects the steady state of the more developed home country will converge or diverge to the world markets' equilibrium.

## 7 Conclusion

This paper has proved two extensions of the Uzawa Lucas framework. It is focused on the introduction of trade with two accumulating factors for a small economy. Two different development stages of this country are analysed. In the first case is about a less relatively developed country with a higher physical capital-human capital rate  $h(t)/k(t)$  than the world market. The second situation look at a more developed country compared to the world market with a smaller  $h(t)/k(t)$  ratio. The less developed country has no physical capital in the education sector whereas the more developed country use physical capital as an additional input in human capital accumulation.

The less developed country improves its economic situation. Trade inducts a higher human capital accumulation. It converges to the balanced growth path of the world market and grew with a higher growth rate than before trade. This country benefits from the import of the consumption good which is produced with relatively more human capital at the world market than at the domestic market. In return a relatively less developed country supports the other region by exporting their physical capital.

It is a bit different in the second case with a more developed country. There is also a convergence to the world markets' growth path, but in a negative way. The export of human capital intensive produced goods change the focus of a relatively more developed country. The education sector becomes less important and for this situation trade makes the situation even worse.

## Mathematical Appendix

### Derivation of the optimal values $C^*(t), k^*(t), h^*(t), u^*(t)$ in a less developed country

The laws of motion for physical and human capital are given by

$$\dot{k}^*(t) = rk^*(t) + wu^*(t)h^*(t) - (\pi + n + g^*)k^*(t) - p(c^*(t) + \bar{c}^*(t)) \quad (36)$$

$$\dot{h}^* = B^*(1 - u^*(t))h^*(t)\bar{c}^*(t) - (\theta + g^*)h^*(t) \quad (37)$$

For the optimization problem of the household's problem Hamiltonian approach is used

$$\begin{aligned} \mathbb{H}^* &= \frac{(c^*(t) + \bar{c}(t))^{1-\sigma}}{1-\sigma} \\ &+ \gamma_1^*(rk^*(t) + wu^*(t)h^*(t) - (\pi + n + g^*)k^*(t) - p(c^*(t) + \bar{c}(t))) \\ &+ \gamma_2^*(B^*(1 - u^*(t))h^*(t)\bar{c}^*(t) - (\theta + g^*)h^*(t)) \end{aligned} \quad (38)$$

The first order conditions are given by

$$\frac{\partial \mathbb{H}^*}{\partial C^*(t)} \stackrel{!}{=} 0 \quad (39)$$

$$\frac{\partial \mathbb{H}^*}{\partial k^*(t)} \stackrel{!}{=} -\dot{\gamma}_1^* - \gamma_1^*(-\rho + n + (1 - \sigma)g^*) \quad (40)$$

$$\frac{\partial \mathbb{H}^*}{\partial h^*(t)} \stackrel{!}{=} -\dot{\gamma}_2^* - \gamma_2^*(-\rho + n + (1 - \sigma)g^*) \quad (41)$$

$$\frac{\partial \mathbb{H}^*}{\partial u^*(t)} \stackrel{!}{=} 0 \quad (42)$$

The calculation of Equation (39) yields

$$(c^*(t) + \bar{c}(t))^{-\sigma} - \gamma_1^* p^* = 0 \quad (43)$$

$$\gamma_1^* = \frac{C^*(t) - 1}{p^*} \quad (44)$$

The law of motion of the shadow price  $\gamma_1^*$  is needed for the Keynes Ramsey Rule. The derivative of Equation (44) with respect to time is

$$\frac{\partial \gamma_1^*}{\partial t} = \frac{-\sigma C^{-\sigma-1}(t) \dot{C}^*(t)}{p^*} \quad (45)$$

$$\dot{\gamma}_1^* = -\sigma \frac{C^{-\sigma}(t)}{p^*} \hat{C}(t) \quad (46)$$

From Equation (40) follows

$$\gamma^*(r - \pi - n - g^*) = -\dot{\gamma}_1^* - \gamma_1^*(-\rho^* + n + (1 - \sigma)g^*) \Leftrightarrow \hat{\gamma}_1^* = \rho + \sigma g^* - r + \pi \quad (47)$$

The calculation of Equation (41) yields

$$\gamma_1^* w u^*(t) + \gamma_2^* (B^* (1 - u^*(t)) \bar{c}(t) - \theta - g^*) = -\dot{\gamma}_2^* - \gamma_2^* (-\rho + n + (1 - \sigma)g^*) \quad (48)$$

$$\Leftrightarrow -\hat{\gamma}_2^* = \frac{\gamma_1^*}{\gamma_2^*} w u^* + B^* (1 - u^*(t)) \bar{c}(t) - \theta - g^* - \rho + n + (1 - \sigma)g^* \quad (49)$$

From Equation (42) yields

$$\gamma_1^* w h^*(t) = \gamma_2^* B^* h^*(t) \bar{c}(t) \Leftrightarrow \frac{\gamma_1^*}{\gamma_2^*} = \frac{B^* \bar{c}(t)}{w} \quad (50)$$

Inserting this in Equation (49), it yields

$$\hat{\gamma}_2^* = \theta - n + \sigma g^* - B^* \bar{c}(t) + \rho \quad (51)$$

To finish the derivation of the consumption growth rate Equation(40), Equation (44) in combination with Equation (52) are needed

$$\frac{C^{-\sigma^*}(t)}{p} (r - \pi - n - g^*) = \sigma \frac{C^{-\sigma^*}(t) \hat{C}(t)}{p} - \frac{C^{-\sigma^*}(t)}{p} - \rho + n + (1 - \sigma)g^* \quad (52)$$

After rearranging it results the Keynes Ramsey Rule

$$\hat{C} = \frac{1}{\sigma} (r - \pi - \sigma g^* - \rho) \quad (53)$$

## Derivation of the optimal values $C^*(t), k^*(t), h^*(t), u^*(t)$ in a more developed country

Physical and human capital change over time as followed

$$\dot{k}^* = r v^* k^*(t) + w u^*(t) h^*(t) - (\pi + n + g^*) k^*(t) - p C^*(t) \quad (54)$$

$$\dot{h}^* = B^* [(1 - u^*(t)) h^*(t)]^{\alpha^*} [(1 - v^*(t)) k^*(t)]^{1 - \alpha^*} - (\theta + \bar{c}(t)) h^*(t) \quad (55)$$

The optimization problem of a representative household is also solved with the Hamiltonian approach

$$\begin{aligned} \mathbb{H}^* &= \frac{((c^*(t) + \bar{c}(t))^{1 - \sigma})}{1 - \sigma} \\ &+ \gamma_1^* (r v^*(t) k^*(t) + w u^*(t) h^*(t) - (\pi + n + g^* - \bar{c}(t)) k^*(t) - p(c^*(t) + \bar{c}(t))) \\ &+ \gamma_2^* (B^* [(1 - u^*(t)) h^*(t)]^{\alpha^*} [(1 - v^*(t)) k^*(t)]^{1 - \alpha^*} - (\theta + \bar{c}(t)) h^*(t)) \end{aligned} \quad (56)$$

The first order conditions are

$$\frac{\partial \mathbb{H}^*}{\partial C^*(t)} \stackrel{!}{=} 0 \quad (57)$$

$$\frac{\partial \mathbb{H}^*}{\partial k^*(t)} \stackrel{!}{=} -\dot{\gamma}_1^* - \gamma_1^*(-\rho + n + (1 - \sigma)g^*) \quad (58)$$

$$\frac{\partial \mathbb{H}^*}{\partial v^*(t)} \stackrel{!}{=} 0 \quad (59)$$

$$\frac{\partial \mathbb{H}^*}{\partial h^*(t)} \stackrel{!}{=} -\dot{\gamma}_2^* - \gamma_2^*(-\rho - n + (1 - \sigma)g^*) \quad (60)$$

$$\frac{\partial \mathbb{H}^*}{\partial u^*(t)} \stackrel{!}{=} 0 \quad (61)$$

$$(62)$$

From Equation (57) yields

$$C^{-\sigma}(t) - \gamma_1^* p = 0 \quad (63)$$

$$\gamma_1^* = \frac{C^{-\sigma}(t)}{p} \quad (64)$$

To build the Keynes Ramsey Rule the derivation with respect to time of Equation (64) is needed

$$\dot{\gamma}_1^* = -\sigma \frac{C^{-\sigma}(t)}{p} \hat{C}^*(t) \quad (65)$$

Equation (58) yields to

$$\begin{aligned} & \gamma_1^*(rv^*(t) - \pi - n - g^* - \bar{c}(t)) \\ & + \gamma_2^*(B^*((1 - u^*(t))h^*(t))^{\alpha^*} (1 - v^*(t))^{1-\alpha^*} (1 - \alpha^*)k^*(t)^{-\alpha^*}) \\ & = -\dot{\gamma}_1^* - \gamma_1^*(-\rho + n + (1 - \sigma)g^*) \end{aligned} \quad (66)$$

$$\begin{aligned} \Leftrightarrow -\hat{\gamma}_1^* & = rv^*(t) - \pi - n - g^* - \bar{c}(t) - \rho + n + (1 - \sigma)g^* \\ & + \frac{\dot{\gamma}_2^*}{\gamma_1^*} (B^*(1 - u^*(t))h^*(t))^{\alpha^*} (1 - v^*(t))^{1-\alpha^*} (1 - \alpha^*)k^{*\alpha^*}(t) \end{aligned} \quad (67)$$

Using and rearranging Equation (59) it results the third first order condition

$$\gamma_1^* r k^*(t) + \gamma_2^* B^*((1 - u^*(t))h^*(t))^{\alpha^*} k^{*1-\alpha^*}(t) (1 - \alpha^*) (-v^*(t))^{*\alpha^*}(t) = 0 \quad (68)$$

$$\Leftrightarrow \gamma_1^* r = \gamma_2^* B^*(1 - u^*(t))^{\alpha^*} \left( \frac{h^*(t)}{k^*(t)} \right)^{\alpha^*} (1 - \alpha^*) v^{*\alpha^*}(t) \quad (69)$$

and is the equivalent to

$$\frac{\gamma_2}{\gamma_1} = \frac{r}{B^* \left( \frac{1-u^*(t)}{v^*(t)} \right)^{\alpha^*} \left( \frac{h^*(t)}{k^*(t)} \right)^{\alpha^*} (1 - \alpha^*)} \quad (70)$$

Inserting Equation (70) in Equation (67) it yields the next first order condition

$$-\hat{\gamma}_1 = rv^*(t) - \pi - \bar{c}(t) - \rho - \sigma g^* + \frac{rB^*(1 - u^*(t))^{\alpha^*} h^+(t) \alpha^* (1 - v^*(t))^{1-\alpha^*} (1 - \alpha^*)}{B^* \left( \frac{1-u^*(t)}{v^*(t)} \right)^{\alpha^*} \left( \frac{h^*(t)}{k^*(t)} \right)^{\alpha^*} (1 - \alpha^*) k^{\alpha^*}(t)} \quad (71)$$

$$\Leftrightarrow \hat{\gamma}_1^* = -rv^*(t) - r(1 - v^*(t))^{1-\alpha^*} v^* \alpha^*(t) + \pi + \bar{c}(t) + \rho + \sigma g^* \quad (72)$$

The calculation of Equation (60) is

$$\begin{aligned} & \gamma_1^* w u^*(t) \\ & + \gamma_2^* (B^* (1 - u^*(t))^{\alpha^*} \alpha^* h^{*\alpha^*-1}(t)) ((1 - v^*(t)) k^*(t))^{1-\alpha^*} - \theta - \bar{c}(t) - g \\ & = -\dot{\gamma}_2^* - \gamma_2^* (-\rho - n + (1 - \sigma) g^*) \end{aligned} \quad (73)$$

$$\Leftrightarrow \frac{\gamma_1^*}{\gamma_2^*} w u^*(t) + B^* (1 - u^*(t))^{\alpha^*} \alpha^* h^{*\alpha^*-1}(t) (1 - v^*(t))^{1-\alpha^*} k^{1-\alpha^*}(t) - \theta - \bar{c}(t) - g \quad (74)$$

The last first order conditions results from Equation(61)

$$\gamma_1^* w h^*(t) + \gamma_2^* B^* h^*(t) \alpha^* \alpha (-u^*(t))^{\alpha-1} ((1 - v^*(t)) k^*(t))^{-\alpha^*} = 0 \quad (75)$$

$$\Leftrightarrow \gamma_1^* w = \gamma_2^* B^* \left( \frac{k^*(t)}{h^*(t)} \right)^{1-\alpha^*} \alpha^* \left( \frac{1 - v^*(t)}{u^*(t)} \right)^{1-\alpha^*} \quad (76)$$

To solve Equation (74) the following term is needed, rearranged from Equation (76)

$$\frac{\gamma_1^*}{\gamma_2^*} = \frac{B^*}{w} \alpha^* \left( \frac{k^*(t)}{h^*(t)} \right)^{1-\alpha^*} \left( \frac{1 - v^*(t)}{u^*(t)} \right)^{1-\alpha^*} \quad (77)$$

$$\Leftrightarrow \hat{\gamma}_2^* = -B^* \alpha^* \left( \frac{k^*(t)}{h^*(t)} \right)^{1-\alpha^*} (1 - v^*(t))^{1-\alpha^*} + \theta + \bar{c}(t) + \rho - n + \sigma g \quad (78)$$

The Keynes Ramsey Rule results from the combination of Equation (58), Equation (64) and Equation(65)

$$\begin{aligned} & \frac{C^{*\sigma}}{p} (rv^*(t) - \pi - n - g - \bar{c}(t)) \\ & + \frac{r C^{*\sigma} B^* (1 - u^*(t))^{\alpha^*} h^*(t) \alpha^* (1 - v^*(t))^{1-\alpha^*} (1 - \alpha^*) k^{*\alpha^*}(t)}{p B^* \left( \frac{1 - u^*(t)}{v^*(t)} \right)^{\alpha^*} \left( \frac{h^*(t)}{k^*(t)} \right)^{\alpha^*} (1 - \alpha^*)} \end{aligned} \quad (79)$$

$$= \sigma \frac{C^{-\sigma}(t) \hat{C}(t)}{p} - \frac{C^{*\sigma}(t)}{p} (-\rho + n + (1 - \sigma) g^*)$$

$$\Leftrightarrow \hat{C}(t) = \frac{1}{\sigma} (rv^*(t) + r(1 - v^*(t))^{1-\alpha^*} v^* \alpha^*(t) - \pi - \sigma g^* - \bar{c}(t) - \rho) \quad (80)$$

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